Completely Positive Maps

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Abstract

These notes are based on my knowledge of completely positive maps that I have gained throughout my studies. Most of the results contained in these notes may be found in Vern Paulsen's *Completely Bounded Maps and Operators Algebras* which is a very complete resource on these topics. However, my notes will focus on those ideas that are essential to be able to study C*-algebras and will take a slightly different point of view towards the subject matter.

These notes will assume that the reader has a basic knowledge of C*-algebras including knowledge of the Continuous Functional Calculus for Normal Operators, a basic knowledge about positive operators, and knowledge of C*-bounded approximate identities for C*-algebras. All C*-algebras will be non-unital unless otherwise specified and all inner products will be linear in the first variable.

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1 Introduction

Completely positive maps are an important collection of morphisms between C*-algebras. These maps have many of the same properties as *-homomorphism even though they are generally not multiplicative. Various interesting properties of C*-algebra can be developed by considering how completely positive maps behave on these C*-algebras. To begin these notes, we will review how positive linear functionals on C*-algebra behave.

Definition 1.1. Let $\mathfrak{A}$ be a C*-algebra. A linear functional $\varphi : \mathfrak{A} \to \mathbb{C}$ is said to be positive if $\varphi(A) \geq 0$ whenever $A \in \mathfrak{A}$ and $A \geq 0$. A linear functional $\varphi : \mathfrak{A} \to \mathbb{C}$ is said to be a state if φ is positive and $\|\varphi\| = 1$.

States are the building blocks of the proof of the GNS theorem; the fact that for every C*-algebra $\mathfrak{A}$ there exists a Hilbert space $\mathcal{H}$ such that $\mathfrak{A}$ may be viewed as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Before we give some examples of positive linear functionals, we first note the following simple yet important observation.

Lemma 1.2. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive linear functional. Then $\varphi(A) \in \mathbb{R}$ whenever A is a self-adjoint element of $\mathfrak{A}$. Moreover $\varphi(A^*) = \overline{\varphi(A)}$ for all $A \in \mathfrak{A}$.*

Proof. Suppose $A \in \mathfrak{A}$ be self-adjoint. By the Continuous Functional Calculus for Normal Operators, $A = A_+ - A_-$ where A_+ and A_- are positive operators. Since φ is a positive linear functional, $\varphi(A_+)$ and $\varphi(A_-)$ are positive scalars and thus $\varphi(A) = \varphi(A_+) - \varphi(A_-) \in \mathbb{R}$.

Next suppose that $A \in \mathfrak{A}$ is an arbitrary operator. Then $\varphi(Re(A)), \varphi(Im(A)) \in \mathbb{R}$ so

$$\begin{aligned}\varphi(A^*) = \varphi((Re(A) + iIm(A))^*) &= \varphi(Re(A) - iIm(A)) \\ &= \varphi(Re(A)) - i\varphi(Im(A)) = \overline{\varphi(Re(A)) + i\varphi(Im(A))} = \overline{\varphi(A)}\end{aligned}$$

as desired. ☐

Examples of positive linear functionals are abundant in mathematics.

Example 1.3. Let X be a compact Hausdorff space and let μ be a probability measure on X. If $\varphi : C(X) \to \mathbb{C}$ is defined by

$$\varphi(f) = \int_X f(x) d\mu(x)$$

for each $f \in C(X)$, then φ is a state on $C(X)$. To see this we first notice that φ is clearly linear by integration theory. Next if $f \in C(X)$ is positive then $f = g^*g$ for some $g \in C(X)$. Whence for all $x \in X$ $f(x) = g^*(x)g(x) = \overline{g(x)}g(x) \geq 0$. Therefore $\varphi(f)$ is the integral of a continuous function that is positive everywhere and thus $\varphi(f) \geq 0$. Moreover

$$|\varphi(f)| \leq \int_X |f(x)| d\mu(x) \leq \int_X \|f\|_\infty \, d\mu(x) \leq \|f\|_\infty \, \mu(X) = \|f\|_\infty$$

for all $f \in C(X)$. Thus $\|\varphi\| \leq 1$. Since $\varphi(I_{C(X)}) = \mu(X) = 1$, $\|\varphi\| = 1$. Whence φ is a state on $C(X)$. In particular the map $f \mapsto f(x)$ is defines a state on $C(X)$ for all $x \in X$.

Example 1.4. Let $\mathfrak{A}$ be a C*-algebra and let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a *-homomorphism. For each $\xi \in \mathcal{H}$ define $\varphi_\xi : \mathfrak{A} \to \mathbb{C}$ by $\varphi_\xi(A) = \langle \pi(A)\xi, \xi\rangle$ for all $A \in \mathfrak{A}$. It is clear that φ_ξ is well-defined and linear. In addition if $A \in \mathfrak{A}$ is positive then $A = B^*B$ for some $B \in \mathfrak{A}$ and thus

$$\varphi_\xi(A) = \langle \pi(B^*B)\xi, \xi\rangle = \langle \pi(B)\xi, \pi(B)\xi\rangle = \|\pi(B)\xi\| \geq 0.$$

Whence each φ_ξ is a positive linear functional. Note that for all $A \in \mathfrak{A}$

$$|\varphi_\xi(A)| \leq \|\pi(A)\xi\| \, \|\xi\| \leq \|A\| \, \|\xi\|^2$$

so $\|\varphi_\xi\| \leq \|\xi\|^2$. Moreover, if $\mathfrak{A}$ is unital and $\pi(I_\mathfrak{A}) = I_\mathcal{H}$ then $\|\varphi_\xi\| = \|\xi\|^2$ as $\varphi_\xi(I_\mathfrak{A}) = \|\xi\|^2$. If φ_ξ is a state, we call φ_ξ a vector state on $\mathfrak{A}$.

In fact it will be shown later that every state on a C*-algebra is a vector state on $\mathfrak{A}$.

Example 1.5. Let $\mathfrak{A} = \mathcal{M}_n(\mathbb{C})$ and define $tr : \mathfrak{A} \to \mathbb{C}$ by $tr([a_{i,j}]) = \frac{1}{n}\sum_{j=1}^n a_{j,j}$ for all $[a_{i,j}] \in \mathcal{M}_n(\mathbb{C})$. It is clear that tr is a linear functional. To see that tr is positive, we notice that for all $A = [a_{i,j}] \in \mathfrak{A}$

$$\begin{aligned} tr(A^*A) &= tr\left(\left[\sum_{k=1}^n \overline{a_{k,i}}a_{k,j}\right]\right) \\ &= \frac{1}{n}\sum_{j=1}^n\sum_{k=1}^n \overline{a_{k,j}}a_{k,j} \\ &= \frac{1}{n}\sum_{j=1}^n\sum_{k=1}^n |a_{k,j}|^2 \geq 0. \end{aligned}$$

Whence tr is a positive linear functional.

We claim that tr is a state on $\mathfrak{A}$. To see this, we notice that $tr(I_n) = 1$ so $\|tr\| \geq 1$. To prove the other inequality, let $A = [a_{i,j}] \in \mathfrak{A}$ be arbitrary and let $E_{i,j}$ be the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$. Since $E_{i,i}$ is a projection, $\|E_{i,i}\| = 1$. Then

$$|a_{i,i}| = \|a_{i,i}E_{i,i}\| = \|E_{i,i}AE_{i,i}\| \leq \|E_{i,i}\|\,\|A\|\,\|E_{i,i}\| = \|A\|\,.$$

Thus

$$|tr(A)| \leq \frac{1}{n}\sum_{j=1}^n |a_{j,j}| \leq \frac{1}{n}\sum_{j=1}^n \|A\| = \|A\|\,.$$

Hence $\|tr\| = 1$ and tr is a state on $\mathfrak{A}$.

Notice that all of the positive linear functionals given in the above examples were continuous even though continuity was not required in Definition 1.1. It turns out that if a linear functional is positive then it is automatically continuous. To prove this, we begin with a lemma about convergence of positive elements in a C*-algebra that the reader may not be familiar with.

Lemma 1.6. *Let $\mathfrak{A}$ be a C*-algebra. Suppose $(A_n)_{n\geq 1} \in \mathfrak{A}$ is a sequence such that $\lim_{n\to\infty} A_n = A \in \mathfrak{A}$ and $A_n \geq 0$ for all $n \in \mathbb{N}$. Then A is positive.*

Proof. By considering the unitization $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$, we may assume that $\mathfrak{A}$ is unital. By the continuity of the adjoint $A^* = \lim_{n\to\infty} A_n^* = \lim_{n\to\infty} A_n = A$ as each A_n is self-adjoint. Let $c := \sup_{n\geq 1}\|A_n\| < \infty$. Thus $\|A\| \leq c$. Since $0 \leq A_n \leq cI_{\mathfrak{A}}$ for all n, $0 \leq 2A_n \leq 2cI_{\mathfrak{A}}$ for all n and thus $-cI_{\mathfrak{A}} \leq 2A_n - cI_{\mathfrak{A}} \leq cI_{\mathfrak{A}}$ for all n. Thus (by the Continuous Functional Calculus) $\|2A_n - cI_{\mathfrak{A}}\| \leq c$ for all $n \in \mathbb{N}$. As $\lim_{n\to\infty} A_n = A$, $\lim_{n\to\infty} 2A_n - cI_{\mathfrak{A}} = 2A - cI_{\mathfrak{A}}$ so $\|2A - cI_{\mathfrak{A}}\| \leq c$. Whence $-cI_{\mathfrak{A}} \leq 2A - cI_{\mathfrak{A}} \leq cI_{\mathfrak{A}}$ and thus $0 \leq A \leq cI_{\mathfrak{A}}$ as desired. □

Proposition 1.7. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive linear functional. Then φ is continuous.*

Proof. First we claim that $\{\varphi(A) \mid A \in \mathfrak{A}, A \geq 0, \|A\| \leq 1\}$ is a bounded subset of $\mathbb{R}_{\geq 0}$ (that is, we claim φ is bounded on the positive elements of norm at most one). To see this, suppose otherwise. Then for each $n \in \mathbb{N}$ there would exists an $A_n \in \mathfrak{A}$ such that $A_n \geq 0$, $\|A_n\| \leq 1$, and $\varphi(A_n) \geq n^2$. Consider $B := \sum_{n\geq 1}^\infty \frac{1}{n^2}A_n$. Notice that B is a well-defined operator in $\mathfrak{A}$ as $(\frac{1}{n^2}A_n)_{n\geq 1}$ is an absolutely summable sequence since $\left\|\frac{1}{n^2}A_n\right\| \leq \frac{1}{n^2}$.

For each $m \in \mathbb{N}$

$$B - \sum_{n=1}^m \frac{1}{n^2}A_n = \sum_{n\geq m+1}^\infty \frac{1}{n^2}A_n \geq 0$$

by Lemma 1.6. Whence

$$\varphi(B) - \sum_{n=1}^{m} \varphi\left(\frac{1}{n^2}A_n\right) = \varphi\left(B - \sum_{n=1}^{m} \frac{1}{n^2}A_n\right) \geq 0$$

so

$$\varphi(B) \geq \sum_{n=1}^{m} \frac{1}{n^2}\varphi(A_n) \geq \sum_{n=1}^{m} 1 = m$$

for every $m \in \mathbb{N}$. As this is an impossibility, we must have that $\{\varphi(A) \mid A \in \mathfrak{A}, A \geq 0, \|A\| \leq 1\}$ is bounded in $\mathbb{R}_{\geq 0}$.

Let

$$M := \sup\{\varphi(A) \mid A \in \mathfrak{A}, A \geq 0, \|A\| \leq 1\}$$

and let $A \in \mathfrak{A}$ be arbitrary. Write $A = Re(A) + iIm(A)$ and recall $\|Re(A)\|, \|Im(A)\| \leq \|A\|$. By the Continuous Functional Calculus, both $Re(A)$ and $Im(A)$ can be written as the difference of two positive elements each with norm at most $\|Re(A)\|$ and $\|Im(A)\|$ respectively. Whence $A = P_1 - P_2 + iP_3 - iP_4$ where $P_j \in \mathfrak{A}$ are positive elements with $\|P_j\| \leq \|A\|$. Thus

$$|\varphi(A)| \leq \sum_{j=1}^{4} |\varphi(P_j)| \leq \sum_{j=1}^{4} M \|P_j\| \leq 4M \|A\|.$$

Thus φ is bounded with $\|\varphi\| \leq 4M$. □

The bounded of $4M$ obtained in the above proposition is never tight. Indeed looking at Examples 1.3, 1.4, and 1.5, we see that $\|\varphi\| = \varphi(I_{\mathfrak{A}})$ in all of these examples. Below we will show that this is not a coincidence. Although Proposition 1.11 will encapsulate Corollary 1.9, the proof of Corollary 1.9 is simple, interesting, and will be used later in a more general setting. We begin with the following proposition for unital C*-algebras.

Proposition 1.8. *Let $\mathfrak{A}$ be a unital C*-algebra. Suppose $\varphi : \mathfrak{A} \to \mathbb{C}$ is a continuous linear functional such that $\|\varphi\| = \varphi(I_{\mathfrak{A}}) = 1$ and suppose $N \in \mathfrak{A}$ is a normal operator. Then $\varphi(N) \in \overline{conv(\sigma(N))}$ where $\sigma(N)$ is the spectrum of N and $\overline{conv(\sigma(N))}$ is the closure of the convex hull of $\sigma(N)$.*

Proof. Suppose that $N \in \mathfrak{A}$ is a normal operator such that $\varphi(N) \notin \overline{conv(\sigma(N))}$. Basic geometry (via the separating version of the Hahn Banach Theorem) shows that there exists a $z \in \mathbb{C}$ and an $r > 0$ such that

$$\overline{conv(\sigma(N))} \subseteq B_r[z] := \{z' \in \mathbb{C} \mid |z - z'| \leq r\}$$

and $|\varphi(N) - z| > r$ (that is $\overline{conv(\sigma(N))}$ is contained in the closed ball of radius r centred at z yet $\varphi(N)$ is not in this ball).

Consider $T := N - zI_{\mathfrak{A}} \in \mathfrak{A}$. Thus T is a normal element such that $\sigma(T) = \sigma(N) - z \subseteq B_r[z] - z = B_r[0]$ where $B_r[0]$ is the closed ball of radius r around the origin. Since T is normal $\|T\| = spr(T) \leq r$. However

$$|\varphi(T)| = |\varphi(N) - z\varphi(I_{\mathfrak{A}})| = |\varphi(N) - z| > r$$

which contradicts the fact that $\|\varphi\| = 1$. □

Corollary 1.9. *Let $\mathfrak{A}$ be a unital C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a linear functional. Then φ is positive if and only if $\|\varphi\| = \varphi(I_{\mathfrak{A}})$.*

Proof. First suppose that φ is positive. We need to show that $|\varphi(A)| \leq \varphi(I_{\mathfrak{A}}) \|A\|$ for all $A \in \mathfrak{A}$. First we will prove the result for self-adjoint elements of $\mathfrak{A}$ and then for a general element of $\mathfrak{A}$.

Let $A \in \mathfrak{A}$ be self-adjoint. Then $\varphi(A) \in \mathbb{R}$ by Lemma 1.2. Since $\|A\| I_{\mathfrak{A}} - A \geq 0$, $\|A\| \varphi(I_{\mathfrak{A}}) - \varphi(A) \geq 0$. Similarly as $\|A\| I_{\mathfrak{A}} + A \geq 0$, $\|A\| I_{\mathfrak{A}} + \varphi(A) \geq 0$. Combining these two inequalities, we obtain that

$$-\|A\| \varphi(I_{\mathfrak{A}}) \leq \varphi(A) \leq \|A\| \varphi(I_{\mathfrak{A}})$$

so $|\varphi(A)| \leq \varphi(I_{\mathfrak{A}}) \|A\|$ as required.

Let $A \in \mathfrak{A}$ be an arbitrary element. Then there exists a $\theta \in [0, 2\pi)$ such that $|\varphi(A)| = e^{i\theta}\varphi(A) = \varphi(e^{i\theta}A)$. Let $B := e^{i\theta}A$. Then

$$0 \leq |\varphi(A)| = \varphi(B) = \varphi(Re(B)) + i\varphi(Im(B)).$$

However, by Lemma 1.2, $\varphi(Re(B))$ and $\varphi(Im(B))$ are real numbers. Therefore $0 \leq \varphi(Re(B)) + i\varphi(Im(B))$ is possible only if $\varphi(Im(B)) = 0$. Thus, since $Re(B)$ is self-adjoint, by applying the above result in the self-adjoint case we obtain that

$$|\varphi(A)| = \varphi(Re(B)) \leq \|Re(B)\| \varphi(I_{\mathfrak{A}}) \leq \|B\| \varphi(I_{\mathfrak{A}}) = \|A\| \varphi(I_{\mathfrak{A}})$$

as desired.

Next suppose that $\|\varphi\| = \varphi(I_{\mathfrak{A}})$. If $\varphi = 0$, we are done so suppose $\|\varphi\| > 0$ and define $\psi : \mathfrak{A} \to \mathbb{C}$ by $\psi(A) = \frac{1}{\|\varphi\|}\varphi(A)$. Then ψ is a linear functional such that $\psi(I_{\mathfrak{A}}) = \|\psi\| = 1$. By Proposition 1.8 $\psi(N) \in \overline{conv(\sigma(N))}$ whenever N is a normal element of $\mathfrak{A}$. In particular, if $A \in \mathfrak{A}$ is a positive element, $\psi(A) \in \overline{conv(\sigma(A))} \subseteq [0, \infty)$. Whence $\varphi(A) = \|\varphi\| \psi(A) \geq 0$. Hence φ is a positive linear functional. $\square$

Remarks 1.10. To prove the analogue of Corollary 1.9 for positive linear functionals on a non-unital C*-algebra, we notice that if $\varphi : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional and we define $[\cdot, \cdot] : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ by

$$[A, B] = \varphi(B^*A)$$

for all $A, B \in \mathfrak{A}$, then $[\cdot, \cdot]$ is a positive sesquilinear form on $\mathfrak{A}$. Indeed $[\cdot, \cdot]$ is clearly linear in the first component, conjugate linear in the second component, and $[A, A] = \varphi(A^*A) \geq 0$ as φ is a positive linear functional. It is a simple proof to show that a positive sesquilinear form on a vector space satisfies the Cauchy-Schwarz inequality (this will be left as an exercise) so that

$$|\varphi(B^*A)| = |\,[A, B]\,| \leq [A, A]^{\frac{1}{2}}\, [B, B]^{\frac{1}{2}} = \varphi(A^*A)^{\frac{1}{2}} \varphi(B^*B)^{\frac{1}{2}}$$

for all $A, B \in \mathfrak{A}$. This observation is essential in the following result along with the incredibly important GNS construction.

Proposition 1.11. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a linear functional. Then the following are equivalent:*

1. *φ is a positive linear functional.*
2. *$\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$ for all C*-bounded approximate identities $(E_\lambda)_\Lambda$ of $\mathfrak{A}$.*
3. *$\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$ where $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity of $\mathfrak{A}$.*

Proof. <u>(1) implies (2)</u> Suppose that φ is a positive linear functional and $(E_\lambda)_\Lambda$ is any C*-bounded approximate identity of $\mathfrak{A}$. If $\varphi = 0$ the result clearly holds. Therefore we will assume that $\|\varphi\| > 0$. Since $(E_\lambda)_\Lambda$ is an increase net of positive elements of $\mathfrak{A}$ and φ is positive, $(\varphi(E_\lambda))_\Lambda$ is an increasing net of positive real numbers and thus converges to $\sup_\Lambda \varphi(E_\lambda)$. Since $\|E_\lambda\| \leq 1$ for all λ, clearly $\|\varphi\| \geq \lim_\Lambda \varphi(E_\lambda)$.

Next suppose $A \in \mathfrak{A}$ is arbitrary. Since $0 \leq E_\lambda$ and $\|E_\lambda\| \leq 1$ for all $\lambda \in \Lambda$, $0 \leq E_\lambda^2 \leq E_\lambda$ by the Continuous Functional Calculus and thus $0 \leq \varphi(E_\lambda^2) \leq \varphi(E_\lambda)$ for all $\lambda \in \Lambda$. Since φ is continuous by Proposition 1.7 and $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity for $\mathfrak{A}$,

$$\begin{aligned}
|\varphi(A)| &= \lim_\Lambda |\varphi(AE_\lambda)| \\
&\leq \lim_\Lambda |\varphi(AA^*)|^{\frac{1}{2}} |\varphi(E_\lambda^* E_\lambda)|^{\frac{1}{2}} && \text{by the Cauchy-Schwarz inequality from Remarks 1.10} \\
&\leq \lim_\Lambda \|\varphi\|^{\frac{1}{2}} \|AA^*\|^{\frac{1}{2}} \varphi(E_\lambda^2)^{\frac{1}{2}} \\
&\leq \|\varphi\|^{\frac{1}{2}} \|A\| \lim_\Lambda \varphi(E_\lambda)^{\frac{1}{2}}.
\end{aligned}$$

As the above inequality holds for all $A \in \mathfrak{A}$, we obtain that

$$\|\varphi\| \leq \|\varphi\|^{\frac{1}{2}} \left(\lim_{\Lambda} \varphi(E_\lambda)\right)^{\frac{1}{2}}$$

and thus, as $\|\varphi\| \neq 0$, $\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$.

(2) implies (3) Clear.

(3) implies (1) Suppose that $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity such that $\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$. First we will show that if $A \in \mathfrak{A}$ is self-adjoint then $\varphi(A) \in \mathbb{R}$ (which is necessary if φ is to be positive by Lemma 1.2). Thus suppose $A \in \mathfrak{A}$ is self-adjoint and write $\varphi(A) = a + bi$ where $a, b \in \mathbb{R}$. We desire to show that $b = 0$. The following may seem extremely obscure, but is a common trick. By exchanging A with $-A$ we can assume that $b \geq 0$. For each $\lambda \in \Lambda$ and $n \in \mathbb{N}$ consider $A_{\lambda,n} := A + inE_\lambda$. Then

$$\begin{aligned}
\liminf_{\Lambda} \|A_{\lambda,n}\|^2 &= \liminf_{\Lambda} \|(A - inE_\lambda)(A + inE_\lambda)\| \\
&\leq \liminf_{\Lambda} \|A\|^2 + n\|AE_\lambda - E_\lambda A\| + n^2 \\
&= \|A\|^2 + n^2.
\end{aligned}$$

So

$$\begin{aligned}
a^2 + (b + n\|\varphi\|)^2 &= |a + (b + n\|\varphi\|)i|^2 \\
&= \lim_{\Lambda} |\varphi(A) + in\varphi(E_\lambda)|^2 \\
&= \lim_{\Lambda} |\varphi(A_{\lambda,n})|^2 \\
&\leq \|\varphi\|^2 \liminf_{\Lambda} \|A_{\lambda,n}\|^2 \\
&\leq \|\varphi\|^2 \|A\|^2 + \|\varphi\|^2 n^2.
\end{aligned}$$

Thus

$$a^2 + b^2 + 2bn\|\varphi\| + \|\varphi\|^2 n^2 \leq \|\varphi\|^2 \|A\|^2 + \|\varphi\|^2 n^2$$

for all $n \in \mathbb{N}$ so

$$a^2 + b^2 + 2bn\|\varphi\| \leq \|\varphi\|^2 \|A\|^2$$

for all $n \in \mathbb{N}$. As $b \geq 0$, this is possible if and only if $b = 0$. Therefore φ maps self-adjoint elements to self-adjoint elements.

To complete the proof that φ is positive, it suffices to show that $\varphi(A) \geq 0$ for all $A \in \mathfrak{A}$ such that $A \geq 0$ and $\|A\| \leq 1$ (as we can scale all other positive elements). Thus suppose $A \in \mathfrak{A}$ is such that $A \geq 0$ and $\|A\| \leq 1$. Let $B_\lambda := A - E_\lambda$. Then

$$-I_{\mathfrak{A}} \leq -E_\lambda \leq A - E_\lambda \leq A \leq I_{\mathfrak{A}}$$

in the unitization of $\mathfrak{A}$. Whence $\|A - E_\lambda\| \leq 1$ so $|\varphi(A) - \varphi(E_\lambda)| \leq \|\varphi\|$ for all $\lambda \in \Lambda$. Since $\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$, by taking a limit we obtain that $|\varphi(A) - \|\varphi\|\,| \leq \|\varphi\|$. However $\varphi(A) \in \mathbb{R}$ so this inequality implies that $0 \leq \varphi(A) \leq \|\varphi\|$ as desired. $\square$

Before we move onto proving the GNS construction, we make an important and necessary detour into the land of extending positive linear functionals. The importance of the Hahn-Banach Theorem is the ability to extend a linear functional f defined on a subspace $\mathfrak{Y}$ of a Banach space $\mathfrak{X}$ to a linear functional $\tilde{f} : \mathfrak{X} \to \mathbb{C}$ such that $\|f\| = \left\|\tilde{f}\right\|$. Thus it is natural to ask whether or not states can be extended to states on C*-algebras. The answer is yes as will be shown below. The proof of this result is made up of a few small extension results that exploit Corollary 1.9 and a few subtleties.

Lemma 1.12. *Let $\mathfrak{A}$ be a non-unital C*-algebra, let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$, and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive linear functional. Then φ has a unique positive extension $\tilde{\varphi} : \tilde{\mathfrak{A}} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. If $\tilde{\varphi} : \tilde{\mathfrak{A}} \to \mathbb{C}$ is a positive extension of φ such that $\|\varphi\| = \|\tilde{\varphi}\|$ then $\tilde{\varphi}(I_{\tilde{\mathfrak{A}}}) = \|\tilde{\varphi}\| = \|\varphi\|$ by Corollary 1.9 (or Proposition 1.11). Thus the only possible way to define $\tilde{\varphi}$ is by $\tilde{\varphi}(\lambda I_{\tilde{\mathfrak{A}}} + A) = \lambda \|\varphi\| + \varphi(A)$. Hence if there is a norm-preserving positive extension of φ it must be unique.

Now define $\tilde{\varphi} : \tilde{\mathfrak{A}} \to \mathbb{C}$ by $\tilde{\varphi}(\lambda I_{\tilde{\mathfrak{A}}} + A) = \lambda \|\varphi\| + \varphi(A)$ for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. Since every element of $\tilde{\mathfrak{A}}$ has a unique decomposition as a sum of an element of $\mathbb{C}I_{\tilde{\mathfrak{A}}}$ and an element of $\mathfrak{A}$, $\tilde{\varphi}$ is a well-defined linear functional. To show that $\tilde{\varphi}$ is positive it suffices by Corollary 1.9 (or Proposition 1.11) to show that $\|\tilde{\varphi}\| = \tilde{\varphi}(I_{\tilde{\mathfrak{A}}})$ (that is; $\tilde{\varphi}$ will be positive if and only if $\|\varphi\| = \|\tilde{\varphi}\|$ so it suffices to show that $\tilde{\varphi}$ is a norm-preserving extension to complete the proof). Let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity for $\mathfrak{A}$. Since φ is positive, Proposition 1.11 implies that $\|\varphi\| = \lim_\Lambda \varphi(E_\lambda)$. Since φ is positive and thus continuous, we have for all $\alpha I_{\tilde{\mathfrak{A}}} + A \in \tilde{\mathfrak{A}}$ that

$$\begin{aligned}
|\tilde{\varphi}(\alpha I_{\tilde{\mathfrak{A}}} + A)| &= |\alpha \|\varphi\| + \varphi(A)| \\
&= \lim_\Lambda |\alpha\varphi(E_\lambda) + \varphi(AE_\lambda)| \\
&= \lim_\Lambda |\varphi(\alpha E_\lambda + AE_\lambda)| \\
&\leq \limsup_\Lambda \|\varphi\| \left\|(\alpha I_{\tilde{\mathfrak{A}}} + A)E_\lambda\right\| \\
&\leq \limsup_\Lambda \|\varphi\| \left\|(\alpha I_{\tilde{\mathfrak{A}}} + A)\right\| \|E_\lambda\| \\
&= \|\varphi\| \left\|(\alpha I_{\tilde{\mathfrak{A}}} + A)\right\|.
\end{aligned}$$

Whence $\|\tilde{\varphi}\| \leq \|\varphi\|$. Since clearly $\|\tilde{\varphi}\| \geq \|\varphi\|$, the result follows. □

Lemma 1.13. *Let $\mathfrak{A}$ be a unital C*-algebra, let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive linear functional, and view $\mathfrak{A}$ as a non-unital C*-subalgebra of $\mathfrak{A} \oplus \mathbb{C}$ canonically. Then φ has a unique positive extension $\tilde{\varphi} : \mathfrak{A} \oplus \mathbb{C} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. If $\tilde{\varphi} : \mathfrak{A} \oplus \mathbb{C} \to \mathbb{C}$ is a positive extension of φ such that $\|\varphi\| = \|\tilde{\varphi}\|$ then

$$\tilde{\varphi}(I_{\mathfrak{A}} \oplus 1) = \|\tilde{\varphi}\| = \|\varphi\| = \varphi(I_{\mathfrak{A}}) = \tilde{\varphi}(I_{\mathfrak{A}} \oplus 0)$$

by Corollary 1.9 (or Proposition 1.11). Hence $\tilde{\varphi}(0 \oplus 1) = 0$. Thus the only possible way to define $\tilde{\varphi} : \mathfrak{A} \oplus \mathbb{C} \to \mathbb{C}$ so that $\tilde{\varphi}$ is positive is by $\tilde{\varphi}(A \oplus \lambda) = \varphi(A)$. Hence if there is a norm-preserving positive extension of φ it must be unique.

Now define $\tilde{\varphi} : \mathfrak{A} \oplus \mathbb{C} \to \mathbb{C}$ by $\tilde{\varphi}(A \oplus \lambda) = \varphi(A)$ for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. As $\|A \oplus \lambda\| = \max\{\|A\|, |\lambda|\}$, and $A \oplus \lambda \geq 0$ if and only if $A \geq 0$ and $\lambda \geq 0$, it is trivial to verify that $\tilde{\varphi}$ is a norm-preserving positive extension of φ. □

Lemma 1.14. *Let $\mathfrak{A} \subseteq \mathfrak{B}$ be C*-algebras with the same unit. If $\varphi : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional then φ has a positive extension $\tilde{\varphi} : \mathfrak{B} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Let $\mathfrak{A} \subseteq \mathfrak{B}$ be C*-algebras with the same unit and suppose $\varphi : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional. By the Hahn-Banach Theorem there exists a linear functional $\tilde{\varphi} : \mathfrak{B} \to \mathbb{C}$ such that $\tilde{\varphi}|_{\mathfrak{A}} = \varphi$ and $\|\varphi\| = \|\tilde{\varphi}\|$. Thus, by Corollary 1.9 (or Proposition 1.11),

$$\|\tilde{\varphi}\| = \|\varphi\| = \varphi(I_{\mathfrak{A}}) = \varphi(I_{\mathfrak{B}}) = \tilde{\varphi}(I_{\mathfrak{B}})$$

Whence $\tilde{\varphi}$ is positive by Corollary 1.9 (or Proposition 1.11). □

Theorem 1.15. *Let $\mathfrak{A} \subseteq \mathfrak{B}$ be C*-algebras and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a positive linear functional. Then there exists a positive linear functional $\psi : \mathfrak{B} \to \mathbb{C}$ such that $\psi|_{\mathfrak{A}} = \varphi$ and $\|\psi\| = \|\varphi\|$.*

Proof. We shall break the proof into two cases that will follow easily from the above lemmas.

Case 1: $\mathfrak{A}$ is not unital Let $\tilde{\mathfrak{B}}$ be the unitization of $\mathfrak{B}$ if $\mathfrak{B}$ is not unital and let $\tilde{\mathfrak{B}}$ be $\mathfrak{B}$ if $\mathfrak{B}$ is unital. Consider the *-algebra

$$\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A} := \{\lambda I_{\tilde{\mathfrak{B}}} + A \mid A \in \mathfrak{A}, \lambda \in \mathbb{C}\} \subseteq \tilde{\mathfrak{B}}.$$

Let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$ and define $\pi : \tilde{\mathfrak{A}} \to \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A}$ by $\pi(\lambda I_{\tilde{\mathfrak{A}}} + A) = \lambda I_{\tilde{\mathfrak{B}}} + A$ for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. It is easy to verify that π is a *-homomorphism. As the domain of π is a C*-algebra and the range of π sits inside the C*-algebra $\tilde{\mathfrak{B}}$, the range of π is a C*-algebra. As $\pi(\tilde{\mathfrak{A}}) = \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A}$, $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A}$ is a C*-subalgebra of $\tilde{\mathfrak{B}}$ with the same unit. Moreover if $\pi(\lambda I_{\tilde{\mathfrak{A}}} + A) = 0$ then $\lambda I_{\tilde{\mathfrak{B}}} = -A \in \mathfrak{A}$. As $\mathfrak{A}$ is not unital, this implies that $\lambda = 0$ and then $A = 0$. Whence π must be injective and thus $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A}$ is *-isomorphic to $\tilde{\mathfrak{A}}$.

By Lemma 1.12, φ extends to a positive linear functional $\tilde{\varphi} : \mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$. Since $\mathbb{C}I_{\tilde{\mathfrak{B}}} + \mathfrak{A} \subseteq \tilde{\mathfrak{B}}$ are C*-algebras with the same unit, Lemma 1.14 implies that $\tilde{\varphi}$ extends to a positive linear functional $\tilde{\psi} : \tilde{\mathfrak{B}} \to \mathbb{C}$ such that $\left\|\tilde{\psi}\right\| = \|\tilde{\varphi}\|$. Let $\psi : \mathfrak{B} \to \mathbb{C}$ be defined by $\psi = \tilde{\psi}|_{\mathfrak{B}}$. Since the restriction of a positive linear functional is clearly positive, ψ is a positive linear functional. Moreover ψ extends φ by construction and $\|\psi\| \leq \left\|\tilde{\psi}\right\| = \|\varphi\| \leq \|\psi\|$ (where the last inequality comes from the fact that ψ extends φ).

<u>*Case 2: $\mathfrak{A}$ is unital*</u> Let $\tilde{\mathfrak{B}}$ be the unitization of $\mathfrak{B}$ if $\mathfrak{B}$ is not unital and let $\tilde{\mathfrak{B}}$ be $\mathfrak{B}$ if $\mathfrak{B}$ is unital. If $\mathfrak{A}$ and $\tilde{\mathfrak{B}}$ have the same unit, the result follows from Lemma 1.14. Else suppose that $I_{\mathfrak{A}} \neq I_{\tilde{\mathfrak{B}}}$. Whence we have that $I_{\mathfrak{A}} \leq I_{\tilde{\mathfrak{B}}}$ (as $\mathfrak{A} \subseteq \tilde{\mathfrak{B}}$) so $P := I_{\tilde{\mathfrak{B}}} - I_{\mathfrak{A}}$ is a non-zero projection in $\tilde{\mathfrak{B}}$. Define $\pi : \mathfrak{A} \oplus \mathbb{C} \to \tilde{\mathfrak{B}}$ by $\pi(A \oplus \lambda) = A + \lambda P$ for all $A \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$. We claim that π is an injective *-homomorphism. It is clear that π is well-defined, linear, and *-preserving. To see that π is multiplicative, we note that $PA = AP = 0$ for all $A \in \mathfrak{A}$ since $I_{\tilde{\mathfrak{B}}}A = AI_{\tilde{\mathfrak{B}}} = A = I_{\mathfrak{A}}A = AI_{\mathfrak{A}}$ for all $A \in \mathfrak{A}$. Therefore

$$\pi(A_1 \oplus \lambda_1)\pi(A_2 \oplus \lambda_2) = (A_1 + \lambda_1 P)(A_2 + \lambda_2 P) = A_1A_2 + \lambda_1\lambda_2 P = \pi((A_1 \oplus \lambda_1)(A_2 \oplus \lambda_2))$$

for all $A_1, A_2 \in \mathfrak{A}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Whence π is a *-homomorphism. To see that π is injective, suppose $\pi(A \oplus \lambda) = 0$. Then $A + \lambda P = 0$ so that $0 = P(A + \lambda P) = \lambda P$. Thus $\lambda = 0$ as $P \neq 0$. Whence $A = 0$ so $A \oplus \lambda = 0$. Thus π is an injective *-homomorphism so $\mathfrak{A} \oplus \mathbb{C}$ can be viewed as a C*-subalgebra of $\tilde{\mathfrak{B}}$. Since $I_{\mathfrak{A}} \oplus 1$ is a unit for $\mathfrak{A} \oplus \mathbb{C}$ and $\pi(I_{\mathfrak{A}} \oplus 1) = I_{\mathfrak{A}} + P = I_{\tilde{\mathfrak{B}}}$, $\mathfrak{A} \oplus \mathbb{C}$ can be viewed as a C*-algebra of $\tilde{\mathfrak{B}}$ with the same unit.

By Lemma 1.13, φ extends to a positive linear functional $\tilde{\varphi} : \mathfrak{A} \oplus \mathbb{C} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$. Since $\mathfrak{A} \oplus \mathbb{C} \subseteq \tilde{\mathfrak{B}}$ are C*-algebras with the same unit, Lemma 1.14 implies that $\tilde{\varphi}$ extends to a positive linear functional $\tilde{\psi} : \tilde{\mathfrak{B}} \to \mathbb{C}$ such that $\left\|\tilde{\psi}\right\| = \|\tilde{\varphi}\|$. Let $\psi : \mathfrak{B} \to \mathbb{C}$ be defined by $\psi = \tilde{\psi}|_{\mathfrak{B}}$. Since the restriction of a positive linear functional is clearly positive, ψ is a positive linear functional. Moreover ψ extends φ by construction and $\|\psi\| \leq \left\|\tilde{\psi}\right\| = \|\varphi\| \leq \|\psi\|$ (where the last inequality comes from the fact that ψ extends φ). □

As a corollary of the above proposition (or simply Lemma 1.12), we obtain a non-unital version of Proposition 1.8.

Proposition 1.16. *Let $\mathfrak{A}$ be a C*-algebra. Suppose $\varphi : \mathfrak{A} \to \mathbb{C}$ is a positive linear functional with $\|\varphi\| = 1$ and $N \in \mathfrak{A}$ is a normal operator. Then $\varphi(N) \in \overline{conv(\sigma(N))}$ where $\sigma(N)$ is the spectrum of N and $\overline{conv(\sigma(N))}$ is the closure of the convex hull of $\sigma(N)$.*

Proof. If $\mathfrak{A}$ is unital, the result was proven in Proposition 1.8. If $\mathfrak{A}$ is not unital, let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$. By Lemma 1.12, φ extends to a positive linear functional $\tilde{\varphi} : \tilde{\mathfrak{A}} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\| = 1$. By Corollary 1.9 (or Proposition 1.11), $\tilde{\varphi}(I_{\tilde{\mathfrak{A}}}) = \|\tilde{\varphi}\| = 1$. Thus if $N \in \mathfrak{A}$ is a normal operator then $\varphi(N) = \tilde{\varphi}(N) \in \overline{conv(\sigma(N))}$ by applying Proposition 1.8 to N and $\tilde{\varphi}$. □

Now that we have demonstrated that positive linear functionals can be extended to positive linear functionals, we will proceed with the GNS construction. To obtain the result that every C*-algebra may be represented on a Hilbert space (Theorem 1.20), it is possible to consider only unital C*-algebras in extending linear functionals and in the following theorem. However, it is instructive to prove the following theorem for non-unital C*-algebras.

Theorem 1.17 (Gelfand-Naimark-Segal Construction)**.** *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a non-zero positive linear functional. Then there exists a Hilbert space $\mathcal{H}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$,* *and a vector* $\xi \in \mathcal{H}$ *such that* $\overline{\pi(\mathfrak{A})\xi} = \mathcal{H}$ *(a vector that satisfies this property is said to be cyclic for* $\pi(\mathfrak{A})$*),* $\|\xi\|_{\mathcal{H}}^2 = \|\varphi\|$*, and* $\langle \pi(A)\xi, \xi\rangle = \varphi(A)$*. If* $\mathfrak{A}$ *is unital then* $\pi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$*.*

Proof. Since φ is a positive linear functional, the map $[\cdot,\cdot] : \mathfrak{A} \times \mathfrak{A} \to \mathbb{C}$ defined by $[A, B] = \varphi(B^*A)$ is a positive sesquilinear form on $\mathfrak{A}$ as seen in Remarks 1.10. We want to turn $\mathfrak{A}$ with this positive sesquilinear form into a Hilbert space. To do this, we must first make this positive sesquilinear form an inner product by removing the elements $A \in \mathfrak{A}$ such that $[A, A] = 0$. Let

$$\mathcal{N} := \{A \in \mathfrak{A} \mid [A, A] = 0\}.$$

We claim that $\mathcal{N}$ is a subspace of $\mathfrak{A}$. Indeed if $A \in \mathcal{N}$ and $B \in \mathfrak{A}$ then

$$|\,[A, B]\,| \leq [A, A]^{\frac{1}{2}}\, [B, B]^{\frac{1}{2}} = 0.$$

Whence

$$\mathcal{N} = \{A \in \mathfrak{A} \mid [A, B] = 0 \text{ for all } B \in \mathfrak{A}\}$$

which is clearly a subspace of $\mathfrak{A}$. Therefore $[\cdot,\cdot]$ defines an inner product $\langle \cdot,\cdot\rangle : (\mathfrak{A}/\mathcal{N}) \times (\mathfrak{A}/\mathcal{N}) \to \mathbb{C}$ by $\langle A + \mathcal{N}, B + \mathcal{N}\rangle = [A, B]$. Let $\mathcal{H}$ be the completion of $\mathfrak{A}/\mathcal{N}$ with respect to the norm induced by this inner product. Whence $\mathcal{H}$ is a Hilbert space.

Next we desire to construct the *-homomorphism π. This will be constructed by letting $\mathfrak{A}$ act on itself by multiplication on the left. Define $\pi'' : \mathfrak{A} \to \mathcal{L}(\mathfrak{A})$ by $\pi''(A)B = AB$. Clearly π'' is a well-defined homomorphism into $\mathcal{L}(\mathfrak{A})$ as each $\pi''(A)$ is clearly linear. We claim that π'' defines a homomorphism from $\mathfrak{A}$ to $\mathcal{L}(\mathfrak{A}/\mathcal{N})$. Indeed if $A, B \in \mathfrak{A}$ then

$$\left[\pi''(A)B, \pi''(A)B\right] = \varphi(B^*A^*AB) \leq \varphi(\|A^*A\|_{\mathfrak{A}}\, B^*B) \leq \|A\|_{\mathfrak{A}}^2\, [B, B]$$

as φ is a positive linear functional. Whence $\pi''(A)B \in \mathcal{N}$ whenever $B \in \mathcal{N}$. Therefore we may define $\pi' : \mathfrak{A} \to \mathcal{L}(\mathfrak{A}/\mathcal{N})$ by $\pi'(A)(B + \mathcal{N}) = AB + \mathcal{N}$. By the above inequality, we obtain that

$$\|\pi'(A)(B + \mathcal{N})\|_{\mathcal{H}} \leq \|A\|_{\mathfrak{A}}\, \|B + \mathcal{N}\|_{\mathcal{H}}$$

Therefore, for each $A \in \mathfrak{A}$ we can extend $\pi'(A)$ to an element $A_\pi \in \mathcal{B}(\mathcal{H})$ by continuity. Define $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by $\pi(A) = A_\pi$ for all $A \in \mathfrak{A}$. We claim that π is a *-homomorphism. To see that π is linear, we notice for all $A, B \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ that $\pi(\lambda A + B) = \lambda\pi(A) + \pi(B)$ on the dense subset $\mathfrak{A}/\mathcal{N}$ of $\mathcal{H}$. Whence π must be linear by continuity. Similarly if $A, B \in \mathfrak{A}$ then $\pi(AB) = \pi(A)\pi(B)$ on the dense subset $\mathfrak{A}/\mathcal{N}$ of $\mathcal{H}$. Whence π must be multiplicative by continuity. Lastly we notice for all $A, B, C \in \mathfrak{A}$ that

$$\begin{aligned}
\langle \pi(A)^*(B + \mathcal{N}), C + \mathcal{N}\rangle &= \langle B + \mathcal{N}, AC + \mathcal{N}\rangle \\
&= \varphi((AC)^*B) \\
&= \varphi(C^*(A^*B)) \\
&= \langle \pi(A^*)(B + \mathcal{N}), C + \mathcal{N}\rangle.
\end{aligned}$$

Since $\mathfrak{A}/\mathcal{N}$ is dense in $\mathcal{H}$, we obtain by continuity of the inner product that $\langle \pi(A)^*\eta, \zeta\rangle = \langle \pi(A^*)\eta, \zeta\rangle$ for all $\eta, \zeta \in \mathcal{H}$. Whence π is a *-homomorphism.

Lastly, we need to construct the vector ξ. In the case that $\mathfrak{A}$ is unital this task is not difficult. We will first present the case when $\mathfrak{A}$ is unital, and then the case when $\mathfrak{A}$ is not unital.

Case 1: $\mathfrak{A}$ is unital Let $\xi := I_{\mathfrak{A}} + \mathcal{N} \in \mathcal{H}$. Then

$$\|\xi\|_{\mathcal{H}}^2 = \langle I_{\mathfrak{A}} + \mathcal{N}, I_{\mathfrak{A}} + \mathcal{N}\rangle = \varphi(I_{\mathfrak{A}}^* I_{\mathfrak{A}}) = \varphi(I_{\mathfrak{A}}) = \|\varphi\|$$

by Corollary 1.9. Moreover $\overline{\pi(\mathfrak{A})\xi} = \overline{\mathfrak{A}/\mathcal{N}} = \mathcal{H}$ so ξ is a cyclic vector. Notice $\pi(I_{\mathfrak{A}})(A + \mathcal{N}) = A + \mathcal{N}$ for all $A \in \mathfrak{A}$. Whence $\pi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$. Lastly

$$\langle \pi(A)\xi, \xi\rangle = \langle A + \mathcal{N}, I_{\mathfrak{A}} + \mathcal{N}\rangle = \varphi(I_{\mathfrak{A}}^* A) = \varphi(A)$$

for all $A \in \mathfrak{A}$. Whence ξ has the desired properties.

Case 2: $\mathfrak{A}$ is not unital Let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity of $\mathfrak{A}$ and let $\xi_\lambda := E_\lambda + \mathcal{N}$. We claim that $(\xi_\lambda)_\Lambda$ is a Cauchy net in $\mathcal{H}$. To begin we notice if $\alpha, \beta \in \Lambda$ then

$$\|\xi_\alpha - \xi_\beta\|_{\mathcal{H}}^2 = \varphi((E_\alpha - E_\beta)^2) = \varphi(E_\alpha^2) + \varphi(E_\beta^2) - \varphi(E_\alpha E_\beta) - \varphi(E_\beta E_\alpha).$$

Note that $\varphi(E_\alpha^2)$ and $\varphi(E_\beta^2)$ are positive as φ is positive and thus $\varphi(E_\alpha E_\beta) + \varphi(E_\beta E_\alpha) \in \mathbb{R}$. Let $\epsilon > 0$. By Proposition 1.11 and the fact that $(\varphi(E_\lambda))_\Lambda$ is an increasing net, there exists an $\alpha_0 \in \Lambda$ such that $\|\varphi\| - \epsilon \leq \varphi(E_\lambda)$ for all $\lambda \geq \alpha_0$. Since $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity, there exists a β_0 such that if $\lambda \geq \beta_0$ then $\|E_\lambda E_{\alpha_0} - E_{\alpha_0}\| < \epsilon$ and $\|E_{\alpha_0} E_\lambda - E_{\alpha_0}\| < \epsilon$. Thus for all $\beta \geq \beta_0$

$$|2\varphi(E_{\alpha_0}) - \varphi(E_{\alpha_0} E_\beta) - \varphi(E_\beta E_{\alpha_0})| < 2\,\|\varphi\|\,\epsilon$$

and thus $-\varphi(E_{\alpha_0} E_\beta) - \varphi(E_\beta E_{\alpha_0}) < 2\,\|\varphi\|\,\epsilon - 2\varphi(E_{\alpha_0})$ as all terms under consideration are real. Therefore

$$\begin{aligned}
\|\xi_{\alpha_0} - \xi_\beta\|_{\mathcal{H}}^2 &= \varphi(E_{\alpha_0}^2) + \varphi(E_\beta^2) - \varphi(E_{\alpha_0} E_\beta) - \varphi(E_\beta E_{\alpha_0}) \\
&\leq 2\,\|\varphi\| + (2\,\|\varphi\|\,\epsilon - 2\varphi(E_{\alpha_0})) \\
&\leq 2\,\|\varphi\| + 2\,\|\varphi\|\,\epsilon - 2(\|\varphi\| - \epsilon) \\
&= 2(\|\varphi\| + 1)\epsilon
\end{aligned}$$

whenever $\beta \geq \beta_0$. Thus if $\beta, \lambda \geq \beta_0$

$$\|\xi_\beta - \xi_\lambda\|_{\mathcal{H}} \leq \|\xi_\beta - \xi_{\alpha_0}\|_{\mathcal{H}} + \|\xi_{\alpha_0} - \xi_\lambda\|_{\mathcal{H}} \leq 2\sqrt{2(\|\varphi\| + 1)\epsilon}$$

As $\epsilon > 0$ was arbitrary, we obtain that $(\xi_\lambda)_\Lambda$ is a Cauchy net in $\mathcal{H}$.

Let $\xi := \lim_\Lambda \xi_\lambda \in \mathcal{H}$. To complete the proof it suffices to show that ξ satisfies the three claims in the theorem. For the first we notice that

$$\|\xi\|_{\mathcal{H}}^2 = \lim_\Lambda \|\xi_\lambda\|_{\mathcal{H}}^2 = \lim_\Lambda \varphi(E_\lambda^2) = \|\varphi\|$$

since $(E_\lambda^2)_\Lambda$ is also a C*-bounded approximate identity for $\mathfrak{A}$ and by Proposition 1.11. Next we claim that $\pi(A)\xi = A + \mathcal{N}$ for all $A \in \mathfrak{A}$. To see this, we notice that

$$\begin{aligned}
\|\pi(A)\xi - (A + \mathcal{N})\|_{\mathcal{H}}^2 &= \lim_\Lambda \|\pi(A)\xi_\lambda - (A + \mathcal{N})\|_{\mathcal{H}}^2 \\
&= \lim_\Lambda \|(AE_\lambda - A) + \mathcal{N}\|_{\mathcal{H}}^2 \\
&\leq \liminf_\Lambda \varphi((AE_\lambda - A)^*(AE_\lambda - A)) \\
&\leq \lim_\Lambda \|\varphi\|\,\|AE_\lambda - A\|^2 = 0
\end{aligned}$$

for all $A \in \mathfrak{A}$. Whence $\pi(A)\xi = A + \mathcal{N}$ for all $A \in \mathfrak{A}$ and thus $\overline{\pi(\mathfrak{A})\xi} = \overline{\mathfrak{A}/\mathcal{N}} = \mathcal{H}$. Lastly we notice that

$$\begin{aligned}
\langle \pi(A)\xi, \xi \rangle &= \langle A + \mathcal{N}, \xi \rangle \\
&= \lim_\Lambda \langle A + \mathcal{N}, E_\lambda + \mathcal{N} \rangle \\
&= \lim_\Lambda \varphi(E_\lambda A) = \varphi(A)
\end{aligned}$$

for all $A \in \mathfrak{A}$ as desired. $\square$

The above theorem is extremely important as it show that each state on a C*-algebra provides a representation of the C*-algebra. The following is an example that we hope removes some mystery of the GNS construction.

Example 1.18. Let X be a compact Hausdorff space, $x \in X$, and $\varphi_x : C(X) \to \mathbb{C}$ be defined by $\varphi_x(f) = f(x)$. It was shown in Example 1.3 that φ_x is a state on $C(X)$. In the GNS construction, we see that $\mathcal{N} = \{f \in C(X) \mid f(x) = 0\}$ so that $f + \mathcal{N} = g + \mathcal{N}$ in $\mathfrak{A}/\mathcal{N}$ if and only if $f(x) = g(x)$. Thus it is easy to see that the map $U : \mathfrak{A}/\mathcal{N} \to \mathbb{C}$ defined by $U(f + \mathcal{N}) = f(x)$ is a well-defined isometry of pre-Hilbert spaces. Whence $\mathfrak{A}/\mathcal{N}$ is complete and isomorphic to $\mathbb{C}$. Then, by identifying the Hilbert space $\mathcal{H}$ in the above construction with $\mathbb{C}$, it is trivial to verify that the *-homomorphism $\pi : C(X) \to \mathcal{B}(\mathbb{C})$ constructed in the above theorem is $\pi(f)\lambda = f(x)\lambda$ for all $\lambda \in \mathbb{C}$ and $f \in C(X)$.

To complete our proof that every C*-algebra can be viewed as a C*-subalgebra of some $\mathcal{B}(\mathcal{H})$, it suffices to construct a faithful *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. To do this we will appeal to Theorem 1.15 and the above theorem.

Lemma 1.19. *Let $\mathfrak{A}$ be a C*-algebra and let $A \in \mathfrak{A}$ be self-adjoint. Then there exists a Hilbert space $\mathcal{H}_A$ and a representation $\pi_A : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_A)$ such that $\|\pi_A(A)\| = \|A\|_{\mathfrak{A}}$. If $\mathfrak{A}$ is unital then we may choose π_A such that $\pi_A(I_{\mathfrak{A}}) = I_{\mathcal{H}_A}$.*

Proof. Let $C^*(A)$ be the C*-subalgebra of $\mathfrak{A}$ generated by A. Then $C^*(A)$ is isomorphic to the continuous functions on $\sigma(A)$ that vanish at zero. Since $\|A\|_{\mathfrak{A}} = spr(A)$, there exists an $x \in \sigma(A)$ such that $|x| = \|A\|_{\mathfrak{A}}$. Define $\varphi : C(\sigma(A)) \to \mathbb{C}$ by $\varphi(f) = f(x)$. By Example 1.3, φ is a state on $C(\sigma(A))$ and thus defines a state $\psi : C^*(A) \to \mathbb{C}$. Since $\psi(A) = x$, $|\psi(A)| = \|A\|_{\mathfrak{A}}$ (alternatively, such a state can be constructed using Gelfand Theorem which is implicitly used to view $C^*(A)$ as continuous functions).

By Theorem 1.15 ψ extends to a state $\tilde{\psi} : \mathfrak{A} \to \mathbb{C}$. By Theorem 1.16 there exists a *-homomorphism $\pi_A : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_A)$ and a $\xi \in \mathcal{H}_A$ such that $\|\xi\| = 1$ and $\langle \pi_A(B)\xi, \xi\rangle = \tilde{\psi}(B)$ for all $B \in \mathfrak{A}$ (and $\pi_A(I_{\mathfrak{A}}) = I_{\mathcal{H}_A}$ if $\mathfrak{A}$ is unital). In particular

$$\|A\|_{\mathfrak{A}} = |\psi(A)| = |\tilde{\psi}(A)| = |\langle \pi_A(A)\xi, \xi\rangle| \leq \|\pi_A(A)\|$$

Since π is a *-homomorphism between C*-algebras, $\|\pi_A(A)\| \leq \|A\|$ so $\|\pi_A(A)\| = \|A\|_{\mathfrak{A}}$. ☐

Theorem 1.20 (GNS Theorem)**.** *Let $\mathfrak{A}$ be a C*-algebra. Then there exists a Hilbert space $\mathcal{H}$ and a faithful *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. Whence every C*-algebra may be viewed as a closed *-subalgebra of $\mathcal{B}(\mathcal{H})$. If $\mathfrak{A}$ is unital then we may choose π such that $\pi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$.*

Proof. For each non-zero positive element $A \in \mathfrak{A}$, let $\mathcal{H}_A$ be a Hilbert space and $\pi_A : \mathfrak{A} \to \mathbb{C}$ be a *-homomorphism from Lemma 1.19 such that $\|\pi_A(A)\| = \|A\|_{\mathfrak{A}}$ (and $\pi_A(I_{\mathfrak{A}}) = I_{\mathcal{H}_A}$ if $\mathfrak{A}$ is unital). Define $\pi : \mathfrak{A} \to \mathcal{B}\left(\bigoplus_{A \in \mathfrak{A}\setminus\{0\}, A \geq 0} \mathcal{H}_A\right)$ by

$$\pi(B) = \bigoplus_{A \in \mathfrak{A}\setminus\{0\}, A \geq 0} \pi_A(B)$$

for all $B \in \mathfrak{A}$. Note that π does map into $\mathcal{B}\left(\bigoplus_{A \in \mathfrak{A}\setminus\{0\}, A \geq 0} \mathcal{H}_A\right)$ since if $B \in \mathfrak{A}$ then $\|\pi_A(B)\| \leq \|B\|$ so that $\pi(B)$ is a direct sum of uniformly bounded operators and thus is bounded. Clearly π is a *-homomorphism being the direct sum of *-homomorphisms (and clearly $\pi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$ if $\mathfrak{A}$ is unital).

To complete the proof it suffices to show that π is isometric. Fix $B \in \mathfrak{A} \setminus \{0\}$. Then $\|\pi_{B^*B}(B^*B)\| = \|B^*B\| = \|B\|^2$ so

$$\|\pi(B)\|^2 \geq \|\pi_{B^*B}(B)\|^2 = \|\pi_{B^*B}(B)^*\pi_{B^*B}(B)\| = \|\pi_{B^*B}(B^*B)\| = \|B^*B\| = \|B\|^2$$

so $\|\pi(B)\| = \|B\|$ as π is contractive.

Since the image of a C*-algebra under a *-homomorphism is closed, the final statement follows. ☐

The above theorem allows us to use the structure of the bounded operators on a Hilbert space to prove many interesting properties of C*-algebras. The first property we will discuss is another characterization of a positive operator in $\mathcal{B}(\mathcal{H})$.

Proposition 1.21. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \geq 0$ if and only if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$.*

Proof. If $T \geq 0$, $T = N^*N$ for some $N \in \mathcal{B}(\mathcal{H})$. Whence

$$\langle T\xi, \xi\rangle = \langle N^*N\xi, \xi\rangle = \|N\xi\|^2 \geq 0$$

for all $\xi \in \mathcal{H}$.

Next suppose that $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$. Thus

$$\langle (T - T^*)\xi, \xi\rangle = \langle T\xi, \xi\rangle - \langle \xi, T\xi\rangle = \langle T\xi, \xi\rangle - \overline{\langle T\xi, \xi\rangle} = 0$$

for all $\xi \in \mathcal{H}$. However, for every $R \in \mathcal{B}(\mathcal{H})$ and every $\zeta, \eta \in \mathcal{H}$, it is trivial to verify the identity

$$\langle R\zeta, \eta\rangle = \frac{1}{4}\left(\langle R(\zeta+\eta), \zeta+\eta\rangle - \langle R(\zeta-\eta), \zeta-\eta\rangle + i\langle R(\zeta+i\eta), \zeta+i\eta\rangle - i\langle R(\zeta-i\eta), \zeta-i\eta\rangle\right).$$

Thus as $\langle (T-T^*)\xi, \xi\rangle = 0$ for all $\xi \in \mathcal{H}$, the above inequality implies that $\langle (T-T^*)\zeta, \eta\rangle = 0$ for all $\zeta, \eta \in \mathcal{H}$ and thus $T = T^*$.

Since T is self-adjoint, we may apply the Continuous Functional Calculus to write $T = T_+ - T_-$ where $T_+, T_- \in \mathcal{B}(\mathcal{H})$ are positive and $T_+T_- = 0 = T_-T_+$. We desire to show that $T_- = 0$. Since $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$, we obtain that

$$\langle T_-\xi, \xi\rangle \leq \langle T_+\xi, \xi\rangle$$

for all $\xi \in \mathcal{H}$. By substituting $T_-\eta$ for ξ in the above expression for any $\eta \in \mathcal{H}$, using the fact that $T_-^3 \geq 0$ by the Continuous Functional Calculus, and the first direction of the proof, we have that

$$0 \leq \langle T_-^3\eta, \eta\rangle \leq \langle T_+T_-\eta, T_-\eta\rangle = 0$$

Whence $\langle T_-^3\eta, \eta\rangle = 0$ for all $\eta \in \mathcal{H}$. As demonstrated above, this implies $T_-^3 = 0$. Whence $T_- = 0$ by the Continuous Functional Calculus and thus $T = T_+ \geq 0$. ☐

Another property that will be essential in Chapter 3 is that if $\mathfrak{A}$ is a C*-algebra, $n \in \mathbb{N}$, and $\mathcal{M}_n(\mathfrak{A})$ is the *-algebra of the $n \times n$ matrices with entries in $\mathfrak{A}$ with the operations $[A_{i,j}] + [B_{i,j}] = [A_{i,j} + B_{i,j}]$, $[A_{i,j}][B_{i,j}] = [\sum_{k=1}^n A_{i,k}B_{k,j}]$, and $[A_{i,j}]^* = [A_{j,i}^*]$ for all $[A_{i,j}], [B_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$, then $\mathcal{M}_n(\mathfrak{A})$ is a C*-algebra. To prove that $\mathcal{M}_n(\mathfrak{A})$ is a C*-algebra, we will use prove that $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is a C*-algebra for every Hilbert space $\mathcal{H}$ and apply the GNS Theorem.

Lemma 1.22. *Let $\mathcal{H}$ be a Hilbert space and $n \in \mathbb{N}$. Then $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is a *-algebra where the operations are as defined above and $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{\oplus n})$ as *-algebras (where $\mathcal{H}^{\oplus n}$ is the direct sum of n copies of $\mathcal{H}$). Whence $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ can be given a C*-norm.*

Proof. For $i \in \{1, \ldots, n\}$, let $E_i : \mathcal{H} \to \mathcal{H}^{\oplus n}$ be defined by

$$E_i\xi = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \xi \oplus 0 \oplus \cdots \oplus 0$$

where the ξ occurs in the i^{th}-coordinate of the direct summand. It is trivial to verify that $E_i^*E_i = I_{\mathcal{H}}$, $E_j^*E_i = 0$ for all $i \neq j$, and $\sum_{i=1}^n E_iE_i^* = I_{\mathcal{H}^{\oplus n}}$. Define $\pi : \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \to \mathcal{B}(\mathcal{H}^{\oplus n})$ by

$$\pi([T_{i,j}]) = \sum_{i,j=1}^n E_iT_{i,j}E_j^*$$

for all $[T_{i,j}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. Note that each $\pi([T_{i,j}])$ is a finite sum of bounded operators and thus is bounded. By the known facts of the E_i and the operations on $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$, π is a *-homomorphism. We claim that π is injective. To see this, we notice that if $\pi([T_{i,j}]) = 0$ then $T_{k,\ell} = E_k^*\pi([T_{i,j}])E_\ell = 0$ (by the properties of the E_i) for all $k, \ell \in \{1, \ldots, n\}$. Whence $[T_{i,j}] = 0$.

Lastly, we claim that π is surjective. To see this, let $T \in \mathcal{B}(\mathcal{H}^{\oplus n})$ be arbitrary. Let $T_{i,j} := E_i^*TE_j \in \mathcal{B}(\mathcal{H})$. Then it is easy to verify (by using the properties of the E_is) that $\pi([T_{i,j}]) = \sum_{i,j=1}^n E_iE_i^*TE_jE_j^* = T$. ☐

From now on we will view $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ and $\mathcal{B}(\mathcal{H}^{\oplus n})$ as the same C*-algebra using either identification as required. The only restriction remaining in showing that $\mathcal{M}_n(\mathfrak{A})$ is a C*-algebra using Theorem 1.20 is completeness. The following lemma solves this problem and is extremely useful.

Lemma 1.23. *Let $[T_{i,j}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ for some $n \in \mathbb{N}$ and some Hilbert space $\mathcal{H}$. Then*

$$\max_{i,j} \|T_{i,j}\| \leq \|[T_{i,j}]\| \leq \left(\sum_{i,j=1}^n \|T_{i,j}\|^2\right)^{\frac{1}{2}} \leq n \max_{i,j} \|T_{i,j}\|.$$

Proof. For $i \in \{1, \ldots, n\}$, let $E_i : \mathcal{H} \to \mathcal{H}^{\oplus n}$ be defined by

$$E_i\xi = 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \xi \oplus 0 \oplus \cdots \oplus 0$$

where the ξ occurs in the i^{th}-coordinate of the direct summand. Suppose $[T_{i,j}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. By the proof of Lemma 1.22, $T_{k,\ell} = E_k^*[T_{i,j}]E_\ell$ for all $k, \ell \in \{1, \ldots, n\}$. Since each E_i is an isometry, $\|E_i\| = 1$ for all $i \in \{1, \ldots, n\}$ so

$$\|T_{k,\ell}\| = \|E_k^*[T_{i,j}]E_\ell\| \leq \|[T_{i,j}]\|$$

for all $k, \ell \in \{1, \ldots, n\}$. Thus $\max_{i,j} \|T_{i,j}\| \leq \|[T_{i,j}]\|$.

For the second inequality, let $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^{\oplus n}$ be arbitrary. Then

$$\begin{aligned}
\|[T_{i,j}]\xi\|^2 = \left\|\left(\sum_{j=1}^n T_{1,j}\xi_j, \ldots, \sum_{j=1}^n T_{n,j}\xi_j\right)\right\|^2 &= \sum_{k=1}^n \left\langle \sum_{i=1}^n T_{k,i}\xi_i, \sum_{j=1}^n T_{k,j}\xi_j \right\rangle \\
&\leq \sum_{k=1}^n \sum_{i,j=1}^n |\langle T_{k,i}\xi_i, T_{k,j}\xi_j\rangle| \\
&\leq \sum_{k=1}^n \sum_{i,j=1}^n \|T_{k,i}\| \|\xi_i\| \|T_{k,j}\| \|\xi_j\| \\
&= \sum_{k=1}^n \left(\sum_{i=1}^n \|T_{k,i}\| \|\xi_i\|\right)^2 \\
&\leq \sum_{k=1}^n \left(\left(\sum_{i=1}^n \|T_{k,i}\|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|\xi_j\|^2\right)^{\frac{1}{2}}\right)^2 \\
&= \sum_{k=1}^n \left(\sum_{i=1}^n \|T_{k,i}\|^2\right) \|\xi\|^2 = \|\xi\|^2 \sum_{i,j=1}^n \|T_{i,j}\|^2.
\end{aligned}$$

Thus, as this holds for all $\xi \in \mathcal{H}^{\oplus n}$, we have $\|[T_{i,j}]\| \leq \left(\sum_{i,j=1}^n \|T_{i,j}\|^2\right)^{\frac{1}{2}}$ as claim.

Finally, it is clear that $\left(\sum_{i,j=1}^n \|T_{i,j}\|^2\right)^{\frac{1}{2}} \leq n \max_{i,j} \|T_{i,j}\|$ as $\sqrt{n^2} = n$. ☐

Theorem 1.24. *Let $\mathfrak{A}$ be a C*-algebra and let $n \in \mathbb{N}$. Then $\mathcal{M}_n(\mathfrak{A})$ is a *-algebra with the operations $[A_{i,j}] + [B_{i,j}] = [A_{i,j} + B_{i,j}]$, $[A_{i,j}][B_{i,j}] = [\sum_{k=1}^n A_{i,k}B_{k,j}]$, and $[A_{i,j}]^* = [A_{j,i}^*]$ for all $[A_{i,j}], [B_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$. Moreover there is a unique C*-norm on $\mathcal{M}_n(\mathfrak{A})$.*

Proof. It is trivial to verify that $\mathcal{M}_n(\mathfrak{A})$ is a *-algebra with the operations indicated above. Moreover, if a C*-norm exists on $\mathcal{M}_n(\mathfrak{A})$ then it must unique by C*-algebra theory.

To show that $\mathcal{M}_n(\mathfrak{A})$ is a C*-algebra, let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a faithful *-homomorphism which exists by Theorem 1.20. Since π is a *-homomorphism between C*-algebras, $\pi(\mathfrak{A})$ is a closed *-subalgebra of

$\mathcal{B}(\mathcal{H})$. Define $\pi_{(n)} : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ by $\pi_{(n)}([A_{i,j}]) = [\pi(A_{i,j})]$. It is trivial to verify that $\pi_{(n)}$ is a *-homomorphism and that $\pi_{(n)}$ is injective since π is injective. Thus $\pi_{(n)}(\mathcal{M}_n(\mathfrak{A}))$ is a *-subalgebra of $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. Thus we can define a C*-norm on $\mathcal{M}_n(\mathfrak{A})$ by

$$\|[A_{i,j}]\|_{\mathcal{M}_n(\mathfrak{A})} = \left\|\pi_{(n)}([A_{i,j}])\right\|_{\mathcal{M}_n(\mathcal{B}(\mathcal{H}))}$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$. Thus it remains only to show that $\mathcal{M}_n(\mathfrak{A})$ is complete. To begin, suppose that $([A_{i,j,k}])_{k\geq 1} \in \mathcal{M}_n(\mathfrak{A})$ is a Cauchy sequence. Then $([\pi(A_{i,j,k})])_{k\geq 1}$ is a Cauchy sequence in $\mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. Since $\max_{i,j} \|T_{i,j}\| \leq \|[T_{i,j}]\|$ for all $[T_{i,j}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$, we obtain that $(\pi(A_{i,j,k}))_{k\geq 1}$ is a Cauchy sequence in $\pi(\mathfrak{A}) \subseteq \mathcal{B}(\mathcal{H})$ for all $i, j \in \{1, \ldots, n\}$. Since $\pi(\mathfrak{A})$ is closed and thus complete, for each $i, j \in \{1, \ldots, n\}$ there exists a $A_{i,j} \in \mathfrak{A}$ such that $\pi(A_{i,j}) = \lim_{k\to\infty} \pi(A_{i,j,k})$. We claim that $([\pi(A_{i,j,k})])_{k\geq 1}$ converges to $\pi_{(n)}([A_{i,j}])$ so that $([A_{i,j,k}])_{k\geq 1}$ converges to $[A_{i,j}]$ in $\mathcal{M}_n(\mathfrak{A})$. To see this, we notice by Lemma 1.23 that

$$\limsup_{k\to\infty} \|[A_{i,j}] - [A_{i,j,k}]\| = \limsup_{k\to\infty} \|[\pi(A_{i,j})] - [\pi(A_{i,j,k})]\| \leq \lim_{k\to\infty} \left(\sum_{i,j=1}^{n} \|\pi(A_{i,j}) - \pi(A_{i,j,k})\|^2\right)^{\frac{1}{2}} = 0.$$

Whence $\mathcal{M}_n(\mathfrak{A})$ is complete with respect to this C*-norm and thus is a C*-algebra. □

Notice that if $\mathfrak{A}$ is a non-unital C*-algebra and $\tilde{\mathfrak{A}}$ is the unitization of $\mathfrak{A}$ then $\mathcal{M}_n(\mathfrak{A}) \subseteq \mathcal{M}_n(\tilde{\mathfrak{A}})$ for all $n \in \mathbb{N}$ although $\mathcal{M}_n(\tilde{\mathfrak{A}})$ is not in general the unitization of $\mathcal{M}_n(\mathfrak{A})$ (as it could be larger). Using $\mathcal{M}_n(\mathfrak{A}) \subseteq \mathcal{M}_n(\tilde{\mathfrak{A}})$ we can often assume that $\mathfrak{A}$ has a unit when dealing with $\mathcal{M}_n(\mathfrak{A})$.

In Chapter 3 maps like $\pi_{(n)}$ will be of the greatest importance.

2 Positive Maps

In the first chapter of these notes we examined positive linear functionals and obtained some important results pertaining to C*-algebras. In this section we will generalize the notion of positive linear functionals to maps between C*-algebras.

Definition 2.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. A linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is said to be positive if $\varphi(A) \geq_{\mathfrak{B}} 0$ whenever $A \in \mathfrak{A}$ and $A \geq_{\mathfrak{A}} 0$.

Example 2.2. Positive linear functionals are positive maps with the complex numbers as their range.

Example 2.3. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a *-homomorphism. Then π is a positive map since

$$\pi(A^*A) = \pi(A^*)\pi(A) = \pi(A)^*\pi(A) \geq_{\mathfrak{B}} 0$$

for all $A \in \mathfrak{A}$.

Example 2.4. Let $\mathfrak{A}$ be a C*-algebra and let $A \in \mathfrak{A}$ be arbitrary. Define $\varphi : \mathfrak{A} \to \mathfrak{A}$ by $\varphi(B) = A^*BA$ for all $B \in \mathfrak{A}$. Then φ is a positive map since

$$\varphi(B^*B) = A^*B^*BA = (BA)^*BA \geq 0$$

for all $B \in \mathfrak{B}$. In general φ is not a *-homomorphism. The map φ is known as the conjugation by A.

Example 2.5. Define $\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ by $\varphi(A) = A^T$ for all $A \in \mathcal{M}_n(\mathbb{C})$ where A^T is the transpose of A. Then φ is a positive map since

$$\varphi(A^*A) = (A^*A)^T = A^T(A^*)^T = A^T\left(A^T\right)^*$$

for all $A \in \mathfrak{A}$ as the transpose and the adjoint commute.

Example 2.6. Let $\varphi_i : \mathfrak{A}_i \to \mathfrak{B}_i$ be positive maps for $i = 1, 2$. Then the map $\varphi_1 \oplus \varphi_2 : \mathfrak{A}_1 \oplus \mathfrak{A}_2 \to \mathfrak{B}_1 \oplus \mathfrak{B}_2$ defined by $(\varphi_1 \oplus \varphi_2)(A_1 \oplus A_2) = \varphi_1(A_1) \oplus \varphi_2(A_2)$ for all $A_i \in \mathfrak{A}_i$ is a positive map. Indeed

$$(\varphi_1 \oplus \varphi_2)((A_1 \oplus A_2)^*(A_1 \oplus A_2)) = \varphi_1(A_1^*A_1) \oplus \varphi_2(A_2^*A_2) \geq 0$$

since φ_i are positive and an element $B_1 \oplus B_2 \in \mathfrak{B}_1 \oplus \mathfrak{B}_2$ is positive if and only if $B_1 \geq 0$ and $B_2 \geq 0$. Thus the direct sum of positive maps is positive.

The most obvious questions are whether or not the theorems developed for positive linear functionals hold for positive maps between C*-algebras. It turns out that the continuity and many (if not all) of the norm properties of positive linear functionals carry-forward to positive maps between C*-algebras. However, the extension properties of positive maps will not hold in general. The proof that positive maps between C*-algebras are continuous is virtually identical to Proposition 1.7 with the addition of the following lemma.

Lemma 2.7. *Let $\mathfrak{A}$ be a C*-algebra and let $A, B \in \mathfrak{A}$. If $0 \leq A \leq B$ then $\|A\| \leq \|B\|$.*

Proof. In the unitization of $\mathfrak{A}$, $0 \leq A \leq B \leq \|B\| I_{\mathfrak{A}}$. Thus $\|A\| \leq \|B\|$ by the Continuous Functional Calculus. ☐

Proposition 2.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebra and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a positive map. Then φ is continuous.*

Proof. We proceed as we did in Proposition 1.7. We claim that $\{\|\varphi(A)\|_{\mathfrak{B}} \mid A \in \mathfrak{A}, A \geq 0, \|A\|_{\mathfrak{A}} \leq 1\}$ is a bounded subset of $\mathbb{R}_{\geq 0}$ (that is, we claim φ is bounded on the positive elements of norm at most one). To see this, suppose otherwise. Then for each $n \in \mathbb{N}$ there would exists an $A_n \in \mathfrak{A}$ such that $A_n \geq 0$, $\|A_n\|_{\mathfrak{A}} \leq 1$, and $\|\varphi(A_n)\|_{\mathfrak{B}} \geq n^3$. Consider $B := \sum_{n\geq 1}^{\infty} \frac{1}{n^2} A_n$. Notice that B is a well-defined operator in $\mathfrak{A}$ as $(\frac{1}{n^2}A_n)_{n\geq 1}$ is an absolutely summable sequence since $\left\|\frac{1}{n^2}A_n\right\|_{\mathfrak{A}} \leq \frac{1}{n^2}$.

For each $m \in \mathbb{N}$

$$B - \sum_{n=1}^{m} \frac{1}{n^2} A_n = \sum_{n \geq m+1}^{\infty} \frac{1}{n^2} A_n \geq 0$$

by Lemma 1.6. Whence

$$\varphi(B) - \sum_{n=1}^{m} \varphi\left(\frac{1}{n^2} A_n\right) = \varphi\left(B - \sum_{n=1}^{m} \frac{1}{n^2} A_n\right) \geq 0$$

so

$$\varphi(B) \geq \sum_{n=1}^{m} \frac{1}{n^2} \varphi(A_n) \geq \frac{1}{m^2} \varphi(A_m)$$

for every $m \in \mathbb{N}$. Since $0 \leq \frac{1}{m^2}\varphi(A_m)$ as φ is a positive map, Lemma 2.7 implies that

$$\|\varphi(B)\|_{\mathfrak{B}} \geq \frac{1}{m^2} \|\varphi(A_m)\|_{\mathfrak{A}} \geq \frac{1}{m^2} m^3 = m$$

for every $m \in \mathbb{N}$. As this is an impossibility, we must have that $\{\|\varphi(A)\|_{\mathfrak{B}} \mid A \in \mathfrak{A}, A \geq 0, \|A\|_{\mathfrak{A}} \leq 1\}$ is bounded in $\mathbb{R}_{\geq 0}$.

Let

$$M := \sup\{\|\varphi(A)\|_{\mathfrak{B}} \mid A \in \mathfrak{A}, A \geq 0, \|A\|_{\mathfrak{A}} \leq 1\}$$

and let $A \in \mathfrak{A}$. Write $A = Re(A) + iIm(A)$ and recall $\|Re(A)\|_{\mathfrak{A}}, \|Im(A)\|_{\mathfrak{A}} \leq \|A\|_{\mathfrak{A}}$. By the Continuous Functional Calculus, both $Re(A)$ and $Im(A)$ can be written as the difference of two positive elements each with norm at most $\|Re(A)\|$ and $\|Im(A)\|$ respectively. Whence $A = P_1 - P_2 + iP_3 - iP_4$ where $P_j \in \mathfrak{A}$ are positive elements with $\|P_j\|_{\mathfrak{A}} \leq \|A\|_{\mathfrak{A}}$ for all j. Thus

$$\|\varphi(A)\|_{\mathfrak{B}} \leq \sum_{j=1}^{4} \|\varphi(P_j)\|_{\mathfrak{B}} \leq \sum_{j=1}^{4} M \|P_j\|_{\mathfrak{A}} \leq 4M \|A\|_{\mathfrak{A}}$$

Thus φ is bounded with $\|\varphi\| \leq 4M$. □

It remains to generalize Proposition 1.11 to positive maps. However, it is clear that the proof of Proposition 1.11 does not follow to the setting of positive maps since one direction uses the Cauchy-Schwarz inequality and the other makes use of properties of the complex numbers. We will see in Chapter 3 that the proof of Proposition 1.11 generalizes to a subcollection of positive maps. It can be shown that Proposition 1.11 generalizes to positive maps but the proof will detour us from our ultimate goal of studying the subcollection of positive maps in Chapter 3. For a quick outline, it is possible to show that a positive map between unital C*-algebras obtains its norm at the identity. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a positive map between non-unital C*-algebras, we recall that the double duals $\mathfrak{A}^{**}$ and $\mathfrak{B}^{**}$ are unital C*-algebras (specifically von Neumann algebras) such that any C*-bounded approximate identity of $\mathfrak{A}$ converges to the identity of $\mathfrak{A}^{**}$ in the weak*-topology. It is then possible to show that the second adjoint of φ, $\varphi^{**} : \mathfrak{A} \to \mathfrak{B}$, is also a positive map that is weak*-continuous. Therefore, it is easy to see that the norm of φ^{**} is obtained at the unit of $\mathfrak{A}^{**}$ and thus is at most the limit inferior of the norms of φ evaluated at the C*-bounded approximate identity of $\mathfrak{A}$.

Before we move on to Chapter 3, we desire to generalize the notion of a positive map. To have a notion of a positive map between vector spaces we need only have a notion of positivity in each space. Therefore we can consider positive maps between vector subspaces of C*-algebras. One small problem is that we would like to be able to decompose every element of these subspaces into a linear combination of positive elements in these subspaces. One way of doing this is to consider special subspaces of unital C*-algebras known as operator systems.

Definition 2.9. An operator system $\mathcal{S}$ is a (not necessarily closed) vector subspace of some unital C*-algebra $\mathfrak{A}$ such that $I_{\mathfrak{A}} \in \mathcal{S}$ and $\mathcal{S}$ is closed under adjoints (that is $\mathcal{S}^* = \{T \in \mathfrak{A} \mid T^* \in \mathcal{S}\} = \mathcal{S}$).

There are a plethora of operator systems so we simple give a few common examples.

Example 2.10. Unital C*-algebras are clearly operator systems.

Example 2.11. Let $\mathfrak{A}$ be a unital C*-algebra and $A \in \mathfrak{A}$ be arbitrary. Then

$$\mathcal{P}(A) := span\{I, \{A^n, (A^*)^n\}_{n\geq 1}\}$$

is an operator system of $\mathfrak{A}$ that is not a *-algebra.

Example 2.12. Let $\mathcal{S} := \{f \in C(\mathbb{T}) \mid f(z) = az + b + c\overline{z}, a, b, c \in \mathbb{C}\}$. Clearly $\mathcal{S}$ is an operator system of $C(\mathbb{T})$ that is not a *-algebra.

Next we demonstrate that operator systems have the desired positivity results mentioned above.

Lemma 2.13. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system. Then every element of $\mathcal{S}$ is a linear combination of four positive elements of $\mathcal{S}$.*

Proof. Let $A \in \mathcal{S}$ be arbitrary. Since $\mathcal{S}$ is closed under adjoints, $A^* \in \mathcal{S}$. Therefore $Re(A), Im(A) \in \mathcal{S}$ are self-adjoint elements (this is one of many reasons why operator systems are required to be closed under adjoints). Thus it suffices to show that every self-adjoint element of $\mathcal{S}$ is a linear combination of two positive elements.

Let $A \in \mathcal{S}$ be self-adjoint. Then $\|A\| I_{\mathfrak{A}} - A$ and $\|A\| I_{\mathfrak{A}} + A$ are positive elements of $\mathfrak{A}$. Since $I_{\mathfrak{A}} \in \mathcal{S}$, $\|A\| I_{\mathfrak{A}} - A, \|A\| I_{\mathfrak{A}} + A \in \mathcal{S}$ (this is one of many reasons why operator systems are required to have units). Moreover $A = \frac{1}{2}(\|A\| I_{\mathfrak{A}} + A) - \frac{1}{2}(\|A\| I_{\mathfrak{A}} - A)$ so A is a linear combination of positive elements of $\mathcal{S}$. ☐

Now we may generalize the notion of positive maps to operator systems.

Definition 2.14. Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and $\mathfrak{B}$ a C*-algebra. A linear map $\varphi : \mathcal{S} \to \mathfrak{B}$ is said to be positive if $\varphi(A) \geq_{\mathfrak{B}} 0$ whenever $A \in \mathcal{S}$ and $A \geq_{\mathfrak{A}} 0$. If $\mathfrak{B} = \mathbb{C}$, we say that φ is a positive linear functional.

As unital C*-algebras are operator systems, we have already seen several positive maps. The following is an example of a positive map on an operator system that is not a C*-algebra.

Example 2.15. Let $\mathcal{S} := \{f \in C(\mathbb{T}) \mid f(z) = az + b + c\overline{z}, a, b, c \in \mathbb{C}\}$. Clearly $\mathcal{S}$ is an operator system of $C(\mathbb{T})$. Define $\varphi : \mathcal{S} \to \mathcal{M}_2(\mathbb{C})$ by

$$\varphi(az + b + c\overline{z}) = \begin{bmatrix} b & 2a \\ 2c & b \end{bmatrix}$$

It is easy to check that φ is well-defined (as every element of $\mathcal{S}$ has a unique representation as $az + b + c\overline{z}$) and is linear. We claim that φ is positive. To see this, suppose that $f(z) = az + b + c\overline{z} \in \mathcal{S}$ is positive. Then $f = f^*$ which implies that $b = b^*$ and $a = c^*$ due to the unique representations of elements of $\mathcal{S}$. Thus it is clear that $\varphi(f)$ is self-adjoint. Next we notice that since $f \geq 0$, $f(z) \geq 0$ for all $z \in \mathbb{T}$. Whence $b + 2Re(az) \geq 0$ for all $z \in \mathbb{C}$. However $2Re(az) \geq -2|a|$ for all $z \in \mathbb{T}$ and equality is obtained for some $z \in \mathbb{T}$ (that is, if $a = re^{i\theta}$, where $r \geq 0$ and $\theta \in [0, 2\pi)$, $Re(a(-e^{-i\theta})) = -|a|$). Since $f \geq 0$ we must have that $b \geq 2|a|$. Since $\varphi(f)$ is self-adjoint, $\varphi(f)$ is positive if and only if its eigenvalues are non-negative. However, the characteristic equation of $\varphi(f)$ is

$$\chi_{\varphi(f)}(\lambda) = \lambda^2 - 2b\lambda + b^2 - 4ac$$

so that $\varphi(f)$ has real eigenvalues if and only if

$$\frac{2b \pm \sqrt{4b^2 - 4(b^2 - 4ac)}}{2} = b \pm 4|ac|$$

are both positive. However $b \geq 2|a| \geq 0$ so $b \geq 0$ and $b^2 \geq 4|a|^2$. Thus $0 \leq \sqrt{b^2 - 4|a|^2} \leq b$. Whence $\varphi(f)$ is positive so φ is a positive linear map.

The next logical question is whether or not all positive maps on operator systems are continuous. The answer is yes, but we will see that the norms of positive maps on operator systems need not be obtained at the identity. First we generalize Lemma 1.2 to show that positive maps on operator systems take self-adjoint elements to self-adjoint elements.

Lemma 2.16. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system or C*-algebra, let $\mathfrak{B}$ be a C*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a positive map. Then $\varphi(A)$ is self-adjoint whenever A is a self-adjoint element of $\mathcal{S}$. Moreover $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{S}$.*

Proof. First suppose $\mathcal{S}$ is a C*-algebra. Let $A \in \mathcal{S}$ be self-adjoint. By the Continuous Functional Calculus, there exists positive elements $A_+, A_- \in \mathcal{S}$ such that $A = A_+ - A_-$. Then $\varphi(A) = \varphi(A_+) - \varphi(A_-)$ is self-adjoint being the difference of two positive operators.

Let $A \in \mathcal{S}$ be arbitrary. Thus

$$\begin{aligned} \varphi(A^*) = \varphi((Re(A) + iIm(A))^*) &= \varphi(Re(A) - iIm(A)) \\ &= \varphi(Re(A)) - i\varphi(Im(A)) \\ &= (\varphi(Re(A)) + i\varphi(Im(A)))^* = \varphi(A)^* \end{aligned}$$

as desired.

Now suppose $\mathcal{S}$ is an operator system. This proof will serve as a great example of why the conditions in the definition of an operator system are necessary. Let $A \in \mathcal{S}$ be self-adjoint. Let $A_+ := \frac{1}{2}(\|A\| I_{\mathfrak{A}} + A)$ and $A_- := \frac{1}{2}(\|A\| I_{\mathfrak{A}} - A)$ which are positive elements of $\mathcal{S}$ with $A = A_+ - A_-$. Then $\varphi(A) = \varphi(A_+) - \varphi(A_-)$ is self-adjoint being the difference of two positive operators.

Let $A \in \mathcal{S}$ be arbitrary. Since $\mathcal{S}$ is closed under adjoints, the self-adjoint elements $Re(A)$ and $Im(A)$ are in $\mathcal{S}$. Thus the computation in the previous case shows $\varphi(A^*) = \varphi(A)^*$ as desired. ☐

Proposition 2.17. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system, let $\mathfrak{B}$ be a C*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a positive map. Then φ is continuous with $\|\varphi\| \leq 2\|\varphi(I_{\mathfrak{A}})\|_{\mathfrak{B}}$. If $\mathfrak{B} = \mathbb{C}$ then $\|\varphi\| = |\varphi(I_{\mathfrak{A}})|$.*

Proof. We will mostly follow the proof of Proposition 2.8 yet we will make use of the unit in $\mathcal{S}$ to simplify the first step. Let $A \in \mathcal{S}$ be positive with $\|A\|_{\mathfrak{A}} \leq 1$. Then $0 \leq A \leq I_{\mathfrak{A}}$ so $0 \leq \varphi(A) \leq \varphi(I_{\mathfrak{A}})$. Whence $\|\varphi(A)\|_{\mathfrak{B}} \leq \|\varphi(I_{\mathfrak{A}})\|_{\mathfrak{B}}$ by Lemma 2.7.

To prove the inequality with the constant 2 (instead of perhaps the constant 4 which we would obtain using the proof of Proposition 2.8) we will need to be a little careful. Consider $A \in \mathfrak{A}$ with A self-adjoint and $\|A\|_{\mathfrak{A}} \leq 1$. Then $\varphi(A)$ is self-adjoint by Lemma 2.16. Let $A_+ := \frac{1}{2}(I_{\mathfrak{A}} + A)$ and $A_- := \frac{1}{2}(I_{\mathfrak{A}} - A)$ which are positive elements of $\mathcal{S}$ with $A = A_+ - A_-$ and $\|A_+\|_{\mathfrak{A}}, \|A_-\|_{\mathfrak{A}} \leq 1$. Whence

$$-A_- \leq A \leq A_+.$$

By applying φ and the above results for positive operators, we obtain that

$$-\|\varphi(I_{\mathfrak{A}})\| I_{\tilde{\mathfrak{B}}} \leq -\varphi(I_{\mathfrak{A}}) \leq -\varphi(A_-) \leq \varphi(A) \leq \varphi(A_+) \leq \varphi(I_{\mathfrak{A}}) \leq \|\varphi(I_{\mathfrak{A}})\| I_{\tilde{\mathfrak{B}}}$$

where $I_{\tilde{\mathfrak{B}}}$ is the unit of the unitization of $\mathfrak{B}$ if $\mathfrak{B}$ is not unital. Therefore $\|\varphi(A)\|_{\mathfrak{B}} \leq \|\varphi(I_{\mathfrak{A}})\|_{\mathfrak{B}}$ by the Continuous Functional Calculus.

Finally suppose $A \in \mathcal{S}$ is an arbitrary element such that $\|A\|_{\mathfrak{A}} \leq 1$. Then $Re(A)$ and $Im(A)$ are self-adjoint elements of $\mathcal{S}$ with norm at most one. Whence

$$\|\varphi(A)\|_{\mathfrak{B}} \leq \|\varphi(Re(A))\|_{\mathfrak{B}} + \|\varphi(Im(A))\|_{\mathfrak{B}} \leq 2\|\varphi(I_{\mathfrak{A}})\|_{\mathfrak{B}}$$

as desired.

Now suppose that $\mathfrak{B} = \mathbb{C}$. We know from above that $|\varphi(A)| \leq \|A\|_{\mathfrak{A}} \varphi(I_{\mathfrak{A}})$ for all $A \in \mathcal{S}$ self-adjoint. Let $A \in \mathcal{S}$ be arbitrary and choose $\theta \in [0, 2\pi)$ such that $e^{i\theta}\varphi(A) = |\varphi(A)|$ and let $B := e^{i\theta}A \in \mathcal{S}$. Then

$$|\varphi(A)| = \varphi(B) = \varphi(Re(B)) + i\varphi(Im(B))$$

(as $Re(B), Im(B) \in \mathcal{S}$ so it makes sense to apply φ to these elements). Since $Re(B)$ and $Im(B)$ are self-adjoint elements of $\mathcal{S}$, $\varphi(Re(B))$ and $\varphi(Im(B))$ are real numbers by Lemma 2.16. However, since $\varphi(Re(B)) + i\varphi(Im(B))$ is a real number, this implies that $\varphi(Im(B)) = 0$ and $\varphi(Re(B)) \geq 0$ so

$$|\varphi(A)| = \varphi(Re(B)) = |\varphi(Re(B))| \leq \|Re(B)\|_{\mathfrak{A}} \, \varphi(I_{\mathfrak{A}}) \leq \|B\|_{\mathfrak{A}} \, \varphi(I_{\mathfrak{A}}) = \|A\|_{\mathfrak{A}} \, \varphi(I_{\mathfrak{A}}).$$

Since $A \in \mathcal{S}$ was arbitrary, $\|\varphi\| \leq \varphi(I_{\mathfrak{A}})$. Since the other inequality is clear, we have completed the proof. $\square$

Remarks 2.18. It turns out that inequality $\|\varphi\| \leq 2\,\|\varphi(I_{\mathfrak{A}})\|_{\mathfrak{B}}$ obtained in the previous proposition is strict for arbitrary positive maps on operator systems. For example, consider the positive map φ from Example 2.15. Then $\varphi(1) = I_2$ has norm one yet if $f(z) = z$ for all $z \in \mathbb{T}$, $f \in \mathcal{S}$ and $\|f\| = 1$ yet

$$\varphi(f) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

has norm two.

However there are many cases where we can obtain a tighter bounded for the norm of a positive map. The first we will demonstrate is when the range of the positive map is an abelian C*-algebra. To prove this we need a simple lemma.

Lemma 2.19. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and let $\mathfrak{B}$ and $\mathfrak{C}$ be C*-algebras. If $\varphi : \mathcal{S} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathbb{C}$ are positive maps then $\psi \circ \varphi : \mathcal{S} \to \mathfrak{C}$ is a positive map.*

Proof. Let $A \in \mathcal{S}$ be a positive operator. Since φ is a positive map, $\varphi(A) \in \mathfrak{B}$ is positive. Since ψ is a positive map, $(\psi \circ \varphi)(A) = \psi(\varphi(A))$ is positive in $\mathfrak{C}$ as desired. $\square$

Proposition 2.20. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system, let X be a compact Hausdorff space, and let $\varphi : \mathcal{S} \to C(X)$ be a positive map. Then $\|\varphi\| = \|\varphi(I_{\mathfrak{A}})\|$.*

Proof. Clearly $\|\varphi\| \geq \|\varphi(I_{\mathfrak{A}})\|_\infty$. The idea for the rest of the proof is to make use of the norm equality from Proposition 2.17 for positive linear functionals. For each $x \in X$ define $\delta_x : C(X) \to \mathbb{C}$ by $\delta_x(f) = f(x)$ for all $f \in C(X)$. Then each δ_x is a positive linear functional by Example 1.3. Moreover $\|f\| = \sup_{x \in X} |\delta_x(f)|$ for all $f \in C(X)$. Let $A \in \mathfrak{A}$ be arbitrary. Then

$$\|\varphi(A)\| = \sup_{x \in X} |\delta_x(\varphi(A))|.$$

However $\delta_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is a positive linear functional for all $x \in X$. Thus $\|\delta_x \circ \varphi\| = \delta_x(\varphi(I_{\mathfrak{A}}))$ by Proposition 2.17. Hence

$$\|\varphi(A)\| \leq \sup_{x \in X} \|\delta_x \circ \varphi\| \, \|A\| = \sup_{x \in X} \|\delta_x(\varphi(I_{\mathfrak{A}}))\| \, \|A\| = \|\varphi(I_{\mathfrak{A}})\| \, \|A\|$$

so $\|\varphi\| \leq \|\varphi(I_{\mathfrak{A}})\|$. $\square$

Next we will show that the norm of a positive map from $C(X)$ (where X is a compact Hausdorff space) is obtained at the identity. First we need a small technical lemma and a common theorem from the theory of $C(X)$ that will not be proven. Note that the following lemma is trivial in the case that $\mathfrak{A} = C(X)$.

Lemma 2.21. *Let $\mathfrak{A}$ be a C*-algebra. Suppose $A_1, \ldots, A_n \in \mathfrak{A}$ are positive elements. Then for all $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ with $|\lambda_i| \leq 1$ for all $i \in \{1, \ldots, n\}$, $\|\sum_{i=1}^n \lambda_i A_i\| \leq \|\sum_{i=1}^n A_i\|$.*

Proof. To prove this lemma, we will Theorem 1.24 and matrix tricks. By considering the unitization of $\mathfrak{A}$ if necessary, we may assume that $\mathfrak{A}$ is unital. Notice in $\mathcal{M}_n(\mathfrak{A})$ that

$$\begin{bmatrix} \sum_{i=1}^n \lambda_i A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} \lambda_1 I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n I \end{bmatrix} \begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, by Lemma 1.23,

$$\left\|\sum_{i=1}^n \lambda_i A_i\right\| \leq \left\|\begin{bmatrix} \sum_{i=1}^n \lambda_i A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right\| \leq \left\|\begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}\right\|^2 \left\|\begin{bmatrix} \lambda_1 I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n I \end{bmatrix}\right\|.$$

It is clear that

$$\left\|\begin{bmatrix} \lambda_1 I & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n I \end{bmatrix}\right\| = \max_{i=1,\dots,n} |\lambda_i| \leq 1$$

(by either viewing $\mathfrak{A}^{\oplus n}$ as a C*-algebra of $\mathcal{M}_n(\mathfrak{A})$ or using the fact that the norm on $\mathcal{M}_n(\mathfrak{A})$ comes from the norm of $\mathcal{M}_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^{\oplus n})$ and the norm in $\mathcal{B}(\mathcal{H}^{\oplus n})$ can easily be check to have this property). Moreover

$$\left\|\begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}\right\|^2 = \left\|\begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}^* \begin{bmatrix} A_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_n^{\frac{1}{2}} & 0 & \cdots & 0 \end{bmatrix}\right\| = \left\|\begin{bmatrix} \sum_{i=1}^n A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right\|.$$

However, by Lemma 1.23 (using the second inequality),

$$\left\|\begin{bmatrix} \sum_{i=1}^n A_i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right\| \leq \left(\left\|\sum_{i=1}^n A_i\right\|^2\right)^{\frac{1}{2}} = \left\|\sum_{i=1}^n A_i\right\|.$$

Whence $\|\sum_{i=1}^n \lambda_i A_i\| \leq \|\sum_{i=1}^n A_i\|$ as desired. □

Theorem 2.22. *Let X be a compact Hausdorff space and let $\{U_i\}_{i=1}^n$ be a finite open covering of X. Then there exists continuous function $g_i : X \to [0,1]$ such that $\sum_{i=1}^n g_i = I_{C(X)}$ and $g_i|_{U_i^c} = 0$. The collection $\{g_i\}_{i=1}^n$ is called a partition of unity of X.*

Proposition 2.23. *Let $\mathfrak{B}$ be a unital C*-algebra, let X be a compact Hausdorff space, and let $\varphi : C(X) \to \mathfrak{B}$ be a positive map. Then $\|\varphi\| = \|\varphi(1)\|$ where 1 is the constant function on X with value 1.*

Proof. Clearly $\|\varphi\| \geq \|\varphi(1)\|$. Fix $f \in C(X)$ with $\|f\| \leq 1$. The idea of the proof is to approximate f using a partition of unity from the above theorem and appeal to the Lemma 2.21 to say that the norm of $\varphi(f)$ is not too large. To begin let $\epsilon > 0$ and for each $y \in X$ let $U_y := f^{-1}(B_\epsilon(f(y)))$ which will be open in X since f is continuous. Since $X = \bigcup_{y\in X} U_y$ and X is compact, there exists a finite open subcover $\{U_{x_i}\}_{i=1}^n$ of X such that for all $x \in U_{x_i}$ $|f(x) - f(x_i)| < \epsilon$. By the above theorem there exists continuous functions $g_i : X \to [0,1]$ such that $\sum_{i=1}^n g_i = 1$ and $g_i(x) = 0$ for all $x \notin U_{x_i}$.

For each $i \in \{1,\dots,n\}$ let $\lambda_i := f(x_i)$. Since $\|f\| \leq 1$, $|\lambda_i| \leq 1$ for all $i \in \{1,\dots,n\}$. Moreover for any $x \in X$

$$\left|f(x) - \sum_{i=1}^n \lambda_i g_i(x)\right| = \left|\sum_{i=1}^n (f(x) - \lambda_i) g_i(x)\right| \leq \sum_{i=1}^n |f(x) - \lambda_i| g_i(x) \leq \sum_{i=1}^n \epsilon g_i = \epsilon$$

since if $x \in U_{x_i}$ then $|f(x) - \lambda_i| = |f(x) - f(x_i)| < \epsilon$ and if $x \notin U_{x_i}$ then $g_i(x) = 0$. Hence $\|f - \sum_{i=1}^n \lambda_i g_i\|_\infty \leq \epsilon$.

Since g_i are positive for all $i \in \{1, \ldots, n\}$ and φ is a positive map, $\varphi(g_i)$ are positive for all $i \in \{1, \ldots, n\}$. Moreover notice that $\|\sum_{i=1}^n \varphi(g_i)\| = \|\varphi(\sum_{i=1}^n g_i)\| = \|\varphi(1)\|$ and $|\lambda_i| \leq 1$ for all $i \in \{1, \ldots, n\}$. Whence $\|\sum_{i=1}^n \lambda_i \varphi(g_i)\| \leq \|\varphi(1)\|$ by Lemma 2.21. Therefore

$$\|\varphi(f)\| \leq \left\|\varphi\left(f - \sum_{i=1}^n \lambda_i g_i\right)\right\| + \left\|\sum_{i=1}^n \lambda_i \varphi(g_i)\right\| \leq \epsilon \|\varphi\| + \|\varphi(1)\|$$

Thus, as $\|\varphi\| < \infty$ is a constant and $\epsilon > 0$ was arbitrary, $\|\varphi(f)\| \leq \|\varphi(1)\|$. Thus, as f was arbitrary, $\|\varphi\| \leq \|\varphi(1)\|$ as desired. ☐

Remarks 2.24. The above proposition allows us prove that there are positive maps from operator systems that cannot be extended to positive maps on the entire C*-algebra. To see this consider $\varphi : \mathcal{S} \to \mathcal{M}_2(\mathbb{C})$ as constructed in Example 2.15. Then φ cannot be extended to a positive map $\tilde{\varphi} : C(\mathbb{T}) \to \mathcal{M}_2(\mathbb{C})$ since this would imply

$$\|\tilde{\varphi}\| = \|\tilde{\varphi}(1)\| = \|\varphi(1)\| = \frac{1}{2}\|\varphi\|$$

which is impossible as $0 < \|\varphi\| \leq \|\tilde{\varphi}\|$ since $\tilde{\varphi}$ extends φ.

One topic we have yet to address is whether positive linear functionals on operator systems have the same properties as positive linear functional on C*-algebras. The answer is yes (except for the GNS construction) and proofs are almost identical to those given in Chapter 1.

Proposition 2.25. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system. Suppose $\varphi : \mathcal{S} \to \mathbb{C}$ is such that $\|\varphi\| = \varphi(I_{\mathfrak{A}}) = 1$ and suppose $N \in \mathcal{S}$ is a normal operator. Then $\varphi(N) \in \overline{conv(\sigma(N))}$ where $\sigma(N)$ is the spectrum of N and $\overline{conv(\sigma(N))}$ is the closure of the convex hull of $\sigma(N)$.*

Proof. Suppose that $N \in \mathcal{S}$ is a normal operator such that $\varphi(N) \notin \overline{conv(\sigma(N))}$. Basic geometry (via the separating version of the Hahn Banach Theorem) shows that there exists a $z \in \mathbb{C}$ and an $r > 0$ such that

$$\overline{conv(\sigma(N))} \subseteq B_r[z] = \{z' \in \mathbb{C} \mid |z - z'| \leq r\}$$

and $|\varphi(N) - z| > r$ (that is $\overline{conv(\sigma(N))}$ is contained in the closed ball of radius r centred at z yet $\varphi(N)$ is not in this ball).

Consider $T := N - zI_{\mathfrak{A}} \in \mathcal{S}$. Thus T is a normal element such that $\sigma(T) = \sigma(N) - z \subseteq B_r[z] - z = B_r[0]$ where $B_r[0]$ is the closed ball of radius r around the origin. Since T is normal $\|T\| = spr(T) \leq r$. However

$$|\varphi(T)| = |\varphi(N) - z\varphi(I_{\mathfrak{A}})| = |\varphi(N) - z| > r$$

which contradicts the fact that $\|\varphi\| = 1$. ☐

Corollary 2.26. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and let $\varphi : \mathcal{S} \to \mathbb{C}$ be a linear functional. Then φ is positive if and only if $\|\varphi\| = \varphi(I_{\mathfrak{A}})$.*

Proof. If φ is positive, Proposition 2.17 implies that $\|\varphi\| = \varphi(I_{\mathfrak{A}})$.

Next suppose that $\|\varphi\| = \varphi(I_{\mathfrak{A}})$. If $\varphi = 0$, we are done so suppose $\|\varphi\| > 0$ and define $\psi : \mathfrak{A} \to \mathbb{C}$ by $\psi(A) = \frac{1}{\|\varphi\|}\varphi(A)$. Then ψ is a linear functional such that $\psi(I_{\mathfrak{A}}) = \|\psi\| = 1$. By Proposition 2.25 $\psi(N) \in \overline{conv(\sigma(N))}$ whenever N is a normal element of $\mathfrak{A}$. In particular, if $A \in \mathfrak{A}$ is a positive element, $\psi(A) \in \overline{conv(\sigma(A))} \subseteq [0, \infty)$. Whence $\varphi(A) = \|\varphi\| \psi(A) \geq 0$. Hence φ is a positive linear functional. ☐

Proposition 2.27. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and let $\varphi : \mathcal{S} \to \mathbb{C}$ be a positive linear functional. Then there exists a positive linear functional $\tilde{\varphi} : \mathfrak{A} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi}|_{\mathcal{S}} = \varphi$.*

Proof. By the Hahn-Banach Theorem there exists a linear functional $\tilde{\varphi} : \mathfrak{A} \to \mathbb{C}$ such that $\|\tilde{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi}|_{\mathcal{S}} = \varphi$. By Proposition 2.17 $\|\tilde{\varphi}\| = \|\varphi\| = \varphi(I_{\mathfrak{A}}) = \tilde{\varphi}(I_{\mathfrak{A}})$. Whence by Corollary 2.26 $\tilde{\varphi}$ is positive. ☐

Lastly we would like to remark that many authors require that operator systems are closed. The following proposition shows why.

Proposition 2.28. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system, let $\mathfrak{B}$ a C^*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a positive map. Then there exists a positive map $\overline{\varphi} : \overline{\mathcal{S}} \to \mathfrak{B}$ extending φ.*

Proof. Since φ is positive, φ is continuous by Proposition 2.17 and thus extends to a linear map $\overline{\varphi} : \overline{\mathcal{S}} \to \mathfrak{B}$. It remains to show that $\overline{\varphi}$ is positive. Let $A \in \overline{\mathcal{S}}$ be positive. Then there exists a sequence $(A'_n)_{n\geq 1} \subseteq \mathcal{S}$ such that $A = \lim_{n\to\infty} A'_n$. Let $A_n := Re(A'_n) \in \mathfrak{A}$. Since A is positive, A is self-adjoint so $A = A^* = \lim_{n\to\infty}(A'_n)^*$ and thus $A = \lim_{n\to\infty} A_n$.

Let $\epsilon > 0$. Since $A = \lim_{n\to\infty} A_n$, by the lower semicontinuity of the spectrum there exists an $N \in \mathbb{N}$ such that $\sigma(A_n) \subseteq [-\epsilon, spr(A) + \epsilon]$ for all $n \geq N$. Therefore $A_n \geq -\epsilon I_{\mathfrak{A}}$ for all $n \geq N$ so that $\varphi(A_n) \geq -\epsilon\varphi(I_{\mathfrak{A}}) = -\epsilon \|\varphi\|$ for all $n \geq N$. Since $\overline{\varphi}(A) = \lim_{n\to\infty} \varphi(A_n) \in \mathbb{R}$, we obtain that $\overline{\varphi}(A) \geq -\epsilon \|\varphi\|$ for all $\epsilon > 0$. Whence $\overline{\varphi}(A) \geq 0$ so $\overline{\varphi}$ is a positive map. $\square$

We close this chapter by noting that there exists an abstract notion of an operator system. For our purposes, viewing operator systems as unital linear subspaces of C*-algebra that are closed under adjoints will be enough.

3 Completely Positive and Completely Bounded Maps

In this chapter we will begin to examine the subcollection of positive maps known as completely positive maps that were mentioned in the previous chapter. Completely positive maps behave significantly better that positive maps and most positive maps that one usually encounters are completely positive. In this chapter we will develop the basic theory behind completely positive maps. As a result, we will also study maps known as completely bounded maps as the basic theory of both types of maps goes hand-in-hand. To obtain the most generality for completely bounded maps, we make the following definition.

Definition 3.1. Let $\mathfrak{A}$ be a C*-algebra. An operator space $\mathcal{S}$ of $\mathfrak{A}$ is a (not necessarily closed) vector subspace of $\mathfrak{A}$. Notice that operator systems and C*-algebras are operator spaces.

The intuition behind studying completely positive and completely bounded maps is that given an operator space $\mathcal{S}$ there are canonical norms on $n \times n$ matrices with entries in $\mathcal{S}$. In particular, if $\mathcal{S} \subseteq \mathfrak{A}$ is an operator space (operator system, C*-subalgebra) then $\mathcal{M}_n(\mathcal{S}) \subseteq \mathcal{M}_n(\mathfrak{A})$ is easily verified to be an operator space (operator system, C*-subalgebra). In the case $\mathcal{S}$ is an operator system or C*-algebra, this allows us to consider positive maps on $\mathcal{M}_n(\mathcal{S})$. Moreover, the proof of Theorem 1.24 showed us a convenient way of taking a linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ and obtaining a linear map $\varphi_{(n)} : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{M}_n(\mathfrak{B})$.

Definition 3.2. Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator space, let $\mathfrak{B}$ be a C*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a linear map. For each $n \in \mathbb{N}$ define $\varphi_{(n)} : \mathcal{M}_n(\mathcal{S}) \to \mathcal{M}_n(\mathfrak{B})$ by

$$\varphi_{(n)}([A_{i,j}]) = [\varphi(A_{i,j})].$$

It is then clear that if $\psi : \mathcal{S} \to \mathfrak{B}$ is a linear map and $\lambda \in \mathbb{C}$ is a scalar then $(\varphi + \lambda\psi)_{(n)} = \varphi_{(n)} + \lambda\psi_{(n)}$.

Define $\|\varphi\|_n = \left\|\varphi_{(n)}\right\|$ and $\|\varphi\|_{cb} = \sup_n \|\varphi\|_n$ (which is known as the completely bounded norm). The linear map φ is said to be completely bounded if $\|\varphi\|_{cb} < \infty$. It is then clear that the completely bounded norm is indeed a norm on the vector space of all completely bounded maps from $\mathcal{S}$ to $\mathfrak{B}$.

Now suppose $\mathcal{S}$ is an operator system or a C*-algebra. For an $n \in \mathbb{N}$, the linear map φ is said to be n-positive if $\varphi_{(n)}$ is a positive map. The linear map φ is said to be completely positive if $\varphi_{(n)}$ is positive for all n.

Remarks 3.3. Traditional what we have denoted $\varphi_{(n)}$ is denoted φ_n in the literature. We have made this non-standard notation in the hopes that students new to the theory will not confuse these maps with sequences of maps and in the hope that those familiar with the theory will be able to easily follow this notation.

Recall that Proposition 2.8 and Proposition 2.17 imply that positive maps (and thus completely positive maps) are norm continuous. We will use this result without statement for the rest of these notes. We begin with some examples and non-examples of completely positive and completely bounded maps.

Example 3.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a *-homomorphism. It is easy to verify that $\pi_{(n)} : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{M}_n(\mathfrak{B})$ is also a *-homomorphism. Therefore $\pi_{(n)}$ is positive for all $n \in \mathbb{N}$ and thus π is completely positive. Moreover, since every *-homomorphism between C*-algebras is contractive, $\left\|\pi_{(n)}\right\| \leq 1$ for all $n \in \mathbb{N}$. Thus $\|\pi\|_{cb} \leq 1$. Whence π is also completely bounded.

Example 3.5. Let $\mathfrak{A}$ be a C*-algebra and let $S \in \mathfrak{A}$ be arbitrary. Define $\varphi : \mathfrak{A} \to \mathfrak{A}$ by $\varphi(A) = S^*AS$ for all $A \in \mathfrak{A}$. Then

$$\varphi_{(n)}([A_{i,j}]) = [S^*A_{ij}S] = diag_n(S)^*[A_{ij}]diag_n(S)$$

(where $diag_n(S)$ is the $n \times n$ matrix in $\mathcal{M}_n(\mathfrak{A})$ with S in each diagonal entry and 0 in each non-diagonal entry). If $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is positive then $[A_{i,j}] = [N_{i,j}]^*[N_{i,j}]$ so that

$$\varphi_{(n)}([A_{i,j}]) = ([N_{i,j}]diag_n(S))^*([N_{i,j}]diag_n(S)) \geq 0.$$

Hence φ is completely positive. Moreover

$$\left\|\varphi_{(n)}([A_{i,j}])\right\| \leq \|diag_n(S^*)\| \, \|[A_{i,j}]\| \, \|diag_n(S)\| = \|S^*\| \, \|S\| \, \|[A_{i,j}]\|$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ and $n \in \mathbb{N}$. Therefore $\|\varphi\|_{cb} \leq \|S^*\| \, \|S\| = \|S\|^2$.

Similarly, if $S, T \in \mathfrak{A}$ we can define $\varphi : \mathfrak{A} \to \mathfrak{A}$ by $\varphi(A) = SAT$. Then φ is a completely bounded map with $\|\varphi\|_{cb} \leq \|S\| \, \|T\|$. However it is possible that φ is not even positive.

More generally, if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then the map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\varphi(T) = V^*TV$ is completely positive by the same arguments.

Example 3.6. Let $\varphi : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_2(\mathbb{C})$ be defined by $\varphi(A) = A^T$ for all $A \in \mathcal{M}_2(\mathbb{C})$. It was shown in Example 2.5 that φ was positive. However φ is not 2-positive. To see this consider $\varphi_{(2)} : \mathcal{M}_2(\mathcal{M}_2(\mathbb{C})) \simeq \mathcal{M}_4(\mathbb{C}) \to \mathcal{M}_2(\mathcal{M}_2(\mathbb{C})) \simeq \mathcal{M}_4(\mathbb{C})$ (where $\mathcal{M}_2(\mathcal{M}_2(\mathbb{C})) \simeq \mathcal{M}_4(\mathbb{C})$ by Lemma 1.22). Let

$$A := \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{M}_2(\mathcal{M}_2(\mathbb{C})).$$

Then A is positive since

$$\langle A(h_1, \ldots, h_4), (h_1, \ldots, h_4) \rangle = (h_1 + h_4)h_1 + (h_1 + h_4)h_4 = (h_1 + h_4)^2 \geq 0$$

(or alternatively that A is clearly self-adjoint and $A^2 = 2A$ so $\sigma(A) \subseteq \{0, 2\}$). However

$$\varphi_{(2)}(A) = \begin{bmatrix} \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) & \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) & \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is not positive as $\langle \varphi_{(2)}(A)(0, 1, -1, 0), (0, 1, -1, 0) \rangle = -2 < 0$ (or since $\lambda = -1$ is an eigenvalue). Hence φ is not 2-positive. In fact, φ is not n-positive for all $n > 1$ as we will see shortly. It turns out that φ is completely bounded with $\|\varphi\|_{cb} = 2$. The proof of this follows from Proposition 7.2 which we will postpone until our further studies of completely bounded maps and the canonical shuffle.

Example 3.7. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $\{e_m\}_{m \geq 1}$. Consider the linear map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ given by $\varphi(T) = T^t$, the transpose map (that is $\langle T^t e_j, e_i \rangle = \overline{\langle T^* e_j, e_i \rangle}$ defines an element of $\mathcal{B}(\mathcal{H})$). It is easy to verify by definitions that $\|T^t\| = \|T^*\| = \|T\|$ (that is $T^t h = \overline{T^* \overline{h}}$ where $\overline{\sum_{m \geq 1} \lambda_m e_m} = \sum_{m \geq 1} \overline{\lambda_m} e_m$). Thus $\|\varphi\| = 1$.

We claim that $\left\|\varphi_{(n)}\right\| = n$ for all $n \in \mathbb{N}$. To see this, for each $i, j \in \mathbb{N}$ let $E_{i,j} \in \mathcal{B}(\mathcal{H})$ be the operator such that $E_{i,j}\left(\sum_{m \geq 1} \lambda_m e_m\right) = \lambda_j e_i$. Then for each $n \in \mathbb{N}$ consider $A_n = [E_{j,i}] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$. We notice that $A_n^2 = [E_{j,i}]^2 = [\sum_{k=1}^n E_{k,i} E_{j,k}] = [\delta_{i,j} P_n]$ where P_n is the projection onto $span\{e_1, \ldots, e_n\}$. Thus A^2 is the finite direct sum of orthogonal projections and hence a projection. Therefore $\left\|A^2\right\| = 1$. However $A^* = [E_{j,i}]^* = [E_{i,j}^*] = [E_{j,i}]$. Therefore A is self-adjoint and thus $\|A\|^2 = \|A^*A\| = 1$. Hence $\|A\| = 1$. However $\varphi_{(n)}(A_n) = [E_{i,j}]$. We claim that $[E_{i,j}]$ has norm n. To see this we note that $[E_{i,j}]^* = [E_{j,i}^*] = [E_{i,j}]$ and $[E_{i,j}]^2 = [\sum_{k=1}^n E_{i,k} E_{k,j}] = n[E_{i,j}]$. Thus $[E_{i,j}]$ is self-adjoint and the function $z^2 - nz$ is equal to 0 on the functional calculus of $[E_{i,j}]$. Thus $n \in \sigma([E_{i,j}])$ as $[E_{i,j}] \neq 0$. Thus $\|[E_{i,j}]\| = \operatorname{spr}([E_{i,j}]) = n$ (we could have also use the fact that $\frac{1}{n}[E_{i,j}]$ is a projection). Hence $\left\|\varphi_{(n)}(A)\right\| = n$ so that $\left\|\varphi_{(n)}\right\| \geq n$. Thus φ is a continuous linear map that is not completely bounded. Later (Proposition 3.10) it will be possible to show that $\|\varphi_n\| \leq n$.

Next we will show that continuous linear functionals are completely bounded and positive linear functionals are completely positive. This provides one motivations for studying completely bounded and completely positive maps; they are a nice generalization of continuous linear functionals and positive linear functionals that share many of the same properties.

Proposition 3.8. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator space and let $\varphi : \mathcal{S} \to \mathbb{C}$ be a continuous linear functional. Then φ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$. Moreover if $\mathcal{S}$ is an operator system or a C*-algebra and φ is positive then φ is completely positive.*

Proof. To show that φ is completely bounded we will show that $\|\varphi\|_{cb} = \|\varphi\|$. Let $A = [A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ be arbitrary. We desire to show that $\left\|\varphi_{(n)}(A)\right\| \leq \|\varphi\| \|A\|$. To see this we recall that $\left\|\varphi_{(n)}(A)\right\| = \sup\{|\langle \varphi_{(n)}(A)x, y\rangle| \mid x, y \in \mathbb{C}^n, \|x\|, \|y\| \leq 1\}$ as $\varphi_{(n)}(A) \in \mathcal{M}_n(\mathbb{C})$. However, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are unit vectors in $\mathbb{C}^n$, then

$$|\langle \varphi_{(n)}(A)x, y\rangle| = \left|\sum_{i,j=1}^{n} \varphi(A_{i,j})x_j\overline{y_i}\right| = \left|\varphi\left(\sum_{i,j=1}^{n} A_{i,j}x_j\overline{y_j}\right)\right| \leq \|\varphi\| \left\|\sum_{i,j=1}^{n} A_{i,j}x_j\overline{y_j}\right\|.$$

Thus it suffices to show that $\left\|\sum_{i,j=1}^{n} A_{i,j}x_j\overline{y_j}\right\| \leq \|A\|$. To see this, we notice that

$$\begin{bmatrix} \sum_{i,j=1}^{n} A_{i,j}x_j\overline{y_j} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{bmatrix} = \begin{bmatrix} \overline{y_1}1 & \ldots & \overline{y_n}1 \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{bmatrix} \begin{bmatrix} x_1 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_n 1 & 0 & \ldots & 0 \end{bmatrix}.$$

By Lemma 1.23 $\left\|\sum_{i,j=1}^{n} A_{i,j}x_j\overline{y_j}\right\|$ is smaller than the norm of the left-most matrix and the other three matrices have norm $\left(\sum_{i=1}^{n} |y_i|^2\right)^{\frac{1}{2}}$, $\|A\|$, and $\left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$ respectively. However, if x and y are unit vectors, $\|y\|_2, \|x\|_2 \leq 1$. Hence $\left\|\sum_{i,j=1}^{n} a_{i,j}x_j\overline{y_j}\right\| \leq \|A\|$ as desired. Hence φ is completely bounded with $\left\|\varphi_{(n)}\right\| = \|\varphi\|$ for all $n \in \mathbb{N}$.

Now suppose that φ is positive. We need to show that $\varphi_{(n)}$ is positive for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and let $A = [A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ be an arbitrary positive element. To show that $\varphi_{(n)}(A)$ is positive we notice for all $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ that

$$\langle \varphi_{(n)}(A)x, x\rangle = \sum_{i,j=1}^{n} \varphi(A_{i,j})x_j\overline{x_i} = \varphi\left(\sum_{i,j=1}^{n} A_{i,j}x_j\overline{x_i}\right).$$

Therefore, by Proposition 1.21, it suffices to show that $\sum_{i,j=1}^{n} A_{i,j}x_j\overline{x_i}$ is a positive element of $\mathfrak{A}$ as φ is positive. However, using a matrix trick similar to that from Lemma 2.21 and viewing $\mathcal{M}_n(\mathcal{S}) \subseteq \mathcal{M}_n(\mathfrak{A}) \subseteq \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$, we see that

$$\sum_{i,j=1}^{n} A_{i,j}x_j\overline{x_i} = \left(\begin{bmatrix} I_{\mathcal{H}} \\ \vdots \\ I_{\mathcal{H}} \end{bmatrix}\right)^* \left(\begin{bmatrix} x_1 1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_n 1 & 0 & \ldots & 0 \end{bmatrix}\right)^* A \begin{bmatrix} x_1 I_{\mathfrak{A}} & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ x_n 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{H}} \\ \vdots \\ I_{\mathcal{H}} \end{bmatrix}.$$

Hence, since A is positive, the product on the right is positive and thus $\sum_{i,j=1}^{n} A_{i,j}x_j\overline{x_j}$ positive. Since $A \in \mathcal{M}_n(\mathfrak{A})$ was arbitrary and since $n \in \mathbb{N}$ was arbitrary, φ is completely positive as desired. □

Proposition 3.9. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator space and let $\mathfrak{B}$ and $\mathfrak{C}$ be C*-algebras. If $\varphi : \mathcal{S} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{C}$ are completely bounded then $\psi \circ \varphi : \mathcal{S} \to \mathfrak{C}$ is a completely bounded map. If $\mathcal{S}$ is an operator system or C*-algebra and φ and ψ are completely positive maps then $\psi \circ \varphi : \mathcal{S} \to \mathfrak{C}$ is a completely positive map.*

Proof. It is trivial to verify that $(\psi \circ \varphi)_{(n)} = \psi_{(n)} \circ \varphi_{(n)}$. Therefore

$$\left\|(\psi \circ \varphi)_{(n)}\right\| \leq \left\|\psi_{(n)}\right\| \left\|\varphi_{(n)}\right\| \leq \|\psi\|_{cb} \|\varphi\|_{cb}$$

for all $n \in \mathbb{N}$ so $\psi \circ \varphi$ is completely bounded if ψ and φ are completely bounded.

By Lemma 2.19 the composition of positive maps is positive and thus $\psi \circ \varphi$ is completely positive whenever φ and ψ are completely positive. $\square$

Our first result is to show that if φ is continuous then $\varphi_{(n)}$ is continuous for all $n \in \mathbb{N}$. However, we will see that the norms need not behave very well.

Proposition 3.10. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator space, let $\mathfrak{B}$ be a C^*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a continuous linear map. Then $\varphi_{(n)}$ is continuous for all n. Moreover $\left\|\varphi_{(n)}\right\| \leq n \|\varphi\|$.*

Proof. First we may assume that $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then for all $A = [A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ $\varphi_{(n)}([A_{i,j}]) = [\varphi(A_{i,j})]$. Then, by Lemma 1.23, we have that

$$\left\|\varphi_{(n)}(A)\right\| \leq \left(\sum_{i,j=1}^{n} \|\varphi(A_{i,j})\|^2\right)^{\frac{1}{2}} \leq \|\varphi\| \left(\sum_{i,j=1}^{n} \|A_{i,j}\|^2\right)^{\frac{1}{2}} \leq \|\varphi\| \left(\sum_{i,j=1}^{n} \|A\|^2\right)^{\frac{1}{2}} = n \|\varphi\| \|A\|.$$

Thus $\varphi_{(n)}$ is continuous and $\left\|\varphi_{(n)}\right\| \leq n \|\varphi\|$. $\square$

Note Example 3.7 shows that the inequality obtained above is tight. Next we will show that the norms of $\varphi_{(n)}$ behave fairly nice and show that the positivity of $\varphi_{(n)}$ behave as expected.

Proposition 3.11. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system or C^*-algebra and let $\mathfrak{B}$ a C^*-algebra. If $\varphi : \mathcal{S} \to \mathfrak{B}$ is linear then $\left\|\varphi_{(k)}\right\| \leq \left\|\varphi_{(n)}\right\|$ for all $k \leq n$. Moreover, if φ is n-positive then φ is k-positive for all $k \leq n$. Consequently φ is not k-positive for all $k \geq n$ if φ is not n-positive.*

Proof. Suppose $k \leq n$. Then each $A \in \mathcal{M}_k(\mathfrak{A})$ we define

$$A' := \begin{bmatrix} A & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & 0_{(n-k)\times(n-k)} \end{bmatrix} \in \mathcal{M}_n(\mathcal{S}).$$

Therefore the map from $\mathcal{M}_k(\mathfrak{A})$ to $\mathcal{M}_n(\mathfrak{A})$ defined by $A \mapsto A'$ is a well-defined, injective *-homomorphism and therefore isometric. Hence $\|A'\| = \|A\|$ for all $A \in \mathcal{M}_k(\mathcal{S})$.

Similarly since

$$\varphi_{(n)}(A') = \begin{bmatrix} \varphi_{(k)}(A) & 0_{k\times(n-k)} \\ 0_{(n-k)\times k} & 0_{(n-k)\times(n-k)} \end{bmatrix} \in \mathcal{M}_n(\mathfrak{B})$$

$\left\|\varphi_{(k)}(A)\right\| = \left\|\varphi_{(n)}(A')\right\|$. Thus

$$\begin{aligned} \left\|\varphi_{(k)}\right\| &= \sup\{\left\|\varphi_{(k)}(A)\right\| \mid A \in \mathcal{M}_k(\mathcal{S}), \|A\| \leq 1\} \\ &= \sup\{\left\|\varphi_{(n)}(A')\right\| \mid A \in \mathcal{M}_k(\mathcal{S}), \|A\| \leq 1\} \\ &\leq \sup\{\left\|\varphi_{(n)}(B)\right\| \mid B \in \mathcal{M}_n(\mathcal{S}), \|B\| \leq 1\} \\ &= \left\|\varphi_{(n)}\right\|. \end{aligned}$$

Thus we have obtained our norm inequality.

Now we desire to show that if φ is n-positive then φ_k is positive whenever $k \leq n$. Let $A \in \mathcal{M}_k(\mathcal{S})$ be positive. Then $A' \geq 0$ since A' is the image of the positive operator A under a *-homomorphism as described above. Thus $\varphi_{(n)}(A') \geq 0$ as φ is n-positive. But then, if we view $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ by the GNS construction, we obtain that for all $h = (h_1, \ldots, h_k) \in \mathcal{H}_2^{\oplus k}$

$$\langle \varphi_{(k)}(A)h, h\rangle = \langle \varphi_{(n)}(A')(h_1, \ldots, h_k, 0, \ldots, 0), (h_1, \ldots, h_k, 0, \ldots, 0)\rangle \geq 0$$

as $\varphi_{(n)}(A') \geq 0$. Hence $\varphi_{(k)}(A) \geq 0$ for all $A \geq 0$, $A \in \mathcal{M}_k(\mathcal{S})$. Hence $\varphi_{(k)}$ is positive. $\square$

We note that the proof of the above result could be slightly simplified if Lemma 3.17 part (3) was proved first although the proof of said lemma is more complicated than the above argument.

So far we have seen that continuous linear functionals are completely bounded and positive linear functionals are completely positive. Our next goal is to develop some similar results where continuity/positivity imply completely bounded/positive.

Let X be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous functions on X. We wish to consider $\mathcal{M}_n(C(X))$ for all $n \in \mathbb{N}$. Notice that each element $F = [f_{i,j}]$ of $\mathcal{M}_n(C(X))$ can be viewed as a continuous matrix-valued function from X (continuous as if $x_n \to x$ in X then each component of F converges (by Lemma 1.23) and hence $[f_{i,j}(x_n)] \to [f_{i,j}(x)]$ in the norm on $\mathcal{M}_n(\mathbb{C})$ (by Lemma 1.23)).

Lemma 3.12. *Let X be a compact Hausdorff space and let $C(X)$ be the C*-algebra of continuous functions on X. Then $\|F\| = \sup\{\|F(x)\|_{\mathcal{M}_n(\mathbb{C})} \mid x \in X\}$ is the unique C*-norm on $\mathcal{M}_n(C(X))$.*

Proof. We note that clearly $\|F\| \geq 0$, $\|F\| = 0$ if and only if $F(x) = [0]$ for all $x \in X$ if and only if $F = [0]$, the triangle inequality and scalar property clearly hold as they hold for $\mathcal{M}_n(\mathbb{C})$, and, since each component of F is bounded on X being a continuous function on a compact set,

$$\|F(x)\| \leq \left(\sum_{i,j=1}^{n} |F_{i,j}(x)|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i,j=1}^{n} \|F_{i,j}\|_\infty^2\right)^{\frac{1}{2}} < \infty$$

by Lemma 1.23. Therefore this is a norm.

We claim that this norm is a C*-norm. First $\mathcal{M}_n(C(X))$ is complete with respect to this norm by applying Lemma 1.23 and the same arguments used in Theorem 1.24. To see that this norm is submultiplicative, we notice for every $F_1, F_2 \in \mathcal{M}_n(C(X))$ and every $x \in X$ that

$$\|F_1F_2(x)\| = \|F_1(x)F_2(x)\| \leq \|F_1(x)\| \, \|F_2(x)\| \leq \|F_1\| \, \|F_2\|$$

as the norm on $\mathcal{M}_n(\mathbb{C})$ is submultiplicative. Hence, as this holds for all $x \in X$, $\|F_1F_2\| \leq \|F_1\| \, \|F_2\|$ so that this norm on $\mathcal{M}_n(C(X))$ is submultiplicative. Lastly, to see that this norm is a C*-norm, we notice that for all $x \in X$ and $F \in \mathcal{M}_n(C(X))$ $\|F^*F(x)\| = \|F^*(x)\| \, \|F(x)\| = \|F(x)\|^2$ as the norm on $\mathcal{M}_n(\mathbb{C})$ is a C*-norm. Hence $\|F^*F\| = \|F\|^2$ so that this norm is a C*-algebra norm. Consequently, as each *-algebra has a unique C*-norm, this norm is the C*-norm on $\mathcal{M}_n(C(X))$. ☐

With this in hand we can easily prove the following lemma and subsequent corollary.

Lemma 3.13. *Let X be a compact Hausdorff space and $A = [f_{i,j}] \in \mathcal{M}_n(C(X))$. Then A is positive in $\mathcal{M}_n(C(X))$ if and only if $A(x) \geq 0$ as an element of $\mathcal{M}_n(\mathbb{C})$ for all $x \in X$.*

Proof. In the case $n = 1$, this is equivalent to saying that a continuous function on a compact Hausdorff space is positive if and only if its range is positive. Since this is true, it seems logical that this lemma should be true. Suppose $A \geq 0$ in $\mathcal{M}_n(C(X))$. Then there exists an $N \in \mathcal{M}_n(C(X))$ such that $A = N^*N$. Thus for all $x \in X$ $A(x) = (N^*N)(x) = N(x)^*N(x) \geq 0$

Now suppose for all $x \in X$ that $A(x) \geq 0$. Therefore $A(x)^* = A(x)$ for all $x \in X$ and thus A is self-adjoint. Moreover, for all $\lambda \in \mathbb{C}$ with $\lambda \notin [0, \infty)$, the matrix $\lambda I_n - A(x)$ is invertible for all $x \in X$. Thus for all $x \in X$ the matrix $F_\lambda(x) = (\lambda I_n - A(x))^{-1}$ exists. By the cofactor expansion of a matrix we can see that $F_\lambda \in \mathcal{M}_n(C(X))$ as each entry with be a rational function of linear combinations of the continuous functions $\lambda\delta_{i,j} - f_{i,j}(x)$ with non-vanishing denominator (this is by Cramer's rule. The denominator is $det(\lambda I_n - A(x)) \neq 0$ as the matrix is invertible). Moreover, for all $x \in X$

$$(\lambda I_n - A(x))F_\lambda(x) = I_n = F_\lambda(x)(\lambda I_n - A(x))$$

and hence

$$(\lambda I_{\mathcal{M}_n(C(X))} - A)F_\lambda = I_{\mathcal{M}_n(C(X))} = F_\lambda(\lambda I_{\mathcal{M}_n(C(X))} - A)$$

as element of $\mathcal{M}_n(C(X))$. Hence $\lambda I_{\mathcal{M}_n(C(X))} - A$ is invertible in $\mathcal{M}_n(C(X))$ for all $\lambda \notin [0, \infty)$ and hence A is positive. ☐

Corollary 3.14. *Let $\mathcal{S}$ be an operator space, let X be a compact Hausdorff space, and let $\varphi : \mathcal{S} \to C(X)$ be a bounded linear map. Then φ is completely bounded and $\|\varphi\|_{cb} = \|\varphi\|$. Moreover, if $\mathcal{S}$ is an operator system or C*-algebra and φ is positive then φ is completely positive and $\|\varphi\|_{cb} = \|\varphi(1)\|$.*

Proof. We shall follow the same idea as Proposition 2.20 and apply the results for linear functionals. Let $x \in X$ be arbitrary. We define the function $\delta_x : C(X) \to \mathbb{C}$ by $\delta_x(f) = f(x)$ for all $f \in C(X)$. Then the map $\delta_x \circ \varphi : \mathcal{S} \to \mathbb{C}$ is a continuous linear functional. Hence $\left\|(\delta_x \circ \varphi)_{(n)}\right\| = \|\delta_x \circ \varphi\|$ for all $n \in \mathbb{N}$ by Proposition 3.8. However, for all $A = [A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$,

$$(\delta_x \circ \varphi)_{(n)}(A) = [\delta_x \circ \varphi(A_{i,j})] = [\varphi(A_{i,j})(x)] = \varphi_{(n)}(A)(x).$$

Hence for all $x \in X$ and $A \in \mathcal{M}_n(\mathcal{S})$ $\left\|\varphi_{(n)}(A)(x)\right\| \leq \|\delta_x \circ \varphi\| \, \|A\|$. Thus

$$\begin{aligned}
\left\|\varphi_{(n)}(A)\right\| &= \sup\{\left\|\varphi_{(n)}(A)(x)\right\| \mid x \in X\} \\
&\leq \|A\| \sup\{\|\delta_x \circ \varphi\| \mid x \in X\} \\
&= \|A\| \sup\{\|\varphi(B)(x)\| \mid x \in X, B \in \mathcal{S}, \|B\| \leq 1\} \\
&= \|A\| \sup\{\|\varphi(B)\| \mid B \in \mathcal{S}, \|B\| \leq 1\} \\
&= \|A\| \, \|\varphi\|
\end{aligned}$$

for all $A \in \mathcal{M}_n(\mathcal{S})$. Thus $\left\|\varphi_{(n)}\right\| \leq \|\varphi\|$ for all $n \in \mathbb{N}$ and hence φ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$.

Suppose that φ is positive. We need to show that $\varphi_{(n)}$ is positive. Thus let $A = [A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ be an arbitrary positive element. Since φ is a positive map and δ_x is a positive linear functional for all $x \in X$ (by Example 1.3), $\delta_x \circ \varphi$ is a positive linear functional for all $x \in X$. Hence $(\delta_x \circ \varphi)_{(n)}$ is a positive map by Proposition 3.8. Since $\varphi_{(n)}(A)(x) = (\delta_x \circ \varphi)_{(n)}(A)$ and $A \geq 0$, $\varphi_{(n)}(A)(x) \geq 0$ for all $x \in X$. Whence $\varphi_{(n)}(A) \geq 0$ by Lemma 3.13. Hence φ is completely positive. □

So far the theory of completely bounded maps and completely positive maps have not been that different. Since our current goal is to study completely positive maps, we will focus on results pertaining to completely positive maps for the remainder of the section. Our first result is that positive maps from abelian C*-algebras are automatically completely positive. To prove this we need a small lemma and to make use of ideas from Proposition 2.23.

Lemma 3.15. *Let $[\alpha_{i,j}] \in \mathcal{M}_n(\mathbb{C})$ be a positive matrix, let $\mathfrak{B}$ be a C*-algebra, and let B be a positive element of $\mathfrak{B}$. Then $[\alpha_{i,j}B]$ is positive in $\mathcal{M}_n(\mathfrak{B})$.*

Proof. By considering the unitization of $\mathfrak{B}$ if $\mathfrak{B}$ is not unital we may assume $\mathfrak{B}$ is unital. First we note that since $[\alpha_{i,j}]$ is positive in $\mathcal{M}_n(\mathbb{C})$ there exists an $[\lambda_{i,j}] \in \mathcal{M}_n(\mathbb{C})$ such that $[\alpha_{i,j}] = [\lambda_{i,j}]^*[\lambda_{i,j}]$. Thus $[\alpha_{i,j}I_{\mathfrak{B}}] = [\lambda_{i,j}I_{\mathfrak{B}}]^*[\lambda_{i,j}I_{\mathfrak{B}}] \in \mathcal{M}_n(\mathfrak{B})$ is a positive matrix.

Since B is positive we may write $B = N^*N$ for some $N \in \mathfrak{B}$. Hence, if $diag(A)$ is the $n \times n$ matrix diagonal matrix with A in each entry of the diagonal,

$$\begin{aligned}
[\alpha_{i,j}B] = [\alpha_{i,j}I_{\mathfrak{B}}]diag(B) &= [\lambda_{i,j}I_{\mathfrak{B}}]^*[\lambda_{i,j}I_{\mathfrak{B}}]diag(N^*N) \\
&= [\lambda_{i,j}I_{\mathfrak{B}}]^*[\lambda_{i,j}I_{\mathfrak{B}}]diag(N^*)diag(N) \\
&= diag(N)^*[\lambda_{i,j}I_{\mathfrak{B}}]^*[\lambda_{i,j}I_{\mathfrak{B}}]diag(N) \\
&= ([\lambda_{i,j}I_{\mathfrak{B}}]diag(N))^*[\lambda_{i,j}I_{\mathfrak{B}}]diag(N) \geq 0
\end{aligned}$$

as the diagonal matrices will commute with $[\lambda_{i,j}I_{\mathfrak{B}}]$ as N commutes with scalar multiples of $I_{\mathfrak{B}}$. □

Theorem 3.16. *Let $\mathfrak{B}$ be a C*-algebra, let X be a compact Hausdorff space, and let $\varphi : C(X) \to \mathfrak{B}$ be a positive map. Then φ is completely positive.*

Proof. Fix $n \in \mathbb{N}$ and let $A \in \mathcal{M}_n(C(X))$ be an arbitrary positive element. It suffices to show that $\varphi_{(n)}(A) \geq 0$. To show this we will follow the ideas used in the proof of Proposition 2.23 and approximate A using a partition of unity. In this manner, let $\epsilon > 0$ and suppose $A = [f_{i,j}]$ where $f_{i,j} \in C(X)$. For each $y \in X$ let $U_y := \bigcap_{i,j=1}^n f_{i,j}^{-1}(B_\epsilon(f_{i,j}(y)))$ (where $B_\epsilon(z) := \{z' \in X \mid |z - z'| < \epsilon\}$). Since each $f_{i,j}$ is continuous and we are taking the intersection of a finite number of open sets, U_y is an open set containing y. Since $X = \bigcup_{y\in X} U_y$ and X is compact there exists a finite open cover $\{U_{x_k}\}_{k=1}^m$ of X such that $|f_{i,j}(x) - f_{i,j}(x_k)| < \epsilon$ for all $x \in U_{x_k}$ and for all i, j. Hence Theorem 2.22 implies that there exists a partition of unity $\{g_k\}_{k=1}^m$ such that $g_k : X \to [0, 1]$ are continuous, $\sum_{k=1}^m g_k = I_{C(X)}$, and $g_k|U_{x_k}^c = 0$.

Let $A_k := A(x_k) = [f_{i,j}(x_k)] = [a_{i,j,k}]$. Then A_k is a positive matrix by Lemma 3.13. Next notice if $x \in U_{x_k}$ then, by an application of Lemma 1.23,

$$\|A(x) - A_k\| \leq \left(\sum_{i,j}^{n} |f_{i,j}(x) - f_{i,j}(x_k)|^2\right)^{\frac{1}{2}} \leq n\epsilon.$$

Consider the element $A' := \sum_k g_k A_k \in \mathcal{M}_n(C(X))$. Then for all $x \in X$

$$\|A(x) - A'(x)\| = \left\|\sum_{k=1}^{m} g_k(x)(A(x) - A_k)\right\| \leq \sum_{k=1}^{m} g_k(x) \|A(x) - A_k\| \leq \sum_{k=1}^{m} g_k(x) n\epsilon = n\epsilon$$

as if $x \notin U_{x_k}$ then $g_k = 0$ and if $x \in U_{x_k}$ then $\|A(x) - A_k\| \leq n\epsilon$. Since this holds for all $x \in X$ $\|A - A'\| \leq n\epsilon$.

We can now use A' to show that $\varphi_{(n)}(A)$ must be positive. First we notice that $\varphi_{(n)}(g_k(x)A_k) = \varphi_{(n)}([g_k(x)a_{i,j,k}]) = [a_{i,j,k}\varphi(g_k(x))]$. However, as the range of g_k was $[0, 1]$, g_k is positive in $C(X)$ and hence $\varphi(g_k)$ is positive as φ is positive. Moreover $A_k = [a_{i,j,k}]$ is a positive scalar matrix. Hence $\varphi_{(n)}(g_k(x)A_k) = [a_{i,j,k}\varphi(g_k(x))]$ is positive by Lemma 3.15. Therefore $\varphi_{(n)}(A')$ is positive. Hence, as $\varphi_{(n)}$ is continuous by Proposition 3.10,

$$\left\|\varphi_{(n)}(A) - \varphi_{(n)}(A')\right\| \leq \left\|\varphi_{(n)}\right\| \|A - A'\| \leq \left\|\varphi_{(n)}\right\| n\epsilon$$

Hence $\varphi_{(n)}(A)$ is within a constant scalar multiple of ϵ of a positive element of $\mathfrak{B}$. Thus $\varphi_{(n)}(A)$ is a limit of positive elements of $\mathfrak{B}$ and thus is positive by Lemma 1.6. ☐

To develop more properties about completely positive maps it would be nice to determine more information about when elements of $\mathcal{M}_n(\mathfrak{A})$ are positive. The following lemma contains the most common results and is essential to studying such positive elements.

Lemma 3.17. *Let $\mathfrak{A}$ be a unital C^*-algebra, let $A, B, P, Q \in \mathfrak{A}$, and suppose $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$. Then*

1. $\begin{bmatrix} I_{\mathfrak{A}} & A \\ A^* & B \end{bmatrix}$ *is positive in $\mathcal{M}_2(\mathfrak{A})$ if and only if $A^*A \leq B$.*

2. *If* $\begin{bmatrix} P & A \\ A^* & P \end{bmatrix}$ *is positive in $\mathcal{M}_2(\mathfrak{A})$ then $A^*A \leq \|P\| P$ and consequently $\|A\| \leq \|P\|$.*

3. $\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix}$ *is positive in $\mathcal{M}_2(\mathcal{B}(\mathcal{H}))$ if and only if $P \geq 0$, $Q \geq 0$, and $|\langle A\eta, \xi\rangle|^2 \leq \langle P\xi, \xi\rangle\langle Q\eta, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.*

Proof. All three parts are basically brute computations using Proposition 1.21. We proceed to prove part (1). Let

$$M := \begin{bmatrix} I_{\mathfrak{A}} & A \\ A^* & B \end{bmatrix}.$$

Suppose $A^*A \leq B$ (and thus B is positive). Then $\langle (B - A^*A)\eta, \eta\rangle \geq 0$ so that $\langle B\eta, \eta\rangle \geq \langle A\eta, A\eta\rangle = \|A\eta\|^2$ for all $\eta \in \mathcal{H}$. Thus for all $\xi \oplus \eta \in \mathcal{H}^{\oplus 2}$

$$\begin{aligned}\langle M(\xi \oplus \eta), \xi \oplus \eta\rangle &= \langle \xi, \xi\rangle + \langle A\eta, \xi\rangle + \langle A^*\xi, \eta\rangle + \langle B\eta, \eta\rangle \\ &\geq \|\xi\|^2 - 2\|A\eta\|\|\xi\| + \langle B\eta, \eta\rangle \\ &\geq \|\xi\|^2 - 2\|A\eta\|\|\xi\| + \|A\eta\|^2 \\ &= (\|\xi\| - \|A\eta\|)^2 \geq 0\end{aligned}$$

(where the first inequality makes sense since M is self-adjoint so $\langle M(\xi, \eta), (\xi, \eta)\rangle$ is real and $\langle \xi, \xi\rangle$ and $\langle B\eta, \eta\rangle$ are positive as B is positive). Hence M is positive when $B \geq A^*A$.

Next suppose that $B \ngeq A^*A$. If B is not positive then there exists an $\eta \in \mathcal{H}$ such that $\langle M(0, \eta), (0, \eta)\rangle = \langle B\eta, \eta\rangle \ngeq 0$ and thus M would not be positive. Thus we may assume that $B \geq 0$ so that $\langle B\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}$. Since $B \ngeq A^*A$ there exists a vector $\eta_0 \in \mathcal{H}$ such that $\langle (B - A^*A)\eta_0, \eta_0\rangle \ngeq 0$. Since $A^*A \geq 0$ this forces $\langle (B - A^*A)\eta_0, \eta_0\rangle$ to be real and strictly less than 0. Thus $0 \leq \langle B\eta_0, \eta_0\rangle < \|A\eta_0\|^2$ for some $\eta_0 \in \mathcal{H}$. As $\|A\eta_0\| \neq 0$, we can let $\eta := \frac{\eta_0}{\|A\eta_0\|} \in \mathcal{H}$. Thus $0 \leq \langle B\eta, \eta\rangle < \|A\eta\|^2 = 1$. Thus, if we let $\xi := -A\eta \in \mathcal{H}$, then $\|\xi\| = 1$ and

$$\begin{aligned}\langle M(\xi \oplus \eta), \xi \oplus \eta\rangle &= \langle \xi, \xi\rangle + \langle A\eta, \xi\rangle + \langle A^*\xi, \eta\rangle + \langle B\eta, \eta\rangle \\ &= \|\xi\|^2 - 2\|A\eta\|^2 + \langle B\eta, \eta\rangle \\ &= 1 - 2 + \langle B\eta, \eta\rangle \\ &< 1 - 2 + 1 = 0\end{aligned}$$

and thus the matrix is not positive when $B \ngeq A^*A$.

To prove (2), we will show that if $A^*A \nleq \|P\| P$ then the desired matrix cannot be positive using the same technique as in part (1). Let

$$M := \begin{bmatrix} P & A \\ A^* & P \end{bmatrix}$$

and suppose that $\|P\| P \ngeq A^*A$. If P is not positive, there exists an $\eta \in \mathcal{H}$ that $\langle M(0, \eta), (0, \eta)\rangle = \langle P\eta, \eta\rangle \ngeq 0$ and thus M would not be positive. Hence we may assume that $P \geq 0$ so that $\langle P\eta, \eta\rangle \geq 0$ for all $\eta \in \mathcal{H}$. Since $\|P\| P \ngeq A^*A$ there exists a vector $\eta_0 \in \mathcal{H}$ such that $\langle (\|P\| P - A^*A)\eta_0, \eta_0\rangle \ngeq 0$. Since $A^*A \geq 0$ this forces $\langle (\|P\| P - A^*A)\eta_0, \eta_0\rangle$ to be real and strictly less than 0. Thus $0 \leq \|P\| \langle P\eta_0, \eta_0\rangle < \|A\eta\|^2$ for some $\eta_0 \in \mathcal{H}$. As $\|A\eta_0\| \neq 0$, we can let $\eta := \frac{\eta_0}{\|A\eta_0\|} \in \mathcal{H}$. Thus $0 \leq \|P\| \langle P\eta, \eta\rangle < \|A\eta\|^2 = 1$. Notice that if $P = 0$ and we let $\xi := -A\eta \in \mathcal{H}$ then

$$\langle M(\xi \oplus \eta), \xi \oplus \eta\rangle = \langle P\xi, \xi\rangle + \langle A\eta, \xi\rangle + \langle A^*\xi, \eta\rangle + \langle P\eta, \eta\rangle = -2\|A\eta\|^2 < 0.$$

Else, if $P \neq 0$ and we let $\xi := -\frac{1}{\|P\|}A\eta \in \mathcal{H}$, then

$$\begin{aligned}\langle M(\xi \oplus \eta), \xi \oplus \eta\rangle &= \langle P\xi, \xi\rangle + \langle A\eta, \xi\rangle + \langle A^*\xi, \eta\rangle + \langle P\eta, \eta\rangle \\ &= \langle P\xi, \xi\rangle - 2\|A\eta\|^2 \frac{1}{\|P\|} + \langle P\eta, \eta\rangle \\ &< \|P\| \|\xi\|^2 - 2\|A\eta\|^2 \frac{1}{\|P\|} + \|A\eta\|^2 \frac{1}{\|P\|} \\ &= \|P\| \left\| -\frac{1}{\|P\|}A\eta \right\|^2 - \|A\eta\|^2 \frac{1}{\|P\|} = 0\end{aligned}$$

and thus the matrix is not positive when $\|P\| P \ngeq A^*A$. Thus if M is positive then $A^*A \leq \|P\| P \leq \|P\|^2 I_{\mathcal{H}}$ and consequently $\|A\|^2 = \|A^*A\| \leq \|P\|^2$.

Lastly, part (3) follows is very similar to part (1). Let

$$M := \begin{bmatrix} P & A \\ A^* & Q \end{bmatrix}.$$

Suppose that P and Q are positive and $|\langle A\eta, \xi\rangle|^2 \leq \langle P\xi, \xi\rangle\langle Q\eta, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. Then for all $\xi \oplus \eta \in \mathcal{H}^{\oplus 2}$

$$\begin{aligned} \langle M(\xi \oplus \eta), \xi \oplus \eta\rangle &= \langle P\xi, \xi\rangle + \langle A\eta, \xi\rangle + \langle A^*\xi, \eta\rangle + \langle Q\eta, \eta\rangle \\ &\geq \langle P\xi, \xi\rangle - 2|\langle A\eta, \xi\rangle| + \langle Q\eta, \eta\rangle \\ &\geq \langle P\xi, \xi\rangle - 2\langle P\xi, \xi\rangle\langle Q\eta, \eta\rangle + \langle Q\eta, \eta\rangle \\ &= (\langle P\xi, \xi\rangle - \langle Q\eta, \eta\rangle)^2 \geq 0 \end{aligned}$$

(where the first inequality holds since $\langle M(\xi, \eta), (\xi, \eta)\rangle$ is real as M is self-adjoint and $\langle P\xi, \xi\rangle$ and $\langle Q\eta, \eta\rangle$ are positive as P and Q are real). Thus M is positive.

Next suppose one of the conditions of part (3) fail. If Q is not positive then there exists an $\eta \in \mathcal{H}$ such that $\langle M(0, \eta), (0, \eta)\rangle = \langle Q\eta, \eta\rangle \not\geq 0$ and thus M would not be positive. Similarly if P is not positive then there exists a $\xi \in \mathcal{H}$ such that that $\langle M(\xi, 0), (\xi, 0)\rangle = \langle P\xi, \xi\rangle \not\geq 0$ and thus M would not be positive. Thus we may assume that $P \geq 0$ and $Q \geq 0$ so that $\langle P\xi, \xi\rangle \geq 0$ and $\langle Q\eta, \eta\rangle \geq 0$ for all $\xi, \eta \in \mathcal{H}$. Lastly suppose $|\langle A\eta_0, \xi\rangle| > \langle P\xi, \xi\rangle^{\frac{1}{2}}\langle Q\eta_0, \eta_0\rangle^{\frac{1}{2}}$ for some $\xi, \eta_0 \in \mathcal{H}$. Let $\theta \in [0, 2\pi)$ be such that

$$|\langle A\eta_0, \xi\rangle| = -e^{i\theta}\langle A\eta_0, \xi\rangle$$

and let $\eta := e^{i\theta}\eta_0 \in \mathcal{H}$. Then

$$-\langle A\eta, \xi\rangle = |\langle A\eta_0, \xi\rangle| > \langle P\xi, \xi\rangle^{\frac{1}{2}}\langle Q\eta_0, \eta_0\rangle^{\frac{1}{2}} = \langle P\xi, \xi\rangle^{\frac{1}{2}}\langle Q\eta, \eta\rangle^{\frac{1}{2}} \geq 0.$$

Suppose $\langle P\xi, \xi\rangle^{\frac{1}{2}} = 0$. Since $\langle A\eta, \xi\rangle < 0$, there exists an $a \in \mathbb{R}$ so that $2a\langle A\eta, \xi\rangle + \langle Q\eta, \eta\rangle < 0$. Then

$$\begin{aligned} \langle M(a\xi \oplus \eta), a\xi \oplus \eta\rangle &= a^2\langle P\xi, \xi\rangle + a\langle A\eta, \xi\rangle + a\langle A^*\xi, \eta\rangle + \langle Q\eta, \eta\rangle \\ &= 2a\langle A\eta, \xi\rangle + \langle Q\eta, \eta\rangle < 0 \end{aligned}$$

and thus M can not be positive. Similarly, suppose $\langle Q\eta, \eta\rangle^{\frac{1}{2}} = 0$. Since $\langle A\eta, \xi\rangle < 0$, there exists an $a \in \mathbb{R}$ so that $2a\langle A\eta, \xi\rangle + \langle P\xi, \xi\rangle < 0$. Then

$$\begin{aligned} \langle M(\xi \oplus a\eta), \xi \oplus a\eta\rangle &= \langle P\xi, \xi\rangle + a\langle A\eta, \xi\rangle + a\langle A^*\xi, \eta\rangle + a^2\langle Q\eta, \eta\rangle \\ &= \langle P\xi, \xi\rangle + 2a\langle A\eta, \xi\rangle < 0 \end{aligned}$$

and thus M cannot be positive. Lastly, suppose $b = \langle P\xi, \xi\rangle^{\frac{1}{2}} > 0$ and $a = \langle Q\eta, \eta\rangle^{\frac{1}{2}} > 0$. Then

$$\begin{aligned} \langle M(a\xi \oplus b\eta), a\xi \oplus b\eta\rangle &= a^2\langle P\xi, \xi\rangle + ab\langle A\eta, \xi\rangle + ab\langle A^*\xi, \eta\rangle + b^2\langle Q\eta, \eta\rangle \\ &= a^2\langle P\xi, \xi\rangle - 2ab\langle A\eta, \xi\rangle + b^2\langle Q\eta, \eta\rangle \\ &< a^2\langle P\xi, \xi\rangle - 2ab\langle P\xi, \xi\rangle^{\frac{1}{2}}\langle Q\eta, \eta\rangle^{\frac{1}{2}} + b^2\langle Q\eta, \eta\rangle \\ &= \left(a\langle P\xi, \xi\rangle^{\frac{1}{2}} - b\langle Q\eta, \eta\rangle^{\frac{1}{2}}\right)^2 = 0 \end{aligned}$$

and thus M cannot be positive. □

It worthwhile to mention that the converse of Lemma 3.17 part (2) fails. Indeed if we let

$$P := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C}) \qquad A := \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \in \mathcal{M}_2(\mathbb{C})$$

then $P \geq 0$ and $A^*A = P = \|P\| P$ yet

$$\left[\begin{array}{cc} P & A \\ A^* & P \end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

is not positive by normal methods (i.e. it has a negative eigenvalue and fails $\langle Bh, h\rangle \geq 0$ for $h = (0, 1, 1, 0)$).

It is also interesting to note that combining parts (1) and (3) give the interesting identity that $A^*A \leq B$ if and only if $|\langle Ax, y\rangle|^2 \leq \|x\|^2 \langle By, y\rangle$ for all $x, y \in \mathcal{H}$.

So far, every completely positive map we have seen has been completely bounded. Lemma 3.17 will allow us to show that every completely positive map is completely bounded. We will deal with the unital case separately as it is easier and more elegant than the non-unital case.

Theorem 3.18. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system, let $\mathfrak{B}$ be a C*-algebra, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a completely positive map. Then φ is completely bounded and $\|\varphi(I_{\mathfrak{A}})\| = \|\varphi\| = \|\varphi\|_{cb}$.*

Proof. From definition of $\|\cdot\|$ and $\|\cdot\|_{cb}$ it is clear that $\|\varphi(I_{\mathfrak{A}})\| \leq \|\varphi\| \leq \|\varphi\|_{cb}$. Thus it suffices to show that $\left\|\varphi_{(n)}\right\| \leq \|\varphi(I_{\mathfrak{A}})\|$ for all $n \in \mathbb{N}$. Fix an arbitrary $A \in \mathcal{M}_n(\mathcal{S})$ with $\|A\| \leq 1$ and let I_n be the identity matrix in $\mathcal{M}_n(\mathcal{S})$. Then the matrix

$$\left[\begin{array}{cc} I_n & A \\ A^* & I_n \end{array}\right]$$

is positive in $\mathcal{M}_2(\mathcal{M}_n(\mathcal{S})) \simeq \mathcal{M}_{2n}(\mathcal{S})$ by Lemma 3.17 part (1) as $\|A\| \leq 1$ so that $A^*A \leq I_n$. Therefore, since φ is completely positive,

$$\varphi_{(2n)}\left(\left[\begin{array}{cc} I_n & A \\ A^* & I_n \end{array}\right]\right) = \left[\begin{array}{cc} \varphi_{(n)}(I_n) & \varphi_{(n)}(A) \\ \varphi_{(n)}(A^*) & \varphi_{(n)}(I_n) \end{array}\right]$$

is a positive matrix. Thus $\left\|\varphi_{(n)}(A)\right\| \leq \left\|\varphi_{(n)}(I_n)\right\|$ by Lemma 3.17 part (2). However $\varphi_{(n)}(I_n)$ is the $n \times n$ diagonal matrix with $\varphi(I_{\mathfrak{A}})$ in each entry along the diagonal so that $\left\|\varphi_{(n)}(A)\right\| = \|\varphi(I_{\mathfrak{A}})\|$. Whence, as $A \in \mathcal{M}_n(\mathfrak{A})$ and $n \in \mathbb{N}$ were arbitrary, $\left\|\varphi_{(n)}\right\| \leq \|\varphi(I_{\mathfrak{A}})\|$ and the result follows. $\square$

We note that Theorem 3.16 and Theorem 3.18 together imply Proposition 2.23.

To deal with the case of completely positive maps with non-unital domains, we would like to have a result similar to Proposition 1.11. However a close investigation of Proposition 1.11 reveals that we need a Cauchy-Schwarz inequality. It was easy to develop such an inequality for positive linear functionals. Next we will develop one such identity for completely positive maps (and another will be developed later).

Lemma 3.19. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a 2-positive map. Then $\|\varphi(A^*B)\|_{\mathfrak{B}} \leq \|\varphi(A^*A)\|_{\mathfrak{B}}^{\frac{1}{2}} \|\varphi(B^*B)\|_{\mathfrak{B}}^{\frac{1}{2}}$ for all $A, B \in \mathfrak{A}$.*

Proof. By the GNS construction we may assume that $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$. Fix $A, B \in \mathfrak{A}$ and consider

$$M := \left[\begin{array}{cc} A^*A & A^*B \\ B^*A & B^*B \end{array}\right] = \left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]^* \left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right] \geq 0.$$

Since φ is 2-positive

$$\varphi_{(2)}(M) = \left[\begin{array}{cc} \varphi(A^*A) & \varphi(A^*B) \\ \varphi(B^*A) & \varphi(B^*B) \end{array}\right] = \left[\begin{array}{cc} \varphi(A^*A) & \varphi(A^*B) \\ \varphi(A^*B)^* & \varphi(B^*B) \end{array}\right]$$

is positive in $\mathfrak{B}$. Hence $|\langle\varphi(A^*B)\eta, \xi\rangle| \leq |\langle\varphi(A^*A)\xi, \xi\rangle|^{\frac{1}{2}} |\langle\varphi(B^*B)\eta, \eta\rangle|^{\frac{1}{2}}$ for all $\xi, \eta \in \mathcal{H}$ by Lemma 3.17 part (3). Thus $\|\varphi(A^*B)\|_{\mathfrak{B}} \leq \|\varphi(A^*A)\|_{\mathfrak{B}}^{\frac{1}{2}} \|\varphi(B^*B)\|_{\mathfrak{B}}^{\frac{1}{2}}$ as desired. $\square$

Theorem 3.20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a completely positive map. For any C*-bounded approximate identity $(E_\lambda)_\Lambda$ of $\mathfrak{A}$ $\|\varphi\|_{cb} = \lim_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}} = \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}} = \|\varphi\|$.*

Proof. Let $(E_\lambda)_\Lambda$ be any C*-bounded approximate identity for $\mathfrak{A}$. Since $E_\lambda \leq E_\alpha$ whenever $\lambda \leq \alpha$ and φ is a positive map, $0 \leq \varphi(E_\lambda) \leq \varphi(E_\alpha)$ whenever $\lambda \leq \alpha$. Whence $0 \leq \|\varphi(E_\lambda)\|_{\mathfrak{B}} \leq \|\varphi(E_\alpha)\|_{\mathfrak{B}} \leq \|\varphi\|$ whenever $\lambda \leq \alpha$. Thus $(\|\varphi(E_\lambda)\|_{\mathfrak{B}})_\Lambda$ is an increasing net of real numbers so $\lim_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}} = \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}} \leq \|\varphi\|$. Thus to complete the proof it suffices to show that $\left\|\varphi_{(n)}\right\| \leq \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}$ for all $n \in \mathbb{N}$.

First we will show that $\|\varphi\| \leq \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}$ and extrapolate the proof for an arbitrary $n \in \mathbb{N}$. Let $A \in \mathfrak{A}$ be arbitrary with $\|A\| \leq 1$. Since φ is 2-positive, the above lemma implies that $\|\varphi(B^*A)\|_{\mathfrak{B}} \leq \|\varphi(A^*A)\|_{\mathfrak{B}}^{\frac{1}{2}} \|\varphi(B^*B)\|_{\mathfrak{B}}^{\frac{1}{2}}$ for all $B \in \mathfrak{A}$. Since $0 \leq E_\lambda^* E_\lambda = E_\lambda^2 \leq E_\lambda$ for all $\lambda \in \Lambda$, $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity, and φ is continuous being a positive map,

$$\begin{aligned}
\|\varphi(A)\|_{\mathfrak{B}} &= \lim_\Lambda \|\varphi(E_\lambda A)\|_{\mathfrak{B}} \\
&\leq \sup_\Lambda \|\varphi(A^*A)\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\varphi(E_\lambda^2)\right\|_{\mathfrak{B}}^{\frac{1}{2}} \\
&\leq \sup_\Lambda \|\varphi\|^{\frac{1}{2}} \|A^*A\|_{\mathfrak{A}}^{\frac{1}{2}} \|\varphi(E_\lambda)\|_{\mathfrak{B}}^{\frac{1}{2}} \\
&= \|\varphi\|^{\frac{1}{2}} \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}^{\frac{1}{2}} .
\end{aligned}$$

Since the above holds for all $A \in \mathfrak{A}$ with $\|A\| \leq 1$, $\|\varphi\| \leq \|\varphi\|^{\frac{1}{2}} \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}^{\frac{1}{2}}$. Whence $\|\varphi\|^{\frac{1}{2}} \leq \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}^{\frac{1}{2}}$ so $\|\varphi\| \leq \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}$.

Fix $n \in \mathbb{N}$. Let $E_\lambda^{(n)} \in \mathcal{M}_n(\mathfrak{A})$ be the $n \times n$ diagonal matrix with E_λ in each entry along the diagonal. Since $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity for $\mathfrak{A}$, it is easy to verify that $\left(E_\lambda^{(n)}\right)_\Lambda$ is a C*-bounded approximate identity for $\mathcal{M}_n(\mathfrak{A})$. Moreover $\varphi_{(n)}\left(E_\lambda^{(n)}\right)$ is the $n \times n$ diagonal matrix with $\varphi(E_\lambda)$ in each entry along the diagonal so that $\left\|\varphi_{(n)}\left(E_\lambda^{(n)}\right)\right\|_{\mathcal{M}_n(\mathfrak{B})} = \|\varphi(E_\lambda)\|_{\mathfrak{B}}$. Since $\varphi_{(n)}$ is completely positive and $\left(E_\lambda^{(n)}\right)_\Lambda$ is a C*-bounded approximate identity for $\mathcal{M}_n(\mathfrak{A})$, the above proof implies that

$$\left\|\varphi_{(n)}\right\| = \sup_\Lambda \left\|\varphi_{(n)}\left(E_\lambda^{(n)}\right)\right\|_{\mathcal{M}_n(\mathfrak{B})} = \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}} .$$

Since the above holds for all $n \in \mathbb{N}$, $\|\varphi\|_{cb} = \sup_\Lambda \|\varphi(E_\lambda)\|_{\mathfrak{B}}$ as desired. ☐

4 Arveson's Extension Theorem and Stinespring's Theorem

So far our study of completely positive maps has been very similar to our study of positive linear functionals in that Theorem 3.20 generalizes the essential direction of Proposition 1.11. As with positive linear functionals we desire to see if we can extend completely positive maps to completely positive maps and if we can generalize the GNS construction. In this section we shall accomplish both. We will proceed in a slightly different order then we did with the theory of positive linear functionals and develop Stinespring's Theorem (the generalization of the GNS construction) before proving Arveson's Extension Theorem (which states that completely positive maps can be extended to completely positive maps of the same norm). This change has been made to simplify the proof of Arveson's Extension Theorem in the non-unital case.

Stinespring's Theorem will tell us that every completely positive map on a C*-algebra has a certain form. Recall by the GNS construction that every C*-algebra can be viewed as a C*-subalgebra of some $\mathcal{B}(\mathcal{H})$ and thus the range of every completely positive map may be viewed as a subset of $\mathcal{B}(\mathcal{H})$ (as, from Example 3.4, *-isomorphisms are invertible completely positive maps). Moreover Example 3.4 showed us that all *-homomorphisms are completely positive and Example 3.5 showed us that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then the map $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\varphi(T) = V^*TV$ for all $T \in \mathcal{B}(\mathcal{H})$ is completely positive. Recall from Proposition 3.9 that the composition of completely positive maps is completely positive so that if $\mathfrak{A}$ is a C*-algebra, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism, and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ then $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ defined by

$$\varphi(A) = V^*\pi(A)V$$

for all $A \in \mathfrak{A}$ is a completely positive map. Stinespring's Theorem tells us that every completely positive map on a C*-algebra has this form.

Theorem 4.1 (Stinespring)**.** *Let $\mathfrak{A}$ be a C*-algebra, let $\mathcal{H}$ be a Hilbert space, and let $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and a bounded linear operator $V : \mathcal{H} \to \mathcal{K}$ with $\|V\|^2 = \|\varphi\|$ such that $\varphi(A) = V^*\pi(A)V$ for all $A \in \mathfrak{A}$. If $\mathfrak{A}$ is unital then π is unital. If $\varphi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$ then V can be taken to be an isometry so we may view $\mathcal{H}$ as a Hilbert subspace of $\mathcal{K}$ and $\varphi(A) = P_{\mathcal{H}}\pi(A)|_{\mathcal{H}}$ where $P_{\mathcal{H}}$ is the projection of $\mathcal{K}$ onto $\mathcal{H}$. Finally, if $\mathfrak{A}$ and $\mathcal{H}$ are separable, then $\mathcal{K}$ can be taken to be separable.*

Before proving Stinespring's Theorem we note that this theorem is actually a generalization of the GNS construction given in Theorem 1.17. To see this we recall that positive linear functionals are completely positive by Proposition 3.8. Therefore, if we apply Stinespring's Theorem to a positive linear functional $\varphi : \mathfrak{A} \to \mathbb{C} = \mathcal{B}(\mathbb{C})$, we obtain a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ and a map $V : \mathbb{C} \to \mathcal{K}$ with $\|V\|^2 = \|\varphi\|$ such that $\varphi(A) = V^*\pi(A)V$ for all $A \in \mathfrak{A}$. Let $\xi := V(1) \in \mathcal{K}$. Thus $V(\lambda) = \lambda\xi$ for all $\lambda \in \mathbb{C}$ so that $\|\varphi\| = \|V\|^2 = \|\xi\|^2$. Moreover

$$\langle \pi(A)\xi, \xi\rangle = \langle V^*\pi(A)V(1), 1\rangle = \langle \varphi(A)1, 1\rangle = \varphi(A)$$

for all $A \in \mathfrak{A}$. Whence Stinespring's Theorem does indeed generalize the GNS construction. Moreover, it is possible in the following proof to see the proof of the GNS construction given in Theorem 1.17.

Proof. To construct π we will mimic the proof of Theorem 1.17 by changing the space upon which $\mathfrak{A}$ will act. We will omit some of the trivial details contained in Theorem 1.17. Those unfamiliar with algebraic tensor products should familiarize themselves with the Universal Property of Algebraic Tensor Products.

Consider the algebraic tensor product $\mathfrak{A} \odot \mathcal{H}$ of $\mathfrak{A}$ and $\mathcal{H}$. We desire to define a positive sesquilinear form $[\cdot, \cdot] : \mathfrak{A} \odot \mathcal{H} \to \mathbb{C}$ such that

$$[A \otimes \xi, B \otimes \eta] = \langle \varphi(B^*A)\xi, \eta\rangle_{\mathcal{H}}$$

for all $A, B \in \mathfrak{A}$ and $\xi, \eta \in \mathcal{H}$. The question is, how can we do this?

To proceed in creating our sesquilinear form, fix $B \in \mathfrak{A}$ and $\eta \in \mathcal{H}$. Define $\phi_{B,\eta} : \mathfrak{A} \times \mathcal{H} \to \mathbb{C}$ by $\phi_{B,\eta}(A, \xi) = \langle \varphi(B^*A)\xi, \eta\rangle_{\mathcal{H}}$ for all $(A, \xi) \in \mathfrak{A} \times \mathcal{H}$. Since φ is linear, it is clear that $\phi_{B,\eta}$ is a bilinear form.

Therefore by the Universal Property of Algebraic Tensor Products there exists a linear map $\phi_{B,\eta} : \mathfrak{A}\odot\mathcal{H} \to \mathbb{C}$ such that $\phi_{B,\eta}(A\otimes\xi) = \phi_{B,\eta}(A,\xi) = \langle \varphi(B^*A)\xi, \eta\rangle_{\mathcal{H}}$. Let G be the space of all conjugate linear functionals on $\mathfrak{A}\odot\mathcal{H}$. Define $\psi : \mathfrak{A}\times\mathcal{H} \to G$ by $\psi(B,\eta) = (\phi_{B,\eta})^*$ where $(\phi_{B,\eta})^*(u) = \overline{\phi_{B,\eta}(u)}$ for all $u \in \mathfrak{A}\odot\mathcal{H}$ (it is clear that $(\phi_{B,\eta})^*$ is a conjugate linear map since $\phi_{B,\eta}$ was a linear map). We claim that ψ is a bilinear form. To see this we notice for all $\lambda \in \mathbb{C}$, $B_1, B_2 \in \mathfrak{A}$, and $\eta \in \mathcal{H}$ that

$$\begin{aligned}
(\phi_{\lambda B_1+B_2,\eta})^*(A\otimes\xi) &= \overline{\phi_{\lambda B_1+B_2,\eta}(A\otimes\xi)} \\
&= \overline{\langle \varphi((\lambda B_1 + B_2)^*A)\xi, \eta\rangle_{\mathcal{H}}} \\
&= \overline{\overline{\lambda}\,\langle \varphi(B_1^*A)\xi, \eta\rangle_{\mathcal{H}} + \langle \varphi((B_2^*A)\xi, \eta\rangle_{\mathcal{H}}} \\
&= \lambda\overline{\langle \varphi(B_1^*A)\xi, \eta\rangle_{\mathcal{H}}} + \overline{\langle \varphi(B_2^*A)\xi, \eta\rangle_{\mathcal{H}}} \\
&= \lambda(\phi_{B_1,\eta})^*(A\otimes\xi) + (\phi_{B_2,\eta})^*(A\otimes\xi)
\end{aligned}$$

for all elementary tensors $A\otimes\xi \in \mathfrak{A}\otimes\mathcal{H}$. Thus, since this holds for all elementary tensors, we obtain by conjugate linearity that $(\phi_{\lambda B_1+B_2,\eta})^* = \lambda(\phi_{B_1,\eta})^* + (\phi_{B_2,\eta})^*$. Thus ψ is linear in the first component. Similarly ψ is linear in the second component so that ψ is a bilinear form. Thus by the Universal Property of Algebraic Tensor Products there exists a $\Psi : \mathfrak{A}\odot\mathcal{H} \to G$ such that $\Psi(B\otimes\eta) = \psi(B,\eta) = (\phi_{B,\eta})^*$ for all elementary tensors $B\otimes\eta \in \mathfrak{A}\odot\mathcal{H}$.

Define $[\cdot,\cdot] : \mathfrak{A}\odot\mathcal{H} \to \mathbb{C}$ by $[u,v] = (\Psi(v)^*)(u)$ for all $u,v \in \mathfrak{A}\odot\mathcal{H}$ (where * represents the same operation on linear/conjugate linear functionals that was used before). Then

$$[A\otimes\xi, B\otimes\eta] = (\Psi(B\otimes\eta)^*)(A\otimes\xi) = ((\phi_{B,\eta})^*)^*(A\otimes\xi) = \phi_{B,\eta}(A\otimes\xi) = \langle \varphi(B^*A)\xi, \eta\rangle_{\mathcal{H}}$$

as desired. To see that $[\cdot,\cdot]$ is a sesquilinear form we notice that each $\Psi(v) \in G$ is conjugate linear so $\Psi(v)^*$ is linear so $[\cdot,\cdot]$ is linear in the first component. Since Ψ is linear, $\Psi(\cdot)^*$ is conjugate linear so $[\cdot,\cdot]$ is conjugate linear in the second component. Thus $[\cdot,\cdot]$ is a sesquilinear form.

To see that $[\cdot,\cdot]$ is a positive sesquilinear form we notice for all $\sum_{j=1}^n A_j\otimes\xi_j \in \mathfrak{A}\odot\mathcal{H}$ that

$$\begin{aligned}
\left[\sum_{j=1}^n A_j\otimes\xi_j, \sum_{i=1}^n A_i\otimes\xi_i\right] &= \sum_{j=1}^n\sum_{i=1}^n \langle \varphi(A_i^*A_j)\xi_j, \xi_i\rangle_{\mathcal{H}} \\
&= \langle [\varphi(A_i^*A_j)](\xi_1\oplus\cdots\oplus\xi_n), \xi_1\oplus\cdots\oplus\xi_n\rangle_{\mathcal{H}^{\oplus n}} \\
&= \left\langle \varphi_{(n)}([A_i^*A_j])(\xi_1\oplus\cdots\oplus\xi_n), \xi_1\oplus\cdots\oplus\xi_n\right\rangle_{\mathcal{H}^{\oplus n}}.
\end{aligned}$$

Thus to show that $\left[\sum_{j=1}^n A_j\otimes\xi_j, \sum_{i=1}^n A_i\otimes\xi_i\right] \geq 0$ it suffices, by Proposition 1.21, to show that $[A_i^*A_j] \geq 0$ as $\varphi_{(n)}$ is positive. However

$$[A_i^*A_j] = \begin{bmatrix} A_1^*A_1 & \dots & A_1^*A_n \\ \vdots & & \vdots \\ A_n^*A_1 & \dots & A_n^*A_n \end{bmatrix} = \left(\begin{bmatrix} A_1 & \dots & A_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}\right)^* \begin{bmatrix} A_1 & \dots & A_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

so $[A_i^*A_j] \geq 0$. Hence $[\cdot,\cdot]$ is a sesquilinear form and thus satisfies the Cauchy-Schwartz inequality; namely

$$|\,[u,v]\,|^2 \leq [u,u]\,[v,v]$$

for all $u,v \in \mathfrak{A}\odot\mathcal{H}$. Therefore, if we define

$$\mathcal{N} := \{u \in \mathfrak{A}\odot\mathcal{H} \mid [u,u] = 0\} = \{u \in \mathfrak{A}\odot\mathcal{H} \mid [u,v] = 0 \text{ for all } v \in \mathfrak{A}\odot\mathcal{H}\}$$

then $\mathcal{N}$ is a subspace of $\mathfrak{A}\odot\mathcal{H}$ by the last term in the above expression. Consider the quotient space $(\mathfrak{A}\odot\mathcal{H})/\mathcal{N}$ and notice that

$$\langle u+\mathcal{N}, v+\mathcal{N}\rangle_{\mathcal{K}} = [u,v]$$

defines inner product on this space. Thus, if $\mathcal{K}$ is the completion of $(\mathfrak{A} \odot \mathcal{H})/\mathcal{N}$ with respect to this inner product, $\mathcal{K}$ is a Hilbert space. Moreover, it is clear that $\mathcal{K}$ is separable if $\mathfrak{A}$ and $\mathcal{H}$ are separable.

Let $A \in \mathfrak{A}$ be arbitrary. Define the linear map $\pi_{00}(A) : \mathfrak{A} \odot \mathcal{H} \to \mathfrak{A} \odot \mathcal{H}$ by

$$\pi_{00}(A)\left(\sum_{j=1}^n A_j \otimes \xi_j\right) = \sum_{j=1}^n AA_j \otimes \xi_j$$

(which is well-defined and exists by universality). We desire to show that $\pi_{00}(A)$ defines a linear map on $(\mathfrak{A} \odot \mathcal{H})/\mathcal{N}$ by showing $\mathcal{N}$ is invariant under $\pi_{00}(A)$. To see this let $\sum_{j=1}^n A_j \otimes \xi_j \in \mathfrak{A} \odot \mathcal{H}$ be arbitrary. By considering the unitization of $\mathfrak{A}$ if necessary, the following computations are valid. Notice $\|A^*A\| I_{\mathfrak{A}} - A^*A$ is a positive element of $\mathfrak{A}$ so $\|A^*A\| I_{\mathfrak{A}} - A^*A = N^*N$ for some $N \in \mathfrak{A}$. Moreover if

$$X := \|A^*A\| [I_{\mathfrak{A}}] - [A^*A] = [\|A^*A\| - A^*A]$$

is the matrix with the positive element $\|A^*A\| - A^*A$ in each entry then $X = [N^*/\sqrt{n}][N/\sqrt{n}]$ is positive. Thus

$$\begin{array}{rcl} [A_i^*A^*AA_j] & = & diag(A_1^*, \dots, A_n^*)[A^*A]diag(A_1, \dots, A_n) \\ & \leq & \|A^*A\| \, diag(A_1^*, \dots, A_n^*)[I_{\mathfrak{A}}]diag(A_1, \dots, A_n) = \|A^*A\| \, [A_i^*A_j]. \end{array}$$

Hence, with the above expression and the fact that $\varphi_{(n)}$ is positive, we obtain for all $\sum_{j=1}^n A_j \otimes h_j \in \mathfrak{A} \odot \mathcal{H}$ that

$$\begin{aligned} \left[\pi_{00}(A)\left(\sum_{j=1}^n A_j \otimes \xi_j\right), \pi_{00}(A)\left(\sum_{i=1}^n A_i \otimes \xi_i\right)\right] &= \sum_{i,j=1}^n [AA_j \otimes \xi_j, AA_i \otimes \xi_i] \\ &= \sum_{i,j=1}^n \langle \varphi(A_i^*A^*AA_j)\xi_j, \xi_i\rangle_{\mathcal{H}} \\ &= \left\langle \varphi_{(n)}([A_i^*A^*AA_j])(\xi_1 \oplus \cdots \oplus \xi_n), \xi_1 \oplus \cdots \oplus \xi_n\right\rangle_{\mathcal{H}^{\oplus n}} \\ &\leq \left\langle \|A^*A\| \, \varphi_{(n)}([A_i^*A_j])(\xi_1 \oplus \cdots \oplus \xi_n), \xi_1 \oplus \cdots \oplus \xi_n\right\rangle_{\mathcal{H}^{\oplus n}} \\ &= \|A^*A\| \sum_{i,j=1}^n \langle \varphi(A_i^*A_j)\xi_j, \xi_i\rangle_{\mathcal{H}} \\ &= \|A\|^2 \left[\sum_{j=1}^n A_j \otimes \xi_j, \sum_{i=1}^n A_i \otimes \xi_i\right]. \end{aligned}$$

Therefore $\pi(A)$ leaves $\mathcal{N}$ invariant as if $[u, u] = 0$ then $[\pi(A)u, \pi(A)u] = 0$. Define $\pi_0(A) : (\mathcal{A} \odot \mathcal{H})/\mathcal{N} \to (\mathcal{A} \odot \mathcal{H})/\mathcal{N}$ by $\pi_0(A)(u + \mathcal{N}) = \pi_{00}(A)u$ which is a well-defined linear map from the above argument. Moreover, the above inequality shows that $\|\pi_0(A)\| \leq \|A\|$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ on $(\mathfrak{A} \odot \mathcal{H})/\mathcal{N}$ and thus we can extend $\pi_0(A)$ to a bounded linear map on $\mathcal{K}$ which we shall denoted A_π.

We claim that the map $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ defined by $\pi(A) = A_\pi$ for all $A \in \mathfrak{A}$ is a *-homomorphism. It is clear that π_{00} was linear and multiplicative and thus since taking the quotient and extending by continuity will preserve these properties, π is linear and multiplicative. Moreover we notice for all $A \in \mathfrak{A}$ and all $u = \sum_{i=1}^n A_i \otimes \xi_i$, $v = \sum_{j=1}^n B_j \otimes \eta_j \in \mathfrak{A} \odot \mathcal{H}$ that

$$\begin{aligned} \langle \pi(A^*)(u + \mathcal{N}), (v + \mathcal{N})\rangle_{\mathcal{K}} &= \sum_{i,j=1}^n \langle \varphi(B_j^*A^*A_i)\xi_i, \eta_j\rangle_{\mathcal{H}} \\ &= \sum_{i,j=1}^n [A_i \otimes \xi_i, AB_j \otimes \eta_j] \\ &= \langle (u + \mathcal{N}), \pi(A)(v + \mathcal{N})\rangle_{\mathcal{K}} = \langle \pi(A)^*(u + \mathcal{N}), (v + \mathcal{N})\rangle_{\mathcal{K}}. \end{aligned}$$

Hence by continuity and the density of $(\mathfrak{A} \odot \mathcal{H})/\mathcal{N}$ in $\mathcal{K}$, π is self-adjoint. Hence $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ is a *-homomorphism.

Now that π has been constructed we divide into the unital and non-unital cases showing that there exists a map $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with the desired properties.

Case 1: $\mathfrak{A}$ is unital Since $\pi_{00}(I_{\mathfrak{A}})$ is the identity map, $\pi_0(I_{\mathfrak{A}})$ is identity map and thus $\pi(I_{\mathfrak{A}})$ is the identity map. Whence π is unital.

Define $V : \mathcal{H} \to \mathcal{K}$ by $V(\eta) = I_{\mathfrak{A}} \otimes \eta + \mathcal{N}$ for all $\eta \in \mathcal{H}$. It is clear that V is linear. Moreover V is bounded as

$$\|V\eta\|_{\mathcal{K}}^2 = \langle I_{\mathfrak{A}} \otimes \eta + \mathcal{N}, I_{\mathfrak{A}} \otimes \eta + \mathcal{N}\rangle_{\mathcal{K}} = \langle \varphi(I_{\mathfrak{A}}^* I_{\mathfrak{A}})\eta, \eta\rangle_{\mathcal{H}} = \langle \varphi(I_{\mathfrak{A}})\eta, \eta\rangle_{\mathcal{H}} \leq \|\varphi\| \, \|\eta\|_{\mathcal{H}}^2$$

with equality if $\varphi(I_{\mathfrak{A}}) = I_{\mathcal{B}(\mathcal{H})}$. Therefore $\|V\|^2 \leq \|\varphi\|$ and V is an isometry when φ is unital (as $\|\varphi\| = \|\varphi(I_{\mathfrak{A}})\| = 1$ when φ is unital by Theorem 3.18).

Next notice for all $\eta, \xi \in \mathcal{H}$ and $A \in \mathfrak{A}$ that

$$\langle V^*\pi(A)V\eta, \xi\rangle_{\mathcal{H}} = \langle \pi(A)(I_{\mathfrak{A}} \otimes \eta + \mathcal{N}), I_{\mathfrak{A}} \otimes \xi + \mathcal{N}\rangle_{\mathcal{K}} = \langle A \otimes \eta, I_{\mathfrak{A}} \otimes \xi\rangle_{\mathcal{K}} = \langle \varphi(A)\eta, \xi\rangle_{\mathcal{H}}.$$

Thus $\varphi(A) = V^*\pi(A)V$ as desired. When V is an isometry it is clear that we may view $\mathcal{H} \simeq Ran(V) \subseteq \mathcal{K}$ and thus $\varphi(A) = P_{\mathcal{H}}\pi(A)|_{\mathcal{H}}$ under this identification. Lastly we notice that

$$\|\varphi(A)\| \leq \|V^*\pi(A)V\| \leq \|V^*\| \, \|\pi(A)\| \, \|V\| \leq \|V\|^2 \, \|A\|$$

for all $A \in \mathfrak{A}$. Whence $\|\varphi\| \leq \|V\|^2$. Combining this inequality with inequality obtained earlier completes the first case.

Case 2: $\mathfrak{A}$ is not unital Let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity of $\mathfrak{A}$ and define $V_\lambda \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ by $V_\lambda(\eta) = E_\lambda \otimes \eta + \mathcal{N}$ for all $\eta \in \mathcal{H}$. Then each V_λ is bounded since

$$\|V_\lambda(\eta)\|_{\mathcal{K}}^2 = \langle E_\lambda \otimes \eta + \mathcal{N}, E_\lambda \otimes \eta + \mathcal{N}\rangle_{\mathcal{K}} = \langle \varphi(E_\lambda^2)\eta, \eta\rangle_{\mathcal{H}} = \left\|\varphi(E_\lambda^2)\right\| |\eta\|_{\mathcal{H}}^2 \leq \|\varphi\| \, \|\eta\|_{\mathcal{H}}^2$$

for all $\eta \in \mathcal{H}$.

We claim that $(V_\lambda(\eta))_\Lambda$ is a Cauchy net in $\mathcal{K}$ for each $\eta \in \mathcal{H}$. To begin we notice if $\alpha, \beta \in \Lambda$ and $\eta \in \mathcal{H}$ then

$$\|V_\alpha(\eta) - V_\beta(\eta)\|_{\mathcal{K}}^2 = \langle(\varphi(E_\alpha^2) + \varphi(E_\beta^2) - \varphi(E_\alpha E_\beta) - \varphi(E_\beta E_\alpha))\eta, \eta\rangle_{\mathcal{H}}.$$

Note that $\langle \varphi(E_\alpha^2)\eta, \eta\rangle_{\mathcal{H}}$ and $\langle \varphi(E_\beta^2)\eta, \eta\rangle_{\mathcal{H}}$ are positive as φ is positive and thus $\langle \varphi(E_\alpha E_\beta)\eta + \varphi(E_\beta E_\alpha)\eta, \eta\rangle_{\mathcal{H}}$ must be real. Let $\epsilon > 0$ and define $\varphi_\eta : \mathfrak{A} \to \mathbb{C}$ by $\varphi_\eta(A) = \langle \varphi(A)\eta, \eta\rangle_{\mathcal{H}}$ for all $A \in \mathfrak{A}$. Since φ is a positive map, φ_η is a positive linear functional. By Proposition 1.11 and the fact that $(\varphi_\eta(E_\lambda))_\Lambda$ is an increasing net of positive numbers, there exists an $\alpha_0 \in \Lambda$ such that $\|\varphi_\eta\| - \epsilon \leq \varphi_\eta(E_\lambda)$ for all $\lambda \geq \alpha_0$. Since $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity, there exists a β_0 such that if $\lambda \geq \beta_0$ then $\|E_\lambda E_{\alpha_0} - E_{\alpha_0}\|_{\mathfrak{A}} < \epsilon$ and $\|E_{\alpha_0} E_\lambda - E_{\alpha_0}\|_{\mathfrak{A}} < \epsilon$. Thus for all $\beta \geq \beta_0$

$$|2\varphi_\eta(E_{\alpha_0}) - \varphi_\eta(E_{\alpha_0}E_\beta) - \varphi_\eta(E_\beta E_{\alpha_0})| < 2\,\|\varphi_\eta\|\,\epsilon$$

and thus $-\varphi_\eta(E_{\alpha_0}E_\beta) - \varphi_\eta(E_\beta E_{\alpha_0}) < 2\,\|\varphi_\eta\|\,\epsilon - 2\varphi_\eta(E_{\alpha_0})$ as all terms under consideration are real numbers. Therefore

$$\begin{aligned}
\|V_{\alpha_0}(\eta) - V_\beta(\eta)\|_{\mathcal{K}}^2 &= \varphi_\eta(E_{\alpha_0}^2) + \varphi_\eta(E_\beta^2) - \varphi_\eta(E_{\alpha_0}E_\beta) - \varphi_\eta(E_\beta E_{\alpha_0}) \\
&\leq 2\,\|\varphi_\eta\| + (2\,\|\varphi_\eta\|\,\epsilon - 2\varphi_\eta(E_{\alpha_0})) \\
&\leq 2\,\|\varphi_\eta\| + 2\,\|\varphi_\eta\|\,\epsilon - 2(\|\varphi_\eta\| - \epsilon) \\
&= 2(\|\varphi_\eta\| + 1)\epsilon
\end{aligned}$$

whenever $\beta \geq \beta_0$. Thus if $\beta, \lambda \geq \beta_0$

$$\|V_\beta(\eta) - V_\lambda(\eta)\| \leq \|V_\beta(\eta) - V_{\alpha_0}(\eta)\| + \|V_{\alpha_0}(\eta) - V_\lambda(\eta)\| \leq 2\sqrt{2(\|\varphi_\eta\| + 1)\epsilon}.$$

As $\epsilon > 0$ was arbitrary, we obtain that $(V_\lambda(\eta))_\Lambda$ is a Cauchy net in $\mathcal{K}$.

Define $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ by $V(\eta) = \lim_\Lambda V_\lambda(\eta)$ for all $\eta \in \mathcal{H}$. It is trivial to verify that V is a well-defined linear map as each $V_\lambda(\eta)$ is a linear map. Moreover

$$\|V\eta\|_{\mathcal{K}}^2 = \lim_\Lambda \|V_\lambda(\eta)\|_{\mathcal{K}}^2 \leq \|\varphi\| \, \|\eta\|_{\mathcal{H}}^2 .$$

Whence V is a bounded linear map with $\|V\|^2 \leq \|\varphi\|$.

To complete the proof it suffices to show that V satisfies the claims in the theorem. For the first we notice that

$$\begin{aligned}
\langle V^*\pi(A)V\xi, \eta\rangle_{\mathcal{H}} &= \lim_\Lambda \langle V_\lambda^*\pi(A)V_\lambda\xi, \eta\rangle_{\mathcal{H}} \\
&= \lim_\Lambda \langle AE_\lambda \otimes \xi + \mathcal{N}, E_\lambda \otimes \eta + \mathcal{N}\rangle_{\mathcal{K}} \\
&= \lim_\Lambda \langle \varphi(E_\lambda A E_\lambda)\xi, \eta\rangle_{\mathcal{H}} \\
&= \langle \varphi(A)\xi, \eta\rangle
\end{aligned}$$

for all $A \in \mathfrak{A}$ and $\xi, \eta \in \mathcal{H}$. Thus $\varphi(A) = V^*\pi(A)V$ for all $A \in \mathfrak{A}$ as desired. Lastly we notice that

$$\|\varphi(A)\| \leq \|V^*\pi(A)V\| \leq \|V^*\| \, \|\pi(A)\| \, \|V\| \leq \|V\|^2 \, \|A\|$$

for all $A \in \mathfrak{A}$. Whence $\|\varphi\| \leq \|V\|^2$. Combining this inequality with inequality obtained earlier completes the proof. ☐

Stinespring's Theorem is very useful in developing theorems about completely positive maps. The following is one example of this which give us another Cauchy-Schwarz inequality for completely positive maps.

Theorem 4.2. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a completely positive map. Then $\varphi(A)^*\varphi(A) \leq \|\varphi\| \, \varphi(A^*A)$ for all $A \in \mathfrak{A}$.*

Proof. By using the GNS construction we may view $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ and thus we can apply Stinespring's Theorem to obtain a Hilbert space $\mathcal{K}$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\|VV^*\| = \|V\|^2 = \|\varphi\|$ and $\varphi(A) = V^*\pi(A)V$ for all $A \in \mathfrak{A}$. Whence for all $A \in \mathfrak{A}$

$$\varphi(A)^*\varphi(A) = V^*\pi(A)^*VV^*\pi(A)V \leq V^*\pi(A)^*(\|VV^*\| \, I_{\mathcal{K}})\pi(A)V = \|\varphi\| \, V^*\pi(A^*A)V = \|\varphi\| \, \varphi(A^*A)$$

as desired. ☐

Another use of Stinespring's Theorem is the ability to determine that some completely positive maps have multiplicative properties. We will present two proofs; the first which is intuitive but only works in the unital case and the second which works in the non-unital case and is a little tricky.

Theorem 4.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebra and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a unital, completely positive map. Suppose $\mathfrak{A}_0$ is a C*-subalgebra of $\mathfrak{A}$ such that $\varphi|_{\mathfrak{A}_0}$ is multiplicative (that is $\varphi(A_1A_2) = \varphi(A_1)\varphi(A_2)$ for all $A_1, A_2 \in \mathfrak{A}_0$). Then $\varphi(AXB) = \varphi(A)\varphi(X)\varphi(B)$ for all $A, B \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$.*

Proof. Without loss of generality we may assume $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$. As φ is unital, Stinespring's Theorem implies there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ such that $\varphi(A) = P_{\mathcal{H}}\pi(A)|_{\mathcal{H}}$ for all $A \in \mathfrak{A}$. First we will show that for each $A \in \mathfrak{A}_0$ $\pi(A)$ decomposes as a diagonal operator with respect to the decomposition $\mathcal{H} \oplus \mathcal{H}^\perp$ of $\mathcal{K}$ with $\varphi(A)$ in the first entry. With respect to the decomposition $\mathcal{H} \oplus \mathcal{H}^\perp$ of $\mathcal{K}$ write

$$\pi(A) = \begin{bmatrix} \varphi(A) & \pi_{1,2}(A) \\ \pi_{2,1}(A) & \pi_{2,2}(A) \end{bmatrix}$$

for all $A \in \mathfrak{A}$ where $\pi_{1,2} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}^\perp, \mathcal{H})$, $\pi_{2,1} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}, \mathcal{H}^\perp)$, and $\pi_{2,2} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}^\perp)$ are linear maps. To begin suppose $A \in \mathfrak{A}_0$ is self-adjoint. Since π is a *-homomorphism and A is self-adjoint, $\pi_{1,2}(A) = \pi_{2,1}(A)^*$. However

$$\begin{bmatrix} \varphi(A^2) & \pi_{1,2}(A^2) \\ \pi_{2,1}(A^2) & \pi_{2,2}(A^2) \end{bmatrix} = \pi(A^2) = \pi(A)^2 = \begin{bmatrix} \varphi(A)^2 + \pi_{1,2}(A)\pi_{2,1}(A) & * \\ * & * \end{bmatrix}$$

(where $*$ represents expressions unimportant to us). Thus $\varphi(A^2) = \varphi(A)^2 + \pi_{1,2}(A)\pi_{2,1}(A)$. Since φ is multiplicative on $\mathfrak{A}_0$, $\varphi(A^2) = \varphi(A)^2$ and thus $0 = \pi_{1,2}(A)\pi_{2,1}(A) = \pi_{1,2}(A)\pi_{1,2}(A)^*$. Thus $\pi_{1,2}(A) = 0 = \pi_{2,1}(A)$. Since the self-adjoint elements of $\mathfrak{A}_0$ span $\mathfrak{A}_0$, we obtain that

$$\pi(A) = \begin{bmatrix} \varphi(A) & 0 \\ 0 & \pi_{2,2}(A) \end{bmatrix}$$

for all $A \in \mathfrak{A}_0$. Whence if $A, B \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$

$$\begin{aligned} \pi(AXB) &= \pi(A)\pi(X)\pi(B) \\ &= \begin{bmatrix} \varphi(A) & 0 \\ 0 & \pi_{2,2}(A) \end{bmatrix} \begin{bmatrix} \varphi(X) & \pi_{1,2}(X) \\ \pi_{2,1}(X) & \pi_{2,2}(X) \end{bmatrix} \begin{bmatrix} \varphi(B) & 0 \\ 0 & \pi_{2,2}(B) \end{bmatrix} \\ &= \begin{bmatrix} \varphi(A)\varphi(X)\varphi(B) & * \\ * & * \end{bmatrix} \end{aligned}$$

so $\varphi(AXB) = P_{\mathcal{H}}\pi(AXB)|_{\mathcal{H}} = \varphi(A)\varphi(X)\varphi(B)$ as desired. □

Theorem 4.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebra and let $\varphi : \mathfrak{A} \to \mathfrak{B}$ be a completely positive map with $\|\varphi\| = 1$. Suppose $\mathfrak{A}_0$ is a C^*-subalgebra of $\mathfrak{A}$ such that $\varphi|_{\mathfrak{A}_0}$ is multiplicative (that is $\varphi(A_1A_2) = \varphi(A_1)\varphi(A_2)$ for all $A_1, A_2 \in \mathfrak{A}_0$). Then $\varphi(AXB) = \varphi(A)\varphi(X)\varphi(B)$ for all $A, B \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$.*

Proof. First we claim that $\varphi(AX) = \varphi(A)\varphi(X)$ for all $A \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$. To begin we notice that $\varphi(AA^*) = \varphi(A)\varphi(A)^*$ for all $A \in \mathfrak{A}_0$ as φ is multiplicative on $\mathfrak{A}_0$ and self-adjoint (being a completely positive map). Since φ is completely positive with $\|\varphi\| = 1$, $\varphi_{(2)}$ is completely positive and $\|\varphi_{(2)}\| = \|\varphi\| = 1$ by Theorem 3.20. Thus, by Theorem 4.2,

$$\varphi_{(2)}\left(\begin{bmatrix} A & 0 \\ X^* & 0 \end{bmatrix}\right)\varphi_{(2)}\left(\begin{bmatrix} A & 0 \\ X^* & 0 \end{bmatrix}\right)^* \leq \|\varphi_{(2)}\|\,\varphi_{(2)}\left(\begin{bmatrix} A & 0 \\ X^* & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ X^* & 0 \end{bmatrix}^*\right).$$

By expanding out both sides we obtain that

$$\begin{bmatrix} \varphi(A)\varphi(A)^* & \varphi(A)\varphi(X) \\ \varphi(X^*)\varphi(A)^* & \varphi(X)^*\varphi(X) \end{bmatrix} \leq \begin{bmatrix} \varphi(AA^*) & \varphi(AX) \\ \varphi(X^*A^*) & \varphi(X^*X) \end{bmatrix}$$

so that the matrix

$$M = \begin{bmatrix} \varphi(AA^*) - \varphi(A)\varphi(A)^* & \varphi(AX) - \varphi(A)\varphi(X) \\ \varphi(X^*A^*) - \varphi(X^*)\varphi(A)^* & \varphi(X^*X) - \varphi(X^*)\varphi(X) \end{bmatrix}$$

is positive. However the $(1,1)$-entry of M is zero. Thus, a simple application of Lemma 3.17 part (3) implies that $\varphi(AX) - \varphi(A)\varphi(X) = 0$. Whence $\varphi(AX) = \varphi(A)\varphi(X)$ for all $A \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$.

To see that $\varphi(XB) = \varphi(X)\varphi(B)$ for all $B \in \mathfrak{A}_0$ and $X \in \mathfrak{A}$ we notice that

$$\varphi(XB) = \varphi(B^*X^*)^* = (\varphi(B^*)\varphi(X^*))^* = \varphi(X)\varphi(B).$$

Whence for all $A, B \in \mathfrak{A}_0$ and $X \in \mathfrak{B}$

$$\varphi(AXB) = \varphi(A)\varphi(XB) = \varphi(A)\varphi(X)\varphi(B)$$

as desired. □

Our next goal is to be able to extend completely positive maps to completely positive maps. If $\varphi : \mathfrak{A} \to \mathfrak{B}$ is a completely positive map, we can apply the GNS construction and the fact that *-homomorphisms are completely positive to view $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Arveson's Extension Theorem will allow us to extend φ to a completely positive map $\tilde{\varphi}$ with the one caveat that the range of $\tilde{\varphi}$ may not sit inside $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$. This is a technical issue that is necessary for the proof of Arveson's Extension Theorem and will be explored in Chapter 5.

First we note that complete positivity is essential as Remarks 2.24 showed that there exists positive maps $\varphi : \mathcal{S} \to \mathfrak{B}$ that cannot be extended to a positive map on $C(\mathbb{T})$.

To begin the proof of Arveson's Extension Theorem we will first show that completely positive maps into $\mathcal{M}_n(\mathbb{C})$ can be extended to completely positive maps for each $n \in \mathbb{N}$. The $\mathcal{B}(\mathcal{H})$-case of Arveson's Extension Theorem will then follow by taking the completely positive map into $\mathcal{B}(\mathcal{H})$, creating a net of completely positive maps into arbitrarily large matrix algebras, considering a certain topology, and taking a cluster point.

To begin the proof, we first prove a simple lemma that will make it easier for us to check that a map is positive.

Lemma 4.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then every positive element of $\mathcal{M}_n(\mathfrak{A})$ is the sum of n positive elements of the form $[A_i^* A_j]$ for some $\{A_1, \ldots, A_n\} \in \mathfrak{A}$.*

Proof. First suppose we have chosen $\{A_i\}_{i=1}^n \in \mathfrak{A}$. Then

$$\begin{bmatrix} A_1^*A_1 & \ldots & A_1^*A_n \\ \vdots & & \vdots \\ A_n^*A_1 & \ldots & A_n^*A_n \end{bmatrix} = \left(\begin{bmatrix} A_1 & \ldots & A_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \right)^* \begin{bmatrix} A_1 & \ldots & A_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix}.$$

Thus $[A_i^* A_j]$ is a positive matrix.

Now suppose $P \in \mathcal{M}_n(\mathfrak{A})$ is a positive matrix. Hence there exists a $B \in \mathcal{M}_n(\mathfrak{A})$ such that $P = B^*B$. Let R_k be an $n \times n$ matrix with its k^{th} row being the k^{th} row of B and 0's elsewhere. Then $R_i^* R_j = 0$ if $i \neq j$. Thus $P = R_1^* R_1 + \ldots + R_n^* R_n$. However each $R_k^* R_k$ is of the form $[A_i^* A_j]$ for some $\{A_1, \ldots, A_n\} \in \mathfrak{A}$ as each R_k is a matrix with only non-zero entries in the k^{th} row (i.e. A_i is the i^{th} entry of the k^{th} row) thus completing the proof. ☐

By the above lemma, to show that $\varphi : \mathfrak{A} \to \mathfrak{B}$ is n-positive it suffices to check that $\varphi_{(n)}([A_i^* A_j])$ is positive for all $A_1, \ldots, A_n \in \mathfrak{A}$.

To show that completely positive maps into $\mathcal{M}_n(\mathbb{C})$ can be extended to completely positive maps for each $n \in \mathbb{N}$, we will make use of the fact that positive linear functionals can be extended and a certain correspondence between linear maps from a space $\mathcal{S}$ into $\mathcal{M}_n(\mathbb{C})$ and linear maps from $\mathcal{M}_n(\mathcal{S})$ into $\mathbb{C}$.

Definition 4.6. Let $\mathcal{S}$ be an operator system and let $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathbb{C}^n$. If $\varphi : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ is linear we define $s_\varphi : \mathcal{M}_n(\mathcal{S}) \to \mathbb{C}$ by

$$s_\varphi([A_{i,j}]) = \frac{1}{n} \sum_{i,j=1}^n \langle \varphi(A_{i,j}) e_j, e_i \rangle_{\mathbb{C}^n}$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$. We shall call s_φ the linear functional associated with φ. Note that s_φ takes a matrix $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ and sums up the $(i,j)^{th}$ entry of φ applied to the $(i,j)^{th}$ entry of $[A_{i,j}]$ (up to an averaging term).

If $s : \mathcal{M}_n(\mathcal{S}) \to \mathbb{C}$ is linear then we define $\varphi_s : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ by

$$\langle \varphi_s(A) e_j, e_i \rangle = n s(A \otimes E_{i,j})$$

for all $A \in \mathfrak{A}$ where $E_{i,j}$ is the matrix in $\mathcal{M}_n(\mathbb{C})$ with a 1 in the $(i,j)^{th}$ entry and 0's elsewhere (by $A \otimes E_{i,j}$, we mean the matrix in $\mathcal{M}_n(\mathcal{S})$ with A in the $(i,j)^{th}$ entry and 0's elsewhere). We shall call φ_s the linear map associated with s.

Remarks 4.7. Consider s_φ. If x denotes the vector in $(\mathbb{C}^n)^{\oplus n} = \mathbb{C}^n \oplus \ldots \oplus \mathbb{C}^n$ given by $x := e_1 \oplus \cdots \oplus e_n$ then

$$s_\varphi([A_{i,j}]) = \frac{1}{n}\sum_{i,j=1}^n \langle \varphi(A_{i,j})e_j, e_i\rangle_{\mathbb{C}^n} = \frac{1}{n}\langle \varphi_{(n)}([A_{i,j}])x, x\rangle_{(\mathbb{C}^n)^{\oplus n}}.$$

It will be useful for later to note that $\|x\|_2 = \sqrt{n}$.

It is also useful to note that the maps $T : \mathcal{L}(\mathcal{S}, \mathcal{M}_n(\mathbb{C})) \to \mathcal{L}(\mathcal{M}_n(\mathcal{S}), \mathbb{C})$ given by $T(\varphi) = s_\varphi$ and $R : \mathcal{L}(\mathcal{M}_n(\mathcal{S}), \mathbb{C}) \to \mathcal{L}(\mathcal{S}, \mathcal{M}_n(\mathbb{C}))$ given by $R(s) = \varphi_s$ are linear maps (it is clear from the definitions that φ_s and s_φ are linear maps). Notice that for all $\psi \in \mathcal{L}(\mathcal{S}, \mathcal{M}_n(\mathbb{C}))$ that

$$RT(\psi)(A) = \varphi_{s_\psi}(A) = n[s_\psi(A \otimes E_{i,j})] = n\left[\frac{1}{n}\langle \psi(A)e_j, e_i\rangle_{\mathbb{C}^n}\right] = [\langle \psi(A)e_j, e_i\rangle_{\mathbb{C}^n}] = \psi(A)$$

for all $A \in \mathcal{S}$. Hence $RT : \mathcal{L}(\mathcal{S}, \mathcal{M}_n(\mathbb{C})) \to \mathcal{L}(\mathcal{S}, \mathcal{M}_n(\mathbb{C}))$ is the identity map. Also, for all $f \in \mathcal{L}(\mathcal{M}_n(\mathcal{S}), \mathbb{C})$

$$\begin{aligned} TR(f)[A_{i,j}] = s_{\varphi_f}[A_{i,j}] &= \tfrac{1}{n}\textstyle\sum_{i,j=1}^n \langle \varphi_f(A_{i,j})e_j, e_i\rangle_{\mathbb{C}^n} \\ &= \tfrac{1}{n}\textstyle\sum_{i,j=1}^n nf(A_{i,j} \otimes E_{i,j}) \\ &= f\left(\textstyle\sum_{i,j=1}^n A_{i,j} \otimes E_{i,j}\right) = f([A_{i,j}]) \end{aligned}$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$. Hence $TR : \mathcal{L}(\mathcal{M}_n(\mathcal{S}), \mathbb{C}) \to \mathcal{L}(\mathcal{M}_n(\mathcal{S}), \mathbb{C})$ is also the identity map. Therefore the maps T and R are mutual inverses.

Example 4.8. Let $\mathcal{S} := \mathbb{C}$ and define $\psi : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ by $\psi(\alpha) = [\alpha]$, the matrix with every entry equal to α. Notice then that $\sigma := s_\psi : \mathcal{M}_n(\mathcal{S}) = \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ is such that $s_\psi([\alpha_{i,j}]) = \frac{1}{n}\sum_{i,j=1}^n \alpha_{i,j}$ which is a scalar multiple of the sum of the entries of $[\alpha_{i,j}]$.

Example 4.9. Let $\mathcal{S} := \mathbb{C}$ and define $tr_m : \mathcal{M}_m(\mathcal{S}) = \mathcal{M}_m(\mathbb{C}) \to \mathbb{C}$ by $tr_m([\alpha_{i,j}]) = \frac{1}{m}\sum_{i=1}^m \alpha_{i,i}$ (i.e. tr_m is the normalized trace). Notice then that $\varphi_{tr_m} : \mathcal{S} = \mathbb{C} \to \mathcal{M}_m(\mathbb{C})$ is such that $\varphi_{tr_m}(\alpha) = [tr_m(\alpha \otimes E_{i,j})] = \alpha I_m$.

Using matrix tricks it is possible to verify that σ and tr_m are positive linear functionals. Moreover notice that $\psi_{(n)}([\alpha_{i,j}]) = [\alpha_{i,j}[1]]$ (where $[1] \in \mathcal{M}_n(\mathbb{C})$ is the matrix with 1s in each entry) and $(\varphi_{tr_m})_{(n)}([\alpha_{i,j}]) = [\alpha_{i,j}I_m]$. Using more matrix tricks it is possible to verify that $\psi_{(n)}$ and $(\varphi_{tr_m})_{(n)}$ are positive maps for all $n \in \mathbb{N}$ and thus ψ and φ_{tr_m} are completely positive. Coincidence, I think not!

Theorem 4.10. *Let $\mathfrak{A}$ be a unital C*-algebra, let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system or C*-algebra, and let $\varphi : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ be a linear map. Then the following are equivalent:*

1. *φ is completely positive.*
2. *φ is n-positive.*
3. *s_φ is positive.*

Proof. It is clear that if φ is completely positive then φ is n-positive. Hence (1) implies (2).

Suppose that φ is n-positive. Then for all $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ we have from Remarks 4.7 that

$$s_\varphi([A_{i,j}]) = \frac{1}{n}\langle \varphi_{(n)}([A_{i,j}])x, x\rangle_{(\mathbb{C}^n)^{\oplus n}}$$

where $x = e_1 \oplus \cdots \oplus e_n$. Therefore if $[A_{i,j}] \geq 0$ then $s_\varphi([A_{i,j}]) \geq 0$ as $\varphi_{(n)}$ was positive (and thus $\langle \varphi_{(n)}([A_{i,j}])x, x\rangle_{(\mathbb{C}^n)^{\oplus n}} \geq 0$ by Theorem 1.21). Therefore (2) implies (3).

Lastly suppose s_φ is positive. By Theorem 1.15 or Proposition 2.27 (depending on whether $\mathcal{S}$ is an operator system or C*-algebra) we may extend s_φ to a positive linear functional $f : \mathcal{M}_n(\mathfrak{A}) \to \mathbb{C}$. Since f extends s_φ, the linear map ψ associated with f extends φ (this easily follows from Remarks 4.7). Thus it suffices to show that ψ is completely positive.

To show that ψ is m-positive it suffices to show that $\psi_{(m)}([A_i^*A_j]) \geq 0$ for all $\{A_1, \ldots, A_m\} \in \mathfrak{A}$ by Lemma 4.5. However, since $\psi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$, $\psi_{(m)} : \mathcal{M}_m(\mathfrak{A}) \to \mathcal{M}_m(\mathcal{M}_n(\mathbb{C})) \simeq \mathcal{M}_{mn}(\mathbb{C})$. Thus it suffices to show that

$$\langle \psi_{(m)}([A_i^*A_j])(x_1 \oplus \cdots \oplus x_m), x_1 \oplus \cdots \oplus x_m \rangle_{(\mathbb{C}^n)^{\oplus m}} \geq 0$$

for all $x_i \in \mathbb{C}^n$. For each j write $x_j = \sum_{k=1}^n \lambda_{j,k} e_k$ where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{C}^n$. Then

$$\begin{aligned} \langle \psi_{(m)}([A_i^*A_j])(x_1 \oplus \cdots \oplus x_m), x_1 \oplus \cdots \oplus x_m \rangle_{(\mathbb{C}^n)^{\oplus m}} &= \sum_{i,j=1}^m \langle \psi(A_i^*A_j)x_j, x_i \rangle_{\mathbb{C}^n} \\ &= \sum_{i,j=1}^m \sum_{k,l=1}^n \lambda_{j,k} \overline{\lambda_{i,l}} \langle \psi(A_i^*A_j)e_k, e_l \rangle_{\mathbb{C}^n} \\ &= n \sum_{i,j=1}^m \sum_{k,l=1}^n \lambda_{j,k} \overline{\lambda_{i,l}} f(A_i^*A_j \otimes E_{l,k}). \end{aligned}$$

Let $B_i \in \mathcal{M}_n(\mathbb{C})$ be the matrix with $(\lambda_{i,1}, \ldots, \lambda_{i,n})$ as its first row and 0's elsewhere. Then

$$B_i^*B_j = \sum_{k,l=1}^n \overline{\lambda_{i,l}} \lambda_{j,k} E_{l,k}.$$

For each $C \in \mathfrak{A}$ and $T = [\alpha_{i,j}] \in \mathcal{M}_n(\mathbb{C})$ let $C \otimes T$ denote the matrix $[\alpha_{i,j}C] \in \mathcal{M}_n(\mathfrak{A})$. Therefore

$$\begin{aligned} \langle \psi_{(m)}([A_i^*A_j])(x_1 \oplus \cdots \oplus x_m), x_1 \oplus \cdots \oplus x_m \rangle_{(\mathbb{C}^n)^{\oplus m}} &= n \sum_{i,j=1}^m f(A_i^*A_j \otimes B_i^*B_j) \\ &= nf\left(\left(\sum_{i=1}^n A_i \otimes B_i\right)^* \left(\sum_{i=1}^n A_i \otimes B_i\right)\right) \geq 0 \end{aligned}$$

as f is positive. Therefore $\psi_{(m)}$ is positive for all $m \in \mathbb{N}$ and hence φ is completely positive. ☐

Note that the above result can be extremely useful as it is much easier to verify that a linear functional is positive than it is to verify that a map is completely positive using definitions. Moreover we know some additional properties of positive linear functionals such as how to extend them. Thus we can use the theory of positive linear functional to extend completely positive maps in this setting.

Theorem 4.11. *Let $\mathfrak{A}$ be a unital C*-algebra, let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system or C*-algebra, and let $\varphi : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ be a completely positive map. Then there exists a completely positive map $\psi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ extending φ.*

Proof. Since φ is completely positive, the linear functional associated with φ, $s_\varphi : \mathcal{M}_n(\mathcal{S}) \to \mathbb{C}$, is positive by Theorem 4.10. Hence, by Theorem 1.15 or Proposition 2.27 (depending on whether $\mathcal{S}$ is an operator system or C*-algebra), s_φ extends to a positive linear functional $f : \mathcal{M}_n(\mathfrak{A}) \to \mathbb{C}$. Let $\psi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ be the linear map associated with f. Hence ψ is completely positive by Theorem 4.10. However $\varphi = \varphi_{s_\varphi} = \varphi_f|_\mathcal{S} = \psi|_\mathcal{S}$ (where the restriction is clearly true by the definitions). Hence we have the desired result. ☐

Notice that if $\mathcal{S}$ and $\mathfrak{A}$ share the same unit then Theorem 3.18 implies that ψ and φ (as in the previous theorem) have the same norm. However if $\mathcal{S}$ does not have the same unit as $\mathfrak{A}$ it is possible that ψ and φ do not have the same norm. One would hope that there is a norm relation between φ and s_φ and there is. However this norm relation depends on n and will cause problems in the proof of Arveson's Extension Theorem in the non-unital case. Nevertheless we will find a way around this.

Now that we have developed a theory of completely positive maps into $\mathcal{M}_n(\mathbb{C})$ we wish to be able to extend these results to completely positive maps into $\mathcal{B}(\mathcal{H})$. To do this we will need to develop a certain topology which will be essential. Below is probably not the quickest way to develop this topology but we develop some nice and more general theory along the way.

Construction 4.12. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces, let $\mathfrak{Y}^*$ be the dual space of $\mathfrak{Y}$, and let $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ be the bounded linear maps from $\mathfrak{X}$ into $\mathfrak{Y}^*$. We desire to show that there exists a Banach space $\mathfrak{Z}$ such that $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*) \simeq \mathfrak{Z}^*$. This will then allow us to equip $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ with a weak*-topology. The idea behind the construction is if $\mathfrak{W}$ is a Banach space, then we desire to extract a certain closed subspace $\mathfrak{Z}$ of $\mathfrak{W}^{***}$; namely the image of $\mathfrak{W}^*$ under the canonical inclusion.

To construct $\mathfrak{Z}$ let $x \in \mathfrak{X}$ and $y \in \mathfrak{Y}$ be arbitrary and define a linear functional $x \otimes y : \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*) \to \mathbb{C}$ by $x \otimes y(L) = L(x)(y)$ for all $L \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$. It is clear that $x \otimes y$ is linear as evaluation in $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ and evaluation in $\mathfrak{Y}^*$ are linear. Moreover it is easy to see that $\|x \otimes y(L)\| = \|L(x)(y)\| \leq \|L\| \|x\| \|y\|$ for all $L \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ so that $\|x \otimes y\| \leq \|x\| \|y\|$. To see that this inequality is actually an equality we can apply the Hahn-Banach Theorem to obtain linear functionals $f_x \in \mathfrak{X}^*$ and $g_y \in \mathfrak{Y}^*$ such that $\|f_x\| = 1 = \|g_y\|$, $f_x(x) = \|x\|$, and $g_y(y) = \|y\|$. Define $L_{x,y} \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ by $L_{x,y}(x') = f_x(x')g_y$. Then it is clear that $L_{x,y}$ is linear and

$$\begin{aligned}\|L_{x,y}\| &= \sup\{\|f_x(x')g_y\| \mid x' \in \mathfrak{X}\, \|x'\| \leq 1\} \\ &= \|g_y\| \sup\{|f_x(x')| \mid x' \in \mathfrak{X}\, \|x'\| \leq 1\} \\ &= \|g_y\| \|f_x\| = 1.\end{aligned}$$

Since $x \otimes y(L_{x,y}) = L_{x,y}(x)(y) = f_x(x)g_y(y) = \|x\| \|y\|$ and $\|L_{x,y}\| = 1$, $\|x \otimes y\| \geq \|x\| \|y\|$ and thus we obtain equality.

We claim that the algebraic tensor product $\mathfrak{X} \odot \mathfrak{Y}$ is isomorphic (as a vector space) to *span*$\{x \otimes y\}_{x\in\mathfrak{X},y\in\mathfrak{Y}} \subseteq \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)^*$. To see this we note that it is easy to verify that

$$(a_1x_1 + a_2x_2) \otimes (b_1y_1 + b_2y_2) = \sum_{i,j=1}^{2} a_ib_j(x_i \otimes y_j)$$

for all $x_1, x_2 \in \mathfrak{X}$, $y_1, y_2 \in \mathfrak{Y}$, and $a_1, a_2, b_1, b_2 \in \mathbb{C}$. Thus, by the Universal Property of the Algebraic Tensor Product, there exists a linear map $\varphi : \mathfrak{X} \otimes \mathfrak{Y} \to \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)^*$ such that $\varphi(x \otimes y) = x \otimes y$ (where $\otimes$ means the appropriate thing in each space). Clearly φ maps onto the span of $\{x\otimes y\}_{x\in\mathfrak{X},y\in\mathfrak{Y}}$ in $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)^*$. To see that φ is injective, suppose $0 \neq t = \sum_{i=1}^n x_i \otimes y_i$ is in the kernel of φ with the x_is and y_is linearly independent. Then for all $L \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ $\sum_{i=1}^n L(x_i)(y_i) = 0$. By the Hahn Banach Theorem there exists linear functions $f_{x_i} \in \mathfrak{X}^*$ and $g_{y_i} \in \mathfrak{Y}^*$ such that $\|f_{x_i}\| = 1 = \|g_{y_i}\|$, $f_{x_i}(x_j) = \delta_{i,j} \|x_i\|$, and $g_{y_i}(y_j) = \delta_{i,j} \|y_i\|$. Define $L_{x_i,y_i} \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ by $L_{x_i,y_i}(x') = f_{x_i}(x')g_{y_i}$. Hence

$$0 = \sum_{j=1}^{n} L_{x_i,y_i}(x_j)(y_j) = \sum_{j=1}^{n} f_{x_i}(x_j)g_{y_i}(y_j) = \|x_i\| \|y_i\|.$$

Hence $x_i = 0$ or $y_i = 0$ which contradicts the fact that the x_is and the y_is are linearly independent. Hence φ is injective and thus we can identify $\mathfrak{X} \odot \mathfrak{Y}$ with this set.

Let $\mathfrak{Z}$ be the closed linear span of the elementary tensors $\{x \otimes y\}_{x\in\mathfrak{X},y\in\mathfrak{Y}}$ in $\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)^*$. Therefore, since $\mathfrak{Z}$ is a closed vector subspace of a Banach space, $\mathfrak{Z}$ is a Banach space. Moreover the following lemma shows us that $\mathfrak{Z}$ satisfies our hopes of performing this construction.

Lemma 4.13. *$\mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ is isometrically isomorphic to $\mathfrak{Z}^*$ by the map $\varphi : \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*) \to \mathfrak{Z}^*$ induced by*

$$\varphi(L)\left(\sum_{i=1}^{n} x_i \otimes y_i\right) = \sum_{i=1}^{n} x_i \otimes y_i(L).$$

Proof. For each fixed $L \in \mathcal{B}(\mathfrak{X},\mathfrak{Y}^*)$ the map $\psi_L : \mathfrak{X} \times \mathfrak{Y} \to \mathbb{C}$ defined by $\psi_L(x,y) = x \otimes y(L)$ is bilinear. Therefore by the Universality Property of the Algebraic Tensor Product ψ_L extends to a linear functional $\varphi_L : \mathfrak{X} \odot \mathfrak{Y} \subseteq \mathfrak{Z} \to \mathbb{C}$ such that $\varphi_L\left(\sum_{i=1}^n x_i \otimes y_i\right) = \sum_{i=1}^n x_i \otimes y_i(L)$. Moreover we notice that φ_L is continuous on $\mathfrak{X} \odot \mathfrak{Y}$ as

$$\begin{aligned}
\|\varphi_L\| &= \sup\left\{\left|\varphi_L\left(\sum_{i=1}^n x_i \otimes y_i\right)\right| \mid \sum_{i=1}^n x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}, \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_{\mathfrak{Z}} \leq 1\right\} \\
&= \sup\left\{\left|\sum_{i=1}^n x_i \otimes y_i(L)\right| \mid \sum_{i=1}^n x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}, \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_{\mathfrak{Z}} \leq 1\right\} \\
&\leq \|L\|.
\end{aligned}$$

Hence φ_L extends by continuity to a linear functional, which we will also call φ_L, on $\mathfrak{Z}$. Thus if we define $\varphi : \mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*) \to \mathfrak{Z}^*$ by $\varphi(L) = \varphi_L$ for all $L \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*)$ then φ is a linear map (linear since a simple computation show that $\varphi_{\lambda L_1 + L_2} = \lambda\varphi_{L_1} + \varphi_{L_2}$ on $\mathfrak{X} \odot \mathfrak{Y}$ and then extend by continuity to obtain this result). Therefore the map φ is well-defined and linear.

To see that φ is isometric we notice that the above computation shows us that $\|\varphi(L)\| \leq \|L\|$. Moreover the above computation show us that

$$\|\varphi(L)\| = \sup\left\{\left|\sum_{i=1}^n L(x_i)(y_i)\right| \mid \sum_{i=1}^n x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}, \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_{\mathfrak{Z}} \leq 1\right\}.$$

However for all $\epsilon > 0$ there exists an $x \in \mathfrak{X}$ such that $\|x\| = 1$ and $\|L(x)\| > \|L\| - \frac{\epsilon}{2}$. Moreover there also exists a $y \in \mathfrak{Y}$ such that $\|y\| = 1$ and $\|L(x)(y)\| > \|L(x)\| - \frac{\epsilon}{2} > \|L\| - \epsilon$. Thus we then have that $\|x \otimes y\| = \|x\| \|y\| = 1$ and $\|L(x)(y)\| > \|L\| - \epsilon$. Thus $\|\varphi(L)\| > \|L\| - \epsilon$ for all $\epsilon > 0$. Thus $\|\varphi(L)\| = \|L\|$ as desired. Therefore φ is a linear isometry and thus is injective.

Now all that remains is to show that φ is surjective. In this regard, let $f \in \mathfrak{Z}^*$ be arbitrary. Then for all $x \in \mathfrak{X}$ define the linear functional $f_x : \mathfrak{Y} \to \mathbb{C}$ by $f_x(y) = f(x \otimes y)$ for all $y \in \mathfrak{Y}$. Since $|f_x(y)| \leq \|f\| \|x\| \|y\|$, $f_x \in \mathfrak{Y}^*$. Define $L \in \mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*)$ by $L(x) = f_x$. Since it is clear that $f_{\lambda x_1 + x_2} = \lambda f_{x_1} + f_{x_2}$, L is linear, and since $\|f_x\| \leq \|f\| \|x\|$, $\|L\| \leq \|f\|$ so that L is indeed bounded. Therefore all that remains is to show that $\varphi(L) = f$ but this is trivial since

$$\varphi(L)(x \otimes y) = L(x)(y) = f_x(y) = f(x \otimes y).$$

Hence by linearity $\varphi(L) = f$ on $\mathfrak{X} \odot \mathfrak{Y}$ and therefore, by continuity, $\varphi(L) = f$ on $\mathfrak{Z}$ as desired. Therefore φ is an isometric isomorphism and the result follows. ☐

From now on we shall drop the φ and write $L(x \otimes y) = L(x)(y)$ interchangeably. Now that we have $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*) \simeq \mathfrak{Z}^*$ we may place a weak*-topology on $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*)$ induced by $\mathfrak{Z}$ which we will call the Bounded Weak topology (or just BW topology). It is useful to note that a net $\{L_\lambda\}_\Lambda$ converges to L in the BW topology if and only if $L_\lambda(z) \to L(z)$ for all $z \in \mathfrak{Z}$ (where we now drop the φ in the preceding lemma). This enables us the slightly stronger result in the case of a bounded net.

Lemma 4.14. *Let $(L_\lambda)_\Lambda$ be a norm bounded net in $\mathcal{B}(\mathfrak{X}, \mathfrak{Y}^*)$. Then L_λ converges to L in the BW topology if and only if $L_\lambda(x)$ converges to $L(x)$ in the weak*-topology on $\mathfrak{Y}^*$ generated by $\mathfrak{Y}$ for all $x \in \mathfrak{X}$.*

Proof. Suppose that L_λ converges to L in the BW topology. Then for each $x \in \mathfrak{X}$ and for all $y \in \mathfrak{Y}$

$$L_\lambda(x)(y) = L_\lambda(x \otimes y) \to L(x \otimes y) = L(x)(y).$$

As this holds for all $y \in \mathfrak{Y}$, $L_\lambda(x) \to L(x)$ in the weak*-topology on $\mathfrak{Y}^*$ for each $x \in \mathfrak{X}$.

Now suppose that $(L_\lambda)_\Lambda$ is a norm bounded net such that $L_\lambda(x)$ converges to $L(x)$ in the weak*-topology on $\mathfrak{Y}^*$ for all $x \in \mathfrak{X}$. Then, by linearity and finiteness,

$$L_\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) \to L\left(\sum_{i=1}^n x_i \otimes y_i\right)$$

for all $\sum_{i=1}^n x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}$. Fix $z \in \mathfrak{Z}$ and let $\epsilon > 0$. Then, due to the density of $\mathfrak{X} \odot \mathfrak{Y}$ in $\mathfrak{Z}$, there exists a $t = \sum_{i=1}^n x_i \otimes y_i \in \mathfrak{X} \odot \mathfrak{Y}$ such that $\|z - t\| < \frac{\epsilon}{M}$ where

$$M := \max\{\sup\{\|L_\lambda\| \mid \lambda \in \Lambda\}, \|L\|, 1\} < \infty$$

as $\{L_\lambda\}_{\lambda\in\Lambda}$ was norm bounded. Since we have shown that $L_\lambda(t) \to L(t)$, there exists $\lambda' \in \Lambda$ such that if $\lambda \geq \lambda'$ then $\|L_\lambda(t) - L(t)\| < \epsilon$. Thus for all $\lambda \geq \lambda'$,

$$\|L_\lambda(z) - L(z)\| \leq \|L_\lambda\| \|z - t\| + \|L_\lambda(t) - L(t)\| + \|L\| \|t - z\| \leq 3\epsilon$$

Therefore $L_\lambda(z) \to L(z)$ for all $z \in \mathfrak{Z}$ and thus $L_\lambda \to L$ in the BW topology. □

The above actually shows something special. Let $\mathcal{H}$ be a Hilbert space. Then, since $\mathcal{H}^* \simeq \mathcal{H}$, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ is a dual space by Lemma 4.13. The question is, "Can we identify this Banach space in another manner?" To do this we shall need to recall some of spectral theory for compact normal operators.

Definition 4.15. A compact operator $K \in \mathcal{B}(\mathcal{H})$ is said to be of trace class if the eigenvalues (including multiplicity) of $|T|$ are summable. Let $\|T\|_1$ denote sum of the eigenvalues (including multiplicity) of $|T|$ and let $\mathcal{C}_1$ denote the set of trace class operators.

Facts 4.16. The set of trace class operators $\mathcal{C}_1$ is a vector subspace of $\mathcal{B}(\mathcal{H})$, $\|\cdot\|_1$ is a norm on $\mathcal{C}_1$, $\mathcal{C}_1$ is complete with respect to $\|\cdot\|_1$, and the finite rank operators are dense in $\mathcal{C}_1$ with respect to this norm. Moreover $TK \in \mathcal{C}_1$ and $KT \in \mathcal{C}_1$ whenever $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_1$.

If $\{e_\lambda\}_\Lambda$ is an orthonormal basis for $\mathcal{H}$, define $Tr : \mathcal{C}_1 \to \mathbb{C}$ by $Tr(K) = \sum_\Lambda \langle Ke_\lambda, e_\lambda\rangle$ for all $K \in \mathcal{C}_1$. Then Tr is a well-defined continuous linear functional that does not depend on the choice of orthonormal basis $\{e_\lambda\}_\Lambda$. Moreover $Tr(TK) = Tr(KT)$ for all $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_1$.

For each $T \in \mathcal{B}(\mathcal{H})$ define $\varphi_T : \mathcal{C}_1 \to \mathbb{C}$ by $\varphi_T(K) = Tr(KT)$ for all $K \in \mathcal{C}_1$. Then φ_T is continuous with $\|\varphi_T\| = \|T\|$. The map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{C}_1^*$ defined by $\Phi(T) = \varphi_T$ is a bijective isometry.

By the above facts and Construction 4.12, $\mathcal{B}(\mathfrak{X}, \mathcal{C}_1^*) = \mathcal{B}(\mathfrak{X}, \mathcal{B}(\mathcal{H}))$ has a BW topology for every Banach space $\mathfrak{X}$. This topology is also called the point-ultraweak topology (as the weak*-topology on $\mathcal{B}(\mathcal{H})$ induced by the trace class operators is called the ultraweak topology and Lemma 4.14 implies that convergence in the BW topology is pointwise convergence in the ultraweak topology). We will abuse the notation and write $\mathcal{B}(\mathfrak{X}, \mathcal{B}(\mathcal{H}))$ for $\mathcal{B}(\mathfrak{X}, \mathcal{C}_1^*)$ (it turns out that $\mathcal{C}_1$ is the only predual of $\mathcal{B}(\mathcal{H})$). On bounded nets, Lemma 4.14 tells us this is the same as the point-WOT.

Proposition 4.17. *Let $\mathfrak{X}$ be a Banach space and $\mathcal{H}$ a Hilbert space. Then a norm bounded net $(L_\lambda)_\Lambda \subseteq \mathcal{B}(\mathfrak{X}, \mathcal{B}(\mathcal{H}))$ converges in the BW topology to L if and only if $\langle (L_\lambda(x))\xi, \eta\rangle$ converges to $\langle (L(x))\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$ and $x \in \mathfrak{X}$.*

Proof. Recall from Lemma 4.14 that L_λ converges to L in the BW topology if and only if $L_\lambda(x)$ converges to $L(x)$ in the weak*-topology on $\mathcal{B}(\mathcal{H})$ (generated by $\mathcal{C}_1$) for all $x \in \mathfrak{X}$. However $L_\lambda(x)$ converges weak* to $L(x)$ in $\mathcal{B}(\mathcal{H}) = (\mathcal{C}_1)^*$ if and only if $(L_\lambda(x))(K) \to (L(x))(K)$ for all $K \in \mathcal{C}_1$ if and only if $Tr((L_\lambda(x))K) \to Tr((L(x))K)$ for all $K \in \mathcal{C}_1$. The idea behind this proof is that the weak*-topology on $\mathcal{B}(\mathcal{H})$ induced by the trace class operators agrees with the WOT on bounded sets as the finite rank operators are dense in the set of trace class operators.

Suppose that L_λ converges to L in the BW topology. For each $\xi, \eta \in \mathcal{H}$, define $\xi\eta^* \in \mathcal{B}(\mathcal{H})$ by $(\xi\eta^*)(\zeta) = \langle \zeta, \eta\rangle\xi$ so that $\xi\eta^*$ is a finite rank operator (and thus a trace class operator). Since $\xi\eta^* \in \mathcal{C}_1$ for all $\xi, \eta \in \mathcal{H}$ it is easy to verify that $Tr(T(\xi\eta^*)) = \langle T\xi, \eta\rangle$ for all $T \in \mathcal{B}(\mathcal{H})$ and thus we obtain that

$$\langle (L_\lambda(x))\xi, \eta\rangle = Tr((L_\lambda(x))(\xi\eta^*)) \to Tr((L(x))(\xi\eta^*)) = \langle (L(x))\xi, \eta\rangle.$$

Thus one direction holds.

For the other direction, suppose $\langle (L_\lambda(x))\xi, \eta\rangle$ converges to $\langle (L(x))\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$ and $x \in \mathfrak{X}$. Therefore $Tr((L_\lambda(x))(\xi\eta^*)) \to Tr((L(x))(\xi\eta^*))$ for all $\xi, \eta \in \mathcal{H}$ and $x \in \mathfrak{X}$. Therefore, by linearity and finiteness, we obtain for all finite rank operators F that $Tr((L_\lambda(x))F) \to Tr((L(x))F)$ for all $x \in \mathfrak{X}$. Fix $x \in \mathfrak{X}$, $K \in \mathcal{C}_1$, and $\epsilon > 0$. By the fact that the finite rank operators are dense in $\mathcal{C}_1$ there exists a finite rank operator F such that $\|K - F\|_1 \leq \frac{\epsilon}{3M(\|x\|+1)}$ where $M := 1 + \sup\{\{\|L_\lambda\| \mid \lambda \in \Lambda\} \cup \|L\|\}$ which is finite as $(L_\lambda)_\Lambda$ is a bounded net. Since F is a finite rank operator there exists a $\lambda_0 \in \Lambda$ such that $|Tr((L(x))F) - Tr((L_\lambda(x))F)| < \frac{\epsilon}{3}$ for all $\lambda \geq \lambda_0$. Hence for all $\lambda \geq \lambda_0$,

$$\begin{aligned}
&|Tr((L(x))K) - Tr((L_\lambda(x))K)| \\
&\leq |Tr((L(x))K - (L(x))F)| + |Tr((L(x))F - (L_\lambda(x))F)| + |Tr((L_\lambda(x))F - (L_\lambda(x))K)| \\
&\leq \|L(x)\| \, \|K - F\|_1 + \frac{\epsilon}{3} + \|L_\lambda(x)\| \, \|F - K\|_1 \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Therefore $Tr((L_\lambda(x))K) \to Tr((L(x))K)$. However, since $K \in \mathcal{C}_1$ and $x \in \mathfrak{X}$ was arbitrary, we have that L_λ converges to L in the BW topology as desired. ☐

Now that we are in the final stretch of our proof of Arveson's Extension Theorem, we need to define a few sets and prove some simple properties of these sets.

Notation 4.18. Let $\mathcal{S} \subseteq \mathfrak{A}$ be operator system or C*-algebra and let $\mathcal{H}$ a Hilbert space. Then for each $r \in [0, \infty)$ we define

$$\begin{aligned}
\mathcal{B}_r(\mathcal{M}, \mathcal{H}) &:= \{L \in \mathcal{B}(\mathcal{M}, \mathcal{B}(\mathcal{H})) \mid \|L\| \leq r\} \text{ and} \\
\mathcal{CP}_r(\mathcal{S}, \mathcal{H}) &:= \{L \in \mathcal{B}(\mathcal{S}, \mathcal{B}(\mathcal{H})) \mid L \text{ is completely positive, } \|L\| \leq r\}.
\end{aligned}$$

With these sets in hand, we shall now begin our first major step in proving the Arveson's Extension Theorem.

Theorem 4.19. *Let $\mathcal{S} \subseteq \mathfrak{A}$ be a closed operator system or C*-algebra. Then each set listed in 4.18 is compact in the BW topology.*

Proof. First note that we require $\mathcal{S}$ to be closed in order to be able to consider the BW topology. Since the BW topology is a weak*-topology, the Banach-Alaoglu Theorem implies that every norm closed ball around 0 is compact in the BW topology. Thus $\mathcal{B}_r(\mathcal{M}, \mathcal{H})$ is BW-compact for all r.

To see that $\mathcal{CP}_r(\mathcal{S}, \mathcal{H})$ is BW-compact, we notice that $\mathcal{CP}_r(\mathcal{S}, \mathcal{H}) \subseteq \mathcal{B}_r(\mathcal{S}, \mathcal{H})$ and thus it suffices to show that $\mathcal{CP}_r(\mathcal{S}, \mathcal{H})$ is BW-closed. Thus let $(L_\lambda)_\Lambda \subseteq \mathcal{CP}_r(\mathcal{S}, \mathcal{H})$ be a convergent net in the BW topology. Let L be the limit of L_λ. Since $L_\lambda \in \mathcal{B}_r(\mathcal{S}, \mathcal{H})$ for all $\lambda \in \Lambda$ and $\mathcal{B}_r(\mathcal{S}, \mathcal{H})$ is a compact set in the BW topology (which is Hausdorff), $\mathcal{B}_r(\mathcal{S}, \mathcal{H})$ is closed. Therefore $L \in \mathcal{B}_r(\mathcal{S}, \mathcal{H})$ and thus $\|L\| \leq r$. Therefore to show that $L \in \mathcal{CP}_r(\mathcal{S}, \mathcal{H})$ it suffices to show that L is completely positive. Fix $n \in \mathbb{N}$ and consider $L_{(n)}$. Then for all $A \in \mathcal{S}$ $\langle (L_\lambda(A))\xi, \eta\rangle_{\mathcal{H}} \to \langle (L(A))\xi, \eta\rangle_{\mathcal{H}}$ for all $\xi, \eta \in \mathcal{H}$ as $(L_\lambda)_\Lambda$ is a bounded net and thus Proposition 4.17 applies. Therefore, by finiteness, we have for all positive $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ and $\xi = \xi_1 \oplus \ldots \oplus \xi_n \in \mathcal{H}^{\oplus n}$ that

$$\langle (L_\lambda)_{(n)}([A_{i,j}])\xi, \xi\rangle_{\mathcal{H}^{\oplus n}} \to \langle L_{(n)}([A_{i,j}])\xi, \xi\rangle_{\mathcal{H}^{\oplus n}}.$$

Hence, as this holds for all positive $[A_{i,j}] \in \mathcal{M}_n(\mathcal{S})$ and $\xi = \xi_1 \oplus \ldots \oplus \xi_n, \in \mathcal{H}^{\oplus n}$, we obtain that $\langle L_{(n)}([A_{i,j}])\xi, \xi\rangle_{\mathcal{H}^{\oplus n}} \geq 0$ since each L_λ is n-positive. Therefore $L_{(n)}$ is positive for all $n \in \mathbb{N}$ so that L is completely positive. Therefore $\mathcal{CP}_r(\mathcal{S}, \mathcal{H})$ is a BW-closed set in a BW-compact set and thus is BW-compact. ☐

Now we just need one more little lemma that may seem a little out of place (in that we could have proven this much earlier).

Lemma 4.20. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be a unital C*-algebra, let $\mathcal{S}$ be an operator system of $\mathfrak{A}$, and let $\varphi : \mathcal{S} \to \mathfrak{B}$ be a completely positive map. Then φ extends by continuity to a completely positive map $\psi : \overline{\mathcal{S}} \to \mathfrak{B}$.*

Proof. From Proposition 2.28 we know that every positive map extends by continuity to a positive map on the norm closure of its domain. Therefore if $\psi : \overline{\mathcal{S}} \to \mathfrak{B}$ is the extension of φ then ψ is positive. Now fix $n \in \mathbb{N}$ and suppose ψ' is the positive extension of $\varphi_{(n)}$ on $\overline{\mathcal{M}_n(\mathcal{S})}$. Notice that $\overline{\mathcal{M}_n(\mathcal{S})} = \mathcal{M}_n(\overline{\mathcal{S}})$ by applying Lemma 1.23. We claim that $\psi' = (\psi)_{(n)}$ so that ψ is n-positive. However for all $A \in \mathcal{M}_n(\mathcal{S})$ $\psi'(A) = \varphi_{(n)}(A) = \psi_{(n)}(A)$. Thus, by continuity, $\psi' = \psi_{(n)}$ so that ψ is n-positive. As this holds for all $n \in \mathbb{N}$, ψ is completely positive and extends φ. □

With this in hand we are now able to prove the Arveson's Extension Theorem. Unfortunately the following proof only works in the unital case as this guarantees (by Theorem 3.18) that the norms of the completely positive maps under consideration are bounded. In the non-unital case we were not able to show that the norms of the completely positive maps under consideration are bounded so the proof does not work. We will deal with the non-unital case afterwards in a fashion similar to that of Chapter 1 for positive linear functionals.

Theorem 4.21 (Arveson's Extension Theorem; Unital Version)**.** *Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and let $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a complete positive map. Then there exists a completely positive map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\psi|_{\mathfrak{B}} = \varphi$ and $\|\psi\| = \|\varphi\|$.*

Proof. Note that as completely positive maps obtain their norm at the identity, any completely positive extension ψ of a completely positive map φ on an operator system must have the property that $\|\psi\| = \|\varphi\|$. Thus it suffices to find a completely positive extension. By Lemma 4.20 we may extend φ to a completely positive map on the norm closure of $\mathcal{S}$. Thus, without loss of generality, we may assume that $\mathcal{S}$ is a closed operator system. The idea of the remainder of the proof is to use what we have developed; construct a net of completely positive maps into matrix algebras so that they can be extended by Theorem 4.11 and then take a cluster point using the BW topology.

Let $\mathcal{F}$ be any finite dimensional subspace of $\mathcal{H}$ and let $P_{\mathcal{F}}$ be the orthogonal projection onto $\mathcal{F}$. Notice that $P_{\mathcal{F}}\mathcal{B}(\mathcal{H})P_{\mathcal{F}}$ is *-isomorphic to $\mathcal{B}(\mathcal{F})$ and we will make this identification. Define $\varphi_{\mathcal{F}} : \mathcal{S} \to \mathcal{B}(\mathcal{F})$ by $\varphi_{\mathcal{F}}(A) = P_{\mathcal{F}}\varphi(A)P_{\mathcal{F}}$ for all $A \in \mathcal{S}$. Note that $\varphi_{\mathcal{F}}$ is completely positive as it is the composition of the conjugation by $P_{\mathcal{F}}$ and a *-isomorphism identifying $P_{\mathcal{F}}\mathcal{B}(\mathcal{H})P_{\mathcal{F}}$ with $\mathcal{B}(\mathcal{F})$ and the composition of completely positive maps is completely positive by Proposition 3.9.

Since $\mathcal{B}(\mathcal{F})$ is isomorphic to $\mathcal{M}_n(\mathbb{C})$ for some $n \in \mathbb{N}$, Theorem 4.11 implies that there exists a completely positive map $\psi_{\mathcal{F}} : \mathfrak{A} \to \mathcal{B}(\mathcal{F})$ that extends $\varphi_{\mathcal{F}}$. Define $\psi'_{\mathcal{F}} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by for each $A \in \mathfrak{A}$ letting $\psi'_{\mathcal{F}}(A)$ be the operator that equals $\psi_{\mathcal{F}}(A)$ on $\mathcal{F}$ and is 0 on $\mathcal{F}^{\perp}$. Therefore, since $\psi'_{\mathcal{F}}$ is the composition of $\psi_{\mathcal{F}}$ and the *-homomorphism $\pi_{\mathcal{F}} : \mathcal{B}(\mathcal{F}) \to \mathcal{B}(\mathcal{H})$ where $\pi_{\mathcal{F}}(T)$ is the operator that equals T on $\mathcal{F}$ and is 0 on $\mathcal{F}^{\perp}$ and since the composition of completely positive maps is completely positive, $\psi'_{\mathcal{F}}$ is completely positive.

Note that for every finite dimensional subspace $\mathcal{F}$ that $\psi'_{\mathcal{F}} \in \mathcal{CP}_{\|\varphi\|}(\mathfrak{A}, \mathcal{H})$ as each $\psi'_{\mathcal{F}}$ is completely positive and thus

$$\|\psi'_{\mathcal{F}}\| = \|\psi'_{\mathcal{F}}(I_{\mathfrak{A}})\| = \|\varphi_{\mathcal{F}}(I_{\mathfrak{A}})\| \leq \|\varphi(I_{\mathfrak{A}})\| = \|\varphi\|$$

by Theorem 3.18. Moreover the set of finite-dimensional subspace of $\mathcal{H}$ form a directed net under inclusion. Hence $(\psi'_{\mathcal{F}})_{\mathcal{F} \subseteq \mathcal{H} \text{ finite}}$ is a net in $\mathcal{CP}_{\|\varphi\|}(\mathfrak{A}, \mathcal{H})$. Since $\mathcal{CP}_{\|\varphi\|}(\mathfrak{A}, \mathcal{H})$ is BW-compact by Theorem 4.19 $(\psi'_{\mathcal{F}})_{\mathcal{F} \subseteq \mathcal{H} \text{ finite}}$ has a convergent subnet. Let $\psi \in \mathcal{CP}_{\|\varphi\|}(\mathfrak{A}, \mathcal{H})$ be the limit of this convergent subnet. Hence ψ is completely positive.

All that remains is to show that this ψ extends φ. To see this let $A \in \mathcal{S}$ be arbitrary. Fix $x, y \in \mathcal{H}$ and let $\mathcal{F}$ be the finite dimensional subspace spanned by x and y. Then for any finite dimensional subspace $\mathcal{F}_1 \supseteq \mathcal{F}$ in our convergent subnet (which exists by the definition of a subnet)

$$\langle \varphi(A)x, y\rangle = \langle \varphi(A)|_{\mathcal{F}_1} x, P_{\mathcal{F}_1} y\rangle = \langle \varphi_{\mathcal{F}_1}(A)x, y\rangle = \langle \psi'_{\mathcal{F}_1}(A)x, y\rangle.$$

However $\psi'_{\mathcal{F}} \to \psi$ in the BW topology and $(\psi'_{\mathcal{F}})_{\mathcal{F} \subseteq \mathcal{H} \text{ finite}}$ is a bounded net. Hence, by Proposition 4.17, the above equality implies for every $A \in \mathcal{S}$ and every $x, y \in \mathcal{H}$ that $\langle \varphi(A)x, y\rangle = \langle \psi(A)x, y\rangle$. Therefore $\psi(A) = \varphi(A)$ for all $A \in \mathcal{S}$. Hence ψ is a completely positive map on $\mathfrak{A}$ that extends φ. □

To prove the non-unital Arveson Extension Theorem, we will prove three results very similar to Lemma 1.12, Lemma 1.13, and Theorem 1.15.

Lemma 4.22. *Let $\mathfrak{B}$ be a non-unital C^*-algebra, let $\tilde{\mathfrak{B}}$ denote the unitization of $\mathfrak{B}$, and let $\varphi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a completely positive map $\tilde{\varphi} : \tilde{\mathfrak{B}} \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\varphi}|_{\mathfrak{B}} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. By Stinespring's Theorem (the non-unital case) there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{B} \to \mathcal{B}(\mathcal{K})$, and a $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\|V\|^2 = \|\varphi\|$ and $\varphi(B) = V^*\pi(B)V$ for all $B \in \mathfrak{B}$. Define $\tilde{\pi} : \tilde{\mathfrak{B}} \to \mathcal{B}(\mathcal{K})$ by $\tilde{\pi}(\lambda I_{\tilde{\mathfrak{B}}} + B) = \lambda I_{\mathcal{K}} + \pi(B)$ for all $\lambda \in \mathbb{C}$ and $B \in \mathfrak{B}$. It is easy to verify that $\tilde{\pi}$ is a *-homomorphism.

Define $\tilde{\varphi} : \tilde{\mathfrak{B}} \to \mathcal{B}(\mathcal{H})$ by

$$\tilde{\varphi}(\lambda I_{\tilde{\mathfrak{B}}} + B) = \lambda V^*V + \varphi(B) = V^*\tilde{\pi}(\lambda I + B)V$$

for all $\lambda I_{\tilde{\mathfrak{B}}} + B \in \tilde{\mathfrak{B}}$. By the remarks preceding Stinespring's Theorem the later expression shows that $\tilde{\varphi}$ is a completely positive map. It is clear that $\tilde{\varphi}$ extends φ. Lastly, by Theorem 3.18,

$$\|\tilde{\varphi}\| = \left\|\tilde{\varphi}(I_{\tilde{\mathfrak{B}}})\right\| = \|V^*V\| = \|V\|^2 = \|\varphi\|$$

as desired. ☐

Lemma 4.23. *Let $\mathfrak{B}$ be a C^*-algebra and let $\varphi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then there exists a completely positive map $\tilde{\varphi} : \mathfrak{B} \oplus \mathbb{C} \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\varphi}|_{\mathfrak{B}} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof. Define $\tilde{\varphi}(B \oplus \lambda) = \varphi(B)$ for all $\lambda \in \mathbb{C}$ and $B \in \mathfrak{B}$. It is trivial to verify that $\tilde{\varphi}$ is completely positive, $\tilde{\varphi}|_{\mathfrak{B}} = \varphi$, and $\|\tilde{\varphi}\| = \|\varphi\|$. ☐

Theorem 4.24 (Arveson's Extension Theorem; Non-Unital Version)**.** *Let $\mathfrak{B} \subseteq \mathfrak{A}$ be C^*-algebras and $\varphi : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be a complete positive map. Then there exists a completely positive map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\psi|_{\mathfrak{B}} = \varphi$ and $\|\psi\| = \|\varphi\|$.*

Proof. We shall break the proof into a two cases that will follow easily from the above lemmas.

<u>*Case 1: $\mathfrak{B}$ is not unital*</u> Let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$ if $\mathfrak{A}$ is not unital and let $\tilde{\mathfrak{A}}$ be $\mathfrak{A}$ if $\mathfrak{A}$ is unital. Consider the *-algebra

$$\mathbb{C}I_{\tilde{\mathfrak{A}}} + \mathfrak{B} := \{\lambda I_{\tilde{\mathfrak{A}}} + B \mid B \in \mathfrak{B}, \lambda \in \mathbb{C}\} \subseteq \tilde{\mathfrak{A}}.$$

Then, as in the proof of Theorem 1.15, $\mathbb{C}I_{\tilde{\mathfrak{A}}} + \mathfrak{B}$ is a C*-algebra that is *-isomorphic to $\tilde{\mathfrak{B}}$.

By Lemma 4.22 φ extends to a completely positive map $\tilde{\varphi} : \mathbb{C}I_{\tilde{\mathfrak{A}}} + \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ such that $\|\tilde{\varphi}\| = \|\varphi\|$. Since $\mathbb{C}I_{\tilde{\mathfrak{A}}} + \mathfrak{B} \subseteq \tilde{\mathfrak{A}}$ are C*-algebras with the same unit, Theorem 4.21 implies that $\tilde{\varphi}$ extends to a completely positive map $\tilde{\psi} : \tilde{\mathfrak{A}} \to \mathcal{B}(\mathcal{H})$ such that $\left\|\tilde{\psi}\right\| = \|\tilde{\varphi}\|$. Let $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be defined by $\psi = \tilde{\psi}|_{\mathfrak{A}}$. Since the restriction of a positive linear functional is clearly positive, ψ is a completely positive map. Moreover, by construction, ψ extends φ and $\|\psi\| \leq \left\|\tilde{\psi}\right\| = \|\varphi\| \leq \|\psi\|$ (where the last inequality comes from the fact that ψ extends φ).

<u>*Case 2: $\mathfrak{B}$ is unital*</u> Let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$ if $\mathfrak{A}$ is not unital and let $\tilde{\mathfrak{A}}$ be $\mathfrak{A}$ if $\mathfrak{A}$ is unital. If $\mathfrak{B}$ and $\tilde{\mathfrak{A}}$ have the same unit, the result follows from Theorem 4.21. Else when $I_{\mathfrak{B}} \neq I_{\tilde{\mathfrak{A}}}$, the proof of Theorem 1.15 shows that $\mathfrak{B} \oplus \mathbb{C}$ can be viewed as a C*-algebra of $\tilde{\mathfrak{A}}$ with the same unit.

By Lemma 4.23 φ extends to a completely positive map $\tilde{\varphi} : \mathfrak{B} \oplus \mathbb{C} \to \mathcal{B}(\mathcal{H})$ such that $\|\tilde{\varphi}\| = \|\varphi\|$. Since $\mathfrak{B} \oplus \mathbb{C} \subseteq \tilde{\mathfrak{A}}$ are C*-algebras with the same unit, Theorem 4.21 implies that $\tilde{\varphi}$ extends to a completely positive map $\tilde{\psi} : \tilde{\mathfrak{A}} \to \mathcal{B}(\mathcal{H})$ such that $\left\|\tilde{\psi}\right\| = \|\tilde{\varphi}\|$. Let $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be defined by $\psi = \tilde{\psi}|_{\mathfrak{A}}$. Since the restriction of a completely positive map is clearly completely positive, ψ is a completely positive map. Moreover, by construction, ψ extends φ and $\|\psi\| \leq \left\|\tilde{\psi}\right\| = \|\varphi\| \leq \|\psi\|$ (where the last inequality comes from the fact that ψ extends φ). ☐

5 Applications of Completely Positive Maps

In this chapter we will examine some applications of completely positive maps. Unfortunately we will not go into much detail as there is a large amount of additional material that would be required.

Our first application comes from the fact that Arveson's Extension Theorem allows us to extend completely positive maps as long as the range is allowed to be contained in $\mathcal{B}(\mathcal{H})$. We desire to determine when a C*-algebra $\mathfrak{B}$ has the property that every completely positive map into $\mathfrak{B}$ has a completely positive extension.

Definition 5.1. A C*-algebra $\mathfrak{B}$ is said to be injective if for every C*-algebra $\mathfrak{A}$, every operator system or C*-subalgebra $\mathcal{S} \subseteq \mathfrak{A}$, and every completely positive map $\varphi : \mathcal{S} \to \mathfrak{B}$ there exists a completely positive map $\tilde{\varphi} : \mathfrak{A} \to \mathfrak{B}$ such that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.

Example 5.2. Let $\mathcal{H}$ be any Hilbert space. The C*-algebra $\mathcal{B}(\mathcal{H})$ is injective by Theorem 4.21 and Theorem 4.24.

For our next example it makes sense to consider the example in full generality.

Lemma 5.3. *Let $\{\mathfrak{A}_n\}_{n\geq 1}$ be a (possibly finite) collection of C*-algebras. Define*

$$\prod_{n\geq 1} \mathfrak{A}_n := \{(A_n)_{n\geq 1} \mid A_n \in \mathfrak{A}_n, \sup_{n\geq 1} \|A_n\|_{\mathfrak{A}_n} < \infty\}.$$

Then $\prod_{n\geq 1} \mathfrak{A}_n$ is a C*-algebra with pointwise addition, multiplication, and involution and the C*-norm $\|(A_n)_{n\geq 1}\| = \sup_{n\geq 1} \|A_n\|_{\mathfrak{A}_n}$. Moreover if $\mathcal{S} \subseteq \mathfrak{A}$ is an operator system or C*-algebra and $\varphi_n : \mathcal{S} \to \mathfrak{A}_n$ are completely positive maps with $\sup_{n\geq 1} \|\varphi_n\| < \infty$ then there exists a well-defined completely positive map $\varphi : \mathcal{S} \to \prod_{n\geq 1} \mathfrak{A}$ such that $\varphi(A) = (\varphi_n(A))_{n\geq 1}$ and $\|\varphi\| = \sup_{n\geq 1} \|\varphi_n\|$.

Proof. It is trivial to verify that $\prod_{n\geq 1} \mathfrak{A}_n$ is a C*-algebra,

$$\mathcal{M}_m\left(\prod_{n\geq 1} \mathfrak{A}_n\right) \simeq \prod_{n\geq 1} \mathcal{M}_m(\mathfrak{A}_n)$$

for all $m \in \mathbb{N}$ via the map $[(A_{i,j,n})_{n\geq 1}] \mapsto ([A_{i,j,n}])_{n\geq 1}$, and that an element $(A_n)_{n\geq 1} \in \prod_{n\geq 1} \mathfrak{A}_n$ is positive if and only if $A_n \geq 0$ for all n.

Suppose $\mathcal{S} \subseteq \mathfrak{A}$ is an operator system or C*-algebra and $\varphi_n : \mathcal{S} \to \mathfrak{A}_n$ are completely positive maps with $\sup_{n\geq 1} \|\varphi_n\| < \infty$. Define $\varphi : \mathcal{S} \to \prod_{n\geq 1} \mathfrak{A}$ by $\varphi(A) = (\varphi_n(A))_{n\geq 1}$. Since $\sup_{n\geq 1} \|\varphi_n\| < \infty$ $(\varphi_n(A))_{n\geq 1} \in \prod_{n\geq 1} \mathfrak{A}_n$ for all $A \in \mathcal{S}$ so that φ is well-defined. Clearly φ is linear. We claim that φ is completely positive. To see this, let $A = [(A_{i,j,n})_{n\geq 1}] \in \mathcal{M}_m(\mathfrak{A})$ be an arbitrary positive element. Then, using the isomorphism $\mathcal{M}_m\left(\prod_{n\geq 1} \mathfrak{A}_n\right) \simeq \prod_{n\geq 1} \mathcal{M}_m(\mathfrak{A}_n)$,

$$\varphi_{(m)}(A) = [\varphi((A_{i,j,n})_{n\geq 1})] = [(\varphi_n(A_{i,j,n}))_{n\geq 1}] \simeq ([\varphi_n(A_{i,j,n})])_{n\geq 1} = ((\varphi_n)_{(m)}([A_{i,j,n}]))_{n\geq 1} \geq 0$$

since each φ_n is completely positive and by the description of the positive elements of $\prod_{n\geq 1} \mathfrak{A}_n$. Whence φ is completely positive. Lastly

$$\begin{aligned}\|\varphi\| &= \sup\{\|(\varphi_n(A))_{n\geq 1}\| \mid A \in \mathcal{S}, \|A\| \leq 1\} \\ &= \sup\{\|\varphi_n(A)\|_{\mathfrak{A}_n} \mid A \in \mathcal{S}, \|A\| \leq 1, n \in \mathbb{N}\} \\ &= \sup_{n\geq 1} \|\varphi_n\|\end{aligned}$$

as desired. □

Proposition 5.4. *Let $\{\mathfrak{A}_n\}_{n\geq 1}$ be a (possibly finite) collection of injective C*-algebras. Then $\prod_{n\geq 1} \mathfrak{A}_n$ is injective.*

Proof. To begin suppose $\mathcal{S} \subseteq \mathfrak{A}$ is an operator system or C*-algebra and $\varphi : \mathcal{S} \to \prod_{n\geq 1} \mathfrak{A}_n$ is completely positive. For each $k \in \mathbb{N}$ define $\pi_k : \prod_{n\geq 1} \mathfrak{A}_n \to \mathfrak{A}_k$ by $\pi((A_n)_{n\geq 1}) = A_k$ for all $(A_n)_{n\geq 1} \in \prod_{n\geq 1} \mathfrak{A}_n$. Clearly π_k is a *-homomorphism. For each $k \in \mathbb{N}$ define $\varphi_k : \mathcal{S} \to \mathfrak{A}_k$ by $\varphi_k = \pi_k \circ \varphi$. Thus φ_k is the composition of completely positive maps and thus is completely positive by Proposition 3.9. Clearly $\|\varphi_k\| \leq \|\varphi\|$ and φ_k is the k^{th} component of φ.

Since each $\mathfrak{A}_k$ is injective, there exist completely positive maps $\tilde{\varphi}_k : \mathfrak{A} \to \mathfrak{A}_k$ with $\|\tilde{\varphi}_k\| = \|\varphi_k\| \leq \|\varphi\|$. Define $\tilde{\varphi} : \mathfrak{A} \to \prod_{n\geq 1} \mathfrak{A}_n$ by $\tilde{\varphi}(A) = (\tilde{\varphi}_n(A))_{n\geq 1}$ for all $A \in \mathfrak{A}$. Then, by Lemma 5.3, $\tilde{\varphi}$ is a well-defined completely positive map such that

$$\|\tilde{\varphi}\| = \sup_{k\geq 1} \|\tilde{\varphi}_k\| = \sup_{k\geq 1} \|\varphi_k\| \leq \|\varphi\|.$$

It is easy to verify that $\tilde{\varphi}|_{\mathcal{S}} = \varphi$ as $\tilde{\varphi}_n|_{\mathcal{S}} = \varphi_n$ for all $n \in \mathbb{N}$. Whence $\prod_{n\geq 1} \mathfrak{A}_n$ is injective. □

Example 5.5. The C*-algebra $\ell_\infty(\mathbb{N}) := \prod_{n\geq 1} \mathbb{C}$ is injective by Proposition 5.4.

Example 5.6. Finite C*-algebras are injective being direct sums of matrix algebras $\mathcal{M}_n(\mathbb{C})$.

Example 5.7. The C*-algebra $c(\mathbb{N})$ of all convergent sequences is not injective. To see this let $\varphi : c(\mathbb{N}) \to c(\mathbb{N})$ be the identity map and notice that $c(\mathbb{N}) \subseteq \ell_\infty(\mathbb{N})$. Thus if $c(\mathbb{N})$ were injective there would exists a completely positive map $\psi : \ell_\infty(\mathbb{N}) \to c(\mathbb{N})$ extending φ. Notice that $I := (1)_{n\geq 1} \in c(\mathbb{N})$ is a unit for both $c(\mathbb{N})$ and $\ell_\infty(\mathbb{N})$. Since ψ is multiplicative on $c(\mathbb{N})$, Theorem 4.4 implies that $\psi(AXB) = \psi(A)\psi(X)\psi(B)$ for all $A, B \in c(\mathbb{N})$ and $X \in \ell_\infty(\mathbb{N})$.

To obtain a contradiction, fix $X = (x_n)_{n\geq 1} \in \ell_\infty(\mathbb{N}) \setminus c(\mathbb{N})$ and let $\psi(A) = (a_n)_{n\geq 1} \in c(\mathbb{N})$. For each $m \in \mathbb{N}$ let E_m be the sequence in $c(\mathbb{N})$ with a one in the m^{th} spot and zeros elsewhere. Then $XE_m \in c(\mathbb{N})$ so that

$$E_m(a_n)_{n\geq 1} = \psi(E_m)\psi(X) = \psi(E_m X) = E_m X = E_m(x_n)_{n\geq 1}$$

for each $m \in \mathbb{N}$ Whence $a_m = x_m$ for each $m \in \mathbb{N}$ so that $X = (a_n)_{n\geq 1} \in c(\mathbb{N})$ which is a contradiction. Thus $c(\mathbb{N})$ is not injective.

The following observation based on the above example is essential in order to obtain more examples of injective and non-injective C*-algebras and for theoretical reasons.

Theorem 5.8. *A C*-algebra $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ is injective if and only if there exists a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{B}$ such that $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$ and $\|\Phi\| = 1$.*

Proof. Suppose that $\mathfrak{B}$ is injective. Then $Id_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$ is completely positive. Since $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ is a C*-subalgebra and $\mathfrak{B}$ is injective, there exists a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{B}$ such that $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$ and $\|\Phi\| = \|Id_{\mathfrak{B}}\| = 1$.

Next suppose there exists a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{B}$ such that $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$ and $\|\Phi\| = 1$. Let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system and let $\varphi : \mathcal{S} \to \mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ be a completely positive map. By the Arveson Extension Theorem there exists a completely positive map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ so that $\psi|_{\mathcal{S}} = \varphi$ and $\|\varphi\| = \|\psi\|$. Let $\tilde{\varphi} : \mathfrak{A} \to \mathfrak{B}$ be defined by $\tilde{\varphi} = \Phi \circ \psi$. Then $\tilde{\varphi}$ is completely positive and

$$\tilde{\varphi}(A) = \Phi(\psi(A)) = \Phi(\psi|_{\mathcal{S}}(A)) = \Phi(\varphi(A)) = \varphi(A)$$

for all $A \in \mathcal{S}$. Whence $\tilde{\varphi}$ is a completely positive extension of φ with $\|\tilde{\varphi}\| = \|\Phi \circ \psi\| \leq \|\Phi\| \|\psi\| = \|\varphi\|$. □

In the case that $I_{\mathcal{H}} \in \mathfrak{B}$ the condition $\|\Phi\| = 1$ can be removed from the above theorem by Theorem 3.18. The above theorem promotes us to make the following definition.

Definition 5.9. Let $\mathfrak{B} \subseteq \mathfrak{A}$ be C*-algebras. A conditional expectation of $\mathfrak{A}$ onto $\mathfrak{B}$ is a completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$ such that $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$ and $\|\Phi\| = 1$.

Thus Theorem 5.8 says that a C*-subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$ is injective if and only if there exists a conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{B}$.

Remarks 5.10. Notice, by Theorem 4.4, that if $\Phi : \mathfrak{A} \to \mathfrak{B}$ is a conditional expectation onto $\mathfrak{B}$ then $\Phi(B_1XB_2) = B_1\Phi(X)B_2$ for all $B_1, B_2 \in \mathfrak{B}$ and $X \in \mathcal{B}(\mathcal{H})$. Moreover, if $\mathfrak{A}$ and $\mathfrak{B}$ are unital C*-algebras, then $\Phi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$. Indeed as Φ is completely positive of norm one, $I_{\mathfrak{B}} \leq I_{\mathfrak{A}}$, and $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$, we must have

$$I_{\mathfrak{B}} = \Phi(I_{\mathfrak{B}}) \leq \Phi(I_{\mathfrak{A}}) \leq \|\Phi(I_{\mathfrak{A}})\| \, I_{\mathfrak{B}} = I_{\mathfrak{B}}$$

and thus $\Phi(I_{\mathfrak{A}}) = I_{\mathfrak{B}}$.

One rather fortunate or unfortunate result of the above theorem depending on your point of view is the following.

Corollary 5.11. *Let $\mathfrak{B}$ be an injective C*-algebra. Then $\mathfrak{B}$ is unital.*

Proof. Suppose $\mathfrak{B}$ is an injective C*-algebra. By the GNS construction we may view $\mathfrak{B}$ as a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Since $\mathfrak{B}$ is injective, there exists a completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{B}$ such that $\Phi|_{\mathfrak{B}} = Id_{\mathfrak{B}}$ and $\|\Phi\| = 1$. By Theorem 4.4 $\Phi(B_1XB_2) = B_1\Phi(X)B_2$ for all $B_1, B_2 \in \mathfrak{B}$ and $X \in \mathcal{B}(\mathcal{H})$. Let $P := \Phi(I_{\mathcal{H}}) \in \mathfrak{B}$. We claim that P is a unit of $\mathfrak{B}$. Indeed for all $B \in \mathfrak{B}$

$$BP = \Phi(B)\Phi(I_{\mathcal{H}}) = \Phi(BI_{\mathcal{H}}) = \Phi(B) = B = \Phi(I_{\mathcal{H}}B) = \Phi(I_{\mathcal{H}})\Phi(B) = PB.$$

Whence $\mathfrak{B}$ is unital. □

Our next example verifies that an important von Neumann algebra is not injective. The arguments used are essential arguments for group von Neumann algebras and are a topic of discussion by themselves. Those unfamiliar with group C*-algebras may skip the following example without loss.

Example 5.12. Let $\mathbb{F}_2$ be the free group on two generators. The von Neumann algebra $L(\mathbb{F}_2)$ is not injective. To see this we will show that there does not exist a completely positive map $\Phi : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to L(\mathbb{F}_2)$ such that $\Phi|_{C^*_\lambda(\mathbb{F}_2)} = Id_{C^*_\lambda(\mathbb{F}_2)}$ where $C^*_\lambda(\mathbb{F}_2) \subseteq L(\mathbb{F}_2)$ is the reduced group C*-algebra. It suffices to prove the non-existence of such a Φ since if $L(\mathbb{F}_2)$ were injective then there would exists a conditional expectation $\Phi : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to L(\mathbb{F}_2)$ by Theorem 5.8.

Suppose exist a completely positive map $\Phi : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to L(\mathbb{F}_2)$ such that $\Phi|_{C^*_\lambda(\mathbb{F}_2)} = Id_{C^*_\lambda(\mathbb{F}_2)}$. For each $A \in \ell_\infty(\mathbb{F}_2)$ define $m_A \in \mathcal{B}(\ell_2(\mathbb{F}_2))$ by $m_A\delta_g = A(g)\delta_g$ for all $g \in \mathbb{F}_2$ (where $A(g)$ is the g^{th} entry of $A \in \ell_\infty(\mathbb{F}_2)$ and $\delta_g \in \ell_2(\mathbb{F}_2)$ is the function that is one at g and zero elsewhere) and by extending by linearity. Clearly m_A is indeed a well-defined bounded operator. Define $\pi : \ell_\infty(\mathbb{F}_2) \to \mathcal{B}(\ell_2(\mathbb{F}_2))$ by $\pi(A) = m_A$ for all $A \in \ell_\infty(\mathbb{F}_2)$. It is trivial to verify that π is a unital *-homomorphism.

Let $\tau : L(\mathbb{F}_2) \to \mathbb{C}$ be defined by $\tau(T) = \langle T\delta_e, \delta_e\rangle$ for all $T \in L_2(\mathbb{F}_2)$ where $e \in \mathbb{F}_2$ is the identity. It is trivial to verify that $\tau(TS) = \tau(ST)$ for all $S, T \in L(\mathbb{F}_2)$ (first verify this for $T = \lambda(g)$ and $S = \lambda(h)$ where $g, h \in \mathbb{F}_2$ (where $\lambda(g)\delta_k = \delta_{g^{-1}k}$ is the left-regular representation) and then use linearity and WOT-density twice) and τ is a positive linear functional.

Define $\mu : \ell_\infty(\mathbb{F}_2) \to \mathbb{C}$ by $\mu(A) = \tau(\Phi(\pi(A)))$ for all $A \in \ell_\infty(\mathbb{F}_2)$. Since μ is the composition of positive maps, μ is a positive linear functional. Moreover

$$\mu(I_{\ell_\infty(\mathbb{F}_2)}) = \tau(\Phi(\pi(I_{\ell_\infty(\mathbb{F}_2)}))) = \tau(\Phi(I_{C^*_\lambda(\mathbb{F}_2)})) = \tau(I_{C^*_\lambda(\mathbb{F}_2)}) = 1$$

so μ is a state on $\ell_\infty(\mathbb{F}_2)$. For each $A \in \ell_\infty(\mathbb{F}_2)$ and $g \in \mathbb{F}_2$ define $g \cdot A \in \ell_\infty(\mathbb{F}_2)$ by $(g \cdot A)(h) = A(g^{-1}h)$. We claim that $\mu(g \cdot A) = \mu(A)$ for all $A \in \ell_\infty(\mathbb{F}_2)$ and $g \in \mathbb{F}_2$. To see this we first notice that

$$\lambda(g)\pi(A)\lambda(g)^{-1}\delta_h = \lambda(g)\pi(A)\delta_{g^{-1}h} = \lambda(g)(A(g^{-1}h)\delta_{g^{-1}h}) = A(g^{-1}h)\delta_h = \pi(g \cdot A)\delta_h$$

for all $h \in \mathbb{F}_2$. Whence $\lambda(g)\pi(A)\lambda(g)^{-1} = \pi(g \cdot A)$ for all $A \in \ell_\infty(\mathbb{F}_2)$ and $g \in \mathbb{F}_2$. Since $\Phi|_{C^*_\lambda(\mathbb{F}_2)} = Id_{C^*_\lambda(\mathbb{F}_2)}$ and Φ is a completely positive map, Theorem 4.4 implies

$$\Phi(\lambda(g)T\lambda(g)^{-1}) = \lambda(g)\Phi(T)\lambda(g)^{-1}$$

for all $g \in \mathbb{F}_2$ and $T \in \mathcal{B}(\ell_2(\mathbb{F}_2))$ (as $\lambda(g) \in C^*_\lambda(\mathbb{F}_2)$). Using this and the fact that $\tau(TS) = \tau(ST)$ for all $S, T \in L(\mathbb{F}_2)$ we obtain that

$$\mu(g \cdot A) = \tau(\Phi(\pi(g \cdot A))) = \tau(\Phi(\lambda(g)\pi(A)\lambda(g)^{-1})) = \tau(\lambda(g)\Phi(\pi(A))\lambda(g)^{-1}) = \tau(\Phi(\pi(A))) = \mu(A)$$

as claimed.

We claim that such a μ cannot possibly exists. To see this let a and b be generators of $\mathbb{F}_2$ and consider the sets

$$X_i := \{w \in \mathbb{F}_2 \mid \text{when written as a reduced word } w \text{ starts with } a^k \text{ where } k \equiv i \bmod 2\}$$

for $i = 0, 1$ and

$$Y_i := \{w \in \mathbb{F}_2 \mid \text{when written as a reduced word } w \text{ starts with } b^k \text{ where } k \equiv j \bmod 3\}$$

for $j = 0, 1, 2$. Clearly X_0 and X_1 are disjoint, Y_0, Y_1, and Y_2 are pairwise disjoint, $X_0 = aX_1$, and $Y_0 = b^2Y_1 = bY_2$. For a subset $Z \subseteq \mathbb{F}_2$ let $\chi_Z \in \ell_\infty(\mathbb{F}_2)$ be the characteristic function of Z (that is $\chi_Z(g) = 1$ if $g \in Z$ and zero otherwise). Notice

$$1 = \mu(I_{\ell_\infty(\mathbb{F}_2)}) = \mu(\chi_{X_0}) + \mu(\chi_{X_1}) = \mu(a \cdot \chi_{X_1}) + \mu(\chi_{X_1}) = 2\mu(\chi_{X_1})$$

so $\mu(\chi_{X_1}) = \frac{1}{2}$. Similarly

$$1 = \mu(\chi_{Y_0}) + \mu(\chi_{Y_1}) + \mu(\chi_{Y_2}) = \mu(\chi_{Y_0}) + \mu(b \cdot \chi_{Y_0}) + \mu(b^2 \cdot \chi_{Y_0}) = 3\mu(\chi_{Y_0})$$

so $\mu(\chi_{Y_0}) = \frac{1}{3}$. However $X_1 \subseteq Y_0$ (as reduced word in X_1 starts with a non-zero power of a) and thus $\chi_{Y_0} \geq \chi_{X_1}$. However μ is positive and $\mu(\chi_{Y_0}) < \mu(\chi_{X_0})$ which is impossible. Hence we have found our contradiction so $L(\mathbb{F}_2)$ cannot possibly be injective.

Our next examples fall in the realm of von Neumann algebras. Again this section may be skipped without loss to the reader that is not familiar with material in the following paragraph.

Remarks 5.13. Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra with $\mathcal{H}$ separable. Then there exists a faithful normal trace τ on $\mathfrak{M}$. Recall that we can give $\mathfrak{M}$ an inner product by $\langle x, y\rangle = \tau(y^*x)$ (this is an actual inner product as τ is faithful). Let $L_2(\mathfrak{M}, \tau)$ be the completion of $\mathfrak{M}$ with respect to this inner product. Thus $L_2(\mathfrak{M}, \tau)$ is a Hilbert space. Notice that for all $x \in \mathfrak{M}$ the map $M_x : \mathfrak{M} \to \mathfrak{M}$ defined by $M_x(y) = xy$ is continuous with respect to the inner product norm as

$$\|xy\|_2^2 = \tau(y^*x^*xy^*) \leq \|x\|^2 \tau(y^*y) = \|x\|^2 \|y\|_2^2$$

and thus extends to an element of $\mathcal{B}(L_2(\mathfrak{M}), \tau)$. Since

$$\langle (M_x)^*y, z\rangle = \langle y, xz\rangle = \tau(z^*x^*y) = \langle x^*y, z\rangle = \langle M_{x^*}y, z\rangle$$

and $M_xM_y = M_{xy}$ for all $x, y, z \in \mathfrak{M}$, the map $\pi : \mathfrak{M} \to \mathcal{B}(L_2(\mathfrak{M}), \tau)$ given by $\pi(x) = M_x$ is a *-isomorphism so we can view $\mathfrak{M} \subseteq \mathcal{B}(L_2(\mathfrak{M}, \tau))$. Moreover $\pi(\mathfrak{M})$ is closed in the WOT and π is normal (ultraweakly continuous).

Using $L_2(\mathfrak{M})$ it is possible to show the following.

Theorem 5.14. *Let $\mathcal{H}$ be a separable Hilbert space, let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra, let τ be a faithful normal trace on $\mathfrak{M}$, and let $\mathfrak{N} \subseteq \mathfrak{M}$ be a von Neumann subalgebra of $\mathfrak{M}$ with $I_{\mathfrak{M}} \in \mathfrak{N}$. Then there exists a unique trace-preserving conditional expectation $E_{\mathfrak{N}} : \mathfrak{M} \to \mathfrak{N}$ of $\mathfrak{M}$ onto $\mathfrak{N}$ (that is a conditional expectation such that $\tau(x) = \tau(E_{\mathfrak{N}}(x))$ for all $x \in \mathfrak{M}$). Specifically $E_{\mathfrak{N}}(T) = P_{\mathfrak{N}}TP_{\mathfrak{N}}$ where $P_{\mathfrak{N}}$ is the projection onto the subspace $L_2(\mathfrak{N}, \tau)$ of $L_2(\mathfrak{M}, \tau)$ generated by $\mathfrak{N}$ (which we can view as a subset of $L_2(\mathfrak{M}, \tau)$ by above). Moreover $E_{\mathfrak{N}}$ is normal.*

Thus we obtain the following.

Theorem 5.15. *Let $\mathcal{H}$ be a separable Hilbert space, let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be an injective finite von Neumann algebra, and let $\mathfrak{N}$ be a von Neumann subalgebra of $\mathfrak{M}$ such that $I_{\mathfrak{M}} \in \mathfrak{N}$. Then $\mathfrak{N}$ is injective.*

Proof. Let $\mathfrak{N} \subseteq \mathfrak{M}$ be as above. Since $\mathfrak{M}$ is injective, there exists a conditional expectation $\Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{M}$ by Theorem 5.8. Therefore $E_{\mathfrak{N}} \circ \Phi : \mathcal{B}(\mathcal{H}) \to \mathfrak{N}$ is a conditional expectation so $\mathfrak{N}$ is injective by Theorem 5.8. □

The following theorem is slightly technical. However this theorem is important.

Theorem 5.16. *Let $\mathcal{H}$ be a separable Hilbert space, let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a finite von Neumann algebra, and let τ be a faithful normal trace on $\mathfrak{M}$. Suppose $(\mathfrak{N}_n)_{n\geq 1}$ be an increasing family of injective von Neumann subalgebras of $\mathfrak{M}$ such that $I_{\mathfrak{M}} \in \mathfrak{N}_n$ for all $n \geq 1$ and $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{M}$. Then $\mathfrak{M}$ is injective.*

Proof. Let $(\mathfrak{N}_n)_{n\geq 1}$ be an increasing family of injective von Neumann subalgebras of $\mathfrak{M}$ such that $I_{\mathfrak{M}} \in \mathfrak{N}_n$ for all $n \geq 1$ and $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{M}$. By Theorem 5.14 there exists conditional expectations $E_n : \mathfrak{M} \to \mathfrak{N}_n$ onto $\mathfrak{N}_n$ for all $n \in \mathbb{N}$. Since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{M}$, $E_n(T) \to T$ ultraweakly for all $T \in \mathfrak{M}$ (to see this recall that the conditional expectations E_n are defined by $E_n(x) = P_n x P_n$ where P_n is the orthogonal projection of $L_2(\mathfrak{M}, \tau)$ onto $L_2(\mathfrak{N}_n, \tau)$. Since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{M}$, $E_n(T) \to T$ in the WOT on $\mathcal{B}(L_2(\mathfrak{M}, \tau))$ for all $T \in \mathfrak{M}$ (i.e. it is easy to show for all $T_n, S_n \in \mathfrak{N}_n$ that $\langle TT_n, S_n \rangle = \langle P_m T P_m T_n, S_n \rangle$ for all $m \geq n$ and $T \in \mathfrak{M}$. Then, since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweak dense in $\mathfrak{M}$, $\bigcup_{n\geq 1} \mathfrak{N}_n \subseteq L_2(\mathfrak{M}, \tau)$ is dense and thus we obtain the result by taking limits). Since the WOT on $\mathcal{B}(L_2(\mathfrak{M}, \tau))$ and ultraweak topology agree on bounded sets, we obtain the desired result).

Since $\mathfrak{N}_n$ is injective for all $n \in \mathbb{N}$, E_n extend to unital completely positive maps $\psi_n : \mathcal{B}(\mathcal{H}) \to \mathfrak{N}_n$. Therefore $(\psi_n)_{n\geq 1}$ is a sequence in $\mathcal{CP}_1(\mathcal{B}(\mathcal{H}), \mathcal{H})$. Since this set is compact in the BW topology by Theorem 4.19, there exists a completely positive map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathfrak{M}$ such that $\|\Psi\| \leq 1$ and $\psi_{n_\lambda}(T) \to \Psi(T)$ in the ultraweak topology for all $T \in \mathcal{B}(\mathcal{H})$ (where n_λ represents a subnet). Since $\mathfrak{M}$ is ultraweakly closed and since $\psi_{n_\lambda}(T) \in \mathfrak{M}$ for all λ and $T \in \mathcal{B}(\mathcal{H})$, $\Psi(T) \in \mathfrak{M}$ for all $T \in \mathcal{B}(\mathcal{H})$. Lastly if $T \in \mathfrak{M}$ then $\psi_{n_\lambda}(T) = E_{n_\lambda}(T) \to T$ in the ultraweak topology. Whence $\Psi(T) = T$ for all $T \in \mathfrak{M}$ so Ψ is a conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{M}$. Whence $\mathfrak{M}$ is injective by Theorem 5.8. □

There is one last von Neumann algebra we desire to consider; the hyperfinite II_1 factor (denoted $\mathfrak{R}$) which is the WOT-closure of a particular faithful representation of a particular approximately finite dimensional C*-algebra. It turns out $\mathfrak{R}$ is a very important von Neumann algebra.

Corollary 5.17. *The hyperfinite II_1 factor $\mathfrak{R}$ is injective.*

Proof. Recall $\mathfrak{R}$ is the increasing limit of finite dimensional C*-algebras (and thus von Neumann algebras) $\{\mathfrak{A}_n\}_{n\geq 1}$ such that $I_{\mathfrak{R}} \in \mathfrak{A}_n$ and $\bigcup_{n\geq 1} \mathfrak{A}_n$ is ultraweakly dense in $\mathfrak{R}$. Since finite dimensional C*-algebras are injective by Example 5.6, $\mathfrak{R}$ is injective by Theorem 5.16. □

Next we leave injective C*-algebras and consider a completely different application of completely positive maps. We begin with a definition.

Definition 5.18. A C*-algebra $\mathfrak{A}$ is said to be nuclear if there exists a net of natural numbers $(n_\lambda)_\Lambda$ and nets of contractive, completely positive maps $\varphi_\lambda : \mathfrak{A} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ such that $\lim_\Lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$ for every $A \in \mathfrak{A}$.

For those familiar with tensor products of C*-algebras and the definition of a nuclear C*-algebra with respect to tensors, the two notions agree although this will not be shown here. We begin with a few examples after proving an equivalent definition.

Proposition 5.19. *Suppose $\mathfrak{A}$ is a C*-algebra such that there exists a net of finite dimensional C*-algebras $\mathfrak{A}_\lambda$ and nets of contractive, completely positive maps $\varphi_\lambda : \mathfrak{A} \to \mathfrak{A}_\lambda$ and $\psi_\lambda : \mathfrak{A}_\lambda \to \mathfrak{A}$ such that*

$$\lim_\lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$$

for every $A \in \mathfrak{A}$. Then $\mathfrak{A}$ is nuclear.

Proof. Since $\mathfrak{A}_\lambda$ is a finite dimensional C*-algebra for every λ there exists $n_\lambda \in \mathbb{N}$ such that $\mathfrak{A}_\lambda \subseteq \mathcal{M}_{n_\lambda}(\mathbb{C})$. Therefore φ_λ can be viewed as a map with codomain $\mathcal{M}_{n_\lambda}(\mathbb{C})$.

Since $\mathfrak{A}_\lambda$ is a finite dimensional C*-algebra, Example 5.6 implies that $\mathfrak{A}_\lambda$ is injective. Thus the identity map $Id_{\mathfrak{A}_\lambda} : \mathfrak{A}_\lambda \to \mathfrak{A}_\lambda$ extends to a completely positive map $E_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}_\lambda$ such that $E_\lambda(A) = A$ for all $A \in \mathfrak{A}_\lambda$ and $\|E_\lambda\| = 1$. Define $\psi'_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ by $\psi'_\lambda = \psi_\lambda \circ E_\lambda$. Thus ψ'_λ is a contractive, completely positive map.

We claim that $\lim_\Lambda \|A - \psi'_\lambda(\varphi_\lambda(A))\| = 0$ for every $A \in \mathfrak{A}$ (which then implies $\mathfrak{A}$ is nuclear). However, since $E_\lambda(A) = A$ for all $A \in \mathfrak{A}_\lambda$ and $\varphi_\lambda(A) \in \mathfrak{A}_\lambda$, $\psi'_\lambda(\varphi_\lambda(A)) = \psi_\lambda(\varphi_\lambda(A))$. Whence the result follows as $\lim_\Lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$ for every $A \in \mathfrak{A}$. ◻

Example 5.20. Finite dimensional C*-algebras are clearly nuclear by the above result.

Example 5.21. Recall that a C*-algebra $\mathfrak{A}$ is said to be approximately finite dimensional if there exists an increasing sequence of finite dimensional C*-subalgebras $\mathfrak{A}_n$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_{n\geq 1} \mathfrak{A}_n}$. For example the compact operators on a separable Hilbert space are approximately finite dimensional.

Every approximately finite dimensional C*-algebra is nuclear. To see this let $\mathfrak{A}$ be approximately finite dimensional. Then there exists an increasing sequence of finite dimensional C*-subalgebras $\mathfrak{A}_n$ of $\mathfrak{A}$ such that $\mathfrak{A} = \overline{\bigcup_{n\geq 1} \mathfrak{A}}$. Since each $\mathfrak{A}_n$ is a finite dimensional C*-algebra, Example 5.6 implies that each $\mathfrak{A}_n$ is injective. Thus the identity map $Id_{\mathfrak{A}_n} : \mathfrak{A}_n \to \mathfrak{A}_n$ extends to a completely positive map $\varphi_n : \mathfrak{A} \to \mathfrak{A}_n$ such that $\varphi_n(A) = A$ for all $A \in \mathfrak{A}_n$ and $\|\varphi_n\| = 1$. Define $\psi_n : \mathfrak{A}_n \to \mathfrak{A}$ to be the inclusion map. Clearly ψ is completely positive and contractive.

We claim that $\lim_{n\to\infty} \|A - \psi_n(\varphi_n(A))\| = 0$ for all $A \in \mathfrak{A}$. If $A \in \mathfrak{A} \cap \mathfrak{A}_m$ for some $m \in \mathbb{N}$ then $A \in \mathfrak{A}_n$ for all $n \geq m$ so that $\varphi_n(A) = A$ for all $n \geq m$. Whence $\psi_n(\varphi_n(A)) = \psi_n(A) = A$ for all $n \geq m$. Thus the claim is true for all $A \in \bigcup_{n\geq 1} \mathfrak{A}_n$. Now suppose $A \in \mathfrak{A}$ is arbitrary and $\epsilon > 0$. Then there exists a $B \in \bigcup_{n\geq 1} \mathfrak{A}_n$ such that $\|A - B\| \leq \epsilon$. Whence

$$\begin{aligned}
\limsup_{n\to\infty} \|A - \psi_n(\varphi_n(A))\| &\leq \limsup_{n\to\infty} \|A - B\| + \|B - \psi_n(\varphi_n(B))\| + \|\psi_n(\varphi_n(B - A))\| \\
&\leq \limsup_{n\to\infty} \epsilon + \|B - \psi_n(\varphi_n(B))\| + \|\psi_n\| \|\varphi_n\| \|B - A\| \\
&\leq \limsup_{n\to\infty} 2\epsilon + \|B - \psi_n(\varphi_n(B))\| = 2\epsilon.
\end{aligned}$$

Thus, as this holds for all $\epsilon > 0$, $\lim_{n\to\infty} \|A - \psi_n(\varphi_n(A))\| = 0$ as desired. Whence $\mathfrak{A}$ is nuclear.

Example 5.22. If X is a compact Hausdorff space then $C(X)$ is nuclear. Indeed for every finite subset $\mathcal{F} \subseteq C(X)$ and $\epsilon > 0$ we will construct unital, completely positive maps $\varphi_{(\mathcal{F},\epsilon)} : C(X) \to \mathbb{C}^{n_{(\mathcal{F},\epsilon)}}$ and $\psi_{(\mathcal{F},\epsilon)} : \mathbb{C}^{n_{(\mathcal{F},\epsilon)}} \to C(X)$ such that $\left\|f - \psi_{(\mathcal{F},\epsilon)}(\varphi_{(\mathcal{F},\epsilon)}(f))\right\|_\infty \leq \epsilon$ for all $f \in \mathcal{F}$. The result will follow by taking the ordering $(\mathcal{F}, \epsilon) \leq (\mathcal{F}', \epsilon')$ if $\mathcal{F} \subseteq \mathcal{F}'$ and $\epsilon' \leq \epsilon$.

The idea behind constructing these maps is the partition of unity argument along with the approximation argument used in Proposition 2.23. Fix $\mathcal{F} \subseteq C(X)$ finite and $\epsilon > 0$. For each $x \in X$ there is an open subset U_x of X such that if $y \in U_x$ then $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{F}$. Since $X = \bigcup_{x\in X} U_x$ and X is compact, there exists a finite subset $\{x_1, \ldots, x_{n_{(\mathcal{F},\epsilon)}}\} \subseteq X$ so that $X = \bigcup_{i=1}^{n_{(\mathcal{F},\epsilon)}} U_{x_i}$. Define $\varphi_{(\mathcal{F},\epsilon)} : C(X) \to \mathbb{C}^{n_{(\mathcal{F},\epsilon)}}$ by $\varphi_{(\mathcal{F},\epsilon)}(f) = (f(x_1), \ldots, f(x_{n_{(\mathcal{F},\epsilon)}}))$ for all $f \in C(X)$. Since point evaluations are positive linear functionals and thus completely positive maps, Lemma 5.3 implies that $\varphi_{(\mathcal{F},\epsilon)}$ is a completely positive map. Clearly $\varphi_{(\mathcal{F},\epsilon)}$ is unital and therefore contractive.

Since $X = \bigcup_{i=1}^{n_{(\mathcal{F},\epsilon)}} U_{x_i}$, Theorem 2.22 implies that there exists $\{g_i\}_{i=1}^{n_{(\mathcal{F},\epsilon)}} \subseteq C(X, [0, 1])$ such that $g_i \geq 0$, $g_i(x_i) = 1$, $\sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} g_i = 1$, and $g_i|_{U^c_{x_i}} = 0$. Since the map $z \in \mathbb{C} \mapsto zg \in C(X)$ is positive when g is positive, the map $\psi_{n_{(\mathcal{F},\epsilon)}} : \mathbb{C}^{n_{(\mathcal{F},\epsilon)}} \to C(X)$ defined by $\psi_{n_{(\mathcal{F},\epsilon)}}((a_1, \ldots, a_{n_{(\mathcal{F},\epsilon)}})) = \sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} a_i g_i$ is the sum of positive maps and thus is positive. By Corollary 3.14, $\psi_{n_{(\mathcal{F},\epsilon)}}$ is automatically completely positive as its range is abelian. Clearly $\psi_{n_{(\mathcal{F},\epsilon)}}$ is unital and thus contractive.

Lastly we notice that $\psi_{n_{(\mathcal{F},\epsilon)}}(\varphi_{n_{(\mathcal{F},\epsilon)}}(f)) = \sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} f(x_i)g_i$ for all $f \in C(X)$. Whence if $f \in \mathcal{F}$ and $x \in X$

$$\begin{aligned} \left|f(x) - \psi_{n_{(\mathcal{F},\epsilon)}}(\varphi_{n_{(\mathcal{F},\epsilon)}}(f))(x)\right| &= \left|\sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} (f(x) - f(x_i))g_i(x)\right| \\ &\leq \sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} |f(x) - f(x_i)|g_i(x) \\ &= \sum_{i=1}^{n_{(\mathcal{F},\epsilon)}} \epsilon g_i(x) = \epsilon \end{aligned}$$

where line 2 to 3 comes from the fact that if $g_i(x) \neq 0$ then $x \in U_{x_i}$ so $|f(x) - f(x_i)| < \epsilon$ (see Proposition 2.23 if more detail are needed). Thus

$$\left\|f - \psi_{n_{(\mathcal{F},\epsilon)}} \circ \varphi_{n_{(\mathcal{F},\epsilon)}}(f)\right\|_\infty < \epsilon$$

for all $f \in \mathcal{F}$ as desired.

In Examples 5.21 and 5.22, it is easy to see that the contractive, completely positive maps implementing the nuclearity of $\mathfrak{A}$ can be chosen to be unital when $\mathfrak{A}$ is unital. It is useful for technical purposes to demonstrate this fact for all unital, nuclear C*-algebras. The proof of this fact will be demonstrated in Proposition 5.24 after the following technical lemma.

Lemma 5.23. *Let $\mathfrak{A}$ be a unital C*-algebra and let $\tilde{\varphi} : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ be a completely positive map. Then there exists a unital, completely positive map $\varphi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ such that*

$$\tilde{\varphi}(A) = \tilde{\varphi}(I_\mathfrak{A})^{\frac{1}{2}}\varphi(A)\tilde{\varphi}(I_\mathfrak{A})^{\frac{1}{2}}$$

for all $A \in \mathfrak{A}$.

Proof. Since $\tilde{\varphi}$ is a completely positive map, $\tilde{\varphi}(I_\mathfrak{A}) \geq 0$ so $\tilde{\varphi}(I_\mathfrak{A})^{\frac{1}{2}}$ makes sense. In the case that $\tilde{\varphi}(I_\mathfrak{A})$ is an invertible element in $\mathcal{M}_n(\mathbb{C})$, it is clear that if we define $\varphi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ by

$$\varphi(A) = \tilde{\varphi}(I_\mathfrak{A})^{-\frac{1}{2}}\tilde{\varphi}(A)\tilde{\varphi}(I_\mathfrak{A})^{-\frac{1}{2}}$$

for all $A \in \mathfrak{A}$, then φ is a completely positive map being the conjugate of a completely positive map by a self-adjoint operator. Moreover, it is clear that $\varphi(I_\mathfrak{A}) = I_n$ and that the conclusion of the lemma holds. Hence the result is complete in the case that $\tilde{\varphi}(I_\mathfrak{A})$ is an invertible element in $\mathcal{M}_n(\mathbb{C})$.

Suppose $\tilde{\varphi}(I_\mathfrak{A})$ is not an invertible element in $\mathcal{M}_n(\mathbb{C})$ and let P be the projection onto the orthogonal complement of the kernel of $\tilde{\varphi}(I_\mathfrak{A})$. Since $\tilde{\varphi}(A) \leq \tilde{\varphi}(I_\mathfrak{A})$ for all $A \in \mathfrak{A}$ such that $0 \leq A \leq I_\mathfrak{A}$, it is easy to see that

$$\tilde{\varphi}(A) = P\tilde{\varphi}(A) = \tilde{\varphi}(A)P$$

for all $A \in \mathfrak{A}$ such that $0 \leq A \leq I_\mathfrak{A}$. By taking linear combinations, we obtain that the above equation holds for all $A \in \mathfrak{A}$. Thus the range of $\tilde{\varphi}$ commutes with P and $\tilde{\varphi}(A)(I_n - P) = 0 = (I_n - P)\tilde{\varphi}(A)$ for all $A \in \mathfrak{A}$.

Define $\tilde{\varphi}_0 : \mathfrak{A} \to P\mathcal{M}_n(\mathbb{C})P$ by $\tilde{\varphi}_0(A) = P\tilde{\varphi}(A)P$ for all $A \in \mathfrak{A}$. Therefore $\tilde{\varphi}_0(I_\mathfrak{A})$ is invertible in $P\mathcal{M}_n(\mathbb{C})P$. Since $P\mathcal{M}_n(\mathbb{C})P \simeq \mathcal{M}_k(\mathbb{C})$ where k is the dimension of the range of P, by the first part of the proof there exists a unital, completely positive map $\varphi_0 : \mathfrak{A} \to P\mathcal{M}_n(\mathbb{C})P$ such that

$$\tilde{\varphi}_0(A) = \tilde{\varphi}_0(I_\mathfrak{A})^{\frac{1}{2}}\varphi_0(A)\tilde{\varphi}_0(I_\mathfrak{A})^{\frac{1}{2}}$$

for all $A \in \mathfrak{A}$.

Let $\phi : \mathfrak{A} \to \mathbb{C}$ be any state on $\mathfrak{A}$ and define $\varphi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ by

$$\varphi(A) = \varphi_0(A) + (I_n - P)\phi(A)$$

for all $A \in \mathfrak{A}$. Since the range of φ_0 is orthogonal to $I_n - P$, it is easy to view φ as the direct sum of completely positive maps and thus is a completely positive map. Moreover we notice that

$$\varphi(I_{\mathfrak{A}}) = \varphi_0(I_{\mathfrak{A}}) + (I_n - P)\phi(I_{\mathfrak{A}}) = P + (I_n - P) = I_n$$

so φ is a unital, completely positive map. Moreover, since $\tilde{\varphi}(A)(I_n - P) = 0 = (I_n - P)\tilde{\varphi}(A)$ for all $A \in \mathfrak{A}$, we obtain that

$$\tilde{\varphi}(I_{\mathfrak{A}})^{\frac{1}{2}}\varphi(A)\tilde{\varphi}(I_{\mathfrak{A}})^{\frac{1}{2}} = \tilde{\varphi}(I_{\mathfrak{A}})^{\frac{1}{2}}P\varphi_0(A)P\tilde{\varphi}(I_{\mathfrak{A}})^{\frac{1}{2}} = \tilde{\varphi}(A)$$

for all $A \in \mathfrak{A}$ as desired. □

Proposition 5.24. *Let $\mathfrak{A}$ be a unital, nuclear C*-algebra. Then there exists a net of natural numbers $(n_\lambda)_\Lambda$ and nets of unital, completely positive maps $\varphi_\lambda : \mathfrak{A} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ such that $\lim_\Lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$ for every $A \in \mathfrak{A}$.*

Proof. Since $\mathfrak{A}$ is nuclear, there exists a net of natural numbers $(n_\lambda)_\Lambda$ and nets of contractive, completely positive maps $\tilde{\varphi}_\lambda : \mathfrak{A} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\tilde{\psi}_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ such that $\lim_\Lambda \left\|A - \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(A))\right\| = 0$ for every $A \in \mathfrak{A}$. By Lemma 5.23 for each $\lambda \in \Lambda$ there exists a unital, completely positive map $\varphi_\lambda : \mathfrak{A} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ such that

$$\tilde{\varphi}_\lambda(A) = \tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}}\varphi_\lambda(A)\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}}$$

for all $A \in \mathfrak{A}$. Thus it remains to correct the maps $\tilde{\psi}_\lambda$.

Since $\lim_\Lambda \left\|I_{\mathfrak{A}} - \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))\right\| = 0$, we obtain for sufficiently large λ that $\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))$ is an invertible element of $\mathfrak{A}$. Moreover, by a simple application of the Continuous Functional Calculus for Normal Operators, we obtain that

$$\lim_\Lambda \left\|I_{\mathfrak{A}} - \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\right\| = 0.$$

For each $\lambda \in \Lambda$ sufficiently large, define $\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ by

$$\psi_\lambda(T) = \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}}T\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}})\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}$$

for all $T \in \mathcal{M}_{n_\lambda}(\mathbb{C})$. Then ψ_λ is a completely positive map being the composition of a conjugation map (which is completely positive) by a self-adjoint operator followed by the composition of a completely positive map followed by the composition of another conjugation map (which is completely positive). Moreover we notice that

$$\psi_\lambda(I_n) = \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}}\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}})\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}} = I_{\mathfrak{A}}$$

so each ψ_λ is a unital, completely positive map. Finally we notice for all $A \in \mathfrak{A}$ that

$$\begin{aligned}\psi_\lambda(\varphi_\lambda(A)) &= \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}}\varphi_\lambda(A)\tilde{\varphi}_\lambda(I_{\mathfrak{A}})^{\frac{1}{2}})\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}} \\ &= \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(A))\tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}.\end{aligned}$$

However, since

$$\lim_\Lambda \left\|I_{\mathfrak{A}} - \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(I_{\mathfrak{A}}))^{-\frac{1}{2}}\right\| = 0.$$

and since $\lim_\Lambda \left\|A - \tilde{\psi}_\lambda(\tilde{\varphi}_\lambda(A))\right\| = 0$ for every $A \in \mathfrak{A}$, we easily obtain that

$$\lim_\Lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$$

as desired. □

In general C*-subalgebras of nuclear C*-algebras need not be nuclear. However the following result says we are not too far off.

Proposition 5.25. *Let $\mathfrak{A}$ be a nuclear C*-algebra and let $\mathfrak{B}$ be a C*-subalgebra of $\mathfrak{A}$ such that there exists a conditional expectation $\Phi : \mathfrak{A} \to \mathfrak{B}$. Then $\mathfrak{B}$ is nuclear.*

Proof. Let $\varphi_\lambda : \mathfrak{A} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{A}$ by contractive, completely positive maps such that $\lim_\Lambda \|A - \psi_\lambda(\varphi_\lambda(A))\| = 0$ for every $A \in \mathfrak{A}$. Let $\varphi'_\lambda : \mathfrak{B} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\psi'_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{B}$ be defined by $\varphi'_\lambda = \varphi_\lambda|_{\mathfrak{B}}$ and $\psi'_\lambda = \Phi \circ \psi_\lambda$. Clearly φ'_λ and ψ'_λ are contractive, completely positive maps. Moreover for all $B \in \mathfrak{B}$

$$\lim_\Lambda \psi'_\lambda(\varphi'_\lambda(B)) = \lim_\Lambda \Phi(\psi_\lambda(\varphi_\lambda(B))) = \Phi(B) = B.$$

Thus $\mathfrak{B}$ is nuclear. □

The following is our easiest example of a C*-algebra that is not nuclear. It will make use of reduced group C*-algebras and the proof given in Example 5.12. Those unfamiliar with group C*-algebras may again skip this example without loss.

Example 5.26. Let $\mathbb{F}_2$ be the free group on two generators. The reduced group C*-algebra $C^*_\lambda(\mathbb{F}_2)$ is not nuclear. To begin we recall that Example 5.12 showed that there does not exist a completely positive map $\Phi : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to L(\mathbb{F}_2)$ such that $\Phi|_{C^*_\lambda(\mathbb{F}_2)} = Id_{C^*_\lambda(\mathbb{F}_2)}$. We will show that if $C^*_\lambda(\mathbb{F}_2)$ were nuclear then such a Φ must exists and thus obtain a contradiction.

Suppose $C^*_\lambda(\mathbb{F}_2)$ is nuclear. Then there exists nets of contractive, completely positive maps $\varphi_\alpha : C^*_\lambda(\mathbb{F}_2) \to \mathcal{M}_{n_\alpha}(\mathbb{C})$ and $\psi_\alpha : \mathcal{M}_{n_\alpha}(\mathbb{C}) \to C^*_\lambda(\mathbb{F}_2)$ such that $\lim_\alpha \|A - \psi_\alpha(\varphi_\alpha(A))\| = 0$ for every $A \in C^*_\lambda(\mathbb{F}_2)$. Since each $\mathcal{M}_{n_\alpha}(\mathbb{C})$ is injective, there exists contractive, completely positive maps $\varphi'_\alpha : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to \mathcal{M}_{n_\alpha}(\mathbb{C})$ extending φ_α. Then $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ is a net of contractive, completely positive maps from $\mathcal{B}(\ell_2(\mathbb{F}_2))$ to $\mathcal{B}(\ell_2(\mathbb{F}_2))$. Therefore, by Theorem 4.19, $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ has a contractive, completely positive map $\Phi : \mathcal{B}(\ell_2(\mathbb{F}_2)) \to \mathcal{B}(\ell_2(\mathbb{F}_2))$ as a cluster point in the BW topology.

We claim that Φ maps into $L(\mathbb{F}_2)$. To see this we notice $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ is a bounded net and thus for all $T \in \mathcal{B}(\ell_2(\mathbb{F}_2))$ $\Phi(T)$ is the WOT-limit of a subnet of $\psi_\alpha(\varphi'_\alpha(T))$ by Proposition 4.17. However $\psi_\alpha(\varphi'_\alpha(T)) \in C^*_\lambda(\mathbb{F}_2)$ for all α so that $\Phi(T) \in \overline{C^*_\lambda(\mathbb{F}_2)}^{WOT} = L(\mathbb{F}_2)$ as desired.

Lastly, if $T \in C^*_\lambda(\mathbb{F})$ then $\Phi(T)$ is the WOT-limit of a subnet of $\psi_\alpha(\varphi'_\alpha(T))$. However $\psi_\alpha(\varphi'_\alpha(T)) = \psi_\alpha(\varphi_\alpha(T))$ converges to T in norm and thus $\Phi(T) = T$ as desired. Whence we have constructed an impossible map Φ. Thus $C^*_\lambda(\mathbb{F}_2)$ cannot be nuclear.

Note that the above proof can also be used to show that $L(\mathbb{F}_2)$ is not a nuclear C*-algebra. However there is another notion of nuclearity for von Neumann algebras that is better.

Definition 5.27. A von Neumann algebra $\mathfrak{M}$ is said to be semidiscrete if there exists a net of natural numbers $(n_\lambda)_\Lambda$ and nets of contractive, completely positive maps $\varphi_\lambda : \mathfrak{M} \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ and $\psi_\lambda : \mathcal{M}_{n_\lambda}(\mathbb{C}) \to \mathfrak{M}$ such that $\psi_\lambda(\varphi_\lambda(T))$ converges to T in the ultraweak topology for every $T \in \mathfrak{M}$.

Since the elements $\psi_\lambda(\varphi_\lambda(T))$ in the above definition are bounded over all λ, concluding that $\psi_\lambda(\varphi_\lambda(T))$ converges to T in the ultraweak topology is the same as saying $\psi_\lambda(\varphi_\lambda(T))$ converges to T in the WOT whenever $\mathfrak{M}$ is a von Neumann subalgebra of some $\mathcal{B}(\mathcal{H})$.

Our first result is the the following which is very similar to the proof given in Example 5.26.

Proposition 5.28. *Every semidiscrete von Neumann algebra is injective.*

Proof. Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a semidiscrete von Neumann algebra. Then there exists nets of contractive, completely positive maps $\varphi_\alpha : \mathfrak{M} \to \mathcal{M}_{n_\alpha}(\mathbb{C})$ and $\psi_\alpha : \mathcal{M}_{n_\alpha}(\mathbb{C}) \to \mathfrak{M}$ such that $\psi_\alpha(\varphi_\alpha(T))$ converges to T ultraweakly for every $T \in \mathfrak{M}$. Since each $\mathcal{M}_{n_\alpha}$ is injective, there exists contractive, completely positive maps $\varphi'_\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{M}_{n_\alpha}$ extending φ_α. Then $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ is a net of contractive, completely positive maps from $\mathcal{B}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. Therefore, by Theorem 4.19, $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ has a contractive, completely positive map $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ as a cluster point in the BW topology.

We claim that Φ maps into $\mathfrak{M}$. To see this, we notice $(\psi_\alpha \circ \varphi'_\alpha)_\Lambda$ is a bounded net and thus for all $T \in \mathcal{B}(\mathcal{H})$ $\Phi(T)$ is the WOT-limit of a subnet of $\psi_\alpha(\varphi'_\alpha(T))$ by Proposition 4.17. However $\psi_\alpha(\varphi'_\alpha(T)) \in \mathfrak{M}$ for all α so that $\Phi(T) \in \overline{\mathfrak{M}}^{WOT} = \mathfrak{M}$ as desired.

Lastly, if $T \in \mathfrak{M}$ then $\Phi(T)$ is the WOT-limit of a subnet of $\psi_\alpha(\varphi'_\alpha(T))$. However $\psi_\alpha(\varphi'_\alpha(T)) = \psi_\alpha(\varphi_\alpha(T))$ converges to T in norm and thus $\Phi(T) = T$ as desired. Thus Φ is a conditional expectation of $\mathcal{B}(\mathcal{H})$ onto $\mathfrak{M}$ and thus $\mathfrak{M}$ is injective by Theorem 5.8. $\square$

It turns out that the converse of the above theorem is true although this is extremely difficult to prove. Using Example 5.12 we obtain the following.

Corollary 5.29. *Let $\mathbb{F}_2$ be the free group on two generators. Then $L(\mathbb{F}_2)$ is not semidiscrete.*

To obtain an example of a semidiscrete von Neumann algebra (without using the fact that every injective von Neumann algebra is semidiscrete) we consider the proof of Theorem 5.16. Again, those unfamiliar with the hyperfinite II_1 factor may skip the following without loss.

Theorem 5.30. *The hyperfinite II_1 factor $\mathfrak{R}$ is semidiscrete.*

Proof. Let $(\mathfrak{N}_n)_{n\geq 1}$ be an increasing family of finite dimensional C*-algebras such that $I_\mathfrak{R} \in \mathfrak{N}_n$ for all $n \geq 1$, $\mathfrak{N}_n \simeq \mathcal{M}_{2^n}(\mathbb{C})$, and $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{M}$. Since $\mathfrak{R}$ has a faithful trace, there exists conditional expectations $\varphi_n : \mathfrak{R} \to \mathfrak{N}_n$ onto $\mathfrak{N}_n$ for all $n \in \mathbb{N}$. Since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{R}$, $\varphi_n(T) \to T$ ultraweakly for all $T \in \mathfrak{R}$ (to see this recall that the conditional expectations E_n are defined by $E_n(x) = P_n x P_n$ where P_n is the orthogonal projection of $L_2(\mathfrak{R}, \tau)$ onto $L_2(\mathfrak{N}_n, \tau)$. Since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweakly dense in $\mathfrak{R}$, $E_n(T) \to T$ in the WOT on $\mathcal{B}(L_2(\mathfrak{R}, \tau))$ for all $T \in \mathfrak{R}$ (i.e. it is easy to show for all $T_n, S_n \in \mathfrak{N}_n$ that $\langle TT_n, S_n\rangle = \langle P_m T P_m T_n, S_n\rangle$ for all $m \geq n$ and $T \in \mathfrak{R}$. Then, since $\bigcup_{n\geq 1} \mathfrak{N}_n$ is ultraweak dense in $\mathfrak{R}$, $\bigcup_{n\geq 1} \mathfrak{N}_n \subseteq L_2(\mathfrak{R}, \tau)$ is dense and thus we obtain the result by taking limits). Since the WOT on $\mathcal{B}(L_2(\mathfrak{R}, \tau))$ and ultraweak topology agree on bounded sets, we obtain the desired result).

Define $\psi_n : \mathfrak{N}_n \to \mathfrak{R}$ to be the inclusion map. Since $\mathfrak{N}_n \simeq \mathcal{M}_{2^n}(\mathbb{C})$ for all n and $\psi_n(\varphi_n(T)) = \varphi_n(T) \to T$ ultraweakly for all $T \in \mathfrak{R}$, $\mathfrak{R}$ is semidiscrete as desired. $\square$

6 Liftings of Completely Positive Maps

In this chapter we will briefly examine the problem of when a completely positive map into a quotient C*-algebra may be lifted to a completely positive map. In particular, Theorem 6.4 will demonstrate that every unital, completely positive map from a unital, nuclear C*-algebra (as described in Chapter 5) into a unital quotient C*-algebra can be lifted to a completely positive map. To prove this fact, we need only two technical results that are interesting by themselves.

To begin the first technical result, we make the following definition.

Definition 6.1. Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be the set of all linear maps from a separable Banach space $\mathfrak{X}$ to a Banach space $\mathfrak{Y}$. The point-norm topology on $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ is the topology such that a sequence $(\psi_n)_{n\geq 1}$ in $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ converges to $\psi \in \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ if and only if $\lim_{n\to\infty} \|\psi(x) - \psi_n(x)\| = 0$ for all $x \in \mathfrak{X}$.

In the above definition, we insist that $\mathfrak{X}$ is separable so that we can consider sequences. Thus we can prove the following result.

Lemma 6.2. *Let $\mathfrak{J}$ be an ideal in a unital C*-algebra $\mathfrak{B}$ and let $\mathcal{S}$ be a separable operator system. The set of contractive, completely positive maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$ with a contractive, completely positive lifting to $\mathfrak{B}$ is closed in the point-norm topology on all bounded linear maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$. Thus the set of unital, completely positive maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$ with a unital, completely positive lifting to $\mathfrak{B}$ is closed in the point-norm topology on all bounded linear maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$.*

Proof. Let $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the canonical quotient map. Let $\varphi : \mathcal{S} \to \mathfrak{B}/\mathfrak{J}$ be a bounded linear map such that there exists contractive (unital), completely positive maps $\psi'_n : \mathcal{S} \to \mathfrak{B}$ such that $(q \circ \psi'_n)_{n\geq 1}$ converges to φ in the point-norm topology. Clearly this implies φ is completely positive and contractive (unital). Let $\{A_k\}_{k\geq 1}$ be a dense subset of $\mathcal{S}$. Therefore, by passing to a subsequence, we may assume that $\|q(\psi'_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n$.

We claim that it suffices to construct a sequence $\psi_n : \mathcal{S} \to \mathfrak{B}$ of contractive (unital), completely positive maps such that $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n$ and $\|\psi_{n+1}(A_k) - \psi_n(A_k)\| < \frac{1}{2^{n-3}}$ for all $k \leq n-1$. If such a sequence exists, then it is clear that $(\psi_n(A_k))_{n\geq 1}$ is a Cauchy sequence for all $k \in \mathbb{N}$ and thus, as $\{A_k\}_{k\geq 1}$ is a dense subset of $\mathcal{S}$, $\psi(A) := \lim_{k\to\infty} \psi_n(A)$ exists for all $A \in \mathcal{S}$. Clearly ψ will be a contractive (unital), completely positive map (being the point-norm limit of contractive (unital), completely positive maps) and, since $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \geq 1$, $q(\psi(A_k)) = \varphi(A_k)$ for all $k \in \mathbb{N}$. Therefore, by density, $q \circ \psi = \varphi$ as desired.

To construct such a sequence, we proceed by induction. Let $\psi_1 := \psi'_1$. Suppose we have constructed $\psi_n : \mathcal{S} \to \mathfrak{B}$ such that $\|q(\psi_n(A_k)) - \varphi(A_k)\| \leq \frac{1}{2^n}$ for all $k \leq n$ and $\|\psi_n(A_k) - \psi_{n-1}(A_k)\| < \frac{1}{2^{n-3}}$ for all $k \leq n-1$. Let $(E_\lambda)_\Lambda$ be a quasicentral C*-bounded approximate identity for $\mathfrak{J}$ in $\mathfrak{B}$. Then

$$\lim_\Lambda \left\| (I_\mathfrak{B} - E_\lambda)^{\frac{1}{2}} \psi_n(A) (I_\mathfrak{B} - E_\lambda)^{\frac{1}{2}} + E_\lambda^{\frac{1}{2}} \psi_n(A) E_\lambda^{\frac{1}{2}} - \psi_n(A) \right\| = 0$$

for all $A \in \mathcal{S}$, and, if $B_k := \psi'_{n+1}(A_k) - \psi_n(A_k)$, then

$$\lim_\Lambda \left\| (I_\mathfrak{B} - E_\lambda)^{\frac{1}{2}} B_k (I_\mathfrak{B} - E_\lambda)^{\frac{1}{2}} \right\| = \|q(B_k)\| < \frac{2}{2^n}$$

if $k \leq n$. Hence there exists an $E := E_\lambda \in \mathfrak{J}$ so that

$$\left\| (I_\mathfrak{B} - E)^{\frac{1}{2}} \psi_n(A_k) (I_\mathfrak{B} - E)^{\frac{1}{2}} + E^{\frac{1}{2}} \psi_n(A_k) E^{\frac{1}{2}} - \psi_n(A_k) \right\| < \frac{1}{2^{n+1}}$$

for all $k \leq n+1$ and $\left\| (I_\mathfrak{B} - E)^{\frac{1}{2}} B_k (I_\mathfrak{B} - E)^{\frac{1}{2}} \right\| < \frac{1}{2^{n-1}}$ for all $k \leq n$. Define $\psi_{n+1} : \mathcal{S} \to \mathfrak{B}$ by

$$\psi_{n+1}(A) := (I_\mathfrak{B} - E)^{\frac{1}{2}} \psi'_{n+1}(A) (I_\mathfrak{B} - E)^{\frac{1}{2}} + E^{\frac{1}{2}} \psi_n(A) E^{\frac{1}{2}}$$

for all $A \in \mathcal{S}$. Clearly ψ_{n+1} is a completely positive map. In the contractive case, to see that ψ_{n+1} is contractive we note that ψ'_{n+1} and ψ_n are contractive maps and $(I_{\mathfrak{B}} - E) + E = I_{\mathfrak{B}}$ so $\|\psi_{n+1}(I_{\mathcal{S}})\| \leq 1$. In the unital case, to see that ψ_{n+1} is unital we note that ψ'_{n+1} and ψ_n are unital maps and $(I_{\mathfrak{B}} - E) + E = I_{\mathfrak{B}}$ so $\psi_{n+1}(I_{\mathcal{S}}) = I_{\mathfrak{B}}$. To see that ψ_{n+1} has the desired properties, we notice that $q \circ \psi_{n+1} = q \circ \psi'_{n+1}$ so $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n+1$. Moreover

$$\begin{aligned} \|\psi_{n+1}(A_k) - \psi_n(A_k)\| &\leq \tfrac{1}{2^{n+1}} + \left\|(I_{\mathfrak{B}} - e)^{\frac{1}{2}} \psi'_{n+1}(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} - (I_{\mathfrak{B}} - E)^{\frac{1}{2}} \psi_n(A_k)(I_{\mathfrak{B}} - E)^{\frac{1}{2}}\right\| \\ &= \tfrac{1}{2^{n+1}} + \|B_k\| \leq \tfrac{1}{2^{n-2}}. \end{aligned}$$

for all $k \leq n$ as desired. □

To complete the proof of Theorem 6.4, we will use Lemma 6.2 and the following lemma that shows that algebraic liftings of unital, completely positive maps from matrix algebras can be taken to be positive.

Lemma 6.3. *Let $\mathfrak{B}$ be a unital C^*-algebra, let $\varphi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}$ be a linear map, and let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard matrix units for $\mathcal{M}_n(\mathbb{C})$. Then the following are equivalent:*

1. *φ is completely positive.*
2. *φ is n-positive.*
3. *$[\varphi(E_{i,j})]$ is positive in $\mathcal{M}_n(\mathfrak{B})$.*

Proof. It is clear that (1) implies (2). To see that (2) implies (3), we notice that $[E_{i,j}] \in \mathcal{M}_n(\mathcal{M}_n(\mathbb{C}))$ is self-adjoint (as $[E_{i,j}]^* = [E^*_{j,i}] = [E_{i,j}]$) and $[E_{i,j}]^2 = [\sum_{k=1}^n E_{i,k}E_{k,j}] = n[E_{i,j}]$. Hence $z^2 - nz = 0$ on $\sigma([E_{i,j}])$ and thus $\sigma([E_{i,j}]) \subseteq \{0, n\}$. Hence $[E_{i,j}]$ is positive. Therefore, since φ is n-positive, $\varphi([E_{i,j}])$ is positive in $\mathcal{M}_n(\mathfrak{B})$.

Suppose (3) holds. Let $k \in \mathbb{N}$ be arbitrary. Without loss of generality we may assume $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. To show that φ is k-positive, let $A_1, \ldots, A_k \in \mathcal{M}_n(\mathbb{C})$ be arbitrary. Since $A_s \in \mathcal{M}_n(\mathbb{C})$, there exists $a_{i,j,s} \in \mathbb{C}$ ($s \in \{1, \ldots k\}$, $i,j \in \{1, \ldots n\}$) such that $A_s = \sum_{i,j=1}^n a_{i,j,s}E_{i,j}$. Thus, a simple computation shows that

$$A_i^* A_j = \left(\sum_{l,m=1}^n \overline{a_{l,m,i}} E_{m,l}\right)\left(\sum_{s,t=1}^n a_{s,t,j}E_{s,t}\right) = \sum_{l,m,t=1}^n \overline{a_{l,m,i}} a_{l,t,j} E_{m,t}.$$

Fix $h = (h_1, h_2, \ldots, h_k) \in \mathcal{H}^{\oplus k}$ and let $x_{l,m} := \sum_{j=1}^k a_{l,m,j}h_j \in \mathcal{H}$ for $l, m = \{1, \ldots n\}$. Then

$$\begin{aligned} \sum_{i,j=1}^k \langle \varphi(A_i^* A_j)h_j, h_i\rangle &= \sum_{i,j=1}^k \sum_{l,m,t=1}^n \langle \varphi(\overline{a_{l,m,i}} a_{l,t,j} E_{m,t})h_j, h_i\rangle \\ &= \sum_{i,j=1}^k \sum_{l,m,t=1}^n \langle \varphi(E_{m,t}) a_{l,t,j}h_j, a_{l,m,i}h_i\rangle \\ &= \sum_{i=1}^k \sum_{l,m,t=1}^n \langle \varphi(E_{m,t}) x_{l,t}, a_{l,m,i}h_i\rangle \\ &= \sum_{l,m,t=1}^n \langle \varphi(E_{m,t}) x_{l,t}, x_{l,m}\rangle \\ &= \sum_{l=1}^n \sum_{m,t=1}^n \langle \varphi(E_{m,t}) x_{l,t}, x_{l,m}\rangle. \end{aligned}$$

However, $[\varphi(E_{i,j})]$ is positive in $\mathcal{M}_n(\mathfrak{B})$ and hence $\sum_{m,t=1}^n \langle \varphi(E_{m,t})x_{l,t}, x_{l,m}\rangle = \langle \phi_{(n)}([E_{m,t}])x, x\rangle \geq 0$ where $x = (x_{l,1}, \ldots, x_{l,n}) \in \mathcal{H}^n$. Hence, since the sum of positive numbers is positive, $\sum_{i,j=1}^k \langle \varphi(A_i^* A_j)h_j, h_i\rangle \geq 0$. Hence φ is k-positive and, as $k \in \mathbb{N}$ was arbitrary, φ is completely positive as desired. □

Theorem 6.4. *Let $\mathfrak{A}$ be a unital, separable, nuclear C^*-algebra, let $\mathfrak{B}$ be a unital C^*-algebra, let $\mathfrak{J}$ be an ideal of $\mathfrak{B}$, and let $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the canonical quotient map. Then for every unital, completely positive map $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ there exists a unital, completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$ such that $q \circ \Phi = \varphi$.*

Proof. Let $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ be a unital, completely positive map. Since $\mathfrak{A}$ is separable, Lemma 6.2 implies that the set of unital, completely positive maps from $\mathfrak{A}$ into $\mathfrak{B}/\mathfrak{J}$ that have liftings is closed in the point-norm topology. Thus it suffices to show that φ is a point-norm limit of unital, completely positive maps into $\mathfrak{B}/\mathfrak{J}$ with unital, completely positive liftings. Since $\mathfrak{A}$ is unital and nuclear, Proposition 5.24 implies that φ is a point-norm limit of unital completely positive maps of the form $\psi \circ \phi$ where $\phi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ and $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}/\mathfrak{J}$ are unital, completely positive maps. If we can show that ψ has a lifting to a unital, completely positive map, then $\psi \circ \phi$ has a lifting to a unital, completely positive map and thus we are done by Lemma 6.2.

To see that ψ has a completely positive lifting, let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard matrix units for $\mathcal{M}_n(\mathbb{C})$. Note that $[\psi(E_{i,j})] \in \mathcal{M}_n(\mathfrak{B}/\mathfrak{J}) \simeq \mathcal{M}_n(\mathfrak{B})/\mathcal{M}_n(\mathfrak{J})$ is positive. Therefore, standard functional calculus results imply that there exists a positive matrix $[B_{i,j}] \in \mathcal{M}_n(\mathfrak{B})$ such that $q_n([B_{i,j}]) = [\varphi(E_{i,j})]$. Define $\Psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}$ by $\Psi([a_{i,j}]) = \sum_{i,j=1}^n a_{i,j}B_{i,j}$ for all $[a_{i,j}] \in \mathcal{M}_n(\mathbb{C})$. Clearly Ψ is a linear map. Notice that $\Psi_n([E_{i,j}]) = [B_{i,j}] \geq 0$ so Ψ is a completely positive map by Lemma 6.3. Moreover

$$q(\Psi([a_{i,j}])) = q\left(\sum_{i,j=1}^n a_{i,j}B_{i,j}\right) = \sum_{i,j=1}^n a_{i,j}\varphi(E_{i,j}) = \psi([a_{i,j}])$$

so Ψ is a lifting of ψ.

However, Ψ need not be unital. To fix this, we notice that $q(\Psi(I_n)) = \psi(I_n) = I_{\mathfrak{B}/\mathfrak{J}}$. Since $\Psi(I_n)$ is self-adjoint, $\Psi(I_n) = I_{\mathfrak{B}} + A$ where $A \in \mathfrak{J}_{sa}$. Using the Continuous Functional Calculus, write $A = A_+ - A_-$ where $A_+, A_- \in \mathfrak{J}_+$ are such that $A_+A_- = 0$. Let $f : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ be any state on $\mathcal{M}_n(\mathbb{C})$ and define $\Psi' : \mathcal{M}_n(\mathbb{C})) \to \mathfrak{B}$ by

$$\Psi'(T) := (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}(\Psi(T) + f(T)A_-)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}$$

for all $T \in \mathcal{M}_n(\mathbb{C}))$. Clearly

$$\Psi'(I_n) = (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}(\Psi(I_n) + A_-)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} = (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}(I_{\mathfrak{B}} + A_+)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} = I_{\mathfrak{B}}$$

so Ψ' is a unital, completely positive map (completely positive being the sum of completely positive maps as the sum of positive operators is positive). Since $q\left((I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}\right) = I_{\mathfrak{B}/\mathfrak{J}}$ and

$$q(\Psi(T) + f(T)A_-) = q(\Psi(T)) = \psi(T)$$

for all $T \in \mathcal{M}_n(\mathbb{C})$, Ψ' is the desired unital, completely positive lifting of ψ. □

7 Wittstock's Theorem

So far we have mainly studied completely positive maps and have ignored completely bounded maps except in the basic theory where the two notions went hand-in-hand. The purpose of this chapter is to develop some important results pertaining to completely bounded maps. Recall that every continuous linear functional on a C*-algebra is completely bounded. Moreover it is possible to show that if $\varphi : \mathfrak{A} \to \mathbb{C}$ is a continuous linear functional then there exists a Hilbert space $\mathcal{H}$, *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$, and vectors $\xi, \eta \in \mathcal{H}$ with $\|\xi\| \, \|\eta\| = \|\varphi\|$ such that $\varphi(A) = \langle \pi(A)\xi, \eta\rangle$. We will develop a similar theorem for completely bounded maps which will be very similar to Stinespring's Theorem. In particular, to prove this theorem we will show that if $\varphi : \mathfrak{A}_1 \to \mathfrak{B}$ is a completely bounded map, then there exists a completely positive map $\Phi : \mathfrak{A}_2 \to \mathcal{M}_2(\mathfrak{B})$ such that φ is the $(1, 2)$-entry of this map. This will enable us to show that every completely bounded map (and thus every continuous linear functional) is the linear combination of four completely positive maps and that every completely bounded map can be extended to a completely bounded map with the same completely bounded norm.

To begin it helps to develop the 'canonical shuffle' and 'scalar matrices'.

Remarks 7.1. Let $\mathfrak{A}$ be a C*-algebra and consider $[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell} \in \mathcal{M}_n(\mathcal{M}_m(\mathfrak{A}))$ where $[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell}$ means the element of $\mathcal{M}_n(\mathcal{M}_m(\mathfrak{A}))$ whose $(k, \ell)^{th}$-matrix entry is the $n \times n$ matrix whose $(i, j)^{th}$-entry is $A_{i,j,k,\ell}$. Notice that $\mathcal{M}_n(\mathcal{M}_m(\mathfrak{A})) \simeq \mathcal{M}_{nm}(\mathfrak{A})$ so we can view $[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell}$ as an element of $\mathcal{M}_{nm}(\mathfrak{A})$. By conjugating $[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell}$ by a permutation matrix in $\mathcal{M}_{nm}(\mathfrak{A})$ (which is a unitary matrix) we can obtain the matrix $[[A_{i,j,k,\ell}]_{k,\ell}]_{i,j} \in \mathcal{M}_m(\mathcal{M}_n(\mathfrak{A}))$. This operation is called the canonical shuffle.

Since conjugating by a unitary preserves positivity and norms, we see that $[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell}$ is positive if and only if $[[A_{i,j,k,\ell}]_{k,\ell}]_{i,j}$ is positive and $\|[[A_{i,j,k,\ell}]_{i,j}]_{k,\ell}\| = \|[[A_{i,j,k,\ell}]_{k,\ell}]_{i,j}\|$. Whence we can use the canonical shuffle to aid us when dealing with matrix algebras of matrix algebras.

For an example of the use of the canonical shuffle, we first present the following result that provides information about the completely bounded norm of an operator into a matrix algebra.

Proposition 7.2. *Let $\mathcal{M}$ be an operator space and let $\phi : \mathcal{M} \to \mathcal{M}_n(\mathbb{C})$ be bounded. Then $\|\phi\|_{cb} \leq n \|\phi\|$.*

Proof. Let $\{E_{i,j}\}_{i,j=1}^n$ be the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$ and for each $A \in \mathcal{M}$ write

$$\phi(A) = \sum_{i,j=1}^{n} \phi_{i,j}(A) \otimes E_{i,j} \in \mathbb{C} \otimes \mathcal{M}_n(\mathbb{C}) \simeq \mathcal{M}_n(\mathbb{C})$$

where $\phi_{i,j} : \mathcal{M} \to \mathbb{C}$ are continuous linear functionals. This is possible since ϕ is linear and continuous in each component by Lemma 1.23. Using the matrix units $E_{i,j}$, we notice that $E_{1,k}\phi(A)E_{\ell,1}$ is the matrix with $\phi_{k,\ell}(A)$ in the first entry and zeros elsewhere. Thus

$$\|\phi_{k,\ell}(A)\| = \|E_{1,k}\phi(A)E_{\ell,1}\| \leq \|E_{1,k}\| \, \|\phi(A)\| \, \|E_{\ell,1}\| = \|\phi(A)\|$$

for all $A \in \mathfrak{A}$. Thus $\|\phi_{i,j}\| \leq \|\phi\|$. Moreover, since each $\phi_{i,j}$ is a continuous linear functional, $\|\phi_{i,j}\|_{cb} = \|\phi_{i,j}\| \leq \|\phi\|$ by Proposition 3.8.

Fix $m \in \mathbb{N}$ and suppose $X = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M})$. By performing the canonical shuffle on $\phi_{(m)}(X)$, $\phi_{(m)}(X)$ becomes $\sum_{i,j=1}^n (\phi_{i,j})_{(m)}(X) \otimes E_{i,j}$. Hence $\|\phi_m(X)\| = \left\|\sum_{i,j=1}^n (\phi_{i,j})_{(m)}(X) \otimes E_{i,j}\right\|$. Hence, by Lemma 1.23,

$$\begin{aligned}
\left\|\phi_{(m)}(X)\right\| &\leq \left(\sum_{i,j=1}^{n} \left\|(\phi_{i,j})_{(m)}(X)\right\|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i,j=1}^{n} \left\|(\phi_{i,j})_{(m)}\right\|^2 \|X\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

$$\le \left(\sum_{i,j=1}^{n} \|\phi\|^2 \|X\|^2 \right)^{\frac{1}{2}}$$
$$= n \|\phi\| \|X\| .$$

However, this holds for all $X = [A_{i,j}] \in \mathcal{M}_m(\mathcal{M})$ so $\|\phi\|_m \le n\|\phi\|$. Hence, as this holds for all $m \in \mathbb{N}$, $\|\phi\|_{cb} \le n\|\phi\|$ as desired. □

Now that we have developed the canonical shuffle, we may continue our study of completely bounded maps. We begin with the following observation.

Remarks 7.3. Suppose $\mathfrak{A}$ is a unital C*-algebra. Then we may define a *-homomorphism $\pi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathfrak{A})$ by $\pi([\alpha_{i,j}]) = [\alpha_{i,j} I_{\mathfrak{A}}]$. Clearly π is a *-homomorphism so the range of π is a C*-subalgebra of $\mathcal{M}_n(\mathfrak{A})$. The C*-subalgebra $\pi(\mathcal{M}_n(\mathbb{C}))$ is known as the scalar matrices of $\mathcal{M}_n(\mathfrak{A})$. As π is a *-homomorphism, the scalar matrices of $\mathcal{M}_n(\mathfrak{A})$ behave in an identical way to matrices in $\mathcal{M}_n(\mathbb{C})$ with regards to norms, positivity, invertibility, and functional calculus.

Our first result will give us the ability to take a contractive, completely bounded map and obtain a completely positive map. The following theorem is essential to the remainder of the chapter.

Theorem 7.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras, let $\mathcal{M}$ be a subspace of $\mathfrak{A}$, and let $\varphi : \mathcal{M} \to \mathfrak{B}$ be a linear map. Define an operator system $\mathcal{S}_{\mathcal{M}} \subseteq \mathcal{M}_2(\mathfrak{A})$ by*

$$\mathcal{S}_{\mathcal{M}} = \left\{ \begin{bmatrix} \lambda I_{\mathfrak{A}} & A \\ B^* & \mu I_{\mathfrak{A}} \end{bmatrix} \mid \lambda, \mu \in \mathbb{C}, A, B \in \mathcal{M} \right\}$$

and $\Phi : \mathcal{S}_{\mathcal{M}} \to \mathcal{M}_2(\mathfrak{B})$ by

$$\Phi\left(\begin{bmatrix} \lambda I_{\mathfrak{A}} & A \\ B^* & \mu I_{\mathfrak{A}} \end{bmatrix} \right) = \begin{bmatrix} \lambda I_{\mathfrak{B}} & \varphi(A) \\ \varphi(B)^* & \mu I_{\mathfrak{B}} \end{bmatrix}.$$

If $\|\varphi\|_{cb} \le 1$ then Φ is completely positive.

Proof. The idea of this proof is to use Lemma 3.19 and the canonical shuffle to show that Φ is completely positive. Fix an arbitrary $n \in \mathbb{N}$, let $[S_{i,j}] \in \mathcal{M}_n(\mathcal{S}_{\mathcal{M}})$, and write

$$S_{i,j} = \begin{bmatrix} \lambda_{i,j} I_{\mathfrak{A}} & A_{i,j} \\ B^*_{i,j} & \mu_{i,j} I_{\mathfrak{A}} \end{bmatrix}.$$

Since $\mathcal{M}_2(\mathcal{S}_{\mathcal{M}})$ is a subspace of $\mathcal{M}_n(\mathcal{M}_2(\mathfrak{A}))$, we may perform the canonical shuffle to view $[S_{i,j}]$ as an element of $\mathcal{M}_2(\mathcal{M}_n(\mathfrak{A}))$. In this form, we can write

$$[S_{i,j}] = \begin{bmatrix} H & A \\ B^* & K \end{bmatrix} \qquad (*)$$

where $H := [\lambda_{i,j} I_{\mathfrak{A}}]$, $K := [\mu_{i,j} I_{\mathfrak{A}}]$, $A := [A_{i,j}]$, and $B := [B_{i,j}]$ are elements of $\mathcal{M}_n(\mathfrak{A})$. If we apply a similar process to $\Phi_{(n)}([S_{i,j}])$, we obtain that

$$\Phi_{(n)}([S_{i,j}]) = \begin{bmatrix} H' & \varphi_{(n)}(A) \\ \varphi_{(n)}(B)^* & K' \end{bmatrix} \qquad (**)$$

where $H' := [\lambda_{i,j} I_{\mathfrak{B}}]$ and $K' := [\mu_{i,j} I_{\mathfrak{B}}]$ are elements of $\mathcal{M}_n(\mathfrak{B})$. Therefore to show that Φ is completely positive it suffices to show that if $(*)$ is positive then $(**)$ is positive.

Suppose that $(*)$ is positive. We notice that if $(*)$ is positive then $A = B$ in order for $(*)$ to be self-adjoint. Whence $(**)$ is self-adjoint. Moreover H and K must be positive elements by Lemma 3.17 part (3). Therefore H' and K' are also positive as H and K are scalar matrices (see Remarks 7.3). To show that $\Phi_{(n)}([S_{i,j}])$ is positive we need to apply a small trick so we may assume that H and K are invertible.

Let $\epsilon > 0$ and consider $H_\epsilon = \epsilon I_{\mathcal{M}_n(\mathfrak{A})} + H$, $K_\epsilon = \epsilon I_{\mathcal{M}_n(\mathfrak{A})} + K$, $H'_\epsilon := \epsilon I_{\mathcal{M}_n(\mathfrak{B})} + H'$, and $K'_\epsilon := \epsilon I_{\mathcal{M}_n(\mathfrak{B})} + K'$. Since H, K, H', and K' are positive scalar matrices, H_ϵ, K_ϵ, H'_ϵ, and K'_ϵ are invertible scalar matrices. Thus we can consider the scalar matrices $H_\epsilon^{-\frac{1}{2}}$, $K_\epsilon^{-\frac{1}{2}}$, $(H'_\epsilon)^{-\frac{1}{2}}$, and $(K'_\epsilon)^{-\frac{1}{2}}$ to obtain that

$$\begin{bmatrix} I_{\mathcal{M}_n(\mathfrak{A})} & H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}} \\ K_\epsilon^{-\frac{1}{2}} A^* H_\epsilon^{-\frac{1}{2}} & I_{\mathcal{M}_n(\mathfrak{A})} \end{bmatrix} = \begin{bmatrix} H_\epsilon^{-\frac{1}{2}} & 0 \\ 0 & K_\epsilon^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} H_\epsilon & A \\ A^* & K_\epsilon \end{bmatrix} \begin{bmatrix} H_\epsilon^{-\frac{1}{2}} & 0 \\ 0 & K_\epsilon^{-\frac{1}{2}} \end{bmatrix}$$

is a positive matrix since the matrix on the right is a conjugation the positive matrix $\epsilon I_{\mathcal{M}_{2n}(\mathfrak{A})} + (*)$. Thus Lemma 3.19 part (2) implies that $\left\| H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}} \right\| \leq 1$. Moreover, since $\varphi_{(n)}$ is linear and $H_\epsilon^{-\frac{1}{2}}$, $K_\epsilon^{-\frac{1}{2}}$, $(H'_\epsilon)^{-\frac{1}{2}}$, and $(K'_\epsilon)^{-\frac{1}{2}}$ are scalar matrices, a simple computation shows that

$$\varphi_{(n)}(H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}}) = (H'_\epsilon)^{-\frac{1}{2}} \varphi_{(n)}(A) (K'_\epsilon)^{-\frac{1}{2}}.$$

Therefore

$$\begin{aligned} & \Phi_{(n)}([S_{i,j}]) + \epsilon I_{\mathcal{M}_{2n}(\mathfrak{B})} \\ = & \begin{bmatrix} H'_\epsilon & \varphi_{(n)}(A) \\ \varphi_{(n)}(A)^* & K'_\epsilon \end{bmatrix} \\ = & \begin{bmatrix} (H'_\epsilon)^{\frac{1}{2}} & 0 \\ 0 & (K'_\epsilon)^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I_{\mathcal{M}_n(\mathfrak{B})} & \varphi_{(n)}(H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}}) \\ \varphi_{(n)}(H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}})^* & I_{\mathcal{M}_n(\mathfrak{B})} \end{bmatrix} \begin{bmatrix} (H'_\epsilon)^{\frac{1}{2}} & 0 \\ 0 & (K'_\epsilon)^{\frac{1}{2}} \end{bmatrix}. \end{aligned}$$

Since $\|\varphi\|_{cb} \leq 1$, $\left\| H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}} \right\| \leq 1$ implies that $\left\| \varphi_{(n)}(H_\epsilon^{-\frac{1}{2}} A K_\epsilon^{-\frac{1}{2}}) \right\| \leq 1$. Therefore, by Lemma 3.19 part (1), the middle matrix on the right hand side is positive. Consequently the matrix on the left hand side is positive. Whence $\sigma(\Phi_{(n)}([S_{i,j}])) \cap (-\infty, -\epsilon) = \emptyset$ for all $\epsilon > 0$ so that $\Phi_{(n)}([S_{i,j}])$ is positive. Thus Φ is completely positive. $\square$

With the above theorem and the Arveson Extension Theorem it is not difficult to show that completely bounded maps may be extended to completely bounded maps.

Theorem 7.5 (Wittstock's Extension Theorem)**.** *Let $\mathfrak{A}$ be a unital C*-algebra, let $\mathcal{M}$ be a subspace of $\mathfrak{A}$, and let $\varphi : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. Then there exists a completely bounded map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\psi|_{\mathcal{M}} = \varphi$ and $\|\psi\|_{cb} = \|\varphi\|_{cb}$.*

Proof. To prove this result we will take the completely positive map obtained in Theorem 7.4 and extend it to a completely positive map by Arveson's Extension Theorem. Then, using an application of Lemma 1.23 and the canonical shuffle, we will show that the $(1, 2)$-entry of this completely positive map is the desired completely bounded extension of φ.

If $\varphi = 0$ then the result is trivial. Otherwise we may scale φ so that $\|\varphi\|_{cb} = 1$ as it is easy to check that $\|a\varphi\|_{cb} = |a| \|\varphi\|_{cb}$ for all $a \in \mathbb{C}$. Let $\mathcal{S}_{\mathcal{M}}$ and Φ be as in Theorem 7.4. Since $\|\varphi\|_{cb} = 1$, Φ is completely positive and unital so $\|\Phi\| = 1$. By the Arveson Extension Theorem there exists a completely positive map $\Psi : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{M}_2(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ such that $\Psi|_{\mathcal{S}_{\mathcal{M}}} = \Phi$ and $\|\Psi\|_{cb} = \|\Psi\| = \|\Phi\| = 1$.

Define $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by

$$\Psi\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} * & \psi(A) \\ * & * \end{bmatrix}$$

(where $*$ represents something we do not care about) or, to be more specific,

$$\psi(A) = P\Psi\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} |_{\mathcal{H} \oplus 0}$$

where P is the projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H} \oplus \{0\} \simeq \mathcal{H}$. It is clear by construction that ψ is well-defined and linear. Moreover, since Ψ extends Φ, ψ extends φ. Moreover, since $\|\Psi\|_{cb} = 1$, we obtain that

$$\|\psi(A)\| \leq \|P\| \left\| \Psi\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right) \right\| \leq \|\Psi\| \left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\| \leq \|\Psi\| \, \|A\| \leq \|A\|$$

(by applying Lemma 1.23 for the matrix norm inequality). Hence ψ is contractive.

It remains only to to show that ψ is completely contractive (which implies since it extends φ that $\|\psi\|_{cb} = 1 = \|\varphi\|_{cb}$ as desired). The proof that $\|\psi\|_{cb} \leq 1$ will follow easily by applying the canonical shuffle. Let $A = [A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ be arbitrary. Then

$$\Psi_{(n)}\left(\left[\begin{bmatrix} 0 & A_{i,j} \\ 0 & 0 \end{bmatrix} \right] \right) = \left[\begin{bmatrix} * & \psi(A_{i,j}) \\ * & * \end{bmatrix} \right].$$

By performing the canonical shuffle on the matrix of matrices on the right we obtain the matrix

$$\begin{bmatrix} * & \psi_{(n)}(A) \\ * & * \end{bmatrix}.$$

Therefore, by applying Lemma 1.23, we obtain that

$$\left\|\psi_{(n)}(A)\right\| \leq \left\| \begin{bmatrix} * & \psi_{(n)}(A) \\ * & * \end{bmatrix} \right\| \leq \left\|\Psi_{(n)}\right\| \left\| \left[\begin{bmatrix} 0 & A_{i,j} \\ 0 & 0 \end{bmatrix} \right] \right\| = \left\| \left[\begin{bmatrix} 0 & A_{i,j} \\ 0 & 0 \end{bmatrix} \right] \right\|.$$

However, by another application of the canonical shuffle,

$$\left[\begin{bmatrix} 0 & A_{i,j} \\ 0 & 0 \end{bmatrix} \right]$$

becomes

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

and thus

$$\left\| \left[\begin{bmatrix} 0 & A_{i,j} \\ 0 & 0 \end{bmatrix} \right] \right\| = \left\| \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \right\| \leq \|A\|$$

by applying the fact that the canonical shuffle preserves the norm and by applying Lemma 1.23 to obtain the inequality. Hence $\left\|\psi_{(n)}(A)\right\| \leq \|A\|$. Whence $\|\psi\|_n \leq 1$ for all $n \in \mathbb{N}$ and thus, as ψ extends φ, $\|\psi\|_{cb} = \|\varphi\|_{cb} = 1$. ☐

Our next goal is to yet again apply the completely positive map obtained in Theorem 7.4 to show that every completely bounded map can be written in a certain way. To begin we show that the desired maps are completely bounded.

Example 7.6. Let $\mathfrak{A}$ be a C*-algebra and let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ be a *-homomorphism, let $V, W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be arbitrary, and define $\varphi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ by $\varphi(T) = W^*TV$ for all $T \in \mathcal{B}(\mathcal{K})$. Since π is completely positive, π is completely bounded. Moreover φ is completely bounded by Example 3.5. Therefore, by Proposition 3.9, the map $\psi := \varphi \circ \pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ defined by $\psi(A) = W^*\pi(A)V$ for all $A \in \mathfrak{A}$ is completely bounded.

We will show that every completely bounded map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is contained in the above example. To begin we need to show that the proof of Theorem 7.5 can be improved to obtain more information about the extension of the map Φ. To simplify technical reasons, we will only deal with unital C*-algebras and our final result will apply to non-unital C*-algebras as we can extend every completely bounded map on a non-unital C*-algebra into $\mathcal{B}(\mathcal{H})$ to a completely bounded map on the unitization.

Lemma 7.7. *Let $\mathfrak{A}$ be a unital C^*-algebra, and let $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be completely bounded. Then there exists completely positive maps $\varphi_i : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ with $\|\varphi_i\|_{cb} = \|\varphi\|_{cb}$ for $i = 1, 2$ such that the map $\Psi : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ given by*

$$\Psi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(a) & \varphi(b) \\ \varphi^*(c) & \varphi_2(d) \end{bmatrix}$$

is completely positive ($\varphi^*(c) = \varphi(c^*)^*$). Moreover if $\|\varphi\|_{cb} = 1$ then we may take $\varphi_1(I_{\mathfrak{A}}) = \varphi_2(I_{\mathfrak{A}}) = I_{\mathcal{H}}$.

Proof. By scaling φ if necessary, we may assume that $\|\varphi\|_{cb} = 1$ (after making this assumption, we can scale Ψ back and the φ_i will still have the correct properties). Since $\|\varphi\|_{cb} = 1$ we may apply Theorem 7.4 to obtain a completely positive map $\Phi : \mathcal{S}_{\mathfrak{A}} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ where $\mathcal{S}_{\mathfrak{A}} \subseteq \mathcal{M}_2(\mathfrak{A})$. By the Arveson's Extension Theorem Φ extends to a completely positive map $\Psi : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$.

Firstly we notice that for all $B, C \in \mathfrak{A}$ that

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathcal{S}_{\mathfrak{A}}$$

and consequently

$$\Psi\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & \varphi(B) \\ \varphi(C^*)^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & \varphi(B) \\ \varphi^*(C) & 0 \end{bmatrix}.$$

Thus to show that Ψ has the desired form we need only create φ_1 and φ_2 and use the linearity of Ψ.

To create φ_1 we desired to show that if we restrict Ψ to elements whose only non-zero entry is in the $(1, 1)$-entry then we get matrices in $\mathcal{M}_2(\mathcal{B}(\mathcal{H}))$ where only the $(1, 1)$-entry is non-zero. Let $P \in \mathfrak{A}$ be an arbitrary positive element with $P \leq I_{\mathfrak{A}}$. Then it is clear that

$$\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} I_{\mathfrak{A}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq \Psi\left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right) \leq \Psi\left(\begin{bmatrix} I_{\mathfrak{A}} & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} I_{\mathfrak{A}} & 0 \\ 0 & 0 \end{bmatrix} \qquad (*)$$

as Ψ is positive. However if

$$\Psi\left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

then, by Lemma 3.19 part (3), the first inequality in $(*)$ implies that $P_4 \geq 0$ and the second inequality in $(*)$ implies that $-P_4 \geq 0$. Hence $P_4 = 0$. Thus $P_2 = P_3 = 0$ by applying Lemma 3.19 part (3). Whence

$$\Psi\left(\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

for some $P_1 \in \mathcal{B}(\mathcal{H})$. Hence, since $\mathfrak{A}$ is the span of its positive elements, for each $A \in \mathfrak{A}$

$$\Psi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} A_{1,1} & 0 \\ 0 & 0 \end{bmatrix}$$

for some $A_{1,1} \in \mathcal{B}(\mathcal{H})$. Thus we may define the linear map $\varphi_1 : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that

$$\Psi\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \varphi_1(A) & 0 \\ 0 & 0 \end{bmatrix}$$

(φ_1 is linear as Ψ is linear) for all $A \in \mathfrak{A}$. However we need to show that φ_1 is completely positive. To see this notice if $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is positive then

$$\left[\begin{bmatrix} A_{i,j} & 0 \\ 0 & 0 \end{bmatrix}\right] \in \mathcal{M}_n(\mathcal{M}_2(\mathfrak{A}))$$

is positive by the canonical shuffle and Lemma 3.19 part (3). Then, since $\Psi_{(n)}$ is completely positive,

$$\Psi_{(n)}\left(\left[\left[\begin{array}{cc} A_{i,j} & 0 \\ 0 & 0 \end{array}\right]\right]\right) = \left[\left[\begin{array}{cc} \varphi_1(A_{i,j}) & 0 \\ 0 & 0 \end{array}\right]\right]$$

is positive. Hence, by another canonical shuffle,

$$\left[\begin{array}{cc} (\varphi_1)_{(n)}([A_{i,j}]) & 0 \\ 0 & 0 \end{array}\right] \in \mathcal{M}_2(\mathcal{M}_n(\mathfrak{A}))$$

is positive. Whence $(\varphi_1)_{(n)}([A_{i,j}])$ is positive by Lemma 3.19 part (3). Hence φ_1 is completely positive.

By similar arguments, we can construct a completely positive map $\varphi_2 : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that we get our final decomposition. Moreover, since $\varphi_1(I_{\mathfrak{A}}) = \varphi_2(I_{\mathfrak{A}}) = I_{\mathcal{H}}$ as Ψ is unital and since φ_i are completely bounded, we obtain that $\|\varphi_i\|_{cb} = 1 = \|\varphi\|_{cb}$ as desired. □

Notice the beauty of the above lemma which can be interpreted as saying that every completely bounded map into $\mathcal{B}(\mathcal{H})$ can be viewed as the $(1,2)$-entry of a completely positive map into $\mathcal{M}_2(\mathcal{B}(\mathcal{H}))$.

Now that we know the extension Ψ of Φ has this nice matrix form we can use this to prove a decomposition theorem for completely bounded maps. First we prove a simple yet technical lemma.

Lemma 7.8. *Let $\mathfrak{A}$ be a unital C*-algebra and let $\pi_1 : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{B}(\mathcal{K}_1)$ a unital *-homomorphism. Then there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and a unitary $W : \mathcal{K}_1 \to \mathcal{K} \oplus \mathcal{K}$ such that*

$$\pi_1\left(\left[\begin{array}{cc} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{array}\right]\right) = W^*\left[\begin{array}{cc} \pi(A_{1,1}) & \pi(A_{1,2}) \\ \pi(A_{2,1}) & \pi(A_{2,2}) \end{array}\right] W$$

for all $A_{i,j} \in \mathfrak{A}$.

Proof. We desire to construct π from π_1. To do this we notice that

$$P_1 := \pi_1\left(\left[\begin{array}{cc} I_{\mathfrak{A}} & 0 \\ 0 & 0 \end{array}\right]\right) \quad \text{and} \quad P_4 := \pi_1\left(\left[\begin{array}{cc} 0 & 0 \\ 0 & I_{\mathfrak{A}} \end{array}\right]\right)$$

are orthogonal projections that sum to the identity (as π_1 is unital *-homomorphism). Moreover

$$U := \pi_1\left(\left[\begin{array}{cc} 0 & I_{\mathfrak{A}} \\ I_{\mathfrak{A}} & 0 \end{array}\right]\right)$$

is a self-adjoint unitary and $P_1 = UP_4U$ so P_1 and P_4 are unitarily equivalent. Let $\mathcal{K} := P_1(\mathcal{K}_1)$ which is a Hilbert space as P_1 is an orthogonal projection. Since $P_1U = UP_4$, for all $\eta \in P_4(\mathcal{K}_1)$,

$$U\eta = UP_4\eta = P_1U\eta \in \mathcal{K}.$$

Similarly if $\xi \in \mathcal{K}$,

$$U\xi = UP_1\xi = P_4U\xi \in P_4(\mathcal{K}_1).$$

Hence, as U is a unitary, $U|_{\mathcal{K}}$ is a unitary that takes $\mathcal{K} = P_1(\mathcal{K}_1)$ onto $P_4(\mathcal{K}_1)$. Therefore the map $W : \mathcal{K}_1 = P_1(\mathcal{K}_1) \oplus P_4(\mathcal{K}_1) \to \mathcal{K} \oplus \mathcal{K}$ defined by $W(\xi + \eta) = \xi \oplus U\eta$ for all $\xi \in P_1(\mathcal{K}_1)$ and $\eta \in P_4(\mathcal{K}_1)$ is a well-defined unitary with $W^* : \mathcal{K} \oplus \mathcal{K} \to P_1(\mathcal{K}_1) \oplus P_4(\mathcal{K}_1) \simeq \mathcal{K}$ by $W^*(\xi \oplus \eta) = \xi + U\eta$.

Since

$$\pi_1\left(\left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right]\right) = P_1\pi_1\left(\left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right]\right) P_1$$

the map $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ defined by

$$\pi(A)\xi = \pi_1\left(\left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right]\right)(\xi)$$

for all $\xi \in \mathcal{K}$ is well-defined as the right hand side may be viewed as an element of $\mathcal{K}$. It is easy to verify that π is a *-homomorphism and

$$\pi(A)\xi = \pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) P_1(\xi+\eta) = \pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)(\xi+\eta)$$

for all $\xi \in \mathcal{K}$ and $\eta \in P_4(\mathcal{K}_1)$. Moreover π is unital as $\pi(I_{\mathfrak{A}}) = I_{\mathcal{K}}$.

It remains only to show the equation relating π, π_1, and W described in the theorem. To see this notice for all $\xi \in \mathcal{K}$, $\eta \in P_4(\mathcal{K}_1)$, and $A \in \mathfrak{A}$ that

$$\begin{aligned} W^* \begin{bmatrix} \pi(A) & 0 \\ 0 & 0 \end{bmatrix} W(\xi+\eta) &= W^* \begin{bmatrix} \pi(A) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ U\eta \end{bmatrix} \\ &= W^*((\pi(A)\xi) \oplus 0) \\ &= \pi(A)\xi \\ &= \pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right)(\xi+\eta) \end{aligned}$$

and

$$\begin{aligned} W^* \begin{bmatrix} 0 & \pi(A) \\ 0 & 0 \end{bmatrix} W(\xi+\eta) &= W^* \begin{bmatrix} 0 & \pi(A) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ U\eta \end{bmatrix} \\ &= W^*(\pi(A)U\eta \oplus 0) \\ &= \pi(A)U\eta \\ &= \pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) P_1(U\eta + U\xi) \qquad \text{as } U\xi \in P_4(\mathcal{K}_1) \\ &= \pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) U(\eta+\xi) \\ &= \pi_1\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)(\eta+\xi) \end{aligned}$$

and

$$\begin{aligned} W^* \begin{bmatrix} 0 & 0 \\ 0 & \pi(A) \end{bmatrix} W(\xi+\eta) &= W^* \begin{bmatrix} 0 & 0 \\ 0 & \pi(A) \end{bmatrix} \begin{bmatrix} \xi \\ U\eta \end{bmatrix} \\ &= W^*(0 \oplus \pi(A)U\eta) \\ &= U\pi(A)U\eta \\ &= U\pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) P_1(U\eta + U\xi) \qquad \text{as } U\xi \in P_4(\mathcal{K}_1) \\ &= U\pi_1\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) U(\eta+\xi) \\ &= \pi_1\left(\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}\right)(\eta+\xi) \end{aligned}$$

and

$$W^* \begin{bmatrix} 0 & \\ \pi(A) & 0 \end{bmatrix} W = \left(W^* \begin{bmatrix} 0 & \pi(A) \\ 0 & 0 \end{bmatrix} W\right)^* = \pi_1\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right)^* = \pi_1\left(\begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix}\right).$$

Whence, using linearity and combining the above expressions, we obtain the desired equation relating π, π_1, and W. □

Theorem 7.9 (Wittstock)**.** *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. Then there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$ (for $i = 1, 2$) such that $\|\varphi\|_{cb} = \|V_1\| \|V_2\|$ and $\varphi(A) = V_1^*\pi(A)V_2$ for all $A \in \mathfrak{A}$. Moreover if $\|\varphi\|_{cb} = 1$ then V_1 and V_2 may be taken to be isometries. In $\mathfrak{A}$ is unital, π can be taken to be unital.*

Proof. If $\varphi = 0$ the result is trivial. As we have done several times in this chapter, by scaling (and unscaling at the end) we may assume that $\|\varphi\|_{cb} = 1$. First suppose $\mathfrak{A}$ is a unital C*-algebra. Let φ_1, φ_2, and Ψ be as in Lemma 7.7 where the φ_is are unital as $\|\varphi\|_{cb} = 1$. Since Ψ is completely positive and unital, by Stinespring's Theorem there exists a Hilbert space $\mathcal{K}_1$, a unital *-homomorphism $\pi_1 : \mathcal{M}_2(\mathfrak{A}) \to \mathcal{B}(\mathcal{K}_1)$, and an isometry $V_0 : \mathcal{H} \oplus \mathcal{H} \to \mathcal{K}_1$ such that $\Psi([A_{i,j}]) = V_0^* \pi_1([A_{i,j}])V_0$ for all $[A_{i,j}] \in \mathcal{M}_2(\mathfrak{A})$.

By Lemma 7.8 there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and a unitary $W : \mathcal{K}_1 \to \mathcal{K} \oplus \mathcal{K}$ such that

$$\pi_1\left(\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}\right) = W^* \begin{bmatrix} \pi(A_{1,1}) & \pi(A_{1,2}) \\ \pi(A_{2,1}) & \pi(A_{2,2}) \end{bmatrix} W.$$

Thus

$$\begin{bmatrix} \varphi_1(A_{1,1}) & \varphi(A_{1,2}) \\ \varphi^*(A_{2,1}) & \varphi_2(A_{2,2}) \end{bmatrix} = \Psi([A_{i,j}]) = V_0^* W^* \begin{bmatrix} \pi(A_{1,1}) & \pi(A_{1,2}) \\ \pi(A_{2,1}) & \pi(A_{2,2}) \end{bmatrix} WV_0 = V^* \begin{bmatrix} \pi(A_{1,1}) & \pi(A_{1,2}) \\ \pi(A_{2,1}) & \pi(A_{2,2}) \end{bmatrix} V$$

where $V := WV_0 : \mathcal{H} \oplus \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}$ is the composition of an isometry and a unitary and thus is an isometry. All that remains is to extract V_1 and V_2 from V.

Notice for any $\xi \in \mathcal{H}$

$$\begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi_1(I_{\mathfrak{A}}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix} = V^* \begin{bmatrix} \pi(I_{\mathfrak{A}}) & 0 \\ 0 & 0 \end{bmatrix} V \begin{bmatrix} \xi \\ 0 \end{bmatrix} = V^* \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} V \begin{bmatrix} \xi \\ 0 \end{bmatrix}.$$

Hence if $V \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \in \mathcal{K} \oplus \mathcal{K}$ then

$$\|\xi\|^2 = \left\langle V^* \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} V \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle_{\mathcal{H}\oplus\mathcal{H}} = \left\langle \begin{bmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \right\rangle_{\mathcal{K}\oplus\mathcal{K}} = \|\eta_1\|^2.$$

Since V is an isometry, the above equation implies that $\eta_2 = 0$ so that

$$V \begin{bmatrix} \xi \\ 0 \end{bmatrix} = \begin{bmatrix} \eta_1 \\ 0 \end{bmatrix}.$$

Therefore, as this holds for all $\xi \in \mathcal{H}$, there exists a linear map $V_1 : \mathcal{H} \to \mathcal{K}$ such that

$$\begin{bmatrix} V_1\xi \\ 0 \end{bmatrix} = V \begin{bmatrix} \xi \\ 0 \end{bmatrix}.$$

Also, since V is an isometry, V_1 is an isometry. Similarly there exists an isometry $V_2 : \mathcal{H} \to \mathcal{K}$ such that

$$\begin{bmatrix} 0 \\ V_2\xi \end{bmatrix} = V \begin{bmatrix} 0 \\ \xi \end{bmatrix}.$$

As V is linear, we obtain that $V = V_1 \oplus V_2$ and thus $V^* = V_1^* \oplus V_2^*$. Hence

$$\begin{bmatrix} \varphi_1(A) & \varphi(B) \\ \varphi^*(C) & \varphi_2(D) \end{bmatrix} = V^* \begin{bmatrix} \pi(A) & \pi(B) \\ \pi(C) & \pi(D) \end{bmatrix} V = \begin{bmatrix} V_1^*\pi(A)V_1 & V_1^*\pi(B)V_2 \\ V_2^*\pi(C)V_1 & V_2^*\pi(D)V_2 \end{bmatrix}$$

which completes the proof (in the unital case) as then $\varphi(B) = V_1^*\pi(B)V_2$ for all $B \in \mathfrak{A}$.

If $\mathfrak{A}$ is not unital let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$. By Theorem 7.5 there exists a completely bounded map $\tilde{\varphi} : \tilde{\mathfrak{A}} \to \mathcal{B}(\mathcal{H})$ such that $\tilde{\varphi}|_{\mathfrak{A}} = \varphi$ and $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$. By the above proof there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\tilde{\pi} : \tilde{\mathfrak{A}} \to \mathcal{B}(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$ (for $i = 1, 2$) such that $\|\tilde{\varphi}\|_{cb} = \|V_1\| \|V_2\|$ and $\tilde{\varphi}(A) = V_1^*\tilde{\pi}(A)V_2$ for all $A \in \tilde{\mathfrak{A}}$ (where V_1 and V_2 are isometries if $\|\varphi\|_{cb} = \|\tilde{\varphi}\|_{cb} = 1$). If $\pi = \tilde{\pi}|_{\mathfrak{A}}$, we obtain that $\varphi(A) = \tilde{\varphi}(A) = V_1^*\pi(A)V_2$ for all $A \in \mathfrak{A}$ and thus the result follows. $\square$

As a corollary to the above theorem we obtain a standard result about continuous linear functionals on a C*-algebra.

Corollary 7.10. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a continuous linear functional. Then there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and vectors $\xi, \eta \in \mathcal{K}$ such that $\|\varphi\| = \|\xi\| \|\eta\|$ and $\varphi(A) = \langle \pi(A)\xi, \eta\rangle_{\mathcal{K}}$ for all $A \in \mathfrak{A}$.*

Proof. Since φ is a continuous linear functional, Proposition 3.8 implies that $\varphi : \mathfrak{A} \to \mathbb{C} = \mathcal{B}(\mathbb{C})$ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$. By Theorem 7.9 there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$ (for $i = 1, 2$) such that $\|\varphi\|_{cb} = \|V_1\| \|V_2\|$ and $\varphi(A) = V_1^* \pi(A) V_2$ (as bounded operators on $\mathbb{C}$) for all $A \in \mathfrak{A}$.

Let $\xi := V_2(1)$ and let $\eta := V_1(1)$. Since $V_1(\lambda) = \lambda\eta$ and $V_2(\lambda) = \lambda\xi$ for all $\lambda \in \mathbb{C}$, it is trivial to verify that $\|\eta\| = \|V_1\|$ and $\|\xi\| = \|V_2\|$ so that $\|\varphi\| = \|\varphi\|_{cb} = \|\xi\| \|\eta\|$. Moreover

$$\varphi(A) = \langle \varphi(A)1, 1\rangle_{\mathbb{C}} = \langle \pi(A)V_2(1), V_1(1)\rangle_{\mathcal{K}} = \langle \pi(A)\xi, \eta\rangle_{\mathcal{K}}$$

for all $A \in \mathfrak{A}$ as desired. □

Remarks 7.11. Notice that if $\mathfrak{A}$ is a C*-algebra, $\varphi_i : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ are completely positive maps, and $\lambda_i \in \mathbb{C}$ are scalars, the map $\psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ given by

$$\psi(A) = \sum_{i=1}^{m} \lambda_i \varphi_i(A)$$

is completely bounded. Indeed $\|\psi\|_n \leq \sum_{i=1}^m |\lambda_i| \|\varphi_i\|_n = \sum_{i=1}^m |\lambda_i| \|\varphi_i\|$ for all $n \in \mathbb{N}$. We desire to show that every completely bounded map on a C*-algebra is a linear combination of completely positive maps. This will follow easily from Theorem 7.9. This will also allow us to show that every continuous linear functional is a linear combination of positive linear functionals whose norms satisfy a certain condition.

Theorem 7.12. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. Then there exists completely positive maps $\varphi_j : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\|\varphi_j\| \leq 2\|\varphi\|_{cb}$ for all j and $\varphi(A) = \sum_{j=1}^4 i^j \varphi_j(A)$ for all $A \in \mathfrak{A}$. If $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathfrak{A}$ we may take $\varphi_1 = \varphi_3 = 0$.*

Proof. Let $\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a completely bounded map. By Theorem 7.9 there exists Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$ (for $i = 1, 2$) such that $\|\varphi\|_{cb} = \|V_1\| \|V_2\|$ and $\varphi(A) = V_1^* \pi(A) V_2$ for all $A \in \mathfrak{A}$. By scaling V_1 and V_2 appropriately, we may assume that $\|V_1\| = \|V_2\| = \|\varphi\|_{cb}^{\frac{1}{2}}$. For each $j \in \{1, 2, 3, 4\}$ define $\varphi_j : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by

$$\varphi_j(A) = \left(\frac{V_2 + i^j V_1}{2}\right)^* \pi(A) \left(\frac{V_2 + i^j V_1}{2}\right)$$

for all $A \in \mathfrak{A}$ which is a well-defined, completely positive map by the remarks preceding Theorem 4.1. Moreover

$$\|\varphi_j\| \leq \frac{1}{4}\left\|V_2 + i^j V_1\right\|^2 \leq \frac{1}{4}(\|V_2\| + \|V_1\|)^2 = \|\varphi\|_{cb}.$$

Moreover we notice (by the same equation as the polarization identity) that

$$\begin{aligned} \varphi(A) &= V_1^* \pi(A) V_2 \\ &= \sum_{j=1}^4 \left(\frac{V_1 + i^j V_2}{2}\right)^* \pi(A) \left(\frac{V_1 + i^j V_2}{2}\right) \\ &= \sum_{j=1}^4 i^j \varphi_j(A) \end{aligned}$$

for all $A \in \mathfrak{A}$ as desired.

Lastly suppose that $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathfrak{A}$. If $A \in \mathfrak{A}$ is self-adjoint then

$$i\varphi_1(A) - \varphi_2(A) - i\varphi_3(A) + \varphi_4(A) = \varphi(A) = \varphi(A)^* = -i\varphi_1(A) - \varphi_2(A) + i\varphi_3(A) + \varphi_4(A)$$

as each φ_j is a positive map so $\varphi_j(A)^* = \varphi_j(A)$ as A is self-adjoint. By rearranging the equation we obtain that $\varphi_1(A) = \varphi_3(A)$ for all $A \in \mathfrak{A}$ self-adjoint and thus (as the self-adjoint elements span a C*-algebra) $\varphi_1 = \varphi_3$. Whence for all $A \in \mathfrak{A}$

$$\varphi(A) = i\varphi_1(A) - \varphi_2(A) - i\varphi_3(A) + \varphi_4(A) = \varphi_4(A) - \varphi_2(A)$$

so we may take $\varphi_1 = \varphi_3 = 0$. $\square$

Theorem 7.13. *Let $\mathfrak{A}$ be a C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a continuous linear functional. Then there exists positive linear functionals $\varphi_j : \mathfrak{A} \to \mathbb{C}$ such that $\sum_{j=1}^4 \|\varphi_j\| \leq 2\|\varphi\|$ and $\varphi(A) = \sum_{j=1}^4 i^j \varphi_j(A)$ for all $A \in \mathfrak{A}$. Moreover if $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathfrak{A}$ then we may take $\varphi_1 = \varphi_3 = 0$ and $\|\varphi_2\| + \|\varphi_3\| = \|\varphi\|$.*

Proof. The proof of this theorem is basically the same as Theorem 7.12 except for the norm condition that will follow from the parallelogram law. Suppose that $\varphi : \mathfrak{A} \to \mathbb{C}$ is a continuous linear functional. By Corollary 7.10 there exists a Hilbert space $\mathcal{K}$, a *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and vectors $\xi, \eta \in \mathcal{K}$ such that $\|\varphi\| = \|\xi\| \|\eta\|$ and $\varphi(A) = \langle \pi(A)\xi, \eta\rangle$ for all $A \in \mathfrak{A}$. By scaling we may assume that $\|\xi\| = \|\eta\| = \|\varphi\|^{\frac{1}{2}}$. For each $j \in \{1, 2, 3, 4\}$ define $\varphi_j : \mathfrak{A} \to \mathbb{C}$ by

$$\varphi_j(A) = \left\langle \pi(A)\left(\frac{\xi + i^j\eta}{2}\right), \frac{\xi + i^j\eta}{2}\right\rangle$$

for all $A \in \mathfrak{A}$ which is a well-defined, positive linear functional by Example 1.4. Moreover

$$\|\varphi_j\| \leq \left\|\frac{\xi + i^j\eta}{2}\right\|^2 = \frac{1}{4}\left\|\xi + i^j\eta\right\|^2$$

so that (by the parallelogram law)

$$\sum_{j=1}^4 \|\varphi_j\| \leq \frac{1}{4}\sum_{j=1}^4 \left\|\xi + i^j\eta\right\|^2 = \frac{1}{4}\left((2\|\xi\|^2 + 2\|\eta\|^2) + (2\|\xi\|^2 + 2\|i\eta\|^2\right) = 2\|\varphi\|.$$

Moreover we notice (by the same equation as the polarization identity) that

$$\begin{aligned}\varphi(A) &= \langle \pi(A)\xi, \eta\rangle \\ &= \sum_{j=1}^4 \left\langle \pi(A)\left(\frac{\xi + i^j\eta}{2}\right), \frac{\xi + i^j\eta}{2}\right\rangle \\ &= \sum_{j=1}^4 i^j\varphi_j(A)\end{aligned}$$

for all $A \in \mathfrak{A}$ as desired.

Lastly suppose that $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathfrak{A}$. If $A \in \mathfrak{A}$ is self-adjoint then

$$i\varphi_1(A) - \varphi_2(A) - i\varphi_3(A) + \varphi_4(A) = \varphi(A) = \varphi(A)^* = -i\varphi_1(A) - \varphi_2(A) + i\varphi_3(A) + \varphi_4(A)$$

as each φ_j is a positive map so $\varphi_j(A)^* = \varphi_j(A)$ as A is self-adjoint. By rearranging the equation, we obtain that $\varphi_1(A) = \varphi_3(A)$ for all $A \in \mathfrak{A}$ self-adjoint and thus (as the self-adjoint elements span a C*-algebra) $\varphi_1 = \varphi_3$. Whence for all $A \in \mathfrak{A}$

$$\varphi(A) = i\varphi_1(A) - \varphi_2(A) - i\varphi_3(A) + \varphi_4(A) = \varphi_4(A) - \varphi_2(A)$$

so we may take $\varphi_1 = \varphi_3 = 0$. Moreover

$$\|\varphi_2\| + \|\varphi_4\| \leq \frac{1}{4}\left(\|\xi - \eta\|^2 + \|\xi + \eta\|^2\right) = \frac{1}{4}\left(2\|\xi\|^2 + 2\|\eta\|^2\right) = \|\varphi\|.$$

Since clearly

$$\|\varphi\| = \|\varphi_2 - \varphi_4\| \leq \|\varphi_2\| + \|\varphi_4\|,$$

the norm condition follows. □

Notice that the above (along with the Riesz Representation Theorem for positive linear functionals of the continuous functions on a compact Hausdorff space) shows that every finite signed measure μ on a compact Hausdorff space X can be written as $\mu = \mu_1 - \mu_2$ where μ_1 and μ_2 are positive measures with $\|\mu\| = \mu_1(X) + \mu_2(X)$.

To complete this chapter, we will present a result that shows certain completely bounded maps are close to completely positive maps thus further intertwining the theory of these types of maps.

Theorem 7.14. *Let $\mathfrak{A}$ be a unital C^*-algebra, let $\mathcal{S} \subseteq \mathfrak{A}$ be an operator system, let $\mathcal{H}$ be a Hilbert space, and let $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be a unital, self-adjoint, completely bounded map. Then there exists a unital, completely positive map $\Psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\|\Psi|_{\mathcal{S}} - \Phi\|_{cb} \leq 2(\|\Phi\|_{cb} - 1)$. In particular, if $\|\Phi\|_{cb} = 1$ then Φ is completely positive.*

Proof. Since Φ is unital, $\|\Phi\|_{cb} \geq 1$. By Theorem 7.5 there exists a completely bounded map $\Psi_0 : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\Psi_0|_{\mathcal{S}} = \Phi$ and $\|\Psi_0\|_{cb} = \|\Phi\|_{cb}$. By Theorem 7.9 there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and isometries $V_i : \mathcal{H} \to \mathcal{K}$ (for $i \in \{1, 2\}$) such that $\Psi_0(A) = \|\Phi\|_{cb} V_1^*\pi(A)V_2$ for all $A \in \mathfrak{A}$.

Since Φ is self-adjoint, we obtain that

$$\Phi(A) = \Psi_0(A) = \|\Phi\|_{cb} V_1^*\pi(A)V_2 = \|\Phi\|_{cb} V_2^*\pi(A)V_1$$

for all $A \in \mathcal{S}$. Define $\Psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by

$$\Psi(A) = \frac{1}{2}(V_1^*\pi(A)V_1 + V_2^*\pi(A)V_2)$$

for all $A \in \mathfrak{A}$. Clearly Ψ is a completely positive map (being the sum of completely positive maps) with $\Psi(I_{\mathfrak{A}}) = \frac{1}{2}(V_1^*V_1 + V_2^*V_2) = I_{\mathcal{H}}$. Moreover we notice for all $A \in \mathcal{S}$ that

$$\begin{aligned}\frac{1}{2}\|\Phi\|_{cb}(V_1 - V_2)^*\pi(A)(V_1 - V_2) &= \frac{1}{2}\|\Phi\|_{cb}(V_1^*\pi(A)V_1 + V_2^*\pi(A)V_2 - V_1^*\pi(A)V_2 - V_2^*\pi(A)V_1) \\ &= \|\Phi\|_{cb}\Psi(A) - \Phi(A).\end{aligned}$$

Therefore

$$\|\Psi|_{\mathcal{S}} - \Phi\|_{cb} \leq \|\Psi - \|\Phi\|_{cb}\Psi\|_{cb} + \|\|\Phi\|_{cb}\Psi|_{\mathcal{S}} - \Phi\|_{cb} \leq (\|\Phi\|_{cb} - 1) + \frac{1}{2}\|\Phi\|_{cb}\|V_1 - V_2\|^2.$$

However, since $\Phi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$, $I_{\mathcal{H}} = \|\Phi\|_{cb} V_1^*V_2$ so

$$\frac{1}{2}\|\Phi\|_{cb}\|V_1 - V_2\|^2 = \frac{1}{2}\|\|\Phi\|_{cb} I_{\mathcal{H}} - 2I_{\mathcal{H}} + \|\Phi\|_{cb} I_{\mathcal{H}}\| = \|\Phi\|_{cb} - 1$$

and thus

$$\|\Psi|_{\mathcal{S}} - \Phi\|_{cb} \leq 2(\|\Phi\|_{cb} - 1)$$

as desired. □

References

[BO] N. Brown and N. Ozawa, *C^*-Algebras and Finite Dimensional Approximations*, American Mathematical Society, Graduate Studies in Mathematics, (2008).

[EH] E. Effros and U. Haagerup, *Lifting Problems and Local Reflexivity for C^*-Algebras*, Duke Mathematics Journal, Volume 52, Number 1, (1985).

[Pa] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, Cambridge Studies in Advanced Mathematics, (2002).