Trace Class Operators

Paul Skoufranis

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Abstract

The purpose of these notes is to develop (without mention of the Hilbert-Schmidt operators) the ideal of trace class operators in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is an infinite dimensional Hilbert space. In addition, we will develop the facts that the trace class operators are the dual of the compact operators and the predual of $\mathcal{B}(\mathcal{H})$. We will proceed to develop the theory in the way that we feel it most intuitive and direct.

In these notes, $\mathcal{H}$ will alway be an infinite (but not necessarily separable) Hilbert space and $\mathfrak{K}(\mathcal{H})$ will denote the set of compact operators. These notes will assume that the reader has a basic knowledge of the Continuous Functional Calculus for Normal Operators, compact operators, the spectral theorem for positive compact operators, and the polar decomposition of an operator. All inner products will be linear in the first component.

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The set of trace class operators on a Hilbert space $\mathcal{H}$, denoted $\mathcal{C}_1(\mathcal{H})$, have many several interesting and important properties. To begin to understand the importance of the trace class operators, recall that $\mathcal{B}(\mathcal{H})$ can be viewed as a non-commutative version of $\ell_\infty(I)$ where I is an infinite set and $\mathfrak{K}(\mathcal{H})$ can be viewed as a non-commutative version of $c_0(I)$. Carrying forth this analogy, we will see that $\mathcal{C}_1(\mathcal{H})$ is really a non-commutative analog of $\ell_1(I)$. Moreover, it is well-known that $c_0(I)^* \simeq \ell_1(I)$ and $\ell_1(I)^* \simeq \ell_\infty(I)$. As such, it will be show that $\mathfrak{K}(\mathcal{H})^* \simeq \mathcal{C}_1(\mathcal{H})$ (Theorem 25) and $\mathcal{C}_1(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H})$ (Theorem 23). Moreover, the fact that $\mathcal{C}_1(\mathcal{H})^* \simeq \mathcal{B}(\mathcal{H})$ allows us to construct a weak*-topology on $\mathcal{B}(\mathcal{H})$ that is very important to the study of von Neumann algebras (although this will not be developed here).

In these notes there are several times that we want to take rather a m-tuple (finite sum, finite set) or a sequence (countable sum, countable set). As such the notation $(a_n)_{n\geq 1}$ will denote either a finite m-tuple (for some $m \in \mathbb{N}$) or a countable sequence. Similarly the notation $\sum_{n\geq 1} a_n$ will denote a countable sum that is possibly a finite sum. Moreover, if multiple similar objects are considered together, they are all 'finite' or all 'countable'. This convenience is made to accommodate finite rank operators simultaneously with compact operators that are not of finite rank.

To begin a study of the trace class operators it is necessary to develop the theory of compact operators at least up to the spectral theorem of compact normal operators. We begin with the statement of the afore mentioned theorem although we will only need the case that compact operator under consideration is self-adjoint.

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space and let $N \in \mathcal{B}(\mathcal{H})$ be a compact, normal operator. Suppose $(\lambda_k)_{k\geq 1}$ are the distinct non-zero eigenvalues of N (recall $\lambda_k \to 0$ if there are infinitely many eigenvalues) and that $P_{\mathcal{M}_k}$ is the orthonormal projection of $\mathcal{H}$ onto $\mathcal{M}_k = \ker(N - \lambda_k I)$. Then each $\mathcal{M}_k$ is a finite dimensional Hilbert space, $P_{\mathcal{M}_k} P_{\mathcal{M}_j} = 0 = P_{\mathcal{M}_j} P_{\mathcal{M}_k}$ if $j \neq k$, and*

$$N = \sum_{k\geq 1} \lambda_k P_{\mathcal{M}_k}$$

where the series converges in the norm topology on $\mathcal{B}(\mathcal{H})$.

Using the above theorem and the polar decomposition of an element of $\mathcal{B}(\mathcal{H})$ it is possible to write every compact operator as a norm convergent sum of rank one operators. Before we show this, we will make some useful notation.

Notation 2. Let $\xi, \eta \in \mathcal{H}$. We denote the rank one operator in $\mathcal{B}(\mathcal{H})$ that takes η to ξ by $\xi\eta^*$ (that is, $\xi\eta^*(\zeta) = \langle \zeta, \eta \rangle \xi$ for all $\zeta \in \mathcal{H}$).

Note that it is easy to verify that if $T \in \mathcal{B}(\mathcal{H})$, $\alpha \in \mathbb{C}$, and $\xi, \xi', \eta, \eta' \in \mathcal{H}$ then $T \circ (\xi\eta^*) = (T\xi)\eta^*$, $(\xi\eta^*)^* = \eta\xi^*$, $(\xi + \alpha\xi')\eta^* = (\xi\eta^*) + \alpha(\xi'\eta^*)$, and $\xi(\eta + \alpha\eta')^* = (\xi\eta^*) + \overline{\alpha}(\xi(\eta')^*)$ as operator in $\mathcal{B}(\mathcal{H})$.

Corollary 3. *Let $K \in \mathfrak{K}(\mathcal{H})$ be such that $K = K^*$. Then*

$$K = \sum_{n \geq 1} \lambda_n \eta_n \eta_n^*$$

where the sum converging in norm, $(\lambda_n)_{n\geq 1}$ are the real non-zero eigenvalues counting multiplicity, and $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ is an orthonormal set.

Proof: Fix $K \in \mathfrak{K}(\mathcal{H})$ such that $K = K^*$. Thus Theorem 1 implies that we may write $K = \sum_{k\geq 1} \lambda'_k P_{\mathcal{M}_k}$ (the sum converging in norm) where $(\lambda'_k)_{k\geq 1}$ is a sequence of non-zero eigenvalues of K that converges to zero, each $\mathcal{M}_k$ is a finite dimensional Hilbert space, and $P_{\mathcal{M}_k} P_{\mathcal{M}_j} = 0 = P_{\mathcal{M}_j} P_{\mathcal{M}_k}$ if $j \neq k$.

Let $\{\eta_\alpha\}_{\alpha \in \Delta_k}$ be an orthonormal basis for $\mathcal{M}_k$. Since each $\mathcal{M}_k$ is a finite dimensional Hilbert space, Δ_k is a finite indexing set and $P_{\mathcal{M}_k} = \sum_{\alpha \in \Delta_k} \eta_\alpha \eta_\alpha^*$ is norm convergent. Thus $\bigcup_{k\geq 1} \{\eta_\alpha\}_{\alpha \in \Delta_k}$ is a countable orthonormal set of vectors as $\mathcal{M}_k$ and $\mathcal{M}_j$ are orthogonal if $j \neq k$. By ordering $\bigcup_{k\geq 1} \{\eta_\lambda\}_{\lambda \in \Lambda_k}$ we obtain an orthonormal set of vectors $\{\eta_n\}_{n\geq 1}$ such that $K = \sum_{n\geq 1} \lambda_n \eta_n \eta_n^*$ is a norm convergent sum where $\lambda_n = \lambda'_k$ for the specific k such that $\eta_n \in \mathcal{M}_k$. Since K is self-adjoint, the eigenvalues of K are real so each λ_n is a real number. $\square$

Corollary 4. *Let $K \in \mathfrak{K}(\mathcal{H})$. Then*

$$K = \sum_{n \geq 1} s_n \xi_n \eta_n^*$$

where the sum converging in norm, $(s_n)_{n\geq 1}$ is a non-increasing sequence of strictly positive real numbers converging to 0, and $\{\xi_n\}_{n\geq 1} \subseteq \mathcal{H}$ and $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ are orthonormal sets (not necessarily orthogonal to each other).

Proof: Fix $K \in \mathfrak{K}(\mathcal{H})$. By the polar decomposition of operators there exists a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that $K = V|K|$ where $|K| = (K^*K)^{\frac{1}{2}}$. Since $|K| \in C^*(K)$ and $C^*(K) \subseteq \mathfrak{K}(\mathcal{H})$, $|K|$ is a positive compact operator. Thus Corollary 3 implies that there exists an orthonormal set $\{\eta_n\}_{n\geq 1}$ such that $|K| = \sum_{n\geq 1} s_n \eta_n \eta_n^*$ is a norm convergent sum where $(s_n)_{n\geq 1}$ are the non-zero eigenvalues of $|K|$ (counting multiplicity). Since $|K|$ is a positive operator, $s_n \geq 0$ for all n. Since $\lim_{n\to\infty} s_n = 0$ as $|K|$ is compact, we may rearrange the order of the η_ns such that $\lim_{n\to\infty} s_n = 0$ and $s_n \geq s_{n+1}$ for all n.

To obtain the final expression for K, recall that V is an isometry on $\overline{Ran(|K|)}$ and let $\xi_n = V\eta_n$. Since $\eta_n \in Ran(|K|)$ for all n, each ξ_n is a unit vector and ξ_k is orthogonal to ξ_j whenever $j \neq k$. Whence

$$K = V|K| = V\left(\sum_{n\geq 1} s_n \eta_n \eta_n^*\right) = \sum_{n\geq 1} s_n V(\eta_n \eta_n^*) = \sum_{n\geq 1} s_n (V\eta_n)\eta_n^* = \sum_{n\geq 1} s_n \xi_n \eta_n^*$$

is such a desired decomposition. $\square$

Notice, given a compact operator K, that the above representation is not unique since we could have chosen different orthonormal bases for each $\mathcal{M}_k$. However the s_ns are unique as they were simply the eigenvalues of $|K|$ (including multiplicity) arranged in decreasing order. To capture this information, we make the following definition.

Definition 5. Let $K \in \mathfrak{K}(\mathcal{H})$. The singular values of K, denoted $(s_n(K))_{n\geq 1}$, are the non-zero eigenvalues of $|K|$ (including multiplicity) arranged in decreasing order (with $s_n(K) = 0$ for all $n > k_0$ if $|K|$ only has k_0 non-zero eigenvalues counting multiplicity). Let $\|K\|_1 := \sum_{n\geq 1} s_n(K)$.

We have seen in Corollary 4 that the singular values appear in a specific decomposition of a compact operator. It turns out that the singular values appear in any such decomposition.

Lemma 6. *Let $K \in \mathfrak{K}(\mathcal{H})$ and suppose*

$$K = \sum_{n\geq 1} t_n \xi_n \eta_n^*$$

where the sum converging in norm, $(t_n)_{n\geq 1}$ is a non-increasing sequence of strictly positive real numbers converging to 0, and $\{\xi_n\}_{n\geq 1} \subseteq \mathcal{H}$ and $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ are orthonormal sets (not necessarily orthogonal to each other). Then $t_n = s_n(K)$. Moreover $\|K\| = \sup_{n\geq 1} t_n = t_1$.

Proof: To determine the singular values of K, we need to compute $|K|$. However we notice for all $\zeta \in \mathcal{H}$ that

$$\begin{aligned}
K^*K\zeta &= \left(\sum_{m\geq 1} t_m \eta_m \xi_m^*\right)\left(\sum_{n\geq 1} t_n \xi_n \eta_n^*\right)\zeta \\
&= \left(\sum_{m\geq 1} t_m \eta_m \xi_m^*\right)\left(\sum_{n\geq 1} t_n \langle \zeta, \eta_n\rangle \xi_n\right) \\
&= \sum_{m\geq 1}\sum_{n\geq 1} t_m t_n \langle \zeta, \eta_n\rangle \langle \xi_n, \xi_m\rangle \eta_m \\
&= \sum_{m\geq 1} t_m^2 \langle \zeta, \eta_m\rangle \eta_m \\
&= \left(\sum_{m\geq 1} t_m^2 \eta_m \eta_m^*\right)\zeta
\end{aligned}$$

Therefore $K^*K = \sum_{m\geq 1} t_m^2 \eta_m \eta_m^*$ as this sum clearly converges in norm as $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ is an orthonormal set and thus K^*K can be viewed as a diagonal operator by extending $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ to an orthonormal basis of $\mathcal{H}$. Similarly $T = \sum_{m\geq 1} t_m \eta_m \eta_m^*$ defines an element of $\mathcal{B}(\mathcal{H})$ as the sum converges in norm. Since $t_n \geq 0$ for all n, T is a positive operator (as it can be viewed as a diagonal operator with positive diagonal entries). Moreover, as $\{\eta_n\}_{n\geq 1} \subseteq \mathcal{H}$ is an orthonormal set, it is clear that $T^2 = K^*K$ so that $T = |K|$ by the uniqueness of the square root of a positive operator. Whence the eigenvalues of $|K|$ counting multiplicity are $(t_n)_{n\geq 1}$. Since $(t_n)_{n\geq 1}$ is a non-increasing sequence and by the definition of the singular values, $s_n(K) = t_n$ as desired.

Lastly, to see that $\|K\| = \sup_{n\geq 1} t_n$, we notice that

$$\|K\| = \|K^*K\|^{\frac{1}{2}} = \||K|\|$$

by the Continuous Functional Calculus. Since $|K| = \sum_{m\geq 1} t_m \eta_m \eta_m^*$, $|K|$ can be viewed as a diagonal operator in $\mathcal{B}(\mathcal{H})$ with diagonal entries contained in the set $\{t_m\}_{m\geq 1} \cup \{0\}$, the result trivially follows. $\square$

Corollary 7. *Let $K \in \mathfrak{K}(\mathcal{H})$. Then $\|K\| \leq \|K\|_1$.*

Proof: Let $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ be a decomposition from Corollary 4. Then $\|K\| = s_1(K) \leq \|K\|_1$ by Lemma 6. $\square$

Now that we have developed the singular values of a compact operator, we can finally define what it means for a compact operator to be a trace class operator.

Definition 8. A compact operator K is said to be a trace class operator if $\|K\|_1 = \sum_{n\geq 1} s_n(K) < \infty$. We shall denote the collection of trace class operators by $\mathcal{C}_1(\mathcal{H})$ and $\|\cdot\|_1 : \mathcal{C}_1(\mathcal{H}) \to [0, \infty)$ is called the trace norm.

Now we shall proceed with proving a series of lemmas and theorems in order to show that $\mathcal{C}_1(\mathcal{H})$ is a Banach space, $\mathcal{B}(\mathcal{H}) \simeq (\mathcal{C}_1(\mathcal{H}))^*$, and obtain more information about trace class operators. We begin by showing certain series essential to the theory of trace class operators are absolutely summable.

Lemma 9. *Let $\{\zeta_\alpha\}_{\alpha\in\Delta}$ and $\{\omega_\alpha\}_{\alpha\in\Delta}$ be orthonormal sets in $\mathcal{H}$ indexed by the same set Δ and let $K \in \mathfrak{K}(\mathcal{H})$. Then $\sum_{\alpha\in\Delta} |\langle K\zeta_\alpha, \omega_\alpha\rangle| \leq \|K\|_1$.*

Proof: Let $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ be a decomposition from Corollary 4. Then

$$\begin{aligned}
\sum_{\alpha\in\Delta} |\langle K\zeta_\alpha, \omega_\alpha\rangle| &= \sum_{\alpha\in\Delta} \left| \left\langle \sum_{n\geq 1} s_n(K)\langle \zeta_\alpha, \eta_n\rangle \xi_n, \omega_\alpha \right\rangle \right| \\
&= \sum_{\alpha\in\Delta} \left| \sum_{n\geq 1} s_n(K)\langle \zeta_\alpha, \eta_n\rangle \langle \xi_n, \omega_\alpha\rangle \right| \\
&\leq \sum_{n\geq 1} s_n(K) \sum_{\alpha\in\Delta} |\langle \zeta_\alpha, \eta_n\rangle||\langle \xi_n, \omega_\alpha\rangle| \\
&\leq \sum_{n\geq 1} s_n(K) \left(\sum_{\alpha\in\Delta} |\langle \zeta_\alpha, \eta_n\rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{\alpha\in\Delta} \langle e_n, y_\alpha\rangle|^2 \right)^{\frac{1}{2}} \\
&\leq \sum_{n\geq 1} s_n(K)\, \|\eta_n\|\, \|\xi_n\| = \sum_{n\geq 1} s_n(K) = \|K\|_1
\end{aligned}$$

Therefore the inequality holds as desired. $\square$

Combining Corollary 4 and Lemma 9 allows us to develop another expression for the trace class norm of a compact operator.

Lemma 10. *If $K \in \mathfrak{K}(\mathcal{H})$ then*

$$\|K\|_1 = \sup \left\{ \sum_{\alpha\in\Delta} |\langle K\zeta_\alpha, \omega_\alpha\rangle| \;\middle|\; \{\zeta_\alpha\}_{\alpha\in\Delta}, \{\omega_\alpha\}_{\alpha\in\Delta} \text{ orthonormal subsets of } \mathcal{H} \right\}$$

Moreover there exists countable orthonormal subsets $\{\zeta_n\}_{n\geq 1}$ and $\{\omega_n\}_{n\geq 1}$ of $\mathcal{H}$ such that $\langle K\zeta_n, \omega_n\rangle \geq 0$ for all n and $\sum_{n\geq 1}\langle K\zeta_n, \omega_n\rangle = \|K\|_1$.

Proof: By Lemma 9

$$\sup \left\{ \sum_{\alpha\in\Delta} |\langle K\zeta_\alpha, \omega_\alpha\rangle| \;\middle|\; \{\zeta_\alpha\}_{\alpha\in\Delta}, \{\omega_\alpha\}_{\alpha\in\Delta} \text{ orthonormal subsets of } \mathcal{H} \right\} \leq \|K\|_1$$

To prove the other inequality and the fact that the supremum is obtained, let $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ be a decomposition from Corollary 4. Choose $\theta_m \in [0, 2\pi)$ such that $\langle K\eta_m, e^{i\theta_m}\xi_m\rangle = |\langle K\eta_m, \xi_m\rangle| \geq 0$. Let $\zeta_m = e^{i\theta_m}\xi_m$. Then $\{\zeta_m\}_{m\geq 1}$ and $\{\eta_m\}_{m\geq 1}$ are orthonormal subsets of $\mathcal{H}$ and

$$\begin{aligned}
\sum_{m\geq 1}\langle K\eta_m, \zeta_m\rangle &= \sum_{m\geq 1} |\langle K\eta_m, \xi_m\rangle| \\
&= \sum_{m\geq 1}\left|\sum_{n\geq 1} s_n(K)\langle(\xi_n\eta_n^*)\eta_m, \xi_m\rangle\right| \\
&= \sum_{m\geq 1}\left|\sum_{n\geq 1} s_n(K)\langle\langle\eta_m, \eta_n\rangle\xi_n, \xi_m\rangle\right| \\
&= \sum_{m\geq 1} s_m(K) = \|K\|_1
\end{aligned}$$

as desired. □

Proposition 11. *$\mathcal{C}_1(\mathcal{H})$ is a vector space and the trace norm $\|\cdot\|_1$ is a norm on $\mathcal{C}_1(\mathcal{H})$.*

Proof: First we notice that $0 \in \mathcal{C}_1(\mathcal{H})$ since 0 is a compact operator with $\|0\|_1 = 0$. Next notice that if $K \in \mathcal{C}_1(\mathcal{H})$ and $\lambda \in \mathbb{C}\setminus\{0\}$ then $\lambda K \in \mathfrak{K}(\mathcal{H})$. Moreover, since $|\lambda K| = (|\lambda|^2 K^*K)^{\frac{1}{2}} = |\lambda||K|$, $s_n(\lambda K) = |\lambda| s_n(K)$ for all $n \in \mathbb{N}$. Whence $\|\lambda K\|_1 = |\lambda|\,\|K\|_1 < \infty$ so $\lambda K \in \mathcal{C}_1(\mathcal{H})$ when $\lambda \neq 0$ (and clearly when $\lambda = 0$).

Notice that if $K_1, K_2 \in \mathcal{C}_1(\mathcal{H})$ then $K_1 + K_2 \in \mathfrak{K}(\mathcal{H})$ and, by Lemma 10,

$$\begin{aligned}
\|K_1 + K_2\|_1 &= \sup\left\{\sum_{\alpha\in\Delta} |\langle(K_1 + K_2)\zeta_\alpha, \omega_\alpha\rangle| \;\middle|\; \{\zeta_\alpha\}_{\alpha\in\Delta}, \{\omega_\alpha\}_{\alpha\in\Delta} \text{ orthonormal subsets of } \mathcal{H}\right\} \\
&\leq \sup\left\{\sum_{\alpha\in\Delta} |\langle K_1\zeta_\alpha, \omega_\alpha\rangle| + \sum_{\alpha\in\Delta} |\langle K_2\zeta_\alpha, \omega_\alpha\rangle| \;\middle|\; \{\zeta_\alpha\}_{\alpha\in\Delta}, \{\omega_\alpha\}_{\alpha\in\Delta} \text{ orthonormal subsets of } \mathcal{H}\right\} \\
&\leq \|K_1\|_1 + \|K_2\|_1 < \infty
\end{aligned}$$

Thus $K_1 + K_2 \in \mathcal{C}_1(\mathcal{H})$. Whence $\mathcal{C}_1(\mathcal{H})$ is a vector space.

To complete the proof that $\|\cdot\|_1$ is a norm on $\mathcal{C}_1(\mathcal{H})$, we notice that if $K \in \mathcal{C}_1(\mathcal{H})$ is such that $\|K\|_1 = 0$ then $s_n(K) = 0$ for all n. Thus all eigenvalues of $|K|$ are zero. Whence $|K| = 0$ so $K = 0$ (as $K = V|K|$ for some partial isometry V). □

Our next goal is to show that $\mathcal{C}_1(\mathcal{H})$ is complete with respect to the trace norm and thus a Banach space. To prove this, we first will develop more information about the singular values of a compact operator and develop some important consequences. Our first result enables us a way to compute the n^{th} singular value of a compact operator.

Lemma 12. *Let K be a compact operator. Then $s_n(K) = \inf\{\|K - F\| \mid \text{rank}(F) \leq n - 1\}$.*

Proof: Fix $n \in \mathbb{N}$. Let $K = \sum_{k\geq 1} s_k(K)\xi_k\eta_k^*$ be a decomposition from Corollary 4. Let $F = \sum_{k=1}^{n-1} s_k(K)\xi_k\eta_k^*$. Then F is a rank $n - 1$ operator and

$$\|K - F\| = \left\|\sum_{k\geq n} s_k(K)\xi_k\eta_k^*\right\| = s_n(K)$$

by Lemma 6. Hence $s_n(K) \geq \inf\{\|K - F\| \mid \text{rank}(F) \leq n - 1\}$.

Let F be an arbitrary finite rank operator with rank at most $n-1$. Then $codim(ker(F)) \leq n-1$ and thus $\ker(F) \cap \operatorname{span}(\eta_1, \ldots, \eta_n) \neq \{0\}$. Let $\zeta \in \ker(F) \cap \operatorname{span}(\eta_1, \ldots, \eta_n) \setminus \{0\}$ be such that $\|\zeta\| = 1$. Since $\zeta \in \operatorname{span}(\eta_1, \ldots, \eta_n)$, ζ is orthogonal to η_k for all $k \geq n$. Therefore

$$\|K-F\|^2 \geq \|(K-F)(\zeta)\|^2 = \|K\zeta\|^2 = \left\|\sum_{k=1}^n s_k(K)\langle \zeta, \eta_k\rangle \xi_k\right\|^2 = \sum_{k=1}^n s_k(K)^2 |\langle \zeta, \eta_k\rangle|^2 \geq s_n(K)^2 \sum_{k=1}^n |\langle \zeta, \eta_k\rangle|^2 = s_n(K)^2$$

Thus the result follows. □

Lemma 13. *Let $K \in \mathfrak{K}(\mathcal{H})$ and let $T \in \mathcal{B}(\mathcal{H})$. Then $s_n(TK) \leq \|T\| s_n(K)$ and $s_n(KT) \leq s_n(K) \|T\|$.*

Proof: First recall that $TK, KT \in \mathfrak{K}(\mathcal{H})$ as the compact operators form an ideal. By Lemma 12 we have for all operators F of rank at most $n-1$ that $s_n(TK) \leq \|TK - TF\| \leq \|T\| \|K-F\|$ and $s_n(KT) \leq \|KT - FT\| \leq \|K - F\| \|T\|$ as TF and FT will also be finite rank operators with rank at most $n-1$. Thus, as this holds for all such finite rank operators, $s_n(TK) \leq \|T\| s_n(K)$ and $s_n(KT) \leq s_n(K) \|T\|$ by Lemma 12. □

With the above result in hand, we can prove the following important result relating the algebraic structures of $\mathcal{C}_1(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$.

Proposition 14. *Let $A, B \in \mathcal{B}(\mathcal{H})$ and let $K \in \mathcal{C}_1(\mathcal{H})$. Then $\|AKB\|_1 \leq \|A\| \|K\|_1 \|B\|$. Moreover $\mathcal{C}_1(\mathcal{H})$ is a self-adjoint algebraic ideal of $\mathcal{B}(\mathcal{H})$.*

Proof: Clearly AKB is compact. Moreover $s_n(AKB) \leq \|A\| s_n(K) \|B\|$ for all n by Corollary 13. Therefore

$$\sum_{n\geq 1} s_n(AKB) \leq \|A\| \left(\sum_{n\geq 1} s_n(K)\right) \|B\| = \|A\| \|K\|_1 \|B\| < \infty$$

Therefore, since $\mathcal{C}_1(\mathcal{H})$ is a vector subspace of $\mathcal{B}(\mathcal{H})$, this implies that $\mathcal{C}_1(\mathcal{H})$ is an algebraic ideal of $\mathcal{B}(\mathcal{H})$.

To see that $\mathcal{C}_1(\mathcal{H})$ is self-adjoint we notice that if $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ then $K^* = \sum_{n\geq 1} s_n(K)\eta_n\xi_n^*$ and thus $s_n(K^*) = s_n(K)$ for all n by Lemma 6. Therefore $\|K^*\|_1 = \|K\|_1$ and thus $\mathcal{C}_1(\mathcal{H})$ is self-adjoint. □

Using the fact that $\mathcal{C}_1(\mathcal{H})$ is self-adjoint allows us to decompose each element of $\mathcal{C}_1(\mathcal{H})$ into a linear combination of positive elements of $\mathcal{C}_1(\mathcal{H})$ as we would in a C*-algebra.

Lemma 15. *Every element of $\mathcal{C}_1(\mathcal{H})$ is a linear combination of four positive elements of $\mathcal{C}_1(\mathcal{H})$.*

Proof: First suppose that $K \in \mathcal{C}_1(\mathcal{H})$ is self-adjoint. Let $K = \sum_{n\geq 1} \lambda_n\eta_n\eta_n^*$ be a decomposition of K as given by Corollary 3 (where $\lambda_n \in \mathbb{R}$). Since $|K| = (K^*K)^{\frac{1}{2}} = (K^2)^{\frac{1}{2}}$ and it is easy to verify that

$$K^2 = \sum_{n\geq 1} \lambda_n^2 \eta_n\eta_n^*$$

we obtain that $|K| = \sum_{n\geq 1} |\lambda_n|\eta_n\eta_n^*$. Therefore, as $K \in \mathcal{C}_1(\mathcal{H})$, $\sum_{n\geq 1} |\lambda_n| < \infty$. Define

$$K_+ = \sum_{n\geq 1} \max\{0, \lambda_n\} h_n h_n^* \quad \text{and} \quad K_- = \sum_{n\geq 1} (-\min\{\lambda_n, 0\}) h_n h_n^*$$

It is easy to check that K_+ and K_- are well-defined compact operators such that $K = K_+ - K_-$ and $K_+, K_- \geq 0$. Since $\sum_{n\geq 1} |\lambda_n| < \infty$, we obtain that $K_+, K_- \in \mathcal{C}_1(\mathcal{H})$. Therefore every self-adjoint element of $\mathcal{C}_1(\mathcal{H})$ is a difference of two positive elements of $\mathcal{C}_1(\mathcal{H})$.

Next suppose $K \in \mathcal{C}_1(\mathcal{H})$ was arbitrary. Then $K^* \in \mathcal{C}_1(\mathcal{H})$ by Proposition 14. Since $\mathcal{C}_1(\mathcal{H})$ is a vector space, $Re(K), Im(K) \in \mathcal{C}_1(\mathcal{H})$ and $K = Re(K) + iIm(K)$. Since $Re(K)$ and $Im(K)$ are self-adjoint elements of $\mathcal{C}_1(\mathcal{H})$, each is the difference of two positive elements of $\mathcal{C}_1(\mathcal{H})$. Whence K is a linear combination of four positive elements of $\mathcal{C}_1(\mathcal{H})$. $\square$

The following lemma is the last result we need in order to prove that $\mathcal{C}_1(\mathcal{H})$ is a Banach space.

Lemma 16. *Let K be a compact operator. If $\{\zeta_k\}_{k=1}^n$ and $\{\omega_k\}_{k=1}^n$ are orthonormal families, then $\sum_{k=1}^n |\langle K\zeta_k, \omega_k\rangle| \leq \sum_{k=1}^n s_k(K)$.*

PROOF: Let U be the rank n partial isometry such that $U\omega_k = \zeta_k$ for $1 \leq k \leq n$. By Lemma 13 $s_k(KU) \leq \|U\|\, s_k(K) = s_k(K)$ for $1 \leq k \leq n$. Moreover and $s_k(KU) = 0$ if $k > n$ as $\mathrm{rank}(KU) \leq n$ so that $\mathrm{rank}(|KU|) \leq n$ so KU can have at most n non-zero singular values. Thus, as KU is compact, we obtain by Lemma 10 that

$$\sum_{k=1}^n |\langle K\zeta_k, \omega_k\rangle| = \sum_{k=1}^n |\langle KU\omega_k, \omega_k\rangle| \leq \|KU\|_1 = \sum_{k=1}^n s_k(KU) \leq \sum_{k=1}^n s_k(K)$$

as desired. $\square$

Theorem 17. *$\mathcal{C}_1(\mathcal{H})$ is complete in the trace norm and the ideal of finite rank operators is dense in $\mathcal{C}_1(\mathcal{H})$.*

PROOF: To see that $\mathcal{C}_1(\mathcal{H})$ is complete, let $(K_n)_{n\geq 1}$ be any Cauchy sequence in $\mathcal{C}_1(\mathcal{H})$. Then we notice that $\|K_n - K_m\| \leq \|K_n - K_m\|_1$ by Corollary 7. Therefore $(K_n)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{K}(\mathcal{H})$. Hence, since the compact operators are complete, $K_n \to K \in \mathfrak{K}(\mathcal{H})$ in the operator norm. Thus we need now show only two things: that $\|K\|_1 < \infty$ and $\|K - K_n\|_1 \to 0$.

Since $\|K_n - K\| \to 0$, $\langle K\zeta, \omega\rangle = \lim_{n\to\infty}\langle K_n\zeta, \omega\rangle$ for all $\zeta, \omega \in \mathcal{H}$. Let $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ be a decomposition of K given by Corollary 4. It is a trivial computation to verify that $s_n(K) = \langle K\eta_n, \xi_n\rangle$ for all $n \in \mathbb{N}$ as $\{\xi_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ are orthogonal sets. Therefore for all $N \in \mathbb{N}$

$$\begin{aligned}
\sum_{n=1}^N s_n(K) &= \sum_{n=1}^N |\langle K\eta_n, \xi_n\rangle| \\
&= \lim_{j\to\infty} \sum_{n=1}^N |\langle K_j\eta_n, \xi_n\rangle| \\
&\leq \limsup_{j\to\infty} \sum_{n=1}^N s_n(K_j) \qquad \text{by Lemma 16} \\
&\leq \limsup_{j\to\infty} \|K_j\|_1 < \infty
\end{aligned}$$

since $(\|K_j\|_1)_{j\geq 1}$ forms a Cauchy sequence in $\mathbb{R}$ by the reverse triangle inequality. Consequently $\sum_{n=1}^N s_n(K) \leq \limsup_{j\to\infty} \|K_j\|_1$ for all $N \in \mathbb{N}$ and thus $\|K\|_1 < \infty$. Hence $K \in \mathcal{C}_1(\mathcal{H})$.

Using the same proof as above, it can be shown that $\|K - K_\ell\|_1 \leq \limsup_{j\to\infty} \|K_j - K_\ell\|_1$ for all $\ell \in \mathbb{N}$. Since $\limsup_{j\to\infty} \|K_j - K_\ell\|_1 \to 0$ as $\ell \to \infty$, $\lim_{\ell\to\infty} \|K - K_\ell\|_1 = 0$. Whence $K_n \to K$ in $\mathcal{C}_1(\mathcal{H})$ so that $\mathcal{C}_1(\mathcal{H})$ is complete.

To see that the ideal of finite rank operators is dense in $\mathcal{C}_1(\mathcal{H})$ we first need to show that each finite rank operator is in $\mathcal{C}_1(\mathcal{H})$. However, if F is a finite rank operator, F is clearly compact and $|F|$ is a positive finite rank operator. Since $|F|$ is a positive finite rank operator, $|F|$ only has a finite number of non-zero eigenvalues counting multiplicity. Therefore only a finite number of singular values of F are non-zero so $F \in \mathcal{C}_1(\mathcal{H})$.

To see that the finite rank operators are dense in $\mathcal{C}_1(\mathcal{H})$, let $K \in \mathcal{C}_1(\mathcal{H})$ and let $K = \sum_{n\geq 1} s_n(K)\xi_n\eta_n^*$ be a decomposition of K from Corollary 4. Then $F_N = \sum_{n=1}^N s_n(K)\xi_n\eta_n^*$ is a finite rank operator and $\|K - F_N\|_1 = \sum_{n\geq N+1} s_n(K)$ by Lemma 6 as $K - F_n = \sum_{n\geq N+1} s_n(K)\xi_n\eta_n^*$. Since $\sum_{n\geq N+1} s_n(K) \to 0$ as $N \to \infty$, $(F_N)_{N\geq 1}$ converges to K in $\mathcal{C}_1(\mathcal{H})$. Therefore the finite rank operators are dense in $\mathcal{C}_1(\mathcal{H})$. $\square$

The next important development in the theory of trace class operator is the trace (hence the name!). The above results allow us to make the following definition.

Definition 18. Let $\{\xi_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis for $\mathcal{H}$. We define a trace $Tr : \mathcal{C}_1(\mathcal{H}) \to \mathbb{C}$ by $Tr(K) = \sum_{\lambda\in\Lambda}\langle K\xi_\lambda, \xi_\lambda\rangle$. Note that the sum converges, Tr is linear, and $\|Tr\| \leq 1$ by Lemma 9.

Based on the above definition, it is possible that we get a different trace for each orthonormal basis of $\mathcal{H}$. We will show below that this is not true; that is, every orthonormal basis defines the same trace. This trace will be essential in our proofs that $\mathfrak{K}(\mathcal{H})^* \simeq \mathcal{C}_1(\mathcal{H})$ and $(\mathcal{C}_1(\mathcal{H}))^* \simeq \mathcal{B}(\mathcal{H})$.

Traditional, a trace τ on a C*-algebra $\mathfrak{A}$ has the property that $\tau(A_1A_2) = \tau(A_2A_1)$ for all $A_1, A_2 \in \mathfrak{A}$. We will now show that Tr satisfies this property.

Proposition 19. *Each trace Tr given in Definition 18 is independent of the choice of basis (and thus they are all the same linear functional). Moreover $Tr(KT) = Tr(TK)$ for all $K \in \mathcal{C}_1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$.*

PROOF: Fix an orthonormal basis $\{\xi_\lambda\}_{\lambda\in\Lambda}$ and define the trace by $Tr(K) = \sum_{\lambda\in\Lambda}\langle K\xi_\lambda, \xi_\lambda\rangle$ for all $K \in \mathcal{C}_1(\mathcal{H})$. Fix $K \in \mathcal{C}_1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$. We would like that the following computation is valid:

$$\begin{aligned}
Tr(TK) &= \sum_{\lambda\in\Lambda}\langle TK\xi_\lambda, \xi_\lambda\rangle \\
&= \sum_{\lambda\in\Lambda}\langle K\xi_\lambda, T^*\xi_\lambda\rangle \\
&= \sum_{\lambda\in\Lambda}\sum_{\alpha\in\Lambda}\langle K\xi_\lambda, \xi_\alpha\rangle\langle\xi_\alpha, T^*\xi_\lambda\rangle \\
&= \sum_{\alpha\in\Lambda}\sum_{\lambda\in\Lambda}\langle K\xi_\lambda, \xi_\alpha\rangle\langle\xi_\alpha, T^*\xi_\lambda\rangle \\
&= \sum_{\alpha\in\Lambda}\sum_{\lambda\in\Lambda}\langle \xi_\lambda, K^*\xi_\alpha\rangle\langle T\xi_\alpha, \xi_\lambda\rangle \\
&= \sum_{\alpha\in\Lambda}\langle T\xi_\alpha, K^*\xi_\alpha\rangle \\
&= \sum_{\alpha\in\Lambda}\langle KT\xi_\alpha, \xi_\alpha\rangle = Tr(KT)
\end{aligned}$$

but it is difficult to see why we can interchange the sums (i.e. we would need absolute convergence of the double sums which is not apparent). We will be clever to get around this.

First suppose that $T = U$ is a unitary and K is a positive operator in $\mathcal{C}_1(\mathcal{H})$. For each $\lambda \in \Lambda$ let $f_\lambda = Ue_\lambda$

so that $\{f_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis for $\mathcal{H}$. Then $U^*KU \in \mathcal{C}_1(\mathcal{H})$ by Proposition 14 and

$$\begin{aligned}
Tr(U^*KU) = \sum_{\lambda\in\Lambda}\langle U^*KUe_\lambda, e_\lambda\rangle &= \sum_{\lambda\in\Lambda}|\langle Kf_\lambda, f_\lambda\rangle| \\
&= \sum_{\lambda\in\Lambda}\left\|K^{\frac{1}{2}}f_\lambda\right\|^2 \\
&= \sum_{\lambda\in\Lambda}\sum_{\alpha\in\Lambda}|\langle K^{\frac{1}{2}}f_\lambda, e_\alpha\rangle|^2 \\
&= \sum_{\alpha\in\Lambda}\sum_{\lambda\in\Lambda}|\langle f_\lambda, K^{\frac{1}{2}}e_\alpha\rangle|^2 \\
&= \sum_{\alpha\in\Lambda}\left\|K^{\frac{1}{2}}e_\alpha\right\|^2 \\
&= \sum_{\alpha\in\Lambda}\langle Ke_\alpha, e_\alpha\rangle = Tr(K)
\end{aligned}$$

where the interchanging of sums is valid as all terms in the sum are positive.

Next suppose $K \in \mathcal{C}_1(\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{H})$ is a unitary. By Lemma 15 we may write $K = \sum_{i=1}^4 a_iP_i$ where $P_i \in \mathcal{C}_1(\mathcal{H})$ are positive and $a_i \in \mathbb{C}$. Since $U^*KU, U^*P_iU, P_i, K \in \mathcal{C}_1(\mathcal{H})$ by Proposition 14 and since the trace is linear on $\mathcal{C}_1(\mathcal{H})$,

$$\begin{aligned}
Tr(U^*KU) &= Tr\left(U^*\left(\sum_{i=1}^4 a_iP_i\right)U\right) \\
&= \sum_{i=1}^4 a_iTr(U^*P_iU) \\
&= \sum_{i=1}^4 a_iTr(P_i) \\
&= Tr(K)
\end{aligned}$$

Therefore $Tr(U^*KU) = Tr(K)$ for all $K \in \mathcal{C}_1(\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{H})$ unitary. Moreover if $K \in \mathcal{C}_1(\mathcal{H})$ and U is unitary then $UK \in \mathcal{C}_1(\mathcal{H})$ so $Tr(KU) = Tr(U^*(UK)U) = Tr(UK)$. Whence $Tr(KU) = Tr(UK)$ for all $K \in \mathcal{C}_1(\mathcal{H})$ and $U \in \mathcal{B}(\mathcal{H})$ unitary.

Next suppose $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_1(\mathcal{H})$ are arbitrary. Then we may write $T = \sum_{i=1}^4 b_iU_i$ where U_i are unitaries and $b_i \in \mathbb{C}$. Since Tr is linear and $KU_i, U_iK \in \mathcal{C}_1(\mathcal{H})$ for all i,

$$Tr(TK) = Tr\left(\sum_{i=1}^4 \beta_iU_iK\right) = \sum_{i=1}^4 b_iTr(U_iK) = \sum_{i=1}^4 b_iTr(KU_i) = Tr(KT)$$

as desired. Hence $Tr(TK) = Tr(KT)$ for all $K \in \mathcal{C}_1(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$.

Now suppose $\{f_\lambda\}_{\lambda\in\Lambda}$ is another orthonormal basis for $\mathcal{H}$. Define $U \in \mathcal{B}(\mathcal{H})$ by $Ue_\lambda = f_\lambda$ and extend by linearity. It is clear that U is a unitary operator. Moreover, for all $K \in \mathcal{C}_1(\mathcal{H})$, $Tr(U^*KU) = Tr(U^*(KU)) = Tr((KU)U^*) = Tr(K)$. Therefore

$$Tr(K) = Tr(U^*KU) = \sum_{\lambda\in\Lambda}\langle U^*KUe_\lambda, e_\lambda\rangle = \sum_{\lambda\in\Lambda}\langle Kf_\lambda, f_\lambda\rangle$$

Thus Tr is independent of the choice of basis. $\square$

It turns out that the trace class operators can be defined via the above trace as the next lemma will demonstrate. In addition, the next lemma will prove a useful fact about the trace of finite rank operators.

Lemma 20. *If $K \in \mathcal{C}_1(\mathcal{H})$ is positive then $Tr(K) = \|K\|_1$. Moreover $Tr(\xi\eta^*) = \langle \xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.*

Proof: Suppose $K \in \mathcal{C}_1(\mathcal{H})$ is positive. Let $K = \sum_{n\geq 1} t_n \eta_n \eta_n^*$ be a decomposition from Corollary 3. Since K is a positive operator, $t_n \geq 0$ for all n and thus we may rearrange this decomposition so that $t_n \geq t_{n+1}$ for all $n \in \mathbb{N}$. Therefore $t_n = s_n(K)$ by Lemma 6 so that $\|K\|_1 = \sum_{n\geq 1} t_n$. Extend $\{\eta_n\}_{n\geq 1}$ to an orthonormal basis $\{\eta_n\}_{n\geq 1} \cup \{\xi_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{H}$. Since the trace is independent of the basis selected, we obtain that

$$\begin{aligned}
Tr(K) &= \sum_{n\geq 1} \langle T\eta_n, \eta_n\rangle + \sum_{\lambda\in\Lambda} \langle T\xi_\lambda, \xi_\lambda\rangle \\
&= \sum_{n\geq 1} t_n \langle \eta_n, \eta_n\rangle \qquad \eta_n \perp \xi_\lambda, \eta_m \perp \eta_n \text{ for all } \lambda \in \Lambda, n \neq m \\
&= \|K\|_1
\end{aligned}$$

as claimed.

Fix $\xi, \eta \in \mathcal{H}$. If $\eta = 0$ then $\xi\eta^* = 0$ so $Tr(\xi\eta^*) = 0 = \langle \xi, \eta\rangle$. If $\eta \neq 0$ extend $\left\{\frac{\eta}{\|\eta\|}\right\}$ to an orthonormal basis $\left\{\frac{\eta}{\|\eta\|}\right\} \cup \{\xi_\lambda\}_{\lambda\in\Lambda}$ for $\mathcal{H}$. Since the trace is independent of the basis selected, we obtain that

$$\begin{aligned}
Tr(\xi\eta^*) &= \left\langle (\xi\eta^*)\frac{\eta}{\|\eta\|}, \frac{\eta}{\|\eta\|} \right\rangle + \sum_{\lambda\in\Lambda} \langle (\xi\eta^*)\xi_\lambda, \xi_\lambda\rangle \\
&= \left\langle \frac{\|\eta\|^2}{\|\eta\|}\xi, \frac{\eta}{\|\eta\|} \right\rangle \qquad \eta \perp \xi_\lambda \text{ for all } \lambda \in \Lambda \\
&= \langle \xi, \eta\rangle
\end{aligned}$$

as claimed. □

Now we are finally ready to begin the proof that $(\mathcal{C}_1(\mathcal{H}))^* \simeq \mathcal{B}(\mathcal{H})$. First we will use the trace to show that every trace class operator defines a linear functional on $\mathcal{B}(\mathcal{H})$.

Lemma 21. *Let $T \in \mathcal{B}(\mathcal{H})$ and define $\varphi_T : \mathcal{C}_1(\mathcal{H}) \to \mathbb{C}$ by $\varphi_T(K) = Tr(TK)$. Then φ_T is linear and $\|\varphi_T\| = \|T\|$.*

Proof: First φ_T is well-defined as $TK \in \mathcal{C}_1(\mathcal{H})$ whenever $T \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{C}_1(\mathcal{H})$ by Proposition 14. Moreover, since Tr is a linear functional, φ_T is also a linear functional. Moreover, since $\|Tr\| \leq 1$, we obtain for all $K \in \mathcal{C}_1(\mathcal{H})$ that

$$|\varphi_T(K)| = |Tr(TK)| \leq \|TK\|_1 \leq \|T\| \, \|K\|_1$$

where the last inequality is by Proposition 14. Therefore $\|\varphi_T\| \leq \|T\|$.

To see that $\|\varphi_T\| = \|T\|$, let $\epsilon > 0$. If $T = 0$, the result follows immediately since $\varphi_0 = 0$. Thus we may assume that $\|T\| > 0$ and $\epsilon < \|T\|$. Choose a $\zeta \in \mathcal{H}$ with $\|\zeta\| \leq 1$ such that $\|T\zeta\| > \|T\| - \epsilon > 0$. Consider $\zeta' = \frac{T\zeta}{\|T\zeta\|}$. Then $\|\zeta'\| = 1$. Define $F = \zeta(\zeta')^* \in \mathcal{B}(\mathcal{H})$. Therefore $\|F\| = \|F(\zeta')\| = \|\zeta\| \leq 1$, and F is a rank one operator. Therefore $F \in \mathcal{C}_1(\mathcal{H})$. However, by Lemma 20,

$$|\varphi_T(F)| = |Tr(T(\zeta(\zeta')^*))| = |Tr((T\zeta)(\zeta')^*)| = |\langle T\zeta, \zeta'\rangle| = \left| \left\langle T\zeta, \frac{T\zeta}{\|T\zeta\|} \right\rangle \right| = \|T\zeta\| > \|T\| - \epsilon$$

Therefore, since $\|F\| \leq 1$, we have that $\|\varphi_T\| \geq \|T\| - \epsilon$. However, as this holds for all $\|T\| > \epsilon > 0$, $\|\varphi_T\| = \|T\|$ as desired. □

Lemma 22. *Let $\varphi \in (\mathcal{C}_1(\mathcal{H}))^*$. Then there exists a $A \in \mathcal{B}(\mathcal{H})$ such that $\varphi = \varphi_A$ and $\|A\| = \|\varphi\|$.*

Proof: Recall for all $\xi, \eta \in \mathcal{H}$ that $\xi\eta^*$ is of finite rank and thus in $\mathcal{C}_1(\mathcal{H})$. Whence we may define a sesquilinear form on $\mathcal{H}$ by $[\xi, \eta] = \varphi(\xi\eta^*)$. It is clear that $[\xi, \eta]$ is linear in the first component and conjugate linear in the second component so this is indeed a sesquilinear form.

We claim that this sesquilinear form is 'bounded'. To see this, we claim that $\|\xi\eta^*\|_1 = \|\xi\| \|\eta\|$ for all $\xi, \eta \in \mathcal{H}$. To see this, we notice that the result is trivial if $\xi = 0$ or $\eta = 0$. Otherwise

$$\xi\eta^* = (\|\xi\| \|\eta\|) \left(\frac{\xi}{\|\xi\|}\right) \left(\frac{\eta}{\|\eta\|}\right)^*$$

and thus Lemma 6 implies that $\|\xi\| \|\eta\|$ is the only non-zero singular value of $\xi\eta^*$.

To see that the sesquilinear form is 'bounded', we notice for all $\xi, \eta \in \mathcal{H}$ that

$$|[\xi, \eta]| = |\varphi(\xi\eta^*)| \leq \|\varphi\| \|\xi\eta^*\|_1 = \|\varphi\| \|\xi\| \|\eta\|$$

(the above statement is what is means for a sesquilinear form to be bounded).

Now we will use this bounded sesquilinear form to create our operator A. To begin we notice that, since the sesquilinear form is bounded, the linear functional $\phi_\eta : \mathcal{H} \to \mathbb{C}$ defined by $\phi_\eta(\xi) = [\xi, \eta]$ is bounded for each fixed $\eta \in \mathcal{H}$. Hence, by the Riesz Representation Theorem, there exists a vector $A^*_\eta \in \mathcal{H}$ so that

$$\varphi(\xi\eta^*) = (\xi, \eta) = \phi_\eta(\xi) = \langle \xi, A^*_\eta \rangle$$

for all $\xi \in \mathcal{H}$. Define a map $A^* : \mathcal{H} \to \mathcal{H}$ by $A^*\eta = A^*_\eta$. Since $[\xi, \eta]$ is conjugate linear in the second component, A^* is a linear map. Moreover

$$|\langle \xi, A^*\eta \rangle| = |[\xi, \eta]| \leq \|\varphi\| \|\xi\| \|\eta\|$$

for all $\xi, \eta \in \mathcal{H}$ so that A^* is bounded with norm at most $\|\varphi\|$. Therefore there exists an $A \in \mathcal{B}(\mathcal{H})$ with norm at most $\|\varphi\|$ such that (by applying Lemma 20)

$$\varphi(\xi\eta^*) = [\xi, \eta] = \langle A\xi, \eta \rangle = Tr((A\xi)\eta^*) = Tr(A(\xi\eta^*))$$

Therefore, by the linearity of the trace and φ, we obtain that $\varphi(F) = Tr(AF) = \varphi_A(F)$ for all F of finite rank. However, since φ_A and φ are continuous and the finite rank operators are dense in $\mathcal{C}_1(\mathcal{H})$ by Theorem 17, we obtain that $\varphi_A = \varphi$. Thus $\|A\| = \|\varphi_A\| = \|\varphi\|$ by Lemma 21. □

Theorem 23. *$\mathcal{B}(\mathcal{H})$ is the dual space of $\mathcal{C}_1(\mathcal{H})$.*

Proof: Lemmas 20 and 21 together show that the map $\Phi : \mathcal{B}(\mathcal{H}) \to (\mathcal{C}_1(\mathcal{H}))^*$ given by $\Phi(T) = \varphi_T$ is isometric and onto. Moreover, due to the linearity of the trace, $\varphi_{\lambda T+S}(K) = Tr((\lambda T + S)(K)) = \lambda Tr(TK) + Tr(SK) = (\lambda\varphi_T + \varphi_S)(K)$ for all $K \in \mathcal{C}_1(\mathcal{H})$, $\lambda \in \mathbb{C}$, and $T, S \in \mathcal{B}(\mathcal{H})$. Hence Φ is an isometric isomorphism and therefore the result follows. □

Our next goal is to show that $(\mathfrak{K}(\mathcal{H}))^* \simeq \mathcal{C}_1(\mathcal{H})$. Before we can show this, we need to know another fact about the trace class operators. Recall in Lemma 10 that we developed another definition of the trace class norm $\|\cdot\|_1$ for compact operators by considering supremums over all pairs of orthonormal sets. The following lemma shows that if this supremum is finite for a $T \in \mathcal{B}(\mathcal{H})$ then T is automatically compact and thus a trace class operator.

Lemma 24. *Let $T \in \mathcal{B}(\mathcal{H})$. Suppose*

$$\sup \left\{ \sum_{n\geq 1} |\langle T\zeta_n, \omega_n \rangle| \;\middle|\; \{\zeta_n\}_{n\geq 1}, \{\omega_n\}_{n\geq 1} \text{ orthonormal subsets of } \mathcal{H} \right\} < \infty$$

Then T is a trace class operator.

PROOF: Let $M_0 = \sup\left\{\sum_{n\geq 1}|\langle T\zeta_n, \omega_n\rangle| \mid \{\zeta_n\}_{n\geq 1}, \{\omega_n\}_{n\geq 1} \text{ orthonormal subsets of } \mathcal{H}\right\} < \infty$. If $T \in \mathfrak{K}(\mathcal{H})$ then $T \in \mathcal{C}_1(\mathcal{H})$ by the expression for the trace class norm from Lemma 10. Thus it suffices to show that T is compact.

To begin, we claim that $\sum_{n\geq 1}\|T\xi_n\|^2 \leq \alpha M_0$ for any orthonormal sequence $\{\xi_n\}_{n\geq 1}$ where α is some finite constant independent of the sequence selected. To see this, we recall that $T = \sum_{j=1}^4 \alpha_j U_j$ where U_j are unitaries. Whence

$$\begin{aligned}
\sum_{n\geq 1}\|T\xi_n\|^2 &= \sum_{n\geq 1}|\langle T\xi_n, T\xi_n\rangle| \\
&\leq \sum_{n\geq 1}\sum_{j=1}^{4}|\alpha_j||\langle T\xi_n, U_j\xi_n\rangle| \\
&= \sum_{j=1}^{4}|\alpha_j|\left(\sum_{n\geq 1}|\langle T\xi_n, U_i\xi_n\rangle|\right) \\
&\leq \left(\sum_{j=1}^{4}|\alpha_j|\right)M_0
\end{aligned}$$

for every orthonormal sequence $\{\xi_n\}_{n\geq 1}$ (since if U_j is a unitary, $\{U_j\xi_n\}_{n\geq 1}$ is an orthonormal sequence).

Let $M = \left(\sum_{i=1}^4|\alpha_i|\right)M_0$. Let $\{\eta_\lambda\}_{\lambda\in\Lambda}$ be an orthonormal basis for $(ker(T))^\perp$. Then $\|T\eta_\lambda\| > 0$ for all λ. Since $\sum_{n\geq 1}\|T\xi_n\|^2 \leq M$ for any orthonormal set $\{\xi_n\}_{n\geq 1}$, for all $k \in \mathbb{N}$ $\{\lambda \mid \|T\eta_\lambda\| > \frac{1}{k}\}$ must be countable. Whence $\{\eta_\lambda\}_{\lambda\in\Lambda}$ is a countable orthonormal basis for $(ker(T))^\perp$.

Let $\{\eta_n\}_{n\geq 1}$ be an orthonormal basis for $(ker(T))^\perp$ and let $M' = \sum_{n\geq 1}\|T\eta_n\|^2 \leq M$. Suppose that $\{\zeta_n\}_{n\geq 1}$ is another basis for $(ker(T))^\perp$. By restricting our focus to $(ker(T))^\perp = ker(|T|)^\perp = ker(|T|^{\frac{1}{2}})^\perp = \overline{ran(|T|^{\frac{1}{2}})}$, we see that

$$\begin{aligned}
M' &= \sum_{n\geq 1}\|T\eta_n\|^2 \\
&= \sum_{n\geq 1}|\langle T^*T\eta_n, \eta_n\rangle| \\
&= \sum_{n\geq 1}\left\||T|^{\frac{1}{2}}\eta_n\right\|^2 \\
&= \sum_{n\geq 1}\sum_{k\geq 1}|\langle |T|^{\frac{1}{2}}\eta_n, \zeta_k\rangle|^2 \qquad \{\zeta_k\}_{k\geq 1} \text{ is an orthonormal basis for } (ker(T))^\perp = \overline{ran(|T|^{\frac{1}{2}})} \\
&= \sum_{k\geq 1}\sum_{n\geq 1}|\langle \eta_n, |T|^{\frac{1}{2}}\zeta_k\rangle|^2 \\
&= \sum_{k\geq 1}\left\||T|^{\frac{1}{2}}\zeta_k\right\|^2 \qquad \{\eta_n\}_{n\geq 1} \text{ is an orthonormal basis for } (ker(T))^\perp \\
&= \sum_{k\geq 1}|\langle T^*T\zeta_k, \zeta_k\rangle| \\
&= \sum_{k\geq 1}\|T\zeta_k\|^2
\end{aligned}$$

(where we can interchange the sums as every term is positive).

We claim that $F_m := \sum_{n=1}^m (T\eta_n)\eta_n^* \to T$ in norm as $m \to \infty$. Since each F_m is a finite rank operator, this will force T to be compact as desired. To see that $F_m \to T$, let $\epsilon > 0$. We notice since $\sum_{n\geq 1} \|T\eta_n\|^2 = M'$ there exists an $N \in \mathbb{N}$ such that $M' - \sum_{n=1}^N \|T\eta_n\|^2 < \epsilon^2$. If $\zeta \in (ker(T))^\perp \setminus \{0\}$ is orthogonal to $\{\eta_1, \ldots, \eta_N\}$, then by extending $\left\{\eta_1, \ldots, \eta_N, \frac{\zeta}{\|\zeta\|}\right\}$ to an orthonormal basis for $(ker(T))^\perp$ and using the above we obtain $\|T\zeta\|^2 \leq \|\zeta\|^2 \left(M' - \sum_{n=1}^N \|T\eta_n\|^2\right) < \epsilon^2 \|\zeta\|^2$. Let $\xi \in \mathcal{H}$ be such that $\|\xi\| \leq 1$ and write $\xi = \sum_{n\geq 1} \alpha_n \eta_n + \omega$ where $\omega \in ker(T)$. Since $\omega \in ker(T)$, $\omega \perp \eta_n$ for all n so $F_m \eta = 0$ for all m. Then if $k \geq N$

$$\begin{aligned}
\|(T - F_k)\xi\| &= \left\| \sum_{n\geq 1} \alpha_n (T\eta_n - F_k\eta_n) \right\| \\
&= \left\| \sum_{n\geq 1}^{k} \alpha_n (T\eta_n - F_k\eta_n) + \sum_{n\geq k+1} \alpha_n (T\eta_n - F_k\eta_n) \right\| \\
&= \left\| \sum_{n\geq 1}^{k} \alpha_n (T\eta_n - T\eta_n) + \sum_{n\geq k+1} \alpha_n (T\eta_n - 0) \right\| \\
&= \left\| \sum_{n\geq k+1} \alpha_n T\eta_n \right\| \\
&= \left\| T\left(\sum_{n\geq k+1} \alpha_n \eta_n \right) \right\|
\end{aligned}$$

Since $\sum_{n\geq k+1} \alpha_n \eta_n \in (ker(T))^\perp$ converges and is orthogonal to $\{\eta_1, \ldots, \eta_N\}$ as $k \geq N$,

$$\|(T - F_k)\xi\| = \left\| T\left(\sum_{n\geq k+1} \alpha_n \eta_n \right) \right\| \leq \left\| \sum_{n\geq k+1} \alpha_n \eta_n \right\| \epsilon \leq \|\xi\| \epsilon \leq \epsilon$$

As $\xi \in \mathcal{H}$ with $\|\xi\| \leq 1$ was arbitrary, $\|T - F_k\| \leq \epsilon$ for all $k \geq N$. Hence $F_m \to T$ as desired. $\square$

Theorem 25. *The dual of the compact operators is $\mathcal{C}_1(\mathcal{H})$.*

SOLUTION: For each $T \in \mathcal{C}_1(\mathcal{H})$ define $\varphi_T \in \mathfrak{K}^*$ by $\varphi_T(K) = Tr(TK)$ (which makes sense as T a trace class operator). Since $|\varphi_T(K)| \leq \|TK\|_1 \leq \|K\| \|T\|_1$, each φ_T is continuous with $\|\varphi_T\| \leq \|T\|_1$. Since T is a trace class operator, there exists orthonormal sets $\{\xi_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ such that $\|T\|_1 = \sum_{n\geq 1} \langle T\xi_n, \eta_n\rangle$ by Lemma 10. For each $m \in \mathbb{N}$ let $F_m = \sum_{n=1}^m \xi_n \eta_n^*$. Then F_m is a finite rank operator with $\|F_m\| = 1$ (as $F_m^* F_m = \sum_{n=1}^m \eta_n \eta_n^*$ is a projection). However

$$\|\varphi_T\| \geq |\varphi_T(F_m)| = \left| \sum_{n=1}^m Tr(T(\xi_n \eta_n^*)) \right| = \left| \sum_{n=1}^m \langle T\xi_n, \eta_n\rangle \right|$$

where the last inequality is by Lemma 20. Since $\sum_{n=1}^m \langle T\xi_n, \eta_n\rangle \to \|T\|_1$ as $m \to \infty$, $\|\varphi_T\| = \|T\|_1$.

Define $\Phi : \mathcal{C}_1(\mathcal{H}) \to \mathfrak{K}(\mathcal{H})^*$ by $\Phi(T) = \varphi_T$. Since Tr is linear, $\varphi_{\lambda T_1 + T_2} = \lambda \varphi_{T_1} + \varphi_{T_2}$ so Φ is linear. Moreover Φ is isometric from above. To show that the dual of the compact operators is $\mathcal{C}_1(\mathcal{H})$, we simply need to show that Φ is surjective.

Let $\psi \in \mathfrak{K}(\mathcal{H})^*$. Since $\mathcal{C}_1(\mathcal{H}) \subseteq \mathfrak{K}(\mathcal{H})$ (as sets) and $\|K\| \leq \|K\|_1$ for all $K \in \mathcal{C}_1(\mathcal{H})$ by Corollary 7, $|\psi(K)| \leq \|\psi\| \|K\| \leq \|\psi\| \|K\|_1$. Whence $\psi \in \mathcal{C}_1(\mathcal{H})^*$. By Theorem 23 there exists a $T \in \mathcal{B}(\mathcal{H})$ such that $\psi(K) = \varphi_T(K)$ for all $K \in \mathcal{C}_1(\mathcal{H})$. Moreover $|\varphi_T(K)| \leq \|\psi\| \|K\|$ for all $K \in \mathcal{C}_1(\mathcal{H})$. We desire to show that $T \in \mathcal{C}_1(\mathcal{H})$ by applying Lemma 24.

Suppose $\{\xi_n\}_{n\geq 1}$ and $\{\eta_n\}_{n\geq 1}$ are arbitrary orthonormal set. For each n choose $\theta \in [0, 2\pi)$ such that $|\langle T\xi_n, \eta_n\rangle| = e^{i\theta_n}\langle T\xi_n, \eta_n\rangle$. For each $m \in \mathbb{N}$ define $F_m = \sum_{n=1}^{m} e^{i\theta_n}\xi_n\xi_n^* \in \mathbb{N}$. Then F_m is a finite rank operator, $\|F_m\| = 1$, and

$$\sum_{n=1}^{m} |\langle T\xi_n, \eta_n\rangle| = \left|\sum_{n=1}^{m} e^{i\theta_n}\langle T\xi_n, \eta_n\rangle\right| = |\varphi_T(F_m)| \leq \|\psi\|$$

By letting $m \to \infty$ we obtain that $\sum_{n\geq 1} |\langle T\xi_n, \eta_n\rangle| \leq \|\psi\|$ for every pair of orthonormal sequences $\{\xi_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$. Thus $T \in \mathcal{C}_1(\mathcal{H})$ by Lemma 24.

Since T is of trace class, $\psi = \varphi_T$ on $\mathcal{C}_1(\mathcal{H})$. Since $\mathcal{C}_1(\mathcal{H})$ contains the finite rank operators and the finite rank operators are dense in $\mathfrak{K}(\mathcal{H})$, $\psi = \Phi(T)$ on $\mathfrak{K}(\mathcal{H})$. Whence Φ is surjective as desired. $\square$

Refernces

K.R. Davidson, *Nest algebras: triangular forms for operator algebras on Hilbert space*, Longman Scientific and Technical, 1988