Cross Product C*-Algebras by Locally Compact Groups

Paul Skoufranis

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Abstract

The purpose of these notes is to construct the cross product C*-algebras of a C*-algebra by a locally compact group. The interested reader can observe how the theory we develop is analogous to the construction of the cross product C*-algebras of a C*-algebra by a countable discrete group. We will assume that the reader has a basic knowledge of Harmonic Analysis on locally compact groups, but we will provide a summary. We will also assume that the reader is familiar with the basics of the Bochner integral for continuous functions of compact support (although we will avoid full use of the Bochner integral whenever possible).

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To begin, let us quickly review some basic Harmonic Analysis on locally compact groups.

Preliminaries 1. Let G be a locally compact group. We recall that G has a Haar measure μ (that is a positive, inner regular, Borel measure such that $\mu(tE) = \mu(E)$ for all $t \in G$ and $E \subseteq G$ Borel) that is unique up to scaling. Moreover there is a continuous homomorphism $\Delta : G \to \mathbb{R}^+$ such that $\mu(Es) = \Delta(s)\mu(E)$ for all $E \subseteq G$ measureable.

A reader familiar with Harmonic Analysis will recall that $L_1(G)$ (the absolutely integrable functions with respect to the Haar measure) is a Banach *-algebra with the operations

$$f * g(s) = \int_G f(t)g(s^{-1}t)dt \qquad f^*(t) = \Delta(t)^{-1}\overline{f(t^{-1})}$$

where dt represents $d\mu(t)$. Moreover, we recall that $L_1(G)$ has a norm one approximate identity of positive, self-adjoint functions.

Theorem 2. *If G is a locally compact group then $L_1(G)$ has a norm one approximate identity of positive, self-adjoint functions.*

Proof: Let $\{U_\lambda\}_\Lambda$ be a neighbourhood basis of e with $m(U_\lambda) \in (0, \infty)$ for all λ, $\overline{U_\lambda}$ compact for all λ, and $U_\lambda = U_\lambda^{-1}$ (since m is a Haar measure, $m(U_\lambda) > 0$ and the measure of every compact set is finite. Since G is locally compact, we can find a neighbourhood basis of such sets). Let $g_\lambda(t) = \Delta(t)^{-\frac{1}{2}}\chi_{U_\lambda}(t)$ for all $t \in G$. Since $\overline{U_\lambda}$ is compact and Δ is a continuous homomorphism of G into $\mathbb{R}^+$, $\inf\{\Delta(t)^{-\frac{1}{2}} \mid t \in U_\lambda\} > 0$. Thus since $\chi_{U_\lambda} \in L_1(G)$, $g_\lambda \in L_1(G)$. Notice that

$$\begin{aligned} g_\lambda^*(t) &= \Delta(t)^{-1}\overline{\Delta(t^{-1})^{-\frac{1}{2}}\chi_{U_\lambda}(t^{-1})} \\ &= \Delta(t)^{-1}\Delta(t^{-1})^{-\frac{1}{2}}\chi_{U_\lambda}(t^{-1}) \\ &= \Delta(t)^{-1}\Delta(t)^{\frac{1}{2}}\chi_{U_\lambda^{-1}}(t) \\ &= \Delta(t)^{-\frac{1}{2}}\chi_{U_\lambda}(t) = g_\lambda(t) \end{aligned}$$

Hence $g_\lambda^* = g_\lambda$. Lastly, we notice that $\|g_\lambda\|_1 \geq m(U_\lambda)\inf\{\Delta(t)^{-\frac{1}{2}} \mid t \in U_\lambda\} > 0$.

Let $f_\lambda = \frac{1}{\|g_\lambda\|_1} g_\lambda$ so $f_\lambda \in L_1(G)$, $f_\lambda^* = f_\lambda$, $f_\lambda(s) \geq 0$ for all $s \in G$, and $\|f_\lambda\| = 1$. We claim that $\{f_\lambda\}_\Lambda$ is a norm one approximate identity for $L_1(G)$. To see this, fix $h \in L_1(G)$. If h is a continuous function with compact support, then

$$\begin{aligned}
\|h - f_\lambda * h\|_1 &= \int_G \left| h(t) - \int_G f_\lambda(s) h(s^{-1}t) ds \right| dt \\
&= \int_G \left| \int_G (h(t) - h(s^{-1}t)) f_\lambda(s) ds \right| dt \qquad f_\lambda \text{ is positive with norm } 1 \\
&\leq \int_G \int_G |h(t) - h(s^{-1}t)| f_\lambda(s) ds dt \\
&= \int_G \int_G |h(t) - h(s^{-1}t)| f_\lambda(s) dt ds \qquad \text{Fubini's Theorem as the domains of integration are compact} \\
&= \int_G \|h_s - h\|_1 f_\lambda(ts) ds \\
&\leq \sup\{\|h_s - h\|_1 \mid s \in U_\lambda\}
\end{aligned}$$

where $h_s(t) = h(s^{-1}t)$. Since $h \in C_c(G)$, $\sup\{\|h_s - h\|_1 \mid s \in U_\lambda\}$ tends to zero so $\|h - h * f_\lambda\|_1 \to 0$. Next if $g \in L_1(G)$ is arbitrary, for all $\epsilon > 0$ we can find a continuous function with compact support h so that $\|g - h\|_1 < \epsilon$. Then $\|f_\lambda * g - f_\lambda * g\| < \epsilon$ so we can apply the above estimate to conclude $\|g - f_\lambda * g\|_1 \to 0$. Hence $\{f_\lambda\}_\Lambda$ is a left norm one approximate identity for $L_1(G)$. Since $f_\lambda^* = f_\lambda$, $\lim_\Lambda \|f_\lambda * g - g\|_1 = 0$ for all $g \in L_1(G)$, and $L_1(G)$ is a Banach *-algebra, $\lim_\Lambda \|g * f_\lambda - g\|_1 = \lim_\Lambda \|f_\lambda * g^* - g^*\|_1 = 0$ for all $g \in L_1(G)$. Whence $\{f_\lambda\}_\Lambda$ is a norm one approximate identity for $L_1(G)$. □

Our first goal is to generalize the notion of $L_1(G)$ to allow functions to take values in a C*-algebra. The following structure allows us to do this.

Definition 3. A C*-dynamical system $(\mathfrak{A}, G, \alpha)$ consists of a C*-algebra $\mathfrak{A}$, a locally compact group G, and a group homomorphism $\alpha : G \to Aut(\mathfrak{A})$ such that $t \mapsto \alpha(t)(A)$ is continuous for all $A \in \mathfrak{A}$. For each $t \in G$ we denote $\alpha(t)$ by α_t.

For a discrete group, the condition $t \mapsto \alpha(t)(A)$ is continuous for all $A \in \mathfrak{A}$ is satisfied by every group homomorphism $\alpha : G \to Aut(\mathfrak{A})$ and thus is not necessary.

Now we will show that every C*-dynamical system $(\mathfrak{A}, G, \alpha)$ generates an $L_1(G)$ like structure with functions taking values in $\mathfrak{A}$.

Construction 4. Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. Consider $C_c(G, \mathfrak{A})$; the set of continuous functions from G into $\mathfrak{A}$ with compact support. Define a norm on $C_c(G, \mathfrak{A})$ by $\|f\|_1 = \int_G \|f(t)\|_\mathfrak{A} dt$. It is trivial to verify that $\|\cdot\|_1$ is indeed a norm. Next define a twisted convolution on $C_c(G, \mathfrak{A})$ by

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))dt$$

To see that $f(t)\alpha_t(g(t^{-1}s))$ is actually integrable, notice for all $t, r \in G$

$$\left\|\alpha_r\left(g(r^{-1}s)\right) - \alpha_t\left(g(t^{-1}s)\right)\right\| \leq \left\|\alpha_r\left(g(r^{-1}s) - g(t^{-1}s)\right)\right\| + \left\|\alpha_r\left(g(t^{-1}s)\right) - \alpha_t\left(g(t^{-1}s)\right)\right\|$$

Thus if $\epsilon > 0$ there exists a neighbourhood U of t so that if $r \in U$ then $\left\|\alpha_r(g(t^{-1}s)) - \alpha_t(g(t^{-1}s))\right\| < \frac{\epsilon}{2}$ as $r \mapsto \alpha_r(A)$ is continuous for all $A \in \mathfrak{A}$ by the definition of a C*-dynamical system. Moreover, since g is continuous, there exists a neighbourhood V of t so that if $r \in V$ then $\left\|g(r^{-1}s) - g(t^{-1}s)\right\| < \frac{\epsilon}{2}$. Thus if $r \in V \cap U$ then

$$\left\|\alpha_r(g(r^{-1}s)) - \alpha_t(g(t^{-1}s))\right\| \leq \left\|g(r^{-1}s) - g(t^{-1}s)\right\| + \left\|\alpha_r(g(t^{-1}s)) - \alpha_t(g(t^{-1}s))\right\| < \epsilon$$

Whence $t \mapsto f(t)\alpha_t(g(t^{-1}s))$ is a continuous function with compact support and therefore is integrable.

For $*$ to be a multiplication on $C_c(G, \mathfrak{A})$, we need to show that $f * g$ is continuous and has compact support. Notice if $s, s' \in G$ then

$$\|f * g(s) - f * g(s')\| \leq \int_G \left\|f(t)(\alpha_t(g(t^{-1}s) - g(t^{-1}s')))\right\| dt \leq \int_G \|f(t)\| \left\|g(t^{-1}s) - g(t^{-1}s')\right\| dt$$

Since g has compact support and thus is uniformly continuous, for all $\epsilon > 0$ there exists an open neighbourhood of e, U_ϵ, such that if $a^{-1}b \in U_\epsilon$, then $\|g(a) - g(b)\| < \epsilon$. Whence if $s^{-1}s' \in U_\epsilon$,

$$\|f * g(s) - f * g(s')\| \leq \epsilon \|f\|_1$$

so $f * g$ is a continuous function. We notice that $f(t)\alpha_t(g(t^{-1}s)) = 0$ unless $t \in supp(f)$ and $t^{-1}s \in supp(g)$. Thus if $s \notin supp(f)supp(g)$, $f * g(s) = 0$. As $supp(f)supp(g)$ is the product of two compact sets and thus is compact, $f * g$ has compact support.

We claim this multiplication turns $C_c(G, \mathfrak{A})$ into an algebra. The only non-trivial property to check is associativity. Thus if $f, g, h \in C_c(G, \mathfrak{A})$, then

$$\begin{aligned}
(f * (g * h))(s) &= \int_G f(t)\alpha_t((g * h)(t^{-1}s))dt \\
&= \int_G f(t)\alpha_t\left(\int_G g(r)\alpha_r(h(r^{-1}t^{-1}s))dr\right)dt \\
&= \int_G f(t)\int_G \alpha_t(g(r))\alpha_{tr}(h(r^{-1}t^{-1}s))drdt \\
&= \int_G f(t)\int_G \alpha_t(g(t^{-1}r))\alpha_r(h(r^{-1}s))drdt \qquad r \mapsto t^{-1}r \\
&= \int_G\int_G f(t)\alpha_t(g(t^{-1}r))\alpha_r(h(r^{-1}s))dtdr \qquad \text{Fubini as the functions have compact support} \\
&= \int_G (f * g)(r)\alpha_r(h(r^{-1}s))dr \\
&= ((f * g) * h)(s)
\end{aligned}$$

Whence $C_c(G, \mathfrak{A})$ is an algebra.

Next we claim the norm on $C_c(G, \mathfrak{A})$ is submultiplicative. Indeed if $f, g \in C_c(G, \mathfrak{A})$ then

$$\begin{aligned}
\|f * g\|_1 &= \int_G \|f * g(s)\| ds \\
&= \int_G \left\|\int_G f(t)\alpha_t(g(t^{-1}s))dt\right\| ds \\
&\leq \int_G\int_G \|f(t)\| \left\|g(t^{-1}s)\right\| dtds \\
&= \int_G\int_G \|f(t)\| \|g(s)\| dsdt \\
&= \int_G \|f(t)\| \|g\|_1 dt \\
&= \|f\|_1 \|g\|_1
\end{aligned}$$

as desired.

Next define an adjoint operation * on $C_c(G, \mathfrak{A})$ by $f^*(s) = \Delta(s)^{-1}\alpha_s(f(s^{-1}))^*$ for all $f \in C_c(G, \mathfrak{A})$. We claim that $f^* \in C_c(G, \mathfrak{A})$ for all $f \in C_c(G, \mathfrak{A})$. To see this, we know that $s \mapsto \Delta(s)^{-1}$ is continuous and

$$\begin{aligned}
\left\|\alpha_s(f(s^{-1}))^* - \alpha_t(f(t^{-1}))^*\right\| &\leq \left\|\alpha_s(f(s^{-1}) - f(t^{-1}))\right\| + \left\|\alpha_s(f(t^{-1})) - \alpha_t(f(t^{-1}))\right\| \\
&\leq \left\|f(s^{-1}) - f(t^{-1})\right\| + \left\|\alpha_s(f(t^{-1})) - \alpha_t(f(t^{-1}))\right\|
\end{aligned}$$

Fix $t \in G$ and let $\epsilon > 0$. Since $f \in C_c(G, \mathfrak{A})$ there exists an open neighbourhood U_1 of t such that if $s \in U_1$ then $\left\|f(s^{-1}) - f(t^{-1})\right\| < \frac{\epsilon}{2}$. Since $s \mapsto \alpha_s(f(t^{-1}))$ is continuous, there exists an open neighbourhood U_2 of t such that if $s \in U_2$ then $\left\|\alpha_s(f(t^{-1})) - \alpha_t(f(t^{-1}))\right\| < \frac{\epsilon}{2}$. Whence if $s \in U_1 \cap U_2$ then $\left\|\alpha_s(f(s^{-1}))^* - \alpha_t(f(t^{-1}))^*\right\| < \epsilon$. Thus $s \mapsto \alpha_s(f(s^{-1}))^*$ is continuous so f^* is continuous. Since f has compact support, clearly f^* has compact support.

To show that * is an involution, we note that * is clearly conjugate linear. Next if $f, g \in C_c(G)$ then

$$\begin{aligned}
(g^* * f^*)(s) &= \int_G g^*(t)\alpha_t(f^*(t^{-1}s))dt \\
&= \int_G \Delta(t^{-1})\alpha_t(g(t^{-1}))^*\alpha_t\left(\Delta((t^{-1}s)^{-1})\alpha_{t^{-1}s}(f(s^{-1}t))^*\right)dt \\
&= \int_G \Delta(s^{-1}t)\Delta(t^{-1})\alpha_t(g(t^{-1}))^*\alpha_s(f(s^{-1}t))^*dt \\
&= \int_G \Delta(s^{-1})(\alpha_s(f(s^{-1}t))\alpha_t(g(t^{-1})))^*dt \\
&= \int_G \Delta(s^{-1})(\alpha_s(f(t))\alpha_{st}(g(t^{-1}s^{-1})))^*dt \\
&= \Delta(s^{-1})\alpha_s\left(\int_G f(t)\alpha_t(g(t^{-1}s^{-1}))dt\right)^* \\
&= \Delta(s^{-1})\alpha_s\left((f * g)(s^{-1})\right)^* = (f * g)^*(s)
\end{aligned}$$

so * is anti-multiplicative. Lastly

$$(f^*)^*(s) = \Delta(s^{-1})\alpha_s(f^*(s^{-1})) = \Delta(s^{-1})\alpha_s(\Delta(s)\alpha_{s^{-1}}(f(s))) = f(s)$$

Whence * is an involution on $C_c(G, \mathfrak{A})$.

Lastly we claim that $\|f^*\|_1 = \|f\|_1$. To see this, we notice that each α_t is an isometry so

$$\begin{aligned}
\|f^*\|_1 &= \int_G \Delta(t^{-1})\left\|\alpha_t(f(t^{-1}))^*\right\|dt \\
&= \int_G \Delta(t^{-1})\left\|f(t^{-1})\right\|dt \\
&= \int_G \|f(t)\|\,dt = \|f\|_1
\end{aligned}$$

as desired.

Let $L_1(G, \mathfrak{A}, \alpha)$ be the completion of $C_c(G, \mathfrak{A})$ with respect to the above norm and operations. Whence $L_1(G, \mathfrak{A}, \alpha)$ is a Banach *-algebra.

It is trivial to verify that $(\mathbb{C}, G, 1)$ is a C*-dynamical system (where $1(g) = Id_{\mathbb{C}}$) and $L_1(G, \mathbb{C}, 1) = L_1(G)$ as the continuous functions of compact support are dense in $L_1(G)$. Notice that if $f \in L_1(G)$ and $A \in \mathfrak{A}$ then the function $g(t) = Af(t)$ is an element of $L_1(G, \mathfrak{A}, \alpha)$ (since if $f \in L_1(G)$, there exists continuous function of compact support $\{f_n\}$ that converge to f in $L_1(G)$. Thus $\{Af_n\}$ is a Cauchy sequence in $L_1(G, \mathfrak{A}, \alpha)$ and then converges to an element $g \in L_1(G, \mathfrak{A}, \alpha)$. We denote g by Af). It is possible to get around some of the technicalities that we will face by knowledge of the Bochner integral, but for the sake of the reader we will not do this.

Before we discuss the representation theory of $L_1(G, \mathfrak{A}, \alpha)$, we get most of the tedious technical results out of the way.

Lemma 5. *Let (A, G, α) be a C^*-dynamical system. Then there exists a bounded homomorphism $\lambda : \mathfrak{A} \to \mathcal{B}(L_1(G, \mathfrak{A}, \alpha))$ such that $\lambda(A)f \in C_c(G, \mathfrak{A})$ for all $f \in C_c(G, \mathfrak{A})$ and $A \in \mathfrak{A}$ with $(\lambda(A)f)(t) = Af(t)$ for all $t \in G$, $(\lambda(A)f)^*(s) = f^*(s)\alpha_s(A^*)$ for all $f, g \in C_c(G, \mathfrak{A})$, $s \in G$, and $A \in \mathfrak{A}$, $\lambda(A)(f * g) = (\lambda(A)f) * g$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$, and $(\lambda(A)f)^* * g = f^* * (\lambda(A^*)g)$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$.*

Proof: For each $A \in \mathfrak{A}$ and $f \in C_c(G, \mathfrak{A})$ define $(\lambda(A)f)(s) = Af(s)$ for all $s \in G$. Clearly $\lambda(A)f \in C_c(G, \mathfrak{A})$ and $\lambda(A)$ is linear for all $A \in \mathfrak{A}$. Notice

$$\|\lambda(A)f\|_1 = \int_G \|Af(t)\|\, dt \leq \|A\|\, \|f\|_1$$

so $\lambda(A)$ extends to a bounded linear map on $L_1(G, \mathfrak{A}, \alpha)$. Define $\lambda : \mathfrak{A} \to \mathcal{B}(L_1(G, \mathfrak{A}, \alpha))$ by $A \mapsto \lambda(A)$. Clearly λ is linear and multiplicative when restricted to $C_c(G, \mathfrak{A})$ and thus λ is a homomorphism.

Notice that

$$((\lambda(A)f) * g)(s) = \int_G Af(t)\alpha_t(g(t^{-1}s))dt = A\int_G f(t)\alpha_t(g(t^{-1}s))dt = A((f * g)(s)) = (\lambda(A)(f * g))(s)$$

for all $f, g \in C_c(G, \mathfrak{A})$. Thus by continuity and density of $C_c(G, \mathfrak{A})$ in $L_1(G, \mathfrak{A}, \alpha)$ $(\lambda(A)f) * g = \lambda(A)(f * g)$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$. Moreover

$$(\lambda(A)f)^*(s) = \Delta(s^{-1})\alpha_s((\lambda(A)f)(s^{-1}))^* = \Delta(s^{-1})\alpha_s(Af(s^{-1}))^* = \Delta(s^{-1})\alpha_s(f(s^{-1}))^*\alpha_s(A^*) = f^*(s)\alpha_s(A^*)$$

Lastly we notice

$$\begin{aligned} ((\lambda(A)f)^* * g)(s) &= \int_G (\lambda(A)f)^*(t)\alpha_t(g(t^{-1}s))dt \\ &= \int_G f^*(t)\alpha_t(A^*)\alpha_t(g(t^{-1}s))dt \\ &= \int_G f^*(t)\alpha_t((\lambda(A)g)(t^{-1}s))dt = (f^* * (\lambda(A^*)g))(s) \end{aligned}$$

for all $f, g \in C_c(G, \mathfrak{A})$. Thus by continuity and density of $C_c(G, \mathfrak{A})$ in $L_1(G, \mathfrak{A}, \alpha)$ $(\lambda(A)f)^* * g = f^* * (\lambda(A^*)g)$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$. $\square$

Lemma 6. *Let (A, G, α) be a C^*-dynamical system. For each $s \in G$ the map $g \mapsto g_s$ on $C_c(G, \mathfrak{A})$ (where $g_s(t) = g(s^{-1}t)$ for all $t \in G$) extends to an invertible linear isometry on $L_1(G, \mathfrak{A}, \alpha)$. Moreover the map $t \mapsto g_t$ is left uniformly continuous for all $g \in L_1(G, \mathfrak{A}, \alpha)$.*

Proof: It is clear that $g \mapsto g_s$ is a linear isometry on $C_c(G, \mathfrak{A})$ and thus extends to a linear isometry on $L_1(G, \mathfrak{A}, \alpha)$. Since $(g_s)_{s^{-1}} = g = (g_{s^{-1}})_s$ for all $g \in C_c(G, \mathfrak{A})$, $g \mapsto g_s$ is invertible.

Let $f \in L_1(G, \mathfrak{A}, \alpha)$ be arbitrary and let $\epsilon > 0$. Choose $g \in C_c(G, \mathfrak{A})$ such that $\|g - f\|_1 < \frac{\epsilon}{3}$. Let $K = \mathrm{supp}(g)$ which is compact. Since g is uniformly continuous, there exists a neighbourhood V of e such that

$$\|g - g_s\|_\infty < \frac{\epsilon}{6(\mu(K) + 1)}$$

for all $s \in V$. Since the support of $g - g_s$ is $K \cup sK$, we have that

$$\begin{aligned} \|g - g_s\|_1 &\leq \int_G \|g(t) - g_s(t)\|\, dt \\ &\leq \int_{K \cup sK} \|g - g_s\|_\infty\, dt \\ &\leq \mu(K \cup sK)\, \|g - g_s\|_\infty \\ &\leq 2\mu(K)\frac{\epsilon}{6(\mu(K) + 1)} < \frac{\epsilon}{3} \end{aligned}$$

for all $s \in V$. Therefore for all $s \in V$,

$$\|f - f_s\|_1 \le \|f - g\|_1 + \|g - g_s\|_1 + \|g_s - f_s\|_1 < \epsilon$$

since $g - f \mapsto g_s - f_s$ is a linear isometry.

Lastly, notice that $h_s - h_t = (h - h_{s^{-1}t})_s$ for all $h \in C_c(G, \mathfrak{A})$ and thus for all $h \in L_1(G, \mathfrak{A}, \alpha)$ by continuity. Therefore $\|f_s - f_t\|_1 = \|(f - f_{s^{-1}t})_s\|_1 < \epsilon$ for all $s^{-1}t \in V$. Hence the map is left uniformly continuous. $\square$

Lemma 7. *Let (A, G, α) be a C^*-dynamical system. For each $s \in G$ there exists an invertible linear isometry $\widetilde{\alpha_s} \in \mathcal{B}(L_1(G, \mathfrak{A}, \alpha))$ defined by $(\widetilde{\alpha_s}(f))(t) = \alpha_s(f(t))$ for all $t \in G$ and $f \in C_c(G, \mathfrak{A})$. Moreover we have the following properties:*

1. *If $A \in \mathfrak{A}$ and $f \in L_1(G)$, then $\widetilde{\alpha_s}(Af) = \alpha_s(A)f$*
2. *$\widetilde{\alpha_t}(((\widetilde{\alpha_{t^{-1}}}f) * g)_t) = f_t * g$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$*
3. *$\widetilde{\alpha_s}((\widetilde{\alpha}_t(g_t))_s) = \widetilde{\alpha}_{st}(g_{st})$ for all $s, t \in G$ and $g \in L_1(G, \mathfrak{A}, \alpha)$.*
4. *$\widetilde{\alpha_t}((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))_t) = \lambda(\alpha_t(A))g$ for all $A \in \mathfrak{A}$, $t \in G$, and $g \in L_1(G, \mathfrak{A}, \alpha)$.*

Proof: Define $\widetilde{\alpha_s} : C_c(G, \mathfrak{A}) \to C_c(G, \mathfrak{A})$ by $(\widetilde{\alpha_s}(f))(t) = \alpha_s(f(t))$ for all $t \in G$ and $f \in C_c(G, \mathfrak{A})$. Since α_s is an isometry, $\widetilde{\alpha_s}(f) \in C_c(G, \mathfrak{A})$ for all $f \in C_c(G, \mathfrak{A})$. Clearly $\widetilde{\alpha_s}$ is linear and

$$\|\widetilde{\alpha_s}(f)\|_1 = \int_G \|\alpha_s(f(t))\| \, dt = \int_G \|f(t)\| \, dt = \|f\|$$

as α_s is an isometry. Whence $\widetilde{\alpha_s}$ extends to a linear isometry on $L_1(G, \mathfrak{A}, \alpha)$. Since $\widetilde{\alpha_s} \circ \widetilde{\alpha_{s^{-1}}} = Id$ on $C_c(G, \mathfrak{A})$, $\widetilde{\alpha_s}^{-1} = \widetilde{\alpha_{s^{-1}}}$.

If $g \in L_1(G, \mathfrak{A}, \alpha)$ is of the form $g(t) = Af(t)$ for some $f \in L_1(G)$ and $A \in \mathfrak{A}$, then there exists $f_n \in C_c(G)$ such that $\lim_{n\to\infty} \|g - Af_n\|_1 = 0$. Since $\widetilde{\alpha_s}(Af_n) = \alpha_s(A)f_n$ for all n, we have $\widetilde{\alpha_s}(g) = \alpha_s(A)g$ by continuity and the definition of how we are viewing $\mathfrak{A} \cdot L_1(G) \subseteq L_1(G, \mathfrak{A}, \alpha)$.

Next suppose $f, g \in C_c(G, \mathfrak{A})$. Then

$$\begin{aligned}
((f_t) * g)(s) &= \int_G f_t(r)\alpha_r(g(r^{-1}s))dr \\
&= \int_G f(t^{-1}r)\alpha_r(g(r^{-1}s))dr \\
&= \int_G f(r)\alpha_{tr}(g(r^{-1}t^{-1}s))dr \\
&= \int_G \alpha_t(\alpha_{t^{-1}}(f(r))\alpha_r(g(r^{-1}t^{-1}s)))dr \\
&= \alpha_t\left(\int_G \alpha_{t^{-1}}(f(r))\alpha_r(g(r^{-1}t^{-1}s))dr\right) \\
&= \alpha_t\left(\int_G (\widetilde{\alpha_{t^{-1}}}(f))(r)\alpha_r(g(r^{-1}t^{-1}s))dr\right) \\
&= \alpha_t\left(((\widetilde{\alpha_{t^{-1}}}(f)) * g)(t^{-1}s)\right) \\
&= \alpha_t\left(((\widetilde{\alpha_{t^{-1}}}(f)) * g)_t(s)\right) \\
&= (\widetilde{\alpha}_t\left(((\widetilde{\alpha_{t^{-1}}}(f)) * g)_t\right))(s)
\end{aligned}$$

Thus $(f_t) * g = \widetilde{\alpha}_t\left(((\widetilde{\alpha_{t^{-1}}}(f)) * g)_t\right)$ for all $t \in G$, $f, g \in C_c(G, \mathfrak{A})$. Thus $\widetilde{\alpha_t}(((\widetilde{\alpha_{t^{-1}}}f) * g)_t) = f_t * g$ for all $f, g \in L_1(G, \mathfrak{A}, \alpha)$ by continuity and density.

Next suppose $g \in C_c(G, \mathfrak{A})$ and $s, t \in G$. Then

$$\begin{aligned}
\widetilde{\alpha_s}((\widetilde{\alpha}_t(g_t))_s)(r) &= \alpha_s((\widetilde{\alpha}_t(g_t))_s(r)) \\
&= \alpha_s((\widetilde{\alpha}_t(g_t))(s^{-1}r)) \\
&= \alpha_s(\alpha_t(g_t(s^{-1}r))) \\
&= \alpha_{st}(g(t^{-1}s^{-1}r)) \\
&= \alpha_{st}(g_{st}(r)) = \widetilde{\alpha_{st}}(g_{st})(r)
\end{aligned}$$

for all $r \in G$. Whence $\widetilde{\alpha_s}((\widetilde{\alpha}_t(g_t))_s) = \widetilde{\alpha}_{st}(g_{st})$ for all $s, t \in G$ and $g \in C_c(G, \mathfrak{A})$. The general result holds by the density of $C_c(G, \mathfrak{A})$ in $L_1(G, \mathfrak{A}, \alpha)$ and continuity.

Lastly suppose $g \in C_c(G, \mathfrak{A})$, $t \in G$, and $A \in \mathfrak{A}$. Then for all $s \in G$

$$\begin{aligned}
\widetilde{\alpha}_t((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))_t)(s) &= \alpha_t((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))_t(s)) \\
&= \alpha_t((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))(t^{-1}s)) \\
&= \alpha_t(A((\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}}))(t^{-1}s))) \\
&= \alpha_t(A)\alpha_t(\alpha_{t^{-1}}(g_{t^{-1}}(t^{-1}s))) \\
&= \alpha_t(A)\alpha_t(\alpha_{t^{-1}}(g(s))) \\
&= \alpha_t(A)g(s) \\
&= (\lambda(\alpha_t(A))g)(s)
\end{aligned}$$

Thus $\widetilde{\alpha}_t((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))_t) = \lambda(\alpha_t(A))g$ for all $g \in C_c(G, \mathfrak{A})$, $t \in G$, and $A \in \mathfrak{A}$. By continuity, $\widetilde{\alpha}_t((\lambda(A)(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}})))_t) = \lambda(\alpha_t(A))g$ for all $g \in L_1(G, \mathfrak{A}, \alpha)$, $t \in G$, and $A \in \mathfrak{A}$. □

The last extremely technical facts we need before moving on is the knowledge of a particular bounded approximate identity of $L_1(G, \mathfrak{A}, \alpha)$ and corollaries of the proof. The following was already proven for $(\mathbb{C}, G, 1)$ and is also trivial when $\mathfrak{A}$ is unital.

Proposition 8. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system, $\{f_\beta\}_\Lambda$ be the norm one approximate identity of positive, self-adjoint functions for $L_1(G)$ from Theorem 2, and let $\{E_\beta\}_\Lambda$ be any C^*-bounded approximate identity for $\mathfrak{A}$ (where both approximate identities are re-indexed by $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$). If we define $g_\beta = E_\beta f_\beta$, $\{g_\beta\}_\Lambda$ is a left bounded (by 1) approximate identity for $L_1(G, \mathfrak{A}, \alpha)$. In addition, for any fixed $s \in G$, $\{\alpha_s(E_\beta)\}_\Lambda$ is a C^*-bounded approximate identity for $\mathfrak{A}$ so if we define $h_\beta = \widetilde{\alpha_s}(g_\beta) = \alpha_s(E_\beta)f_\beta$, then $\{h_\beta\}_\Lambda$ is a left bounded (by 1) approximate identity for $L_1(G, \mathfrak{A}, \alpha)$.*

Proof: Notice

$$\|g_\beta\|_1 = \int_G \|E_\beta f_\beta(t)\| \, dt = \int_G \|E_\beta\| \, f_\beta(t) dt = \|E_\beta\| \leq 1$$

as claimed. Suppose $g \in C_c(G, \mathfrak{A})$ has the form $g(t) = Af(t)$ for some $A \in \mathfrak{A}$ and $f \in C_c(G)$. Since $g_\beta = E_\beta f_\beta$ and f_β is the $L_1(G)$ limit of positive elements of $C_c(G)$ with support contained in $\text{supp}(f_\beta)$ (by Urysohn's Lemma), there exists a net $\{f_{\beta,\gamma}\}_\gamma$ of elements of $C_c(G)$ such that $\text{supp}(f_{\beta,\gamma}) \subseteq \text{supp}(f_\beta)$, $\lim_\gamma \|f_{\beta,\gamma} - f_\beta\|_1 = 0$ (and thus $\lim_\gamma \|E_\beta f_{\beta,\gamma} - g_\beta\|_1 = 0$) and $\|f_{\beta,\gamma}\|_1 = 1$ (by renormalizing our net). But then

$$
\begin{aligned}
\|g_\beta * g - g\|_1 &= \lim_\gamma \|(E_\beta f_{\beta,\gamma}) * g - g\|_1 \\
&= \lim_\gamma \int_G \|((E_\beta f_{\beta,\gamma}) * g)(s) - g(s)\| \, ds \\
&= \lim_\gamma \int_G \left\| \int_G E_\beta f_{\beta,\gamma}(t)\alpha_t(Af(t^{-1}s))dt - Af(s) \right\| ds \\
&= \int_G \left\| \int_G E_\beta \alpha_t(A) f_{\beta,\gamma}(t) f(t^{-1}s)dt - \int_G A f_{\beta,\gamma}(t) f(s) dt \right\| ds \\
&\leq \limsup_\gamma \int_G \left\| \int_G E_\beta \alpha_t(A) f_{\beta,\gamma}(t) f(t^{-1}s)dt - \int_G E_\beta \alpha_t(A) f_{\beta,\gamma}(t) f(s) dt \right\| ds \\
&\quad + \int_G \left\| \int_G E_\beta \alpha_t(A) f_{\beta,\gamma} f(s)dt - \int_G A f_{\beta,\gamma}(t) f(s) dt \right\| ds \\
&\leq \limsup_\gamma \|A\| \int_G \int_G |f_{\beta,\gamma}(t) f(t^{-1}s) - f_{\beta,\gamma}(t) f(s)| dt ds \\
&\quad + \int_G \int_G \|E_\beta \alpha_t(A) - A\| \, |f_{\beta,\gamma}(t) f(s)| dt ds \\
&\leq \limsup_\gamma \|A\| \int_G \int_G f_{\beta,\gamma}(t) |f(t^{-1}s) - f(s)| ds dt \\
&\quad + \int_G \int_G (\|E_\beta \alpha_t(A) - E_\beta A\| + \|E_\beta A - A\|) |f(s)| f_{\beta,\gamma}(t) ds dt \\
&\leq \limsup_\gamma \|A\| \int_G f_{\beta,\gamma}(t) \sup\{\|f_r - f\|_1 \mid r \in supp(f_{\beta,\gamma})\} dt \\
&\quad + \int_G (\|\alpha_t(A) - A\| + \|E_\beta A - A\|) \, \|f\|_1 \, f_{\beta,\gamma}(t) dt \\
&\leq \|A\| \sup\{\|f_r - f\|_1 \mid r \in supp(f_\beta)\} \\
&\quad + \|f\|_1 \sup\{\|\alpha_r(A) - A\| \mid r \in supp(f_\beta)\} + \|f\|_1 \|E_\beta A - A\|
\end{aligned}
$$

Since each of our final terms converges to zero over β (since f_β have compact support indexed such that if U is any open neighbourhood of e, there exists a β' such that $supp(f_\beta) \subseteq U$ for all $\beta \geq \beta'$, $t \mapsto f_t$ is a continuous map from G to $L_1(G)$, $t \mapsto \alpha_t(A)$ is continuous on G (and $\alpha_e(A) = A$), and E_β is a bounded approximate identity for $\mathfrak{A}$), we obtain $\lim_\Lambda g_\beta * g = g$.

Next we claim that $\overline{span\{g \in C_c(G) \mid g(t) = Af(t) \text{ for all } t \in G, A \in \mathfrak{A}, f \in C_c(G)\}}^{\|\cdot\|_\infty} = C_c(G, \mathfrak{A})$. To see this, suppose $f \in C_c(G, \mathfrak{A})$. Let $\epsilon > 0$ and let $K = supp(f)$ which is compact. Since X is a locally compact Hausdorff space there exists a compact set K' that is a neighbourhood for each element of K. Moreover K and $\overline{X \setminus K'}$ are disjoint closed set so there exists a continuous function $f_0 : X \to [0,1]$ such that $f_0|_K = 1$ and $\text{supp}(f_0) = K'$ by Urysohn's Lemma. Thus $f(x) = f(x)f_0(x)$ for all $x \in G$.

Consider the open sets $U_x = \{y \in G \mid \|f(x) - f(y)\| < \epsilon\}$ where $x \in K'$. Since K' is compact, there exist a finite set $\{x_1, \ldots, x_n\} \subseteq K'$ such that $K' \subseteq \bigcup_{i=1}^n U_{x_i}$. Hence for all $x \in K'$ there exists an x_i such that $\|f(x) - f(x_i)\| < \epsilon$.

Choose a partition of unity for K'; that is continuous functions $h_i : G \to [0,1]$ such that $h_i|_{K' \setminus U_{x_i}} = 0$, $\sum_{i=1}^n h_i = 1$ on K' and $\sum_{i=1}^n h_i(x) \in [0,1]$ for all $x \in G \setminus K'$ (i.e. create a partition of unity on the compact Hausdorff space K'. By the Tietz Extension Theorem, we can extend each h_i to a continuous function on G with range in $[0,1]$. Since $\sum h_i$ is continuous on K' and equal to 1, there exists a compact neighbourhood F of K' such that $\sum h_i$ is greater than $\frac{1}{2}$ on F. By Urysohn's Lemma, there exists a function $w : G \to [0,1]$ that is 1 on K' and 0 on $G \setminus F$. Then the functions $(\sum h_i(x))^{-1} w(x) h_i(x)$ (if $x \in F$, 0 elsewhere) will be continuous and have the desired properties).

Let $h = \sum_{i=1}^n f(x_i) f_0 h_i \in C_c(G, \mathfrak{A})$. If $x \in K'$

$$\begin{aligned}\|f(x) - h(x)\| &= \left\| f_0(x) f(x) \left(\sum_{i=1}^n h_i(x) \right) - \sum_{i=1}^n f_0(x) h_i(x) f(x_i) \right\| \\ &= \left\| \sum_{i=1}^n f_0(x)(f(x) - f(x_i)) h_i(x) \right\| \\ &\leq \sum_{i=1}^n \|f(x) - f(x_i)\| f_0(x) h_i(x) \\ &\leq \sum_{i=1}^n \epsilon h_i(x) = \epsilon\end{aligned}$$

as $0 \leq f_0(x) \leq 1$, if $x \notin U_{x_i}$ $h_i(x) = 0$, and if $x \in U_{x_i}$ $\|f(x) - f(x_i)\| \leq \epsilon$. Moreover if $x \notin K'$ then $f(x) = 0$ and $h(x) = \sum_{i=1}^n f_0(x) h_i(x) f(x_i) = 0$ as $f_0(x) = 0$. Therefore $\|f - h\| \leq \epsilon$. Whence $span\{g \in C_c(G) \mid g(t) = Af(t) \text{ for all } t \in G, A \in \mathfrak{A}, f \in C_c(G)\}$ is dense in $C_c(G, \mathfrak{A})$.

Fix $g \in C_c(G, \mathfrak{A})$. By the above proof, for any compact neighbourhood K' of $supp(g)$ and any $\epsilon > 0$ there exists $\{f_i\}_{i=1}^n \in C_c(G)$ with $supp(f_i) \subseteq K'$ and $\{A_i\}_{i=1}^n \subset \mathfrak{A}$ so that $\|g - \sum_{i=1}^n A_i f_i\| < \frac{\epsilon}{\mu(K')+1}$ (compact subsets have finite measure). Therefore

$$\left\| g - \sum_{i=1}^n A_i f_i \right\|_1 = \int_G \left\| g(t) - \sum_{i=1}^n A_i f_i(t) \right\| dt \leq \epsilon$$

Thus $\overline{span\{g \in C_c(G) \mid g(t) = Af(t) \text{ for all } t \in G, A \in \mathfrak{A}, f \in C_c(G)\}}^{\|\cdot\|_1} \supseteq \overline{C_c(G, \mathfrak{A})}^{\|\cdot\|_1} = L_1(G, \mathfrak{A}, \alpha)$.

Fix $g \in L_1(G, \mathfrak{A}, \alpha)$ and let $\epsilon > 0$. Choose $\{f_i\}_{i=1}^n \in C_c(G)$ and $\{A_i\}_{i=1}^n \subset \mathfrak{A}$ such that if $h = \sum_{i=1}^n A_i f_i$ then $\|g - h\|_1 < \frac{\epsilon}{3}$. Since $\lim_\Lambda \|g_\beta * (A f_i) - A f_i\|_1 = 0$, by linearity there exists an $\beta' \in \Lambda$ such that for all $\beta \geq \beta'$, $\|g_\beta * h - h\|_1 < \frac{\epsilon}{3}$. Thus for all $\beta \geq \beta'$

$$\begin{aligned}\|g_\beta * g - g\|_1 &\leq \|g_\beta * g - g_\beta * h\|_1 + \|g_\beta * h - h\|_1 + \|h - g\|_1 \\ &\leq \|g_\beta\|_1 \|g - h\|_1 + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

Thus $\lim_\Lambda g_\beta * g = g$ for all $g \in L_1(G, \mathfrak{A}, \alpha)$. $\square$

Corollary 9. *Let (A, G, α) be a C*-dynamical system. Then $t \mapsto \widetilde{\alpha}_t(g_t)$ is continuous for all $g \in L_1(G, \mathfrak{A}, \alpha)$.*

Proof: Suppose $f \in C_c(G)$, $A \in \mathfrak{A}$, and $s, t \in G$. If $g = Af$ then

$$\begin{aligned}\|\widetilde{\alpha_s}(g_s) - \widetilde{\alpha}_t(g_t)\|_1 &= \|\widetilde{\alpha_s}(g_s) - \widetilde{\alpha_s}(g_t)\|_1 + \|\widetilde{\alpha_s}(g_t) - \widetilde{\alpha}_t(g_t)\|_1 \\ &\leq \|g_s - g_t\|_1 + \int_G \|\alpha_s(A) f(r) - \alpha_t(A) f(r)\| \, dr \\ &\leq \|g_s - g_t\|_1 + \|\alpha_s(A) - \alpha_t(A)\| \, \|f\|_1\end{aligned}$$

Since $\lim_{s \to t} \|g_s - g_t\|_1 = 0$ by left uniform continuity of $t \mapsto g_t$ and $\lim_{s \to t} \|\alpha_s(A) - \alpha_t(A)\| = 0$ by the definition of a C*-dynamical system, $\lim_{s \to t} \|\widetilde{\alpha_s}(g_s) - \widetilde{\alpha}_t(g_t)\|_1 = 0$. Thus $t \mapsto \widetilde{\alpha}_t(g_t)$ is continuous in this case.

By linearity $t \mapsto \widetilde{\alpha}_t(f_t)$ is continuous for all $f \in span\{g \in C_c(G, \mathfrak{A}) \mid g = Af, A \in \mathfrak{A}, f \in C_c(G)\}$. Let $g \in L_1(G, \mathfrak{A}, \alpha)$ and fix $t \in G$ and $\epsilon > 0$. Since $span\{g \in C_c(G, \mathfrak{A}) \mid g = Af, A \in \mathfrak{A}, f \in C_c(G)\}$ is dense in $L_1(G, \mathfrak{A}, \alpha)$ there exists an $f \in span\{g \in C_c(G, \mathfrak{A}) \mid g = Af, A \in \mathfrak{A}, f \in C_c(G)\}$ so that $\|g - f\|_1 < \frac{\epsilon}{3}$. By above there exists a neighbourhood V of t such that if $s \in V$ then $\|\widetilde{\alpha_s}(f_s) - \widetilde{\alpha}_t(f_t)\|_1 < \frac{\epsilon}{3}$. Thus if $s \in V$ then

$$\begin{aligned}\|\widetilde{\alpha_s}(g_s) - \widetilde{\alpha}_t(g_t)\|_1 &\leq \|\widetilde{\alpha_s}(g_s) - \widetilde{\alpha_s}(f_s)\|_1 + \|\widetilde{\alpha_s}(f_s) - \widetilde{\alpha}_t(f_t)\|_1 + \|\widetilde{\alpha}_t(f_t) - \widetilde{\alpha}_t(g_t)\|_1 \\ &\leq \|g_s - f_s\|_1 + \frac{\epsilon}{3} + \|f_t - g_t\|_1 \\ &= \|g - f\|_1 + \frac{\epsilon}{3} + \|f - g\|_1 \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

Whence $\lim_{s\to t} \widetilde{\alpha_s}(g_s) = \widetilde{\alpha}_t(g_t)$ so $t \mapsto \widetilde{\alpha}_t(g_t)$ is continuous for all $g \in L_1(G, \mathfrak{A}, \alpha)$. □

Corollary 10. *Let (A, G, α) be a C*-dynamical system. If $f, g \in C_c(G, \mathfrak{A})$ then $\int_G \lambda(f(t))\widetilde{\alpha}_t(g_t)dt = f * g$.*

Proof: Notice that the left hand side of the desired equation is actually and element of $C_c(G, \mathfrak{A})$ since $t \mapsto \lambda(f(t))\widetilde{\alpha}_t(g_t)$ is a continuous map from G to $C_c(G, \mathfrak{A})$ by the above lemmas and corollary. Therefore $\int_G \lambda(f(t))\widetilde{\alpha}_t(g_t)dt$ is continuous being a Bochner integral of a function with values in $C_0(G, \mathfrak{A})$ (continuous functions from G to $\mathfrak{A}$ that vanish at infinity) which is complete. Moreover, we obtain for all $s \in G$ that

$$\begin{aligned}\left(\int_G \lambda(f(t))\widetilde{\alpha}_t(g_t)dt\right)(s) &= \int_G (\lambda(f(t))\widetilde{\alpha}_t(g_t))(s)dt \\ &= \int_G f(t)(\widetilde{\alpha}_t(g_t)(s))dt \\ &= \int_G f(t)\alpha_t(g_t(s))dt \\ &= \int_G f(t)\alpha_t(g(t^{-1}s))dt \\ &= (f * g)(s)\end{aligned}$$

Whence the result follows. □

We are finally ready to explore the representation theory of $L_1(G, \mathfrak{A}, \alpha)$. We recall that a *-homomorphism $\sigma : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ is contractive being a *-homomorphism from a Banach *-algebra to a C*-algebra. Moreover we recall that a *-homomorphism $\sigma : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ is said to be non-degenerate if $\mathcal{H} = \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}}$ and is said to have trivial null space if $\sigma(L_1(G, \mathfrak{A}, \alpha))\xi = \{0\}$ implies $\xi = 0$. We begin with the following proposition showing it suffices to consider non-degenerate representations. Note that the proposition generalizes to any Banach *-algebra with a left bounded approximate identity.

Proposition 11. *Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system and $\mathcal{H}$ a Hilbert space. Suppose $\sigma : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism and let $\mathcal{K} = \overline{\sigma(\mathfrak{A})\mathcal{H}}$. Let $\sigma' : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{K})$ be defined by $\sigma'(f) = \sigma(f)|_{\mathcal{K}}$ for all $f \in L_1(G, \mathfrak{A}, \alpha)$. Then σ' is a non-degenerate representation of $L_1(G, \mathfrak{A}, \alpha)$. In fact, $\sigma(f)h = 0$ for all $h \in \mathcal{K}^\perp$. Hence $\|\sigma'(f)\| = \|\sigma(f)\|$ for all $f \in L_1(G, \mathfrak{A}, \alpha)$. Moreover, if σ is faithful then σ' is faithful. Lastly σ is non-degenerate if and only if $\sigma(L_1(G, \mathfrak{A}, \alpha))$ has trivial null space.*

Proof: Firstly if $\mathcal{K}$ is a closed subspace then σ' is a well-defined representation as $\mathcal{K}$ is a reducing subspace for $\sigma(L_1(G, \mathfrak{A}, \alpha))$. We shall show that σ' is non-degenerate by showing $\overline{\sigma'(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}} = \mathcal{K}$ and this will also show us that $\mathcal{K}$ is a subspace. Let $k \in \mathcal{K} = \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}}$. Thus there exists $f_n \in L_1(G, \mathfrak{A}, \alpha)$ and $h_n \in \mathcal{H}$ such that $k = \lim_n \sigma(f_n)h_n$. By Proposition 8, $L_1(G, \mathfrak{A}, \alpha)$ has a left bounded (by 1) approximate identity $\{f_\beta\}_\Lambda$. Let $\epsilon > 0$. Since $k = \lim_n \sigma(f_n)h_n$ there exists an $N \in \mathbb{N}$ such that $\|k - \sigma(f_N)h_N\| \leq \frac{\epsilon}{3}$ and since $\{f_\beta\}_\Lambda$ is a left bounded approximate identity for $L_1(G, \mathfrak{A}, \alpha)$, there exists a $\beta' \in \Lambda$ such that for all $\beta \geq \beta'$, $\|f_\beta f_N - f_N\|_1 \leq \frac{\epsilon}{3(\|h_N\|+1)}$. Since σ is a contraction $\|\sigma(f_\beta f_N) - \sigma(f_N)\| \leq \frac{\epsilon}{3(\|h_N\|+1)}$. Hence for all

$\beta \geq \beta'$

$$\begin{aligned}\|k - \sigma(f_\beta)k\| &\leq \|k - \sigma(f_N)h_N\| + \|\sigma(f_N)h_N - \sigma(f_\beta f_N)h_N\| + \|\sigma(f_\beta f_N)h_N - \sigma(e_\beta)k\| \\ &\leq \frac{\epsilon}{3} + \|\sigma(f_N) - \sigma(f_\beta f_N)\| \, \|h_N\| + \|\sigma(f_\beta)\| \, \|\sigma(f_N)h_N - k\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3(\|h_N\|+1)} \|h_N\| + 1\frac{\epsilon}{3} \leq \epsilon\end{aligned}$$

Hence $\sigma(f_\beta)k \to k$ so $k \in \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}} = \overline{\sigma'(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}}$. Moreover, if $k_1, k_2 \in \mathcal{K}$ and $\alpha \in \mathbb{C}$, then $\sigma(f_\beta)(\alpha k_1 + k_2) \to \alpha k_1 + k_2$ so that $\alpha k_1 + k_2 \in \mathcal{K}$. Hence $\mathcal{K}$ is a subspace. Moreover

$$\mathcal{K} \subseteq \overline{\sigma'(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}} = \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}} \subseteq \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}} = \mathcal{K}$$

Hence $\mathcal{K} = \overline{\sigma'(L_1(G, \mathfrak{A}, \alpha))\mathcal{K}}$ so σ' is a well defined, non-degenerate representation.

Suppose $h \in \mathcal{K}^\perp$. We claim that $\sigma(f)h = 0$ for all $f \in L_1(G, \mathfrak{A}, \alpha)$. To see this, suppose $k \in \mathcal{K}$. Then

$$\langle \sigma(f)h, k\rangle = \langle h, \sigma(f^*)k\rangle = 0$$

since $h \in \mathcal{K}^\perp$ and $\sigma(f^*)k \in \mathcal{K}$. Similarly, if $k \in \mathcal{K}^\perp$ then $\langle \sigma(f)h, k\rangle = 0$. Hence $\sigma(f)h = 0$ as claimed. Consequently $\|\sigma(f)\| = \|\sigma'(f)\|$ as $\sigma'(f) = \sigma(f)|_\mathcal{K}$. If σ is faithful and $\sigma'(f_1) = \sigma'(f_2)$, then $\sigma(f_1)|_\mathcal{K} = \sigma(f_2)|_\mathcal{K}$ and since $\sigma(f_1)|_{\mathcal{K}^\perp} = \sigma(f_2)|_{\mathcal{K}^\perp} = 0$, $\sigma(f_1) = \sigma(f_2)$ so $f_1 = f_2$. Hence σ' is faithful.

Lastly suppose σ is non-degenerate so that $\mathcal{H} = \mathcal{K} = \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}}$. Suppose $\sigma(L_1(G, \mathfrak{A}, \alpha))k = \{0\}$. Then, by repeating the first part of this proof, we obtain that $\|k\| = \|k - \sigma(f_\beta)k\| < \epsilon$ for all $\epsilon > 0$ with a suitable choice of β. Hence $k = 0$ so that $\sigma(L_1(G, \mathfrak{A}, \alpha))$ has trivial null space. Now suppose $\sigma(L_1(G, \mathfrak{A}, \alpha))$ has trivial null-space and suppose further that $\mathcal{H} \neq \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}}$. Then there exists a $k \in \mathcal{H} \setminus \{0\}$ such that $k \in \overline{\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}}^\perp$. However, from earlier work, this implies that $\sigma(f)k = 0$ for all $f \in L_1(G, \mathfrak{A}, \alpha)$ which contradicts the fact that $\sigma(L_1(G, \mathfrak{A}, \alpha))$ had trivial null-space. □

We would like to be able to complete $L_1(G, \mathfrak{A}, \alpha)$ with respect to a C*-norm to create a C*-algebra. To do this, we will need to know that a *-representation of $L_1(G, \mathfrak{A}, \alpha)$ exists. This leads us to the concept of a covariant representations of a C*-dynamical system.

Definition 12. Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. A covariant representation of $(\mathfrak{A}, G, \alpha)$ is a pair (π, U) where $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism and $t \mapsto U_t \in \mathcal{B}(\mathcal{H})$ is a unitary representation of G (which we require to be SOT-continuous) so that $U_t\pi(A)U_t^* = \pi(\alpha_t(A))$ for all $t \in G$ and $A \in \mathfrak{A}$.

Proposition 13. *Every C*-dynamical system has a covariant representation.*

Proof: Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system and let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be any *-homomorphism. Consider the Hilbert space $\mathcal{K} = L_2(G, \mathcal{H})$ (which (if the reader is not familiar with) can be viewed as the completion of all continuous functions $f \in C_c(G, \mathcal{H})$ with $\int_G \|f(t)\|_\mathcal{H}^2 \, dt < \infty$ with respect to the inner product $\langle f, g\rangle = \int_G \langle f(t), g(t)\rangle_\mathcal{H} dt)$. Define $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(L_2(G, \mathcal{H}))$ by $(\tilde{\pi}(A)f)(s) = \pi(\alpha_{s^{-1}}(A))f(s)$ and $\Lambda_t \in \mathcal{B}(L_2(G, \mathcal{H}))$ by $(\Lambda_t f)(s) = f(t^{-1}s)$ for all $f \in C_c(G, \mathcal{H})$ and extending by continuity. It is necessary to check $\tilde{\pi}$ and Λ_t are well-defined and have the desired properties.

Fix $A \in \mathfrak{A}$. We notice if $f \in C_c(G, \mathcal{H})$, then since $s \mapsto \alpha_{s^{-1}}(A)$ is continuous for all $A \in \mathfrak{A}$, $\tilde{\pi}(A)f \in C_c(G, \mathcal{H})$. Clearly $\tilde{\pi}(A)$ is linear. Moreover

$$\int_G \|(\tilde{\pi}(A)f)(s)\| \, ds = \int_G \|\pi(\alpha_{s^{-1}}(A))f(s)\| \, ds \leq \|A\| \, \|f\|_2$$

Whence $\tilde{\pi}(A)$ extends to a bounded linear operator on $L_2(G, \mathcal{H})$. Next we notice that

$$(\tilde{\pi}(A)\tilde{\pi}(B)f)(s) = \pi(\alpha_{s^{-1}}(A))((\tilde{\pi}(B)f)(s)) = \pi(\alpha_{s^{-1}}(A))\pi(\alpha_{s^{-1}}(B))f(s) = \pi(\alpha_{s^{-1}}(AB))f(s) = (\tilde{\pi}(AB)f)(s)$$

for all $A, B \in \mathfrak{A}$ and $f \in C_c(G, \mathcal{H})$. Thus $\tilde{\pi}$ extends to a homomorphism. Lastly we notice that

$$\begin{aligned}\langle \tilde{\pi}(A)^* f, g\rangle &= \langle f, \tilde{\pi}(A) g\rangle \\ &= \int_G \langle f(s), \pi(\alpha_{s^{-1}}(A)) g(s)\rangle ds \\ &= \int_G \langle \pi(\alpha_{s^{-1}}(A^*)) f(s), g(s)\rangle ds \\ &= \langle \tilde{\pi}(A^*) f, g\rangle\end{aligned}$$

for all $A \in \mathfrak{A}$ and $f, g \in C_c(G, \mathcal{H})$. Whence $\tilde{\pi}$ extends to a *-homomorphism.

Fix $t \in G$. Since $s \mapsto (\Lambda_t f)(s) = f(t^{-1}s)$ is a continuous function for all $f \in C_c(G, \mathcal{H})$, Λ_t is well-defined. It is clear that Λ_t is linear. Notice

$$\|\Lambda_t f\|_2 = \int_G \left\| f(t^{-1}s) \right\| ds = \int_G \|f(s)\| \, ds = \|f\|_2$$

for all $f \in C_c(G, \mathcal{H})$ so Λ_t is an isometry and thus extends to an isometry on $L_2(G, \mathcal{H})$. Moreover $(\Lambda_t \Lambda_{t^{-1}} f)(s) = (\Lambda_{t^{-1}} f)(t^{-1}s) = f(s)$ for all $f \in C_c(G, \mathcal{H})$ and $t \in G$. Whence Λ_t is invertible with $\Lambda_t^{-1} = \Lambda_{t^{-1}}$ so Λ_t extends to a unitary operator on $L_2(G, \mathcal{H})$. Moreover it is trivial to verify that $\Lambda_t \Lambda_s = \Lambda_{ts}$ on $C_c(G, \mathcal{H})$ for all $s, t \in G$. Thus $t \mapsto \Lambda_t$ is a group homomorphism. Lastly we must verify that $t \mapsto \Lambda_t f$ is continuous for all $f \in L_2(G, \mathcal{H})$. Since each Λ_t is a bounded operator of norm one, it suffices to verify this on $C_c(G, \mathcal{H})$. However, if $f \in C_c(G, \mathcal{H})$, we clearly see that $t \mapsto f_t$ is (uniformly) continuous. Whence $t \mapsto \Lambda_t$ is a unitary representation of G.

Lastly we notice for all $f \in C_c(G, \mathcal{H})$, $A \in \mathfrak{A}$, and $t \in G$ that

$$\begin{aligned}(\Lambda_t \tilde{\pi}(A) \Lambda_t^* f)(s) &= (\tilde{\pi}(A) \Lambda_t^* f)(t^{-1}s) \\ &= \pi(\alpha_{s^{-1}t}(A))(\Lambda_t^* f)(t^{-1}s) \\ &= (\pi(\alpha_{s^{-1}}(\alpha_t(A))) f)(s) \\ &= (\tilde{\pi}(\alpha_t(A)) f)(s)\end{aligned}$$

Since $C_c(G, \mathcal{H})$ is dense in $L_2(G, \mathcal{H})$, we obtain that $\Lambda_t \tilde{\pi}(A) \Lambda_t^* = \tilde{\pi}(\alpha_t(A))$ so $(\tilde{\pi}, \Lambda)$ is a covariant representation of $(\mathfrak{A}, G, \alpha)$. □

To get a *-representation of $L_1(G, \mathfrak{A}, \alpha)$, we have the following result.

Theorem 14. *Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. Suppose $(\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}), U)$ is a covariant representation of $(\mathfrak{A}, G, \alpha)$. Then there exists a *-homomorphism $\sigma_{(\pi,U)} : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ defined by*

$$\sigma_{(\pi,U)}(f) = \int_G \pi(f(t)) U_t dt$$

for all $f \in C_c(G, \mathfrak{A})$. If π is non-degenerate then $\sigma_{(\pi,U)}$ is non-degenerate.

Conversely, if $\sigma : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ is a non-degenerated *-homomorphism, there exists a covariant representation (π, U) such that $\sigma_{(\pi,U)} = \sigma$.

PROOF: Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. Suppose $(\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}), U)$ is a covariant representation of $(\mathfrak{A}, G, \alpha)$. For each $f \in C_c(G, \mathfrak{A})$, define $\sigma_{(\pi,U)}(f) = \int_G \pi(f(t)) U_t dt$. This is clearly well-defined since $f \in C_c(G, \mathfrak{A})$ so $t \mapsto \pi(f(t)) U_t$ is bounded and continuous with respect to the strong operator topology (and thus $\xi \mapsto \int_G \pi(f(t)) U_t \xi dt$ defines a bounded linear operator on $\mathcal{H}$). Clearly $\sigma_{(\pi,U)}(f)$ is linear and since

$$\left\|\sigma_{(\pi,U)}(f)\right\| \leq \int_G \|\pi(f(t))\| \, dt \leq \int_G \|f(t)\| \, dt = \|f\|_1$$

$\sigma_{(\pi,U)}(f)$ is bounded. By continuity $\sigma_{(\pi,U)}$ extends to a linear map from $L_1(G,\mathfrak{A},\alpha)$ to $\mathcal{B}(\mathcal{H})$. Since for all $f,g \in C_c(G,\mathfrak{A})$

$$\begin{aligned}
\sigma_{(\pi,U)}(f*g) &= \int_G \pi((f*g)(s))U_s ds \\
&= \int_G \pi\left(\int_G f(t)\alpha_t(g(t^{-1}s))dt\right) U_s ds \\
&= \int_G \int_G \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))U_s dtds \\
&= \int_G \int_G \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))U_s dsdt \\
&= \int_G \int_G \pi(f(t))U_t\pi(g(t^{-1}s))U_t^*U_s dsdt \\
&= \int_G \int_G \pi(f(t))U_t\pi(g(s))U_{t^{-1}}U_{ts} dsdt \\
&= \int_G \int_G \pi(f(t))U_t\pi(g(s))U_s dsdt \\
&= \int_G \pi(f(t))U_t\sigma_{(\pi,U)}(g)dt \\
&= \sigma_{(\pi,U)}(f)\sigma_{(\pi,U)}(g)
\end{aligned}$$

$\sigma_{(\pi,U)}$ must have extended to be a homomorphism. Moreover for all $f \in C_c(G,\mathfrak{A})$ and $\xi,\eta \in \mathcal{H}$

$$\begin{aligned}
\langle \sigma_{(\pi,U)}(f^*)\xi,\eta\rangle &= \left\langle \left(\int_G \pi(f^*(t))U_t dt\right)\xi,\eta\right\rangle \\
&= \int_G \langle \pi(f^*(t))U_t\xi,\eta\rangle\, dt \\
&= \int_G \Delta(t)^{-1}\left\langle \pi(\alpha_t(f(t^{-1})))^*U_t\xi,\eta\right\rangle dt \\
&= \int_G \Delta(t)^{-1}\left\langle \xi, U_{t^{-1}}\pi(\alpha_t(f(t^{-1})))\eta\right\rangle dt \\
&= \int_G \langle \xi, U_t\pi(\alpha_{t^{-1}}(f(t)))\eta\rangle\, dt \\
&= \overline{\int_G \langle U_t\pi(\alpha_{t^{-1}}(f(t)))\eta,\xi\rangle\, dt} \\
&= \overline{\int_G \langle \pi(f(t))U_t\eta,\xi\rangle\, dt} \\
&= \overline{\langle \sigma_{(\pi,U)}(f)\eta,\xi\rangle} \\
&= \langle \sigma_{(\pi,U)}(f)^*\xi,\eta\rangle
\end{aligned}$$

Thus $\sigma_{(\pi,U)}$ must have extended to be a *-homomorphism as desired.

Next suppose that π is non-degenerate. To show that $\sigma_{(\pi,U)}$ is non-degenerate, it suffices to show that $\sigma_{(\pi,U)}$ has trivial null-space by Proposition 11. Suppose $\xi \in \mathcal{H}$ is such that $\sigma_{(\pi,U)}(L_1(G,\mathfrak{A},\alpha))\xi = \{0\}$. Let $\{f_\beta\}_\Lambda$ be the norm one bounded approximate identity for $L_1(G)$ from Theorem 2. Then for each $A \in \mathfrak{A}$,

$Af_\beta \in L_1(G, \mathfrak{A}, \alpha)$ and

$$\begin{aligned}
\|\pi(A)\xi\| &= \left\|\pi(A)\xi - \sigma_{(\pi,U)}(Af_\beta)\xi\right\| \\
&= \left\|\pi(A)\xi - \int_G \pi(Af_\beta(t))U_t\xi dt\right\| \\
&= \left\|\pi(A)(\xi - \int_G f_\beta(t)U_t\xi dt)\right\| \\
&= \left\|\pi(A)\int_G f_\beta(t)(\xi - U_t\xi)dt\right\| \\
&\leq \|\pi(A)\| \int_G f_\beta(t) \|\xi - U_t\xi\| dt \\
&\leq \|\pi(A)\| \int_G f_\beta(t) \sup\{\|\xi - U_r\xi\| \mid r \in supp(f_\beta)\}dt \\
&= \|\pi(A)\| \sup\{\|\xi - U_r\xi\| \mid r \in supp(f_\beta)\}
\end{aligned}$$

Since $r \mapsto U_r\xi$ is continuous with $U_e\xi = \xi$ and by the construction of the f_β's, we obtain that $\|\pi(A)\xi\| = \lim_\Lambda \|\pi(A)\| \sup\{\|\xi - U_r\xi\| \mid r \in supp(f_\beta)\} = 0$. Whence $\pi(A)\xi = 0$ for all $A \in \mathfrak{A}$. Since π is non-degenerate, $\xi = 0$. Whence $\sigma_{(\pi,U)}$ is non-degenerate by Proposition 11.

The converse direction is significantly harder (even though we have most of the technical preliminaries out of the way) and relies heavily on the left bounded approximate identity for $L_1(G, \mathfrak{A}, \alpha)$ from Proposition 8. Suppose $\sigma : L_1(G, \mathfrak{A}, \alpha) \to \mathcal{B}(\mathcal{H})$ is a non-degenerated *-homomorphism. Therefore $\sigma(L_1(G, \mathfrak{A}, \alpha))\mathcal{H}$ is dense in $\mathcal{H}$. Since $C_c(G, \mathfrak{A})$ is dense in $L_1(G, \mathfrak{A}, \alpha)$, $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$.

Let $\{g_\beta\}_\Lambda$ be the left bounded approximate identity for $L_1(G, \mathfrak{A}, \alpha)$ from Proposition 8 and fix $A \in \mathfrak{A}$. We claim that SOT-$\lim_\Lambda \sigma(\lambda(A)g_\beta)$ exists. Let $\epsilon > 0$. If $g \in L_1(G, \mathfrak{A}, \alpha)$ and $\eta \in \mathcal{H}$, there exists a $\beta' \in \Lambda$ so that if $\beta_1, \beta_2 \geq \beta'$ then $\|g_{\beta_1} * g - g_{\beta_2} * g\| < \frac{\epsilon}{(\|A\|+1)(\|\eta\|+1)}$. Thus if $\beta_1, \beta_2 \geq \beta'$

$$\begin{aligned}
\|\sigma(\lambda(A)g_{\beta_1})\sigma(g)\eta - \sigma(\lambda(A)g_{\beta_2})\sigma(g)\eta\| &= \|\sigma((\lambda(A)g_{\beta_1}) * g - (\lambda(A)g_{\beta_2}) * g)\eta\| \\
&\leq \|(\lambda(A)g_{\beta_1}) * g - (\lambda(A)g_{\beta_2}) * g\| \|\eta\| \\
&\leq \|\lambda(A)(g_{\beta_1} * g - g_{\beta_2} * g)\| \|\eta\| \qquad \text{Lemma 5} \\
&\leq \|A\| \|g_{\beta_1} * g - g_{\beta_2} * g\| \|\eta\| \\
&\leq \|A\| \frac{\epsilon}{(\|A\|+1)(\|\eta\|+1)} \|\eta\| < \epsilon
\end{aligned}$$

Thus $\{\sigma(\lambda(A)g_\beta)(\sigma(g)\eta)\}_\Lambda$ is a Cauchy net in $\mathcal{H}$ and thus converges. Notice $\|\sigma(\lambda(A)g_\beta)\| \leq \|\lambda(A)(g_\beta)\| \leq \|A\| \|g_\beta\| \leq \|A\|$ by Lemma 5. Fix $\xi \in \mathcal{H}$ and let $\epsilon > 0$. Since $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$ there exists a $g \in C_c(G, \mathfrak{A})$ and $\eta \in \mathcal{H}$ so that $\|\xi - \sigma(g)\eta\| \leq \frac{\epsilon}{3(\|A\|+1)}$. By above there exists a $\beta' \in \Lambda$ such that for all $\beta_1, \beta_2 \geq \beta'$ $\|\sigma(\lambda(A)g_{\beta_1})\sigma(g)\eta - \sigma(\lambda(A)g_{\beta_2})\sigma(g)\eta\| < \frac{\epsilon}{3}$. Whence for all $\beta_1, \beta_2 \geq \beta'$

$$\begin{aligned}
\|\sigma(\lambda(A)g_{\beta_1})\xi - \sigma(\lambda(A)g_{\beta_2})\xi\| &\leq \|\sigma(\lambda(A)g_{\beta_1})\xi - \sigma(\lambda(A)g_{\beta_2})\sigma(g)\eta\| + \|\sigma(\lambda(A)g_{\beta_1})\sigma(g)\eta - \sigma(\lambda(A)g_{\beta_2})\sigma(g)\eta\| \\
&\quad + \|\sigma(\lambda(A)g_{\beta_2})\sigma(g)\eta - \sigma(\lambda(A)g_{\beta_2})\xi\| \\
&\leq \|\sigma(\lambda(A)g_{\beta_1})\| \|\xi - \sigma(g)\eta\| + \frac{\epsilon}{3} + \|\sigma(\lambda(A)g_{\beta_2})\| \|\sigma(g)\eta - \xi\| \\
&\leq \|\lambda(A)g_{\beta_1}\| \frac{\epsilon}{3(\|A\|+1)} + \frac{\epsilon}{3} + \|\lambda(A)g_{\beta_2}\| \frac{\epsilon}{3(\|A\|+1)} \\
&\leq \|A\| \frac{\epsilon}{3(\|A\|+1)} + \frac{\epsilon}{3} + \|A\| \frac{\epsilon}{3(\|A\|+1)} < \epsilon
\end{aligned}$$

Thus $\{\sigma(\lambda(A)g_\beta)\xi\}_\Lambda$ is a Cauchy net in $\mathcal{H}$ and thus converges. Therefore $T_A\xi = \lim_\Lambda \sigma(\lambda(A)g_\beta)\xi$ defines a map from $\mathcal{H}$ to itself. Since each $\sigma(\lambda(A)g_\beta)$ is linear so T_A is linear and since $\|\sigma(\lambda(A)g_\beta)\xi\| \leq \|A\| \|\xi\|$ for all β, $T_A \in \mathcal{B}(\mathcal{H})$ with $\|T_A\| \leq \|A\|$.

Define $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by $\pi(A) = \text{SOT-}\lim_\Lambda \sigma(\lambda(A)g_\beta)$. Notice for all $A \in \mathfrak{A}$, $g \in C_c(G, \mathfrak{A})$, and $\eta \in \mathcal{H}$ that

$$\begin{aligned}\pi(A)\sigma(g)\eta &= \lim_\Lambda \sigma(\lambda(A)g_\beta)\sigma(g)\eta \\ &= \lim_\Lambda \sigma((\lambda(A)g_\beta) * g)\eta \\ &= \lim_\Lambda \sigma(\lambda(A)(g_\beta * g))\eta \qquad \text{Lemma 5} \\ &= \sigma(\lambda(A)(g))\eta\end{aligned}$$

From this it is clear that π is linear and multiplicative on $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$ (as λ is linear and multiplicative) and thus on $\mathcal{H}$ by continuity and density. Moreover we notice

$$\begin{aligned}\langle \pi(A)^*\sigma(g)\xi, \sigma(f)\eta\rangle &= \langle \sigma(g)\xi, \pi(A)\sigma(f)\eta\rangle \\ &= \langle \sigma(g)\xi, \sigma(\lambda(A)f)\eta\rangle \\ &= \langle \sigma((\lambda(A)f)^* * g)\xi, \eta\rangle \\ &= \langle \sigma(f^* * (\lambda(A^*)g))\xi, \eta\rangle \qquad \text{Lemma 5} \\ &= \langle \sigma(\lambda(A^*)g)\xi, \sigma(f)\eta\rangle \\ &= \langle \pi(A^*)\sigma(g)\xi, \sigma(f)\eta\rangle\end{aligned}$$

for all $A \in \mathfrak{A}$, $g, f \in C_c(G, \mathfrak{A})$, and $\xi, \eta \in \mathcal{H}$. Since $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$, we obtain that π is a *-homomorphism.

Now we define the unitary representation of G. Notice for each $t \in G$, $g \in C_c(G, \mathfrak{A})$ and $\eta \in \mathcal{H}$ that $\sigma((g_\beta)_t)\sigma(g)\eta = \sigma(\widetilde{\alpha}_t(((\widetilde{\alpha_{t^{-1}}}(g_\beta)) * g)_t))\eta$ by Lemma 7.2. From Proposition $\{\widetilde{\alpha_{t^{-1}}}(g_\beta)\}_\Lambda$ is a left bounded approximate identity for $L_1(G, \mathfrak{A}, \alpha)$. Whence

$$\begin{aligned}\lim_\Lambda \sigma((g_\beta)_t)\sigma(g)\eta &= \lim_\Lambda \sigma(\widetilde{\alpha}_t(((\widetilde{\alpha_{t^{-1}}}(g_\beta)) * g)_t))\eta \\ &= \sigma(\widetilde{\alpha}_t(g_t))\eta \in \mathcal{H}\end{aligned}$$

by earlier continuity results. Since $\|\sigma((g_\beta)_t)\| \leq \|g_\beta\| \leq 1$, we obtain (as in the $\pi(A)$ case) that SOT-$\lim_\Lambda \sigma((g_\beta)_t)$ is a well-defined bounded linear operator with norm at most one.

Define a map from G to $\mathcal{B}(\mathcal{H})$ by $t \mapsto U_t = \text{SOT-}\lim_\Lambda \sigma((g_\beta)_t)$. Notice for all $g \in C_c(G, \mathfrak{A})$ and $\eta \in \mathcal{H}$ that $U_t(\sigma(g)\eta) = \sigma(\widetilde{\alpha}_t(g_t))\eta$. Therefore $U_sU_t(\sigma(g)\eta) = U_s\sigma(\widetilde{\alpha}_t(g_t))\eta = \sigma(\widetilde{\alpha_s}((\widetilde{\alpha}_t(g_t))_s))\eta = \sigma(\widetilde{\alpha_{st}}(g_{st}))\eta = U_{st}(\sigma(g)\eta)$ by Lemma 7.3 (the elimination of U_s holds since $g_t \in C_c(G, \mathfrak{A})$ so $\widetilde{\alpha}_t(g_t) \in C_c(G, \mathfrak{A})$). Since $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$, $U_sU_t = U_{st}$ so $t \mapsto U_t$ is a group homomorphism. Since each U_t has norm at most one, this is a contractive group homomorphism.

Notice for all $g \in C_c(G, \mathfrak{A})$ and $\eta \in \mathcal{H}$ that $U_e\sigma(g)\eta = \lim_\Lambda \sigma(g_\beta)\sigma(g)\eta = \lim_\Lambda \sigma(g_\beta * g)\eta = \sigma(g)\eta$ as $\{g_\beta\}$ is a left bounded approximate identity. Whence $U_e = I_\mathcal{H}$ on a dense subset so $U_e = I_\mathcal{H}$. Therefore $I = U_tU_{t^{-1}} = U_{t^{-1}}U_t$ so each U_t is invertible. Since $\|U_t\| \leq 1$ and $\|U_{t^{-1}}\| \leq 1$, we must have that each U_t is an isometry and thus an invertible isometry. Whence each U_t is a unitary.

To verify that $t \mapsto U_t$ is a unitary representation of G, we must show that $t \mapsto U_t\xi$ is continuous for all $\xi \in \mathcal{H}$. Since $\|U_t\| = 1$ for all t, it suffices to show this on a dense subset of $\mathcal{H}$, namely $\sigma(C_c(G, \mathfrak{A}))\mathcal{H}$. For all $g \in C_c(G, \mathfrak{A})$ and $\eta \in \mathcal{H}$ $U_t(\sigma(g)\eta) = \sigma(\widetilde{\alpha}_t(g_t))\eta$. Since $t \mapsto \widetilde{\alpha}_t(g_t)$ is continuous by Corollary 9 and σ is continuous, we obtain the desired result. Whence $t \mapsto U_t$ is a unitary representation of G.

We claim that (π, U) is a covariant representation of $(\mathfrak{A}, G, \alpha)$. To see this we notice for all $g \in C_c(G, \mathfrak{A})$ and $\xi \in \mathcal{H}$ that

$$\begin{aligned}U_t\pi(A)U_t^*(\sigma(g)\xi) &= U_t\pi(A)\sigma(\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}}))\xi \\ &= U_t\sigma(\lambda(A)\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}}))\xi \\ &= \sigma(\widetilde{\alpha}_t((\lambda(A)\widetilde{\alpha_{t^{-1}}}(g_{t^{-1}}))_t))\xi \\ &= \sigma(\lambda(\alpha_t(A))g)\xi \qquad \text{Lemma 7.4} \\ &= \pi(\alpha_t(A))(\sigma(g)\xi)\end{aligned}$$

(as all functions in consideration are in $C_c(G,\mathfrak{A})$ so any formulae developed for π and U hold). Since $\sigma(C_c(G,\mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$, $U_t\pi(A)U_t^* = \pi(\alpha_t(A))$ for all $t \in G$ and $A \in \mathfrak{A}$. Whence (π, U) is a covariant representation of $(\mathfrak{A}, G, \alpha)$.

It remains only to verify that $\sigma_{(\pi,U)} = \sigma$. Then for all $f, g \in C_c(G,\mathfrak{A})$ and $\xi \in \mathcal{H}$

$$\begin{aligned}\sigma_{(\pi,U)}(f)(\sigma(g)\xi) &= \left(\int_G \pi(f(t))U_t dt\right)(\sigma(g)\xi)\\ &= \int_G \pi(f(t))U_t\sigma(g)\xi dt\\ &= \int_G \pi(f(t))\sigma(\widetilde{\alpha_t}(g_t))\xi dt\\ &= \int_G \sigma(\lambda(f(t))(\widetilde{\alpha_t}(g_t)))\xi dt\\ &= \sigma\left(\int_G \lambda(f(t))(\widetilde{\alpha_t}(g_t))dt\right)\xi\\ &= \sigma(f*g)\,\xi \qquad \text{Corollary 10}\\ &= \sigma(f)(\sigma(g)\xi)\end{aligned}$$

Since $\sigma(C_c(G,\mathfrak{A}))\mathcal{H}$ is dense in $\mathcal{H}$, $\sigma_{(\pi,U)}(f) = \sigma(f)$ for all $f \in C_c(G,\mathfrak{A})$. Since σ and $\sigma_{(\pi,U)}$ are continuous and $C_c(G,\mathfrak{A})$ is dense in $L_1(G,\mathfrak{A},\alpha)$, we obtain $\sigma = \sigma_{(\pi,U)}$ as desired. □

The last result we need is that one of the *-homomorphisms on $L_1(G,\mathfrak{A},\alpha)$ given by Theorem 14 is faithful. This *-homomorphism will be given by one from Proposition 13 assuming the representation of $\mathfrak{A}$ taken is faithful.

Proposition 15. *Let $(\mathfrak{A}, G, \alpha)$ be a C^*-dynamical system. Let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be any faithful representation. If $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(L_2(G,\mathcal{H}))$ and $\Lambda : G \to \mathcal{U}(L_2(G,\mathcal{H}))$ is the covariant representation induced by π in Proposition 13, then the *-homomorphism $\sigma_{(\tilde{\pi},\Lambda)} : L_1(G,\mathfrak{A},\alpha) \to \mathcal{B}(L_2(G,\mathcal{H}))$ from Theorem 14 is injective.*

Proof: Let $f \in L_1(G,\mathfrak{A},\alpha)$ be arbitrary. Unfortunately, we cannot avoid the Bochner integral here. By considering the definition of $\sigma_{(\tilde{\pi},\Lambda)}$ for elements of $C_c(G,\mathfrak{A})$ and how elements of $L_1(G,\mathfrak{A},\alpha)$ are limits of elements of $C_c(G,\mathfrak{A})$, we obtain $\sigma_{(\tilde{\pi},\Lambda)}(f) = \int_G \tilde{\pi}(f(t))\Lambda_t dt$ for all $f \in L_1(G,\mathfrak{A},\alpha)$. Suppose there exists an $f \in L_1(G,\mathfrak{A},\alpha)$ so that $\sigma_{(\tilde{\pi},\Lambda)}(f) = 0$. Let $\{f_\lambda\}$ be the bounded approximate identity for $L_1(G)$ from Theorem 2 and let $\xi \in \mathcal{H}$ be arbitrary. Since f_λ is non-zero only on a compact set, $\xi f_\lambda \in L_2(G,\mathcal{H})$ (you can show that this function is indeed in $L_2(G,\mathcal{H})$ if you take the definition given in Proposition 13). Then $0 = \sigma_{(\tilde{\pi},\Lambda)}(f)(\xi f_\lambda) = \int_G \tilde{\pi}(f(t))\Lambda_t(\xi f_\lambda)dt$ as an element of $L_2(G,\mathcal{H})$. Whence $0 = \left(\int_G \tilde{\pi}(f(t))\Lambda_t(\xi f_\lambda)dt\right)(s) = \int_G \pi(\alpha_{t^{-1}}(f(t)))\xi f_\lambda(t^{-1}s)dt$ for almost all $s \in G$. Since f_λ is a bounded approximate identity for $L_1(G)$, it is trivial to show that

$$\int_G \pi(\alpha_{t^{-1}}(f(t)))\xi f_\lambda(t^{-1}s)dt \to \pi(\alpha_{s^{-1}}(f(s)))\xi$$

by using typical convolution tricks. Whence $0 = \pi(\alpha_{s^{-1}}(f(s)))\xi$ for almost every s (where the almost everywhere set depends on ξ). (Note all of the following functions can be shown to be measurable in the sense of the Bochner integral) Thus, considering $s \mapsto \pi(\alpha_{s^{-1}}(f(s))) \in L_1(G,\mathcal{B}(\mathcal{H}))$, we obtain $\pi(\alpha_{s^{-1}}(f(s))) = 0$ almost everywhere (since it it were not zero as a function in $L_1(G,\mathcal{B}(\mathcal{H})$, it would be non-zero on a positive measure subset and thus its values when applied to ξ would be non-zero in a positive measure set). Since π is a faithful representation (by viewing $s \mapsto \alpha_{s^{-1}}(f(s)) \in L_1(G,\mathfrak{A})$ we obtain $\alpha_{s^{-1}}(f(s)) = 0$ almost everywhere. Thus, since $\alpha_{s^{-1}}$ is faithful, $f(s) = 0$ for almost all s. Whence $f = 0$. Thus $\sigma_{(\tilde{\pi},\Lambda)}$ is injective as desired. □

Using Proposition 13, Theorem 14, and Proposition 15, we see that if $(\mathfrak{A}, G, \alpha)$ is a C*-dynamical system, then there exists a faithful *-homomorphism from $L_1(G, \mathfrak{A}, \alpha)$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since every *-homomorphism from a Banach *-algebra to a C*-algebra is contractive, this allows us to define a C*-norm on $L_1(G, \mathfrak{A}, \alpha)$.

Definition 16. Let $(\mathfrak{A}, G, \alpha)$ be a C*-dynamical system. Define a C*-norm on $L_1(G, \mathfrak{A}, \alpha)$ by

$$\|f\| = \sup_{\sigma} \|\sigma(f)\|$$

where the supremum is taken over all (non-degenerate) *-representations of $L_1(G, \mathfrak{A}, \alpha)$. Based on the theory developed, it is trivial to verify that this is a well-defined C*-norm. Let $\mathfrak{A} \rtimes_\alpha G$ denote the completion of $L_1(G, \mathfrak{A}, \alpha)$ with respect to this norm. Whence $\mathfrak{A} \rtimes_\alpha G$ is a C*-algebra known as the cross product of $\mathfrak{A}$ by G.

Recall how the covariant representation in Proposition 13 was developed from any *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$. We can define another C*-norm on $L_1(G, \mathfrak{A}, \alpha)$ by

$$\|f\|_r = \sup_{\sigma} \|\sigma(f)\|$$

where the supremum is taken over all *-representations of $L_1(G, \mathfrak{A}, \alpha)$ that are created by the proof of Proposition 13. Based on the theory developed, it is trivial to verify that this is a well-defined C*-norm. Let $\mathfrak{A} \rtimes_{\alpha,r} G$ denote the completion of $L_1(G, \mathfrak{A}, \alpha)$ with respect to this norm. Whence $\mathfrak{A} \rtimes_{\alpha,r} G$ is a C*-algebra known as the reduced cross product of $\mathfrak{A}$ by G.

It is trivial to verify that if $\mathfrak{A} = (\mathbb{C}, \|\cdot\|_\infty)$, G is a locally compact group, and $\alpha : G \to Aut(\mathfrak{A})$ by $\alpha(g) = Id$ for all g, then $\mathfrak{A} \rtimes_{\alpha,r} G \simeq C^*_r(G)$ and $\mathfrak{A} \rtimes_\alpha G \simeq C^*(G)$ (every unitary representation of G is a covariant representation of $(\mathbb{C}, G, \alpha)$).

By construction, it is clear that there is a one-to-one correspondence between representations of $L_1(G, \mathfrak{A}, \alpha)$ on Hilbert spaces and between representations of $\mathfrak{A} \rtimes_\alpha G$. By Theorem 14 there is a one-to-one correspondence between non-degenerate representations of $\mathfrak{A} \rtimes_\alpha G$ and non-degenerate covariant representations of $(\mathfrak{A}, G, \alpha)$. Therefore $\mathfrak{A} \rtimes_\alpha G$ encodes the representation theory of $(\mathfrak{A}, G, \alpha)$.

References

[1] K. Davidson, *C*-Algebras by Example*, Fields Institute Monographs 6, American Mathematical Society, Providence, RI, 1996.