Hilbert C*-Bimodules

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Abstract

The purpose of these notes is to develop the basic theory and examples of Hilbert C*-Bimodules. A reader of these notes should be familar with the basics of C*-algebra theory mainly pertaining to the notion of a positive operator. All inner products in these notes will be linear in the second variable.

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We all know how important Hilbert spaces are. The question is "Can we find something more general that Hilbert spaces that have a rich structure?" The first step in this process would be to generalize the inner product. Upon looking at the theory of Hilbert space, we see that most of the basic property of an inner product is that it maps into $\mathbb{C}$ where there is a notion of positivity and we can take adjoints. Thus it makes sense to allow inner products to take on values in a C*-algebra.

We begin with a few definitions. For these notes all inner products will be linear in the second component.

Definition Let X be a vector space and $\mathfrak{B}$ a C*-algebra. A $\mathfrak{B}$-valued positive sesquilinear form on X is a map $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ such that $\langle \cdot, \cdot \rangle$ is linear in the second component, conjugate linear in the first component, and $\langle x, x\rangle \geq_{\mathfrak{B}} 0$ for all $x \in X$.

Remarks If $\langle \cdot, \cdot \rangle$ is a $\mathfrak{B}$-valued positive sesquilinear form on X, then $\langle x, y\rangle^* = \langle y, x\rangle$ for all $x, y \in X$. To see this, we notice since $\langle \cdot, \cdot \rangle$ is sesquilinear

$$\langle x, y\rangle = \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle + i\langle x+iy, x+iy\rangle - i\langle x-iy, x-iy\rangle\right)$$

for all $x, y \in X$. Since $\langle z, z\rangle \geq 0$ for all $z \in X$, $\langle z, z\rangle^* = \langle z, z\rangle$ so

$$\begin{aligned}
\langle x, y\rangle^* &= \frac{1}{4}\left(\langle x+y, x+y\rangle^* - \langle x-y, x-y\rangle^* + (i\langle x+iy, x+iy\rangle)^* - (i\langle x-iy, x-iy\rangle)^*\right) \\
&= \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle - i\langle x+iy, x+iy\rangle + i\langle x-iy, x-iy\rangle\right) \\
&= \frac{1}{4}\left(\langle x+y, x+y\rangle - \langle y-x, y-x\rangle - i\langle -ix+y, -ix+y\rangle + i\langle ix+y, ix+y\rangle\right) \\
&= \langle x, y\rangle
\end{aligned}$$

as claimed.

Definition Let X be a vector space, $\mathfrak{B}$ a C*-algebra and let $\langle \cdot, \cdot \rangle$ be a $\mathfrak{B}$-valued positive sesquilinear form on X. We call $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$-valued inner product on X if $x \in X$ and $\langle x, x\rangle = 0$ implies $x = 0$.

Example If $X = \mathfrak{B}$ is a C*-algebra, we can define $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by $\langle A, B\rangle = A^*B$ or by

$\langle A, B\rangle = BA^*$. It is easy to see that these are $\mathfrak{B}$-valued inner products. Moreover we notice that $\|\langle A, A\rangle\|_{\mathfrak{B}} = \|A^*A\|_{\mathfrak{B}} = \|A\|_{\mathfrak{B}}^2$. Therefore $\|A\|_{\mathfrak{B}} = \sqrt{\|\langle A, A\rangle\|_{\mathfrak{B}}}$. Our next question is whether the quantity on the right is always a norm.

Example Let $\mathfrak{B}$ be a C*-algebra and let $X = \mathfrak{B}^n$ be the set of all n-tuples with entries from $\mathfrak{B}$. We define $\langle\cdot,\cdot\rangle : X \times X \to \mathfrak{B}$ by $\langle (A_1, A_2, \ldots, A_n), (B_1, B_2, \ldots, B_n)\rangle = \sum_{i=1}^n A_i^* B_i$. It is easy to see that this is a $\mathfrak{B}$-inner product. This generalizes $(\mathbb{C}^n, \|\cdot\|_2)$ to C*-algebras.

Example Let $\mathfrak{B}$ be a C*-algebra and I an index set. Let $X = \{(B_i)_{i\in I} \mid \sum_{i\in I} B_i^* B_i \text{ converges in } \mathfrak{B}\}$ where $(B_i)_{i\in I}$ denotes a collection index by I. Define $\langle\cdot,\cdot\rangle : X \times X \to \mathfrak{B}$ by $\langle (A_i)_{i\in I}, (B_i)_{i\in I}\rangle = \sum_{i\in I} A_i^* B_i$. It is not clear that $\langle\cdot,\cdot\rangle$ is well-defined. To see $\langle\cdot,\cdot\rangle$ is well-defined, we can assume that $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ by the GNS construction. Let $(A_i)_{i\in I}, (B_i)_{i\in I} \in X$ be arbitrary. We recall that if $P, Q, A \in \mathcal{B}(\mathcal{H})$ are such that

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \geq 0$$

then $|\langle \eta, A\xi\rangle| \leq |\langle \eta, P\eta\rangle|^{\frac{1}{2}} |\langle \xi, Q\xi\rangle|^{\frac{1}{2}}$ for all $\xi, \eta \in \mathcal{H}$. However, we notice that

$$\begin{bmatrix} A_i^*A_i & A_i^*B_i \\ B_i^*A_i & B_i^*B_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix}^* \begin{bmatrix} A_i & B_i \\ 0 & 0 \end{bmatrix} \geq 0$$

Thus, if $J \subseteq I$ is any finite set, we obtain that

$$\begin{bmatrix} \sum_{i\in J} A_i^*A_i & \sum_{i\in J} A_i^*B_i \\ \sum_{i\in J} B_i^*A_i & \sum_{i\in J} B_i^*B_i \end{bmatrix} \geq 0$$

so

$$\begin{aligned} \left|\left\langle \eta, \sum_{i\in J} A_i^*B_i\xi \right\rangle\right| &\leq \left|\left\langle \eta, \sum_{i\in J} A_i^*A_i\eta \right\rangle\right|^{\frac{1}{2}} \left|\left\langle \xi, \sum_{i\in J} B_i^*B_i\xi \right\rangle\right|^{\frac{1}{2}} \\ &\leq \left\|\sum_{j\in J} A_j^*A_j\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{j\in J} B_j^*B_j\right\|_{\mathfrak{B}}^{\frac{1}{2}} \|\xi\| \, \|\eta\| \end{aligned}$$

for all $\xi, \eta \in \mathcal{H}$ and $J \subseteq I$ finite. Therefore $\left\|\sum_{i\in J} A_i^*B_i\right\|_{\mathfrak{B}} \leq \left\|\sum_{j\in J} A_j^*A_j\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{j\in J} B_j^*B_j\right\|_{\mathfrak{B}}^{\frac{1}{2}}$. Thus, if we order all finite subsets J of I by inclusion, $\left(\sum_{i\in J} A_i^*B_i\right)_{J\subseteq I}$ becomes a Cauchy net by the above inequality. Since $\mathfrak{B}$ is complete, this sum converges so $\sum_{i\in I} A_i^*B_i \in \mathfrak{B}$ is well-defined. Moreover, by taking limits $\left\|\sum_{i\in I} A_i^*B_i\right\|_{\mathfrak{B}} \leq \left\|\sum_{i\in I} A_i^*A_i\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{i\in I} B_i^*B_i\right\|_{\mathfrak{B}}^{\frac{1}{2}}$. It is now easy to verify that this is a $\mathfrak{B}$-inner product. This generalizes the concept of $\ell_2(I)$ to C*-algebras.

Example Let $\mathfrak{B} = C[0,1]$ (the continuous functions on $[0,1]$ with the supremum norm) and let $X = C([0,1]^2)$. Define $\langle\cdot,\cdot\rangle : X \times X \to \mathfrak{B}$ by $\langle f, g\rangle(x) = \int_0^1 \overline{f(x,y)} g(x,y)dy$ for all $x \in [0,1]$ and $f, g \in X$. It is easy to verify that $\langle f, g\rangle \in \mathfrak{B}$ for all $f, g \in X$ and that $\langle\cdot,\cdot\rangle$ is a $C[0,1]$-valued inner product.

Remarks Given a $\mathfrak{B}$-valued inner product on a vector space X, we would like to give X a norm. In order to see that the canonical way of defining a norm on X is in fact a norm, we need the following lemma.

Lemma *Let X be a vector space, $\mathfrak{B}$ a C*-algebra, and $\langle\cdot,\cdot\rangle$ a positive $\mathfrak{B}$-valued sesquilinear form on X. Then $\|Re\langle x, y\rangle\|_{\mathfrak{B}}^2 \leq \|\langle x, x\rangle\|_{\mathfrak{B}} \|\langle y, y\rangle\|_{\mathfrak{B}}$ and $\|Im\langle x, y\rangle\|_{\mathfrak{B}}^2 \leq \|\langle x, x\rangle\|_{\mathfrak{B}} \|\langle y, y\rangle\|_{\mathfrak{B}}$ for all $x, y \in X$. Thus $\|\langle x, y\rangle\|_{\mathfrak{B}} \leq 2 \|\langle x, x\rangle\|_{\mathfrak{B}}^{\frac{1}{2}} \|\langle y, y\rangle\|_{\mathfrak{B}}^{\frac{1}{2}}$.*

PROOF: Apply the GNS construction to view $\mathfrak{B}$ as a subalgebra of $\mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. For each $k \in \mathcal{K}$, we can define $\langle \cdot, \cdot \rangle_k : X \times X \to \mathbb{C}$ by $\langle x, y \rangle_k = \langle k, \langle x, y \rangle k \rangle_{\mathcal{K}}$. Then $\langle \cdot, \cdot \rangle_k$ is a positive sesquilinear form on X so by the Cauchy-Schwarz inequality

$$|\langle x, y \rangle_k|^2 \leq \langle x, x \rangle_k \langle y, y \rangle_k$$

Therefore, if $\|k\|_{\mathcal{K}} \leq 1$,

$$\begin{aligned}
|\langle k, Re(\langle x, y \rangle) k \rangle_{\mathcal{K}}|^2 &= |Re(\langle k, \langle x, y \rangle k \rangle_{\mathcal{K}})|^2 \\
&\leq \langle k, \langle x, x \rangle k \rangle_{\mathcal{K}} \langle k, \langle y, y \rangle k \rangle_{\mathcal{K}} \\
&\leq \|\langle x, x \rangle\|_{\mathfrak{B}} \|\langle y, y \rangle\|_{\mathfrak{B}}
\end{aligned}$$

Since $Re(\langle x, y \rangle)$ is a self-adjoint element of $\mathfrak{B}$, the numerical radius of $Re(\langle x, y \rangle)$ is the same as its norm so we obtain $\|Re\langle x, y \rangle\|_{\mathfrak{B}}^2 \leq \|\langle x, x \rangle\|_{\mathfrak{B}} \|\langle y, y \rangle\|_{\mathfrak{B}}$ as claimed. The same result holds for Im and combining the two results gives us the final inequality. □

Proposition *Let X be a vector space, $\mathfrak{B}$ a C*-algebra, and $\langle \cdot, \cdot \rangle$ is a $\mathfrak{B}$-valued inner product on X. If we define $\|x\|_X = \sqrt{\|\langle x, x \rangle\|_{\mathfrak{B}}}$ then $\|\cdot\|_X$ is a norm on X. Moreover $\langle \cdot, \cdot \rangle$ is continuous in each component with respect to this norm.*

PROOF: First it is clear that $\|\cdot\|_X$ is well-defined, $\|x\|_X \geq 0$, $\|x\|_X = 0$ implies $x = 0$, and $\|\lambda x\|_X = |\lambda| \|x\|_X$. Lastly

$$\begin{aligned}
\|x + y\|_X &= \sqrt{\|\langle x, x \rangle + \langle y, y \rangle + 2Re\langle x, y \rangle\|_{\mathfrak{B}}} \\
&\leq \sqrt{\|\langle x, x \rangle\|_{\mathfrak{B}} + \|\langle y, y \rangle\|_{\mathfrak{B}} + 2 \|\langle x, x \rangle\|_{\mathfrak{B}}^{\frac{1}{2}} \|\langle y, y \rangle\|_{\mathfrak{B}}^{\frac{1}{2}}} \\
&= \sqrt{\left(\|\langle x, x \rangle\|_{\mathfrak{B}}^{\frac{1}{2}} + \|\langle y, y \rangle\|_{\mathfrak{B}}^{\frac{1}{2}} \right)^2} \\
&= \|x\|_X + \|y\|_X
\end{aligned}$$

as claimed. By the above Lemma, $\|\langle x, y \rangle\|_{\mathfrak{B}} \leq 2 \|x\|_X \|y\|_X$ so $\langle \cdot, \cdot \rangle$ is continuous in each component with respect to this norm. □

Definition Let X be a vector space, $\mathfrak{B}$ a C*-algebra, and $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$-valued inner product on X. If X is complete with respect to the above norm, we call X a Hilbert $\mathfrak{B}$-space.

Example We saw earlier that if $X = \mathfrak{B}$ is a C*-algebra, and $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ defined by $\langle A, B \rangle = A^* B$ or by $\langle A, B \rangle = BA^*$, then $\langle \cdot, \cdot \rangle$ is a $\mathfrak{B}$-valued inner product and $\|A\|_{\mathfrak{B}} = \sqrt{\|\langle A, A \rangle\|_{\mathfrak{B}}}$ so that $\mathfrak{B}$ is complete with respect to this norm. Thus $\mathfrak{B}$ has the additional structure of being a Hilbert $\mathfrak{B}$-space.

Example We saw earlier that if $\mathfrak{B}$ be a C*-algebra, $X = \mathfrak{B}^n$, and $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ defined by $\langle (A_1, A_2, \ldots, A_n), (B_1, B_2, \ldots, B_n) \rangle = \sum_{i=1}^n A_i^* B_i$ is a $\mathfrak{B}$-valued inner product. To see that X is complete with respect to the norm $\|(B_1, B_2, \ldots, B_n)\|_X = \sqrt{\|\sum_{i=1}^n B_i^* B_i\|}$, suppose $(x_m) \in X$ is a Cauchy sequence. Write $x_m = \left(A_1^{(m)}, \ldots, A_n^{(m)} \right)$ for all m. Then since $0 \leq \left(A_j^{(m)} - A_j^{(k)} \right)^* \left(A_j^{(m)} - A_j^{(k)} \right) \leq \sum_{i=1}^n \left(A_i^{(m)} - A_i^{(k)} \right)^* \left(A_i^{(m)} - A_i^{(k)} \right)$, $\left\| A_j^{(m)} - A_j^{(k)} \right\|_{\mathfrak{B}} \leq \|x_m - x_k\|_X$ for all $j = 1, \ldots, n$. Thus $\left(A_j^{(m)} \right)_j$ is a Cauchy sequence in $\mathfrak{B}$ for all $j = 1, \ldots, n$. Since $\mathfrak{B}$ is complete, there exists $A_j \in \mathfrak{B}$ such that $A_j^{(m)} \to A_j$ as $m \to \infty$. If $x = (A_1, \ldots, A_n) \in X$, then $\|x - x_m\|_X^2 = \left\| \sum_{i=1}^n \left(A_i^{(m)} - A_i \right)^* \left(A_i^{(m)} - A_i \right) \right\| \to 0$ as $m \to \infty$. Hence $x_m \to x$ in X so X is a Hilbert $\mathfrak{B}$-space.

Example Let $\mathfrak{B}$ be a C*-algebra and I an index set. Let $X = \{(B_i)_{i\in I} \mid \sum_{i\in I} B_i^* B_i$ converges in $\mathfrak{B}\}$ and define $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ by $\langle (A_i)_{i\in I}, (B_i)_{i\in I}\rangle = \sum_{i\in I} A_i^* B_i$ as before. We claim that X is complete with respect to the norm induced by the inner product. To see this, let $(x_n) \in X$ be a Cauchy sequence. Write $x_n = \left(A_i^{(n)}\right)_{i\in I}$ for all n. Then for all $i \in I$ and $n, m \in \mathbb{N}$, $0 \leq \left(A_i^{(n)} - A_i^{(m)}\right)^* \left(A_i^{(n)} - A_i^{(m)}\right) \leq \langle x_n - x_m, x_n - x_m\rangle$ as an infinite sum of positive elements in a C*-algebra is positive. Therefore $\left\|A_i^{(n)} - A_i^{(m)}\right\|_{\mathfrak{B}} \leq \|x_n - x_m\|_X$ so $\left(A_i^{(n)}\right)_n$ is a Cauchy sequence in $\mathfrak{B}$ for all $i \in I$. Since $\mathfrak{B}$ is complete, there exists $A_i \in \mathfrak{B}$ such that $A_i^{(n)} \to A_i$ as $n \to \infty$.

Let $x = (A_i)_{i\in I}$. By the GNS construction, suppose $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$. Then for all $h \in \mathcal{H}$ with $\|h\| \leq 1$,

$$\begin{aligned}\sum_{i\in I}\langle h, A_i^* A_i h\rangle &= \sum_{i\in I}\lim_{n\to\infty}\left\langle h, \left(A_i^{(n)}\right)^* A_i^{(n)} h\right\rangle \\ &= \lim_{n\to\infty}\sum_{i\in I}\left\langle h, \left(A_i^{(n)}\right)^* A_i^{(n)} h\right\rangle \\ &= \lim_{n\to\infty}\langle h, \langle x_n, x_n\rangle h\rangle \\ &\leq \limsup_{n\to\infty}\|\langle x_n, x_n\rangle\| \\ &= \limsup_{n\to\infty}\|x_n\|_X^2\end{aligned}$$

(where we can exchange the sums and limits since we are adding positive numbers). Since $(x_n) \in X$ is a Cauchy sequence, $\|x_n\|_X^2$ is finite. Using the fact that

$$\langle g, Th\rangle = \frac{1}{4}\left(\langle g+h, T(g+h)\rangle - \langle g-h, T(g-h)\rangle + i\langle g+ih, T(g+ih)\rangle - i\langle g-ih, T(g-ih)\rangle\right)$$

for all $T \in \mathcal{B}(\mathcal{H})$ and $g, h \in \mathcal{H}$, we obtain that $\left\|\sum_{i\in I} A_i^* A_i\right\| \leq \limsup_{n\to\infty}\|x_n\|_X^2 < \infty$ so $\sum_{i\in I} A_i^* A_i$ defines an operator in $\mathcal{B}(\mathcal{H})$ (this does not mean the sum converges in norm). Next, by repeating the same arguments with $x - x_m$ for a fixed m, we obtain that $\|x - x_m\|_X \leq \limsup_{n\to\infty}\|x_n - x_m\|_X$. Since $\limsup_{n\to\infty}\|x_n - x_m\|_X \to 0$ as $m \to \infty$, we obtain that $\|x - x_m\|_X \to 0$ as $m \to \infty$. Lastly, we must show that $x \in X$. Fix $\epsilon > 0$ and choose m such that $\|x - x_m\|_X < \epsilon$. Then, if $J \subseteq I$ is finite, then $\left\|\sum_{j\in J}\left(A_j - A_j^{(m)}\right)^*\left(A_j - A_j^{(m)}\right)\right\|_{\mathfrak{B}} \leq \|x - x_m\|_X^2 \leq \epsilon^2$ and then by using the above example, we obtain $\left\|\sum_{j\in J} A_j^* A_j\right\|_{\mathfrak{B}} \leq \epsilon^2 + \left\|\sum_{j\in J}\left(A_j^{(m)}\right)^* A_j^{(m)}\right\|_{\mathfrak{B}}$. Order all finite subsets of I by reverse inclusion. Since $\sum_{i\in I}\left(A_i^{(m)}\right)^* A_i^{(m)}$ converges, there exists a finite subset J_ϵ so that if $J \subseteq I$ is finite with $J_\epsilon \cap J = \emptyset$, then $\left\|\sum_{j\in J}\left(A_j^{(m)}\right)^* A_j^{(m)}\right\|_{\mathfrak{B}} < \epsilon$. Therefore, if $J \subseteq I$ is finite with $J_\epsilon \cap J = \emptyset$ then $\left\|\sum_{j\in J} A_j^* A_j\right\|_{\mathfrak{B}} < \epsilon^2 + \epsilon$. Hence $\left(\sum_{j\in J} A_j^* A_j\right)_{J\subseteq I}$ is a Cauchy sequence in $\mathfrak{B}$ and thus converges. Hence $x \in X$ so X is a Hilbert $\mathfrak{B}$-space. We write $\oplus_I \mathfrak{B}$ for X. If I is finite, we will write $\mathfrak{B}^{|I|}$ for X.

Example Let $\mathfrak{B} = C[0,1]$ (the continuous functions on $[0,1]$ with the supremum norm) and let $X = C([0,1]^2)$. Define $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ by $\langle f, g\rangle(x) = \int_0^1 \overline{f(x,y)} g(x,y)dy$ for all $x \in [0,1]$ and $f, g \in X$ as before. Unfortunately X is not a Hilbert $\mathfrak{B}$-space as it is not complete. To see this, suppose $h_0 \in L_2([0,1], m) \setminus C[0,1]$ where m is the Lebesgue measure. Then there exists a sequence $(f_n) \in C[0,1]$ such that $f_n \to h_0$ in $L_2([0,1], m)$. Let $g_n(x,y) = f_n(y) \in X$ and $h(x,y) = h_0(y) \notin X$. We notice that $|\langle (g_n - g_m), (g_n - g_m)\rangle(x)| = \|f_n - f_m\|_2^2$ for all $x \in [0,1]$ so $\|g_n - g_m\|_X = \|f_n - f_m\|_2$. Thus, since $(f_n) \in C[0,1]$ is Cauchy in $L_2([0,1], m)$, $(g_n) \in X$ is a Cauchy sequence. If $g_n \to g \in X$, then $\int_0^1 |g_n(x,y) - g(x,y)|^2 dy = \int_0^1 |f_n(y) - g(x,y)|^2 dy \to 0$ as $n \to \infty$. Since limits in $L_2([0,1], m)$ are unique, this forces $g(x,y) = h_0(y) = h(x,y)$ for all $x, y \in [0,1]$ which contradicts the fact that $h \notin X$.

Remarks If X is a vector space, $\mathfrak{B}$ a C*-algebra, $\langle \cdot, \cdot \rangle$ a $\mathfrak{B}$-valued inner product on X, and X is not complete with respect to the norm $\|x\|_X = \sqrt{\|\langle x, x\rangle\|_{\mathfrak{B}}}$, then it has a completion that is a Hilbert $\mathfrak{B}$-space. To see this, we can use the method of completion using equivalence classes of Cauchy sequences along with the facts that $\|\langle x, y\rangle\|_{\mathfrak{B}}^2 \leq 4\|\langle x, x\rangle\|_{\mathfrak{B}} \|\langle y, y\rangle\|_{\mathfrak{B}}$ for all $x, y \in X$ and $\mathfrak{B}$ is complete to see that we can extend $\langle \cdot, \cdot \rangle$ to an $\mathfrak{B}$-inner product on a completion on X with $\|x\|_X = \sqrt{\|\langle x, x\rangle\|_{\mathfrak{B}}}$ for all x in the completion.

Our elementary proofs above did not enable us to prove the Cauchy Schwarz inequality in the traditional sense for $\mathfrak{B}$-valued positive sesquilinear forms. However we showed that

$$\left\|\sum_{i\in I} A_i^* B_i\right\|_{\mathfrak{B}} \leq \left\|\sum_{i\in I} A_i^* A_i\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{i\in I} B_i^* B_i\right\|_{\mathfrak{B}}^{\frac{1}{2}}$$

so the Cauchy-Schwarz inequality holds for $\oplus_I \mathfrak{B}$. Similarly for $X = C([0,1]^2)$ with the above $C[0,1]$-valued inner product, we notice for all $x \in [0,1]$ and $f, g \in C([0,1]^2)$ that

$$|\langle f, g\rangle(x)| = \left|\int_0^1 \overline{f(x,y)} g(x,y)dy\right| \leq \left(\int_0^1 |f(x,y)|^2 dy\right)^{\frac{1}{2}} \left(\int_0^1 |g(x,y)|^2 dy\right)^{\frac{1}{2}} \leq \|f\|_X^{\frac{1}{2}} \|g\|_X^{\frac{1}{2}}$$

so the Cauchy-Schwarz inequality holds here as well.

However, there is no reason that Cauchy-Schwarz inequality holds in general. Recall that in a traditional proof of the Cauchy-Schwarz inequality, we are able to pull scalars out of the inner product. The reason that the Cauchy-Schwarz inequality holds for these examples is that we can 'pull out' elements of $\mathfrak{B}$ out of the inner product in a certain way. Since inner products map into $\mathbb{C}$ and are $\mathbb{C}$-linear, it makes sense that $\mathfrak{B}$-valued inner products should be '$\mathfrak{B}$-linear'. Since $\mathfrak{B}$ may not be commutative, we need to decide what we mean by '$\mathfrak{B}$-linear'.

Definition Let $\mathcal{H}$ be a Hilbert $\mathfrak{B}$-space. If there is a linear map $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ (the set of linear maps on $\mathcal{H}$) that is anti-multiplicative (i.e. $\rho(ab) = \rho(b)\rho(a)$ for all $a, b \in \mathfrak{B}$) such that $\langle h, \rho(b)g\rangle = \langle h, g\rangle b$ for all $b \in \mathfrak{B}$ and $g, h \in \mathcal{H}$, then $\mathcal{H}$ is called a right Hilbert $\mathfrak{B}$-module.

Remarks Of course saying $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module is an abuse for the collection $(\mathcal{H}, \langle \cdot, \cdot \rangle, \rho)$.

Example We saw earlier that if $\mathcal{H} = \mathfrak{B}$ is a C*-algebra, and $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by $\langle A, B\rangle = A^*B$, then X was a Hilbert $\mathfrak{B}$-space. If we define $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)A = AB$ for all $A \in X$ and $B \in \mathfrak{B}$ then X is a right Hilbert $\mathfrak{B}$-module. Clearly each $\rho(B)$ is linear, ρ is linear, and $\rho(BC)A = ABC = \rho(C)AB = \rho(C)\rho(B)A$ so ρ is anti-multiplicative. Lastly $\langle A, \rho(C)B\rangle = \langle A, BC\rangle = A^*BC = \langle A, B\rangle C$. Thus $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module. However, if we defined $\langle A, B\rangle = BA^*$, then the above ρ does not work. In a sense, this later example will be a left Hilbert $\mathfrak{B}$-module.

Example We saw earlier that if $\mathfrak{B}$ is a C*-algebra and I an index set, then $\mathcal{H} = \oplus_I \mathfrak{B}$ was a Hilbert $\mathfrak{B}$-space. If we define $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)(A_i)_{i\in I} = (A_i B)_{i\in I}$, then $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module for the same reasons as above.

Example We saw earlier that if $\mathfrak{B} = C[0,1]$, $X = C([0,1]^2)$, $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ by $\langle f, g\rangle(x) = \int_0^1 \overline{f(x,y)} g(x,y)dy$, and $\mathcal{H}$ was the completion of X with respect to the norm induced by this $\mathfrak{B}$-inner product, then $\mathcal{H}$ was a Hilbert $\mathfrak{B}$-space. If we define $\rho : \mathfrak{B} \to \mathcal{L}(X)$ by $(\rho(f)g)(x,y) = f(x)g(x,y)$ for all $g \in X$ and $f \in \mathfrak{B}$, then $\langle h, \rho(f)g\rangle(x) = \int_0^1 \overline{h(x,y)} g(x,y)f(x)dy = \left(\int_0^1 \overline{h(x,y)} g(x,y)dy\right) f(x) = \langle h, g\rangle(x)f(x)$ and ρ is linear and anti-multiplicative. Moreover, for all $g \in X$ and $f \in C[0,1]$

$$\langle \rho(f)g, \rho(f)g\rangle = \langle \rho(f)g, g\rangle f = (\langle g, \rho(f)g\rangle)^* f = (\langle g, g\rangle f)^* f = f^* \langle g, g\rangle f$$

Hence $\|\rho(f)g\|_X \leq \|f\|_{\mathfrak{B}} \|g\|_X$. Hence $\rho : \mathfrak{B} \to \mathcal{B}(X)$ and thus extends to an anti-multiplicative, linear map $\rho' : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ such that $\langle h, \rho(b)g\rangle = \langle h, g\rangle b$ for all $b \in \mathfrak{B}$ and $g, h \in \mathcal{H}$ (due to the continuity of the $\mathfrak{B}$-valued inner product). Thus $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module.

Lemma *If $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module and $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ is the map that induces the action of $\mathfrak{B}$ on $\mathcal{H}$, then $\rho : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ (the bounded linear maps on $\mathcal{H}$). Moreover $\|\rho(b)\| \leq \|b\|_{\mathfrak{B}}$ so ρ is a continuous linear map and the map $\mathcal{H} \times \mathfrak{B} \to \mathcal{H}$ by $(h, b) \mapsto \rho(b)h$ is continuous.*

PROOF: Let $h \in \mathcal{H}$ be arbitrary. Then

$$\langle \rho(b)h, \rho(b)h\rangle = \langle \rho(b)h, h\rangle b = (\langle h, \rho(b)h\rangle)^* b = (\langle h, h\rangle b)^* b = b^* \langle h, h\rangle b$$

The above shows us that $\langle \rho(b)g, h\rangle = b^*\langle g, h\rangle$ for all $b \in \mathfrak{B}$ and $g, h \in \mathcal{H}$. Therefore

$$\|\langle \rho(b)h, \rho(b)h\rangle\|_{\mathfrak{B}} \leq \|b\|_{\mathfrak{B}} \|\langle h, h\rangle\|_{\mathfrak{B}} \|b^*\|_{\mathfrak{B}} = \|b\|_{\mathfrak{B}}^2 \|h\|_{\mathcal{H}}^2$$

which proves the claim. □

Remarks For a Hilbert $\mathfrak{B}$-space, we could not obtain the Cauchy-Schwarz inequality in the traditional sense. However we can for Hilbert $\mathfrak{B}$-modules due to the ability to pull out $\mathfrak{B}$-values out of the inner product. This demonstrates that we should consider $\mathfrak{B}$-linearity when dealing with $\mathfrak{B}$-valued inner products.

Lemma *Suppose $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module. Then for all $\xi, \eta \in \mathcal{H}$, $\langle \eta, \xi\rangle\langle \xi, \eta\rangle \leq \|\xi\|_{\mathcal{H}}^2 \langle \eta, \eta\rangle$. Thus since $\langle \eta, \xi\rangle\langle \xi, \eta\rangle$ is positive, $\|\langle \xi, \eta\rangle\|_{\mathfrak{B}} = \|\langle \eta, \xi\rangle\langle \xi, \eta\rangle\|_{\mathfrak{B}}^{\frac{1}{2}} \leq \|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}}$.*

PROOF: If $\xi = 0$, the result is trivially true. Thus suppose $\xi \neq 0$. First we notice that $\langle \xi, \xi\rangle \leq_{\mathfrak{B}} \|\xi\|_{\mathcal{H}}^2 1$. Hence, since $\langle \eta, \xi\rangle = \langle \xi, \eta\rangle^*$, we obtain that $\langle \eta, \xi\rangle\langle \xi, \xi\rangle\langle \xi, \eta\rangle \leq \|\xi\|_{\mathcal{H}}^2 \langle \eta, \xi\rangle\langle \xi, \eta\rangle$ (it appears we assumed $\mathfrak{B}$ was unital, but we get around this fact by considering $\mathfrak{B}$ as a *-subalgebra of a unital C*-algebra). However we notice that

$$\begin{aligned}
0 &\leq \langle \rho(\langle \xi, \eta\rangle)\xi - \|\xi\|_{\mathcal{H}}^2 \eta, \rho(\langle \xi, \eta\rangle)\xi - \|\xi\|_{\mathcal{H}}^2 \eta\rangle \\
&= \langle \eta, \xi\rangle\langle \xi, \xi\rangle\langle \xi, \eta\rangle - \|\xi\|_{\mathcal{H}}^2 \langle \eta, \xi\rangle\langle \xi, \eta\rangle - \|\xi\|_{\mathcal{H}}^2 \langle \eta, \xi\rangle\langle \xi, \eta\rangle + \|\xi\|_{\mathcal{H}}^4 \langle \eta, \eta\rangle \\
&\leq \|\xi\|_{\mathcal{H}}^4 \langle \eta, \eta\rangle - \|\xi\|_{\mathcal{H}}^2 \langle \eta, \xi\rangle\langle \xi, \eta\rangle
\end{aligned}$$

Since $\xi \neq 0$, $\|\xi\|_{\mathcal{H}} \neq 0$ so we obtain the result as desired. □

Remarks If X is a vector space, $\mathfrak{B}$ a C*-algebra, $\langle \cdot, \cdot\rangle : X \times X \to \mathfrak{B}$ a $\mathfrak{B}$-valued positive sesquilinear form, and $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ is a linear, anti-multiplicative map such that $\langle h, \rho(b)g\rangle = \langle h, g\rangle b$ for all $b \in \mathfrak{B}$ and $g, h \in X$, then the above lemma holds (where the only change is that if $\langle \xi, \xi\rangle = 0$, then $\langle \xi, \eta\rangle = 0$ by the partial Cauchy-Schwarz inequality proven earlier).

Now that we have the Cauchy-Schwarz inequality, our attention moves to orthogonality. In a Hilbert space the inner product induces a rich geometric structure that allows orthogonal direct sums, orthogonal projections, and orthonormal bases. Unfortunately, as our inner products no longer map into $\mathbb{C}$, we lose most of this geometry.

In a Hilbert space $\mathcal{H}$, it is well known that if $\mathcal{K}$ is a closed subspace and $\mathcal{K}^\perp = \{\xi \in \mathcal{H} \mid \langle \eta, \xi\rangle = 0 \text{ for all } \eta \in \mathcal{K}\}$, then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. However orthogonal complements need not exist in right Hilbert modules.

Example Let $\mathfrak{B}$ be any unital C*-algebra that is not $\mathbb{C}$ and place the canonical right Hilbert $\mathfrak{B}$-module structure on $\mathfrak{B}$. If 1 is the unit of $\mathfrak{B}$, then $\langle 1, B\rangle = B$ for all $B \in \mathfrak{B}$. Thus, if $\mathcal{K}$ is the subspace of $\mathfrak{B}$ generated by 1, then $\mathcal{K}^\perp = \{0\}$ so $\mathcal{K} \oplus \mathcal{K}^\perp \neq \mathfrak{B}$.

Remarks We also note that the above shows that if we have a unit vector, there does not exists an orthonormal basis for the module that contains the vector. Thus our next hope is that there exists some orthonormal basis for the space. The easiest case would be to consider finite dimensions. If we were dealing with a standard inner product, we would apply the Gram-Schmidt orthogonalization process. However, a close examination of the process shows that we need to divide by the inner product of a non-zero vector which may not make sense in a C*-algebra.

Our only other hope would be to use the same method for infinite dimensional Hilbert space for which we show that if X is a closed subspace and $y \in \mathcal{H} \setminus X$, there exists a unique element $x_y \in X$ such that $\|x_y - y\| < \|x - y\|$ for all $x \in X \setminus \{x_0\}$. This enables us to take orthogonal projections onto closed subspaces. The proof of the above fact relies heavily on the parallelogram law which state that if $\mathcal{H}$ is a Hilbert space, $2\|x\|_{\mathcal{H}}^2 + 2\|y\|_{\mathcal{H}}^2 = \|x-y\|_{\mathcal{H}}^2 + \|x+y\|_{\mathcal{H}}^2$ for all $x, y \in \mathcal{H}$. This comes straight from the inner product and the definition of the norm on $\mathcal{H}$. Unfortunately the proof does not transfer to our context since we need to apply $\|\cdot\|_{\mathfrak{B}}$ to our $\mathfrak{B}$-valued inner product to obtain our norm.

Remarks We have seen with the development of $\mathfrak{B}$-inner products that it makes more sense to consider $\mathfrak{B}$-linearity. Therefore, instead of taking a complex span, we should consider taking an $\mathfrak{B}$-span. By this we mean if $\{x_i\}_{i\in I}$ is a set of vectors in a right Hilbert $\mathfrak{B}$-module, then the $\mathfrak{B}$-span is the set $\{\sum_{j\in J} \rho(B_j)x_j \mid B_j \in \mathfrak{B}, J \subseteq I \text{ finite}\}$. In this setting, we are more likely to find a set of orthogonal vectors who $\mathfrak{B}$-span is the entire space. Moreover, we notice if $\{x_i\}_{i\in I}$ is a set of orthogonal vectors and $y = \sum_{i\in I} \rho(B_i)x_i$, then $\langle x_i, y\rangle = \langle x_i, x_i\rangle B_i$ and $\langle y, y\rangle = \sum_{i\in I} B_i^* \langle x_i, x_i\rangle B_i$ which is similar to sum obtain for Hilbert spaces. Unfortunately it is possible that $\langle x_i, x_i\rangle B_i$ even though $B_i \neq 0$ so we will not get a unique representation in terms of the orthogonal vectors.

Example Let $\mathfrak{B}$ be a unital C*-algebra and $\mathcal{H} = \oplus_I \mathfrak{B}$. If $\{e_i\}_{i\in I} \in \mathcal{H}$ are the vectors with $1_{\mathfrak{B}}$ in the i^{th} spot and zeros elsewhere, then $\{e_i\}_{i\in I}$ is a set of orthogonal vectors whose $\mathfrak{B}$-span is $\mathcal{H}$.

Example Let $\mathfrak{B} = L_\infty([0,1], m)$ (where m is the Lebesgue measure) and let $\mathcal{H} = \mathfrak{B}$ with the canonical right Hilbert $\mathfrak{B}$-module structure. By the above $f(x) = 1$ for all f is a vector whose $\mathfrak{B}$-span is all of $\mathcal{H}$. If $f_1 = \chi_{[0,0,5]}$ and $f_2 = \chi_{[0.5,1]}$, then $\langle f_1, f_2\rangle = 0$ and the $\mathfrak{B}$-span of $\{f_1, f_2\}$ is $\mathcal{H}$.

Remarks The above example shows that we do not have a well-defined notion of the dimension of a right Hilbert $\mathfrak{B}$-module based on this generalization of orthogonal bases. One idea to rectify this would be to add the condition that if $\langle x, x\rangle B = 0$ for $B \in \mathfrak{B}$ and x in the orthogonal basis, then $B = 0$. This rectifies the above example. However, if $\mathfrak{A}$ is the *-subalgebra of $C[0,1]$ containing all function that vanish at 0, and if we place the canonical right Hilbert $\mathfrak{A}$-structure on $\mathfrak{A}$, it is unclear that such a basis exists. We will not pursue this line as it not of interest to us.

The next major theory done for Hilbert spaces would be to consider the bounded linear maps. These operators form a C*-algebra as every bounded linear operator has an adjoint; that is the adjoint of a bounded linear operator T is a bounded linear operator such that $\langle \xi, T\eta\rangle = \langle T^*\xi, \eta\rangle$ for all η, ξ in our Hilbert space. The main tool for proving that every bounded linear operator has an adjoint is the Riesz Representation Theorem (i.e. for every fixed η and bounded linear operator T, $\xi \mapsto \langle \eta, T\xi\rangle$ is a continuous linear functional so there exists a $T^*\eta$ such that $\langle T^*\eta, \xi\rangle = \langle \eta, T\xi\rangle$ for all ξ). However it is difficult believe that a Riesz Representation Theorem holds for $\mathfrak{B}$-valued inner products as the traditional proof relies on the fact that $dim(\mathbb{C}) = 1$ and the existence of orthogonal complements. In fact, we cannot prove such a theorem as it is not true.

Example Let $\mathcal{H} = \mathcal{M}_2(\mathbb{C}) = \mathfrak{B}$, $\langle\cdot,\cdot\rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by $\langle A, B\rangle = A^*B$, and $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)A = AB$ for all $A \in \mathcal{H}$ and $B \in \mathfrak{B}$. We have seen that $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module. Define $\phi : \mathcal{H} \to \mathfrak{B}$ by $\phi(A) = A^T$. Clearly ϕ is a linear map. We claim that ϕ is not representable by the inner product. To see this, suppose there exists a $B_0 \in \mathcal{H}$ such that $\phi(A) = \langle B_0, A\rangle$ for all $A \in \mathcal{H}$. Then $A^T = B_0^* A$ for all $A \in \mathcal{H}$. Letting $A = I_2$ implies $B_0^* = I$ and this implies $A^T = A$ for all $A \in \mathcal{H}$ which is impossible.

Remarks One of the main issues with the example is that ϕ is not $\mathfrak{B}$-linear. That is $\phi((\rho(B))A) \neq \rho(B)(\phi(A))$ (as $(AB)^T = B^T A^T \neq A^T B$). We will see later that if a bounded linear operator on a right Hilbert $\mathfrak{B}$-module has an adjoint, then it must be $\mathfrak{B}$-linear.

Example Let $\mathfrak{A}$ be the *-subalgebra of $C[0,1]$ consisting of all functions that vanish at 0. Let $\mathcal{H} = \mathfrak{A} \oplus C[0,1]$ and define $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to C[0,1]$ by $\langle (f_1, g_1), (f_2, g_2)\rangle = \overline{f_1} f_2 + \overline{g_1} g_2$. If we define $\rho : C[0,1] \to \mathcal{L}(\mathcal{H})$ by $\rho(h)(f,g) = (fh, gh)$, then $\mathcal{H}$ is a right Hilbert $C[0,1]$-module. Define $T : \mathcal{H} \to \mathcal{H}$ by $T(f,g) = (0,f)$. It is easy to verify that T is $C[0,1]$-linear and continuous on $\mathcal{H}$. Suppose T were adjointable. Let $(g_1, g_2) = T^*(0,1)$ so $g_1 \in \mathfrak{A}$. Then for all $f \in \mathfrak{A}$, $\overline{g_1} f = \langle T^*(0,1), (f,0)\rangle = \langle (0,1), T(f,0)\rangle = \langle (0,1), (0,f)\rangle = f$. Therefore $g_1(x) = 1$ for all $x \in (0,1]$ which is impossible since $g \in \mathfrak{A}$ so $g(0) = 0$ and g is continuous. Hence T does not have an adjoint.

Remarks To get around this issue of the Riesz Representation Theorem, we will only be interested in linear maps that have adjoints. Moreover, this will automatically introduce $\mathfrak{B}$-linearity which we have seen is desired.

Definition Let $\mathcal{H}$ be a Hilbert $\mathfrak{B}$-space. A linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be adjointable if there exists a linear operator T^* such that $\langle \xi, T\eta\rangle = \langle T^*\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. Let $\mathcal{B}_a(\mathcal{H})$ denote the set of adjointable linear maps on $\mathcal{H}$.

If $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module, a linear map $T : \mathcal{H} \to \mathcal{H}$ is said to be $\mathfrak{B}$-linear if $T(\rho(B)h) = \rho(B)(Th)$ for all $B \in \mathfrak{B}$ and $h \in \mathcal{H}$.

Remarks Using the usual Hilbert space arguments, it is easy to show that $(\lambda S + T)^* = \overline{\lambda} S^* + T^*$, and $(ST)^* = T^* S^*$ for all $S, T \in \mathcal{B}_a(\mathcal{H})$. Moreover, if $\langle \xi, T\eta\rangle = \langle T^*\xi, \eta\rangle$ then $\langle \xi, T\eta\rangle^* = \langle T^*\xi, \eta\rangle^*$ so $\langle \xi, T\eta, \xi\rangle = \langle \eta, T^*\xi\rangle$. Hence $(T^*)^* = T$. Therefore $\mathcal{B}_a(\mathcal{H})$ is a *-algebra. We will prove the following lemma before showing that $\mathcal{B}_a(\mathcal{H})$ is a C*-algebra.

Lemma *Let $\mathcal{H}$ be a Hilbert $\mathfrak{B}$-space. If $T \in \mathcal{B}_a(\mathcal{H})$, then T is bounded. If $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module and $T \in \mathcal{B}_a(\mathcal{H})$, T is $\mathfrak{B}$-linear.*

Proof: Let $T \in \mathcal{B}_a(\mathcal{H})$. To see that T is continuous (and thus bounded), we will apply the Closed Graph Theorem. Recall it suffices to show that if $\xi_n \to \xi$ in $\mathcal{H}$ and $T\xi_n \to \eta$ in $\mathcal{H}$, then $\eta = T\xi$. To see this, we notice by the continuity of the inner product that for all $\zeta \in \mathcal{H}$

$$\langle \zeta, T\xi\rangle = \langle T^*\zeta, \xi\rangle = \lim_{n\to\infty} \langle T^*\zeta, \xi_n\rangle = \lim_{n\to\infty} \langle \zeta, T\xi_n\rangle = \langle \zeta, \eta\rangle$$

Letting $\zeta = T\xi - \eta$ and using the definiteness of the $\mathfrak{B}$-valued inner product, we obtain that $T\xi = \eta$ as desired.

Now suppose T is a right Hilbert $\mathfrak{B}$-module. Fix $\xi \in \mathcal{H}$ and $B \in \mathfrak{B}$. Then for all $\eta \in \mathcal{H}$

$$\langle \eta, T(\rho(B)\xi)\rangle = \langle T^*\eta, \rho(B)\xi\rangle = \langle T^*\eta, \xi\rangle B = \langle \eta, T(\xi)\rangle B = \langle \eta, \rho(B)(T(\xi))\rangle$$

Thus $\langle \eta, T(\rho(B)\xi) - \rho(B)(T(\xi))\rangle = 0$. By repeating the same trick as before, $T(\rho(B)x) = \rho(B)(T(\xi))$ as desired. □

Theorem *Let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module. Then $\mathcal{B}_a(\mathcal{H})$ is a C*-algebra with the above involution and operator norm.*

Proof: Before completeness, we will show that the C*-equation holds. First we claim that $\|T\| =$

$\sup\{\|\langle T\xi, \eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\}$. By the Cauchy-Schwarz inequality, we clearly have

$$\sup\{\|\langle T\xi, \eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} \leq \|T\|.$$

For the other direction,

$$\begin{aligned}\sup\{\|\langle T\xi, \eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} &\geq \sup\left\{\left\|\left\langle T\xi, \frac{T\xi}{\|T\xi\|_{\mathcal{H}}}\right\rangle\right\|_{\mathfrak{B}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} = 1, T\xi \neq 0\right\} \\ &= \sup\{\|T\xi\|_{\mathcal{H}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} = 1\} = \|T\|\end{aligned}$$

Thus we have proven the claim.

Next, if $T, S \in \mathcal{B}_a(\mathcal{H})$, $\|ST\xi\|_{\mathcal{H}} \leq \|S\| \|T\xi\|_{\mathcal{H}} \leq \|S\| \|T\| \|\xi\|_{\mathcal{H}}$ so $\|ST\| \leq \|S\| \|T\|$. Next

$$\begin{aligned}\|T^*T\| &= \sup\{\|\langle T^*T\xi, \eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} \\ &\geq \sup\{\|\langle T^*T\xi, \xi\rangle\|_{\mathfrak{B}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} \leq 1\} \\ &= \sup\{\|\langle T\xi, T\xi\rangle\|_{\mathfrak{B}} \mid \xi \in \mathcal{H}, \|\xi\|_{\mathcal{H}} \leq 1\} = \|T\|^2\end{aligned}$$

Also

$$\begin{aligned}\|T^*\| &= \sup\{\|\langle T^*\xi, \eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} \\ &= \sup\{\|\langle \xi, T\eta\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} \\ &= \sup\{\|\langle T\eta, \xi\rangle\|_{\mathfrak{B}} \mid \xi, \eta \in \mathcal{H}, \|\xi\|_{\mathcal{H}}, \|\eta\|_{\mathcal{H}} \leq 1\} = \|T\|\end{aligned}$$

as $\|B^*\| = \|B\|$ for all $B \in \mathfrak{B}$. Therefore $\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$. Thus the C*-equation holds.

From above $\|T\| = \|T^*\|$ for all $T \in \mathcal{B}_a(\mathcal{H})$. To see that $\mathcal{B}_a(\mathcal{H})$ is a C*-algebra, we have already seen that it is a *-algebra with a submultiplicative norm that satisfies the C*-identity. Hence it remains only to see that $\mathcal{B}_a(\mathcal{H})$ is closed subset of $\mathcal{B}(\mathcal{H})$ (since $\mathcal{B}(\mathcal{H})$ is complete as $\mathcal{H}$ is complete). To see this, suppose $\{T_n\}_{n=1}^\infty$ is Cauchy in $\mathcal{B}_a(\mathcal{H})$. Thus $\{T_n^*\}_{n=1}^\infty$ is Cauchy in $\mathcal{B}_a(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is complete, there exist $T, S \in \mathcal{B}(\mathcal{H})$ such that $T_n \to T$ and $T_n^* \to S$ as $n \to \infty$. We claim that S is the adjoint of T. To see this, we notice for all $\xi, \eta \in \mathcal{H}$

$$\langle \eta, T\xi\rangle = \lim_{n\to\infty} \langle \eta, T_n\xi\rangle = \lim_{n\to\infty} \langle T_n^*\eta, \xi\rangle = \langle S\eta, \xi\rangle$$

as $T_n \to T$ and $T_n^* \to S$ as $n \to \infty$ in norm (and thus pointwise) and the $\mathfrak{B}$-valued inner product is continuous in each component. Thus $T \in \mathcal{B}_a(\mathcal{H})$ so $\mathcal{B}_a(\mathcal{H})$ is a C*-algebra. □

Remarks In the above Theorem, we needed $\mathcal{H}$ to be a right Hilbert $\mathfrak{B}$-module in order to use the Cauchy-Schwarz Inequality. Now we compare the notions of positivity and adjoints in $\mathcal{B}_a(\mathcal{H})$ to the standard result for the bounded linear maps on a Hilbert space

Lemma *Let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module and let $T \in \mathcal{B}_a(\mathcal{H})$. Then $T = T^*$ if and only if $\langle \xi, T\xi\rangle = \langle \xi, T\xi\rangle^*$ for all $\xi \in \mathcal{H}$. Moreover $T \geq 0$ in $\mathcal{B}_a(\mathcal{H})$ if and only if $\langle \xi, T\xi\rangle \geq_{\mathfrak{B}} 0$.*

PROOF: Fix $T \in \mathcal{B}_a(\mathcal{H})$. First suppose $T = T^*$. Then for all $\xi \in \mathcal{H}$, $\langle \xi, T\xi\rangle = \langle T^*\xi, \xi\rangle = \langle T\xi, \xi\rangle = \langle \xi, T\xi\rangle^*$. Next suppose $\langle \xi, T\xi\rangle = \langle \xi, T\xi\rangle^*$ for all $\xi \in \mathcal{H}$. Then $\langle \xi, T\xi\rangle = \langle \xi, T\xi\rangle^* = \langle T\xi, \xi\rangle = \langle \xi, T^*\xi\rangle$ so $\langle \xi, (T - T^*)\xi\rangle = 0$ for all $\xi \in \mathcal{H}$. If $S = T - T^*$ then

$$\langle \eta, S\xi\rangle = \frac{1}{4}\left(\langle \eta + \xi, S(\eta + \xi)\rangle - \langle \eta - \xi, S(\eta - \xi)\rangle + i\langle \eta + i\xi, S(\eta + i\xi)\rangle - i\langle \eta - i\xi, S(\eta - i\xi)\rangle\right)$$

for all $\xi, \eta \in \mathcal{H}$. Since each quantity on the right is zero, $\langle \eta, (T - T^*)\xi\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$. Hence $T - T^* = 0$ so $T = T^*$.

Next suppose that $T \geq 0$ in $\mathcal{B}_a(\mathcal{H})$. Then there exists an $S \in \mathcal{B}_a(\mathcal{H})$ such that $T = S^*S$. Whence $\langle \xi, T\xi\rangle = \langle S\xi, S\xi\rangle \geq_{\mathfrak{B}} 0$ for all $\xi \in \mathcal{H}$. Now suppose that $\langle \xi, T\xi\rangle \geq_{\mathfrak{B}} 0$ for all $\xi \in \mathcal{H}$. By the above paragraph,

$T = T^*$. By the Continuous Functional Calculus, there exists $T_+, T_- \in \mathcal{B}_a(\mathcal{H})$ such that $T = T_+ - T_-$, $T_+, T_- \geq 0$, $T_+T_- = T_-T_+ = 0$. Since $T_- \geq 0$, $T_-^3 \geq 0$ and $\langle \xi, T_-^3\xi\rangle \geq 0$ by the above proof. Moreover $0 \leq \langle \xi, T\xi\rangle \leq \langle \xi, T_+\xi\rangle - \langle \xi, T_-\xi\rangle$ for all $\xi \in \mathcal{H}$. If $\eta \in \mathcal{H}$ and we let $\xi = T_-\mathcal{H}\eta$ in the previous inequality, then

$$\langle \eta, T_-^3\eta\rangle = \langle T_-\eta, T_-(T_-\eta)\rangle \leq \langle T_-\eta, T_+T_-\eta\rangle = 0$$

Therefore $\langle \eta, T_-^3\eta\rangle = 0$ for all $\eta \in \mathcal{H}$. By the same polar decomposition of the inner product, we obtain that $T_-^3 = 0$ which implies $T_- = 0$. Hence $T = T_+ \geq 0$ as desired. □

There is no reason that we should only act on a space from one side. We needed the right action of $\mathfrak{B}$ on a right Hilbert $\mathfrak{B}$-module to obtain the Cauchy Schwarz inequality but there is no reason that we require the left action to be by $\mathfrak{B}$.

Definition Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras, let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module, and suppose that there exists a *-homomorphism $\lambda : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$. Then we call $\mathcal{H}$ a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule.

Example Let $\mathcal{H} = \mathfrak{B}$, $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by $\langle A, B\rangle = A^*B$, and $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)A = AB$ for all $A \in \mathcal{H}$ and $B \in \mathfrak{B}$. We have seen that $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module. For each $A \in \mathfrak{B}$, define $\lambda(A)B = AB$. We claim that $\lambda : \mathfrak{B} \to \mathcal{B}_a(\mathcal{H})$ is a unital *-homomorphism. It is clear that each $\lambda(A)$ is linear and that λ is a homomorphism. Also, for all $A \in \mathfrak{B}$ and $B, C \in \mathcal{H}$,

$$\langle C, \lambda(A)B\rangle = \langle C, AB\rangle = C^*AB = (A^*C)^*B = \langle A^*C, B\rangle = \langle \lambda(A^*)C, B\rangle$$

so $\lambda(A) \in \mathcal{B}_a(\mathcal{H})$ with $\lambda(A^*) = \lambda(A)^*$. Hence $\mathcal{H}$ is a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. Notice that λ and ρ are just left and right multiplication respectively. This is the canonical example.

Example Let $\mathfrak{B}$ be a C*-algebra, I an index set, and $\mathcal{H} = \oplus_I \mathfrak{B}$. Define $\lambda : \mathfrak{B} \to \mathcal{B}_a(\mathcal{H})$ by $\lambda(B)(A_i)_{i\in I} = (BA_i)_{i\in I}$. We notice for all i that $A_i^*B^*BA_i \leq \|B\|^2 A_i^*A_i$ so $\sum_{i\in I} A_i^*B^*BA_i$ converges as $\sum_{i\in I} A_i^*A_i$ converges. Thus λ is well-defined. It is clear that λ is a homomorphism, λ commutes with the right action of $\mathfrak{B}$ on $\mathcal{H}$ and that $\lambda(B)^* = \lambda(B^*)$ by the same proof as above. Hence $\mathcal{H}$ is a Hilbert $\mathfrak{B}$-$\mathfrak{B}$-bimodule.

Example Let $\mathfrak{B}$ be a C*-algebra, $\mathcal{H} = \mathfrak{B}^n$, $\langle \cdot, \cdot \rangle : \mathcal{H}\times\mathcal{H} \to \mathfrak{B}$ defined by $\langle (A_1, A_2, \dots, A_n), (B_1, B_2, \dots, B_n)\rangle = \sum_{i=1}^n A_i^*B_i$, and let $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)(A_1, A_2, \dots, A_n) = (A_1B, A_2B, \dots, A_nB)$. We have see that $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module. Let $\mathfrak{A} = \mathcal{M}_n(\mathfrak{B})$ and define $\lambda : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$ by $\lambda([B_{i,j}])(A_1, A_2, \dots, A_n) = \left(\sum_{j=1}^n B_{1,j}A_j, \dots, \sum_{j=1}^n B_{n,j}A_j\right)$. It is clear that λ is a homomorphism as this is simply matrix multiplication. To see that λ is a *-homomorphism, we notice that

$$\begin{aligned}
\langle(\lambda([B_{i,j}]))^*(B_1, B_2, \dots, B_n), (A_1, A_2, \dots, A_n)\rangle &= \langle(B_1, B_2, \dots, B_n), \lambda([B_{i,j}])(A_1, A_2, \dots, A_n)\rangle \\
&= \left\langle (B_1, B_2, \dots, B_n), \left(\sum_{j=1}^n B_{1,j}A_j, \dots, \sum_{j=1}^n B_{n,j}A_j\right)\right\rangle \\
&= \sum_{j,k=1}^n B_k^*B_{k,j}A_j \\
&= \left\langle \left(\sum_{k=1}^n B_{1,k}^*B_k, \dots, \sum_{k=1}^n B_{n,k}^*B_k\right), (A_1, A_2, \dots, A_n)\right\rangle \\
&= \langle \lambda([B_{i,j}]^*)(B_1, B_2, \dots, B_n), (A_1, A_2, \dots, A_n)\rangle
\end{aligned}$$

Hence λ is a *-homomorphism. Moreover

$$\begin{aligned}\lambda([B_{i,j}])\rho(B)(A_1, A_2, \ldots, A_n) &= \lambda([B_{i,j}])(A_1B, A_2B, \ldots, A_nB) \\ &= \left(\sum_{j=1}^{n} B_{1,j}A_jB, \ldots, \sum_{j=1}^{n} B_{n,j}A_jB\right) \\ &= \rho(B)\left(\sum_{j=1}^{n} B_{1,j}A_j, \ldots, \sum_{j=1}^{n} B_{n,j}A_j\right) \\ &= \rho(B)\lambda([B_{i,j}])(A_1, A_2, \ldots, A_n)\end{aligned}$$

Hence $\mathcal{H}$ is a Hilbert $\mathcal{M}_n(\mathfrak{B})$-$\mathfrak{B}$-bimodule. Similarly, if $\mathfrak{A}$ is a closed *-subalgebra of $\mathcal{M}_n(\mathfrak{B})$, $\lambda|_{\mathfrak{A}}$ turns $\mathcal{H}$ into a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. Similarly, we can place other left actions on $\oplus_I \mathfrak{B}$ but we need to be careful about convergence.

Example We saw earlier that if $\mathfrak{B} = C[0,1]$, $X = C([0,1]^2)$, $\langle \cdot, \cdot \rangle : X \times X \to \mathfrak{B}$ by $\langle f, g\rangle(x) = \int_0^1 \overline{f(x,y)}g(x,y)dy$, $\rho : \mathfrak{B} \to \mathcal{L}(X)$ by $(\rho(f)g))(x,y) = f(x)g(x,y)$, and $\mathcal{H}$ was the completion of X with respect to the norm induced by this $\mathfrak{B}$-inner product and the action of ρ on X, $\mathcal{H}$ was a right Hilbert $\mathfrak{B}$-module. Define $\lambda_0 : \mathfrak{B} \to \mathcal{L}(X)$ by $(\lambda_0(f)g)(x,y) = f(y)g(x,y)$ for all $f \in C[0,1]$ and $g \in X$. It is easy to verify that λ_0 is a homomorphism with $\|\lambda(f)\| \leq \|f\|_\infty$. Hence we can extend the definition of λ_0 to $\lambda : \mathfrak{B} \to \mathcal{B}_a(\mathcal{H})$. Moreover, since λ_0 was a homomorphism, λ is as well. It is easy to verify that $\langle \lambda(f^*)g, h\rangle = \langle \lambda(f)^*g, h\rangle$ for all $g, h \in X$ and thus by continuity, λ is a *-homomorphism. Similarly, $\lambda(f)\rho(g)h = \rho(g)\lambda(f)h$ for all $h \in X$ and thus for all $h \in \mathcal{H}$ by continuity. Hence $\mathcal{H}$ has the structure of a Hilbert $C[0,1]$-$C[0,1]$ bimodule.

Remarks Every right Hilbert $\mathfrak{B}$-module $\mathcal{H}$ is a Hilbert $\mathbb{C}$-$\mathfrak{B}$-bimodule where the action of $\mathbb{C}$ on $\mathcal{H}$ is scalar multiplication.

Remarks Now we would like to show an equivalence between completely positive maps and Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodules. The main point of the following theorem is that every completely positive map between C*-algebras gives rise to a Hilbert bimodule. The following is a generalization of the GNS construction.

Theorem *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras and $\varphi : \mathfrak{A} \to \mathfrak{B}$ a completely positive map. Then there exists a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule $\mathcal{H}$ and a $\xi \in \mathcal{H}$ such that $\|\xi\|^2_{\mathcal{H}} = \|\varphi\|_{cb}$ and $\varphi(A) = \langle \xi, \lambda(A)\xi\rangle$. Moreover $\lambda(1_{\mathfrak{A}}) = Id_{\mathcal{H}}$, $\rho(1_{\mathfrak{B}}) = I_{\mathcal{H}}$, and $span\{\lambda(\mathfrak{A})\rho(\mathfrak{B})\xi\}$ is dense in $\mathcal{H}$.*

Remarks If $\mathfrak{B} = \mathbb{C}$, then the above is precisely the GNS theorem.

Proof: Consider $\mathfrak{A} \odot \mathfrak{B}$, the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$. We desire to define a positive sesquilinear form $\langle \cdot, \cdot \rangle : \mathfrak{A} \odot \mathfrak{B} \to \mathfrak{B}$ such that $\langle c \otimes d, a \otimes b\rangle = d^*\varphi(c^*a)b$ for all elementary tensors $a \otimes b, c \otimes d \in \mathfrak{A} \odot \mathfrak{B}$. The question is, how can we do this?

We recall from algebra that if A, B, and C are vector spaces, $\psi : A \times B \to A \odot B$ is the canonical bilinear map, and $\phi : A \times B \to C$ is a bilinear map, there exists a unique linear map $\tilde{\phi} : A \odot B \to C$ such that $\tilde{\phi}(a \otimes b) = \phi(a,b)$ for all elementary tensors $a \otimes b \in A \odot B$. To proceed in creating our sesquilinear form, fix $c \in \mathfrak{A}$ and $d \in \mathfrak{B}$. Define $\phi_{c,d} : \mathfrak{A} \times \mathfrak{B} \to \mathfrak{B}$ by $\phi_{c,d}(a,b) = d^*\varphi(c^*a)b$ (which is well-defined since $c^*a \in \mathfrak{A}$ and $d^*, b, \varphi(c^*a) \in \mathfrak{B}$). Since φ is linear, it is clear that $\phi_{c,d}$ is a bilinear form. Therefore, by the Universal Property of Algebraic Tensor Products, there exists a linear map $\tilde{\phi}_{c,d} : \mathfrak{A} \odot \mathfrak{B} \to \mathfrak{B}$ such that $\tilde{\phi}_{c,d}(a \otimes b) = \phi_{c,d}(a,b) = d^*\varphi(c^*a)b$. Let G be the space of all conjugate linear functionals on $\mathfrak{A} \odot \mathfrak{B}$. Define $\psi : \mathfrak{A} \times \mathfrak{B} \to G$ by $\psi(c,d) = (\tilde{\phi}_{c,d})^*$ where $(\tilde{\phi}_{c,d})^*(u) = (\tilde{\phi}_{c,d}(u))^*$ for all $u \in \mathfrak{A} \odot \mathfrak{B}$ (it is clear that $(\tilde{\phi}_{c,d})^*$ is a conjugate linear map since $\tilde{\phi}_{c,d}$ was a linear map). We claim that ψ is a bilinear form. To see this, we notice for all $\lambda \in \mathbb{C}$ and $c_1, c_2 \in \mathfrak{A}$ that

$$\begin{aligned}(\tilde{\phi}_{\lambda c_1+c_2,d})^*(a\otimes b) &= (\tilde{\phi}_{\lambda c_1+c_2,d}(a\otimes b))^* \\ &= (d^*\varphi((\lambda c_1+c_2)^*a)b)^* \\ &= (\overline{\lambda}d^*\varphi(c_1^*a)b + d^*\varphi(c_2^*a)b)^* \\ &= \lambda(d^*\varphi(c_1^*a)b)^* + (d^*\varphi(c_2^*a)b)^* \\ &= \lambda(\tilde{\phi}_{c_1,d})^*(a\otimes b) + (\tilde{\phi}_{c_2,d})^*(a\otimes b)\end{aligned}$$

for all elementary tensors $a \otimes b \in \mathfrak{A} \otimes \mathfrak{B}$. Thus, since this holds for all elementary tensors, we obtain by conjugate linearity that $(\tilde{\phi}_{\lambda c_1+c_2,d})^* = \lambda(\tilde{\phi}_{c_1,d})^* + (\tilde{\phi}_{c_2,d})^*$. Thus ψ is linear in the first component. Similarly ψ is linear in the second component so that ψ is a bilinear form. Thus, by the Universal Property of Algebraic Tensor Products, there exists a $\Psi : \mathfrak{A} \odot \mathfrak{B} \to G$ such that $\Psi(c \otimes d) = \psi(c, d) = (\tilde{\phi}_{c,d})^*$ for all elementary tensors $c \otimes d \in \mathfrak{A} \odot \mathfrak{B}$.

Define $\langle \cdot, \cdot \rangle : \mathfrak{A} \odot \mathfrak{B} \to \mathfrak{B}$ by $\langle v, u\rangle = (\Psi(v)^*)(u)$ for all $u, v \in \mathfrak{A} \odot \mathfrak{B}$ (where * represents the same operation on linear/conjugate linear functionals that was used before). Then

$$\langle c\otimes d, a\otimes b\rangle = (\Psi(c\otimes d)^*)(a\otimes b) = ((\tilde{\phi}_{c,d})^*)^*(a\otimes b) = \tilde{\phi}_{c,d}(a\otimes b) = d^*\varphi(c^*a)b$$

as desired. To see that $\langle \cdot, \cdot \rangle$ is a sesquilinear form, we notice that each $\Psi(v) \in G$ is conjugate linear so $\Psi(v)^*$ is linear so $\langle \cdot, \cdot \rangle$ is linear in the second component. Since Ψ is linear, $\Psi(\cdot)^*$ is conjugate linear so $\langle \cdot, \cdot \rangle$ is conjugate linear in the first component. Thus $\langle \cdot, \cdot \rangle$ is a sesquilinear form.

We claim that $\langle \cdot, \cdot \rangle$ is positive. To see this, let $u = \sum_{i=1}^n a_i \otimes b_i \in \mathfrak{A} \odot \mathfrak{B}$ be arbitrary. Then

$$\begin{aligned}\langle u, u\rangle &= \left\langle \sum_{i=1}^n a_i\otimes b_i, \sum_{j=1}^n a_j \otimes b_j, \right\rangle \\ &= \sum_{i,j=1}^n b_i^*\varphi(a_i^*a_j)b_j\end{aligned}$$

Thus to show that $\langle u, u\rangle \geq_{\mathfrak{B}} 0$, we need only show that $\sum_{i,j=1}^n b_i^*\varphi(a_i^*a_j)b_j \geq_{\mathfrak{B}} 0$. To see this, we notice that

$$\begin{bmatrix} \sum_{i,j=1}^n b_i^*\varphi(a_i^*a_j)b_j & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}_{n\times n} = \begin{bmatrix} b_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}_{n\times n}^* [\varphi(a_i^*a_j)]_{i,j} \begin{bmatrix} b_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}_{n\times n} \quad (*)$$

However $[\varphi(a_i^*a_j)]_{i,j} = \varphi_n([a_i^*a_j]_{i,j})$. Since

$$[a_i^*a_j]_{i,j} = \begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}^* \begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

$[a_i^*a_j]_{i,j}$ is positive. Since φ is completely positive, $[\varphi(a_i^*a_j)]_{i,j}$ is positive so the matrix on the left in $(*)$ is positive. Hence $\sum_{i,j=1}^n b_i^*\varphi(a_i^*a_j)b_j$ is positive so $\langle u, u\rangle \geq_{\mathfrak{B}} 0$. Hence $\langle \cdot, \cdot \rangle$ is positive.

Let $\mathcal{N} = \{u \in \mathfrak{A} \odot \mathfrak{B} \mid \langle u, u\rangle = 0\}$. We claim that $\mathcal{N}$ is a subspace of $\mathfrak{A} \odot \mathfrak{B}$. To see this, we recall that $\|\langle u, v\rangle\|_{\mathfrak{B}}^2 \leq 4\|\langle u, u\rangle\|_{\mathfrak{B}} \|\langle v, v\rangle\|_{\mathfrak{B}}$ for all $u, v \in \mathfrak{A} \odot \mathfrak{B}$. Thus if $u \in \mathcal{N}$ then $\langle u, v\rangle = 0$ for all $v \in \mathfrak{A} \odot \mathfrak{B}$. Therefore we obtain that $\mathcal{N} = \{u \in \mathfrak{A} \odot \mathfrak{B} \mid \langle u, v\rangle = 0 \text{ for all } v \in \mathfrak{A} \odot \mathfrak{B}\}$ so that $\mathcal{N}$ is a subspace of $\mathfrak{A} \odot \mathfrak{B}$.

Consider $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$ which is a well-defined vector space. Since $\mathcal{N} = \{u \in \mathfrak{A} \odot \mathfrak{B} \mid \langle u, u\rangle = 0\}$, $\langle \cdot, \cdot \rangle$ restricts to a well-defined $\mathfrak{B}$-valued inner product on $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$. Define a norm on $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$ by

$\|h\| = \sqrt{\|\langle h,h\rangle\|_{\mathfrak{B}}}$. Let $\mathcal{H}$ be the completion of $\mathfrak{A}\odot\mathfrak{B}/\mathcal{N}$ with respect to this norm. Hence $\mathcal{H}$ is a Hilbert $\mathfrak{B}$-space.

Now we desire to define the $\mathfrak{B}$-action on $\mathcal{H}$. For each $b\in\mathfrak{B}$, define $\rho(b):\mathfrak{A}\odot\mathfrak{B}\to\mathfrak{A}\odot\mathfrak{B}$ by $\rho(b)(a'\otimes b') = a'\otimes b'b$ which exists as it comes from a well-defined bilinear map on $\mathfrak{A}\times\mathfrak{B}$. Next we notice for all $u=\sum_{i=1}^n a_i\otimes b_i\in\mathfrak{A}\odot\mathfrak{B}$,

$$\begin{aligned}
0\leq \langle\rho(b)u,\rho(b)u\rangle &= \left\langle \sum_{i=1}^n a_i\otimes b_ib, \sum_{j=1}^n a_j\otimes b_jb, \right\rangle \\
&= \sum_{i,j=1}^n b^*b_i^*\varphi(a_i^*a_j)b_jb \\
&= b^*\langle u,u\rangle b
\end{aligned}$$

Hence if $u\in\mathcal{N}$, then $\rho(b)u\in\mathcal{N}$. Thus $\rho(b)$ extends to a well-defined linear map on $\mathfrak{A}\odot\mathfrak{B}/\mathcal{N}$. Moreover, the above computations show that $\|\rho(b)u\|_{\mathcal{H}}\leq\sqrt{\|b\|_{\mathfrak{B}}^2\,\|\langle u,u\rangle\|_{\mathfrak{B}}} = \|b\|_{\mathfrak{B}}\,\|u\|_{\mathcal{H}}$. Thus $\rho(b)$ is bounded in the norm on $\mathfrak{A}\odot\mathfrak{B}/\mathcal{N}$. Since $\mathfrak{A}\odot\mathfrak{B}/\mathcal{N}$ is dense in $\mathcal{H}$, $\rho(b)$ extends to a bounded linear map on $\mathcal{H}$. Define $\rho:\mathfrak{B}\to\mathcal{B}(\mathcal{H})$ by $b\mapsto\rho(b)$. From the definition of $\rho(b)$ on $\mathfrak{A}\odot\mathfrak{B}$, it is clear that ρ is linear and anti-multiplicative. Lastly we notice that if $u=\sum_{i=1}^n a_i\otimes b_i, v=\sum_{j=1}^m c_j\otimes d_j\in\mathfrak{A}\odot\mathfrak{B}$ and $b\in\mathfrak{A}$, then

$$\begin{aligned}
\langle u,\rho(b)v\rangle &= \left\langle \sum_{i=1}^n a_i\otimes b_i, \sum_{j=1}^m c_j\otimes d_jb \right\rangle \\
&= \sum_{i=1}^n\sum_{j=1}^m b_i^*\varphi(a_i^*c_j)d_jb \\
&= \left\langle \sum_{i=1}^n a_i\otimes b_i, \sum_{j=1}^m c_j\otimes d_j \right\rangle b \\
&= \langle u,v\rangle b
\end{aligned}$$

Thus, as the above holds on $\mathfrak{A}\odot\mathfrak{B}$, ρ is bounded with $\|\rho\|\leq 1$, and the $\mathfrak{B}$-valued inner product is continuous in each component, each of these properties extends to $\mathcal{H}$ as desired. Thus $\mathcal{H}$ is a right Hilbert $\mathfrak{B}$-module.

Next we desire to define λ. For each $a\in\mathfrak{A}$, define $\lambda(a):\mathfrak{A}\odot\mathfrak{B}\to\mathfrak{A}\odot\mathfrak{B}$ by $\lambda(a)(a'\otimes b') = aa'\otimes b'$ which exists as it comes from a well-defined bilinear map on $\mathfrak{A}\times\mathfrak{B}$. Next we notice for all $u=\sum_{i=1}^n a_i\otimes b_i\in\mathfrak{A}\odot\mathfrak{B}$,

$$\begin{aligned}
0\leq \langle\lambda(a)u,\lambda(a)u\rangle &= \left\langle \sum_{i=1}^n aa_i\otimes b_i, \sum_{j=1}^n aa_j\otimes b_j, \right\rangle \\
&= \sum_{i,j=1}^n b_i^*\varphi((aa_i)^*aa_j)b_j \\
&= \sum_{i,j=1}^n b_i^*\varphi(a_i^*a^*aa_j)b_j
\end{aligned}$$

However $\|a^*a\|_{\mathfrak{A}}1 - a^*a\geq 0$ so $\|a^*a\|_{\mathfrak{A}}I_n - diag(a^*a)\geq 0$ so

$$\begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix}^* (\|a^*a\|_{\mathfrak{A}}I_n - diag(a^*a)) \begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{bmatrix} \geq 0$$

Hence $\|a^*a\|_{\mathfrak{A}}^2 [a_i^* a_j] \geq [a_i^* a^* a a_j]$. Therefore $\|a^*a\|_{\mathfrak{A}}^2 [\varphi(a_i^* a_j)] \geq [\varphi(a_i^* a^* a a_j)]$ as φ is completely positive so

$$\sum_{i,j=1}^{n} b_i^* (\|a^*a\|_{\mathfrak{A}}^2 - \varphi(a_i^* a^* a a_j)) b_j =$$

$$P_{1,1}\left(\begin{bmatrix} b_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}_{n\times n}^* (\|a^*a\|_{\mathfrak{A}}^2 [\varphi(a_i^* a_j)] - [\varphi(a_i^* a^* a a_j)]) \begin{bmatrix} b_1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}_{n\times n} \right) \geq 0$$

(where $P_{1,1} : \mathcal{M}_n(\mathfrak{B}) \to \mathfrak{B}$ takes the $(1,1)$-entry of the matrix) since $P_{1,1}$ is a positive map and the matrix inside is positive. Therefore

$$0 \leq \langle \lambda(a)u, \lambda(a)u \rangle \leq \|a^*a\|_{\mathfrak{A}}^2 \sum_{i,j=1}^{n} b_i^* \varphi(a_i^* a_j) b_j = \|a\|_{\mathfrak{A}}^2 \langle u, u \rangle$$

Hence if $u \in \mathcal{N}$, then $\lambda(a)u \in \mathcal{N}$. Thus $\lambda(a)$ extends to a well-defined linear map on $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$. Moreover, the above computations show that $\|\lambda(a)u\|_{\mathcal{H}} \leq \sqrt{\|a\|_{\mathfrak{A}}^2 \|\langle u, u \rangle\|_{\mathfrak{B}}} = \|a\|_{\mathfrak{A}} \|u\|_{\mathcal{H}}$. Thus $\lambda(a)$ is bounded in the norm on $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$. Since $\mathfrak{A} \odot \mathfrak{B}/\mathcal{N}$ is dense in $\mathcal{H}$, $\lambda(a)$ extends to a bounded linear map on $\mathcal{H}$. Define $\lambda : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by $a \mapsto \lambda(a)$. From the definition of $\lambda(a)$ on $\mathfrak{A} \odot \mathfrak{B}$, it is clear that λ is a homomorphism. Lastly we notice that if $u = \sum_{i=1}^n a_i \otimes b_i, v = \sum_{j=1}^m c_j \otimes d_j \in \mathfrak{A} \odot \mathfrak{B}$ and $a \in \mathfrak{A}$, then

$$\begin{aligned} \langle \lambda(a^*)u, v \rangle &= \left\langle \sum_{i=1}^{n} a^* a_i \otimes b_i, \sum_{j=1}^{n} c_j \otimes d_j \right\rangle \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} b_j^* \varphi(a_j^*(a^* c_i)) d_j \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} b_j^* \varphi((a a_j)^* c_i) d_j \\ &= \left\langle \sum_{i=1}^{n} a_i \otimes b_i, \sum_{j=1}^{n} a c_j \otimes d_j \right\rangle \\ &= \langle u, \lambda(a)v \rangle \end{aligned}$$

Thus, as the above holds on $\mathfrak{A} \odot \mathfrak{B}$, λ is bounded with $\|\lambda\| \leq 1$, and the $\mathfrak{B}$-valued inner product is continuous in each component, each of these properties extends to $\mathcal{H}$ as desired. Moreover, it is clear that $\lambda(1) = Id_{\mathcal{H}}$.

Let $\xi = 1_{\mathfrak{A}} \otimes 1_{\mathfrak{B}} + \mathcal{N} \in \mathcal{H}$. Then $\langle \xi, \xi \rangle = \varphi(1)$ so $\|\xi\|_{\mathcal{H}}^2 = \|\varphi(1_{\mathfrak{A}})\|_{\mathfrak{B}} = \|\varphi\|_{cb}$. Moreover for all $a \in \mathfrak{A}$,

$$\langle \xi, \lambda(a)\xi \rangle = \langle 1_{\mathfrak{A}} \otimes 1_{\mathfrak{B}}, a \otimes 1_{\mathfrak{B}} \rangle = \varphi(a)$$

as desired. Lastly, we notice that $\rho(b)(\lambda(a)\xi) = a \otimes b + \mathcal{H}$. Thus it is clear that $span\{\lambda(\mathfrak{A})\rho(\mathfrak{B})\xi\}$ is dense in $\mathcal{H}$. □

Remarks Now we would like to prove the converse of the above; that is every map of the form $\varphi(A) = \langle \xi, \lambda(A)\xi \rangle$ in a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule is completely positive. To prove this, it is convenient to prove the following lemma.

Lemma *Let $\mathfrak{A}$ be a C^*-algebra. Then every positive element of $\mathcal{M}_n(\mathfrak{A})$ is the sum of n positive elements of the form $[a_i^* a_j]$ for some $\{a_1, \dots, a_n\} \in \mathfrak{A}$.*

Proof: Suppose $\{a_1, \ldots, a_n\} \in \mathfrak{A}$. Then

$$\begin{bmatrix} a_1^* a_1 & \ldots & a_1^* a_n \\ \vdots & & \vdots \\ a_n^* a_1 & \ldots & a_n^* a_n \end{bmatrix} = \left(\begin{bmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix} \right)^* \begin{bmatrix} a_1 & \ldots & a_n \\ 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \end{bmatrix}$$

Thus $[a_i^* a_j]$ will be a positive matrix.

Suppose $P \in \mathcal{M}_n(\mathfrak{A})$ is a positive matrix. Then there exists a $B \in \mathcal{M}_n(\mathfrak{A})$ such that $P = B^*B$. Let R_k be an $n \times n$ matrix with its k^{th} row being the k^{th} row of B and 0's elsewhere. Then $R_i^* R_j = 0$ if $i \neq j$. Thus $P = R_1^* R_1 + \ldots + R_n^* R_n$. However, each $R_k^* R_k$ is of the form $[a_i^* a_j]$ for some $\{a_1, \ldots, a_n\} \in \mathfrak{A}$ as each R_k is a matrix with only non-zero entries in the k^{th} row (i.e. a_i is the i^{th} entry of the k^{th} row). $\square$

Remarks The above lemma shows use that we need only consider elements of the form $[a_i^* a_j]$ for some $\{a_1, \ldots, a_n\} \in \mathfrak{A}$ when testing when a positive map is completely positive. Before we get to our major theorem, we have the following essential lemma.

Lemma *Let $\mathfrak{A}$ be a C*-algebra. An element $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is positive if and only if $\sum_{i,j=1}^n A_i^* A_{i,j} A_j \geq 0$ for all $A_1, \ldots, A_n \in \mathfrak{A}$.*

Proof: We recall that if $\mathfrak{A}$ is a C*-algebra, $\mathcal{H} = \mathfrak{A}^n$, $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathfrak{A}$ is defined by $\langle (A_1, \ldots, A_n), (B_1, \ldots, B_n) \rangle = \sum_{i=1}^n A_i^* B_i$, $\rho : \mathfrak{A} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)(A_1, \ldots, A_n) = (A_1 B, \ldots, A_n B)$, and $\lambda : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{B}_a(\mathcal{H})$ by $\lambda([B_{i,j}])(A_1, \ldots, A_n) = \left(\sum_{j=1}^n B_{1,j} A_j, \ldots, \sum_{j=1}^n B_{n,j} A_j \right)$, then $\mathcal{H}$ is a $\mathcal{M}_n(\mathfrak{A})$-$\mathfrak{A}$-bimodule.

Let $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ be fixed. Since $\lambda : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{B}_a(\mathcal{H})$ is a *-homomorphism $[A_{i,j}]$ is positive in $\mathcal{M}_n(\mathfrak{A})$ if and only if $\lambda([A_{i,j}])$ is positive in $\mathcal{B}_a(\mathcal{H})$. By a previous lemma, $\lambda([A_{i,j}])$ is positive in $\mathcal{B}_a(\mathcal{H})$ if and only if $\langle h, \lambda([A_{i,j}])h \rangle \geq 0$ for all $h \in \mathcal{H}$. However, if $h = (A_1, A_2, \ldots, A_n) \in \mathcal{H}$ is arbitrary

$$\langle h, \lambda([A_{i,j}])h \rangle_{\mathcal{H}} = \left\langle (A_1, A_2, \ldots, A_n), \left(\sum_{j=1}^n A_{1,j} A_j, \ldots, \sum_{j=1}^n A_{n,j} A_j \right) \right\rangle = \sum_{i,j=1}^n A_i^* A_{i,j} A_j$$

so the result follows. $\square$

Corollary *Let $\mathfrak{B}$ be a C*-algebras and let $\mathcal{H}$ be a right Hilbert $\mathfrak{B}$-module. For any $\eta_1, \ldots, \eta_n \in \mathcal{H}$ the matrix $[\langle \eta_i, \eta_j \rangle]_{i,j} \in \mathcal{M}_n(\mathfrak{B})$ is positive.*

Proof: We will apply the above lemma. Suppose $B_1, \ldots, B_n \in \mathfrak{B}$ are arbitrary. Then

$$\begin{aligned} \sum_{i,j=1}^n B_i^* \langle \eta_i, \eta_j \rangle B_j &= \sum_{i,j=1}^n \langle \rho(B_i)(\eta_i), \rho(B_j)(\eta_j) \rangle \\ &= \sum_{j=1}^n \left\langle \sum_{i=1}^n \rho(B_i)(\eta_i), \rho(B_j)(\eta_j) \right\rangle \\ &= \left\langle \sum_{i=1}^n \rho(B_i)(\eta_i), \sum_{j=1}^n \rho(B_j)(\eta_j) \right\rangle \geq 0 \end{aligned}$$

Hence the result follows from the above lemma. $\square$

Corollary *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\mathcal{H}$ be a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. For any $\eta_1, \ldots, \eta_n \in \mathcal{H}$ and $a \in \mathfrak{A}$, $[\langle \lambda(a)\eta_i, \lambda(a)\eta_j \rangle]_{i,j} \leq \|a\|_{\mathfrak{A}}^2 [\langle \eta_i, \eta_j \rangle]_{i,j} \in \mathcal{M}_n(\mathfrak{B})$.*

Proof: We will apply the above lemma. Suppose $B_1, \ldots, B_n \in \mathfrak{B}$ are arbitrary. Then

$$\begin{aligned}\sum_{i,j=1}^{n} B_i^*(\|a\|_{\mathfrak{A}}^2 \langle \eta_i, \eta_j\rangle - \langle \lambda(a)\eta_i, \lambda(a)\eta_j\rangle)B_j &= \|a\|_{\mathfrak{A}}^2 \left\langle \sum_{i=1}^{n} \rho(B_i)(\eta_i), \sum_{j=1}^{n} \rho(B_j)(\eta_j)\right\rangle \\ &\quad - \left\langle \sum_{i=1}^{n} \rho(B_i)(\lambda(a)\eta_i), \sum_{j=1}^{n} \rho(B_j)(\lambda(a)\eta_j)\right\rangle \\ &= \|a\|_{\mathfrak{A}}^2 \left\langle \sum_{i=1}^{n} \rho(B_i)(\eta_i), \sum_{j=1}^{n} \rho(B_j)(\eta_j)\right\rangle \\ &\quad - \left\langle \lambda(a)\left(\sum_{i=1}^{n} \rho(B_i)(\eta_i)\right), \lambda(a)\left(\sum_{j=1}^{n} \rho(B_j)(\eta_j)\right)\right\rangle \\ &= \|a\|_{\mathfrak{A}}^2 \langle u, u\rangle - \langle \lambda(a)u, \lambda(a)u\rangle\end{aligned}$$

where $u = \sum_{i=1}^{n} \rho(B_i)(\eta_i) \in \mathcal{H}$. Thus is suffices to show that $\|a\|_{\mathfrak{A}}^2 \langle u, u\rangle - \langle \lambda(a)u, \lambda(a)u\rangle \geq 0$ for any $u \in \mathcal{H}$. However $\lambda : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$ is a *-homomorphism and thus a contractive positive map. Since $a^*a \geq 0$, $\lambda(a^*a) \geq 0$ and $0 \leq \lambda(a^*a) \leq \|\lambda(a^*a)\| I \leq \|a^*a\|_{\mathfrak{A}} I \leq \|a\|_{\mathfrak{A}}^2 I$. Therefore, since $I \in \mathcal{B}_a(\mathcal{H})$, by a earlier lemma $\langle u, (\|a\|_{\mathfrak{A}}^2 I - \lambda(a^*a))u\rangle \geq 0$ so $0 \leq \langle \lambda(a)u, \lambda(a)u\rangle = \langle u, \lambda(a^*a)u\rangle \leq \|a\|_{\mathfrak{A}}^2 \langle u, u\rangle$ as desired. □

Theorem *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $\mathcal{H}$ be a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. If $\xi \in \mathcal{H}$ and we define $\varphi_\xi : \mathfrak{A} \to \mathfrak{B}$ by $\varphi_\xi(a) = \langle \xi, \lambda(a)\xi\rangle$, then φ_ξ is a completely positive map.*

Proof: By the first of the above lemmas, it suffice to show that $\varphi_n([a_i^*a_j]) = [\langle \xi, \lambda(a_i^*a_j)\xi\rangle] = [\langle \lambda(a_i)\xi, \lambda(a_j)\xi\rangle] \in \mathcal{M}_n(\mathfrak{B})$ is positive for all $\{a_1, \ldots, a_n\} \in \mathfrak{A}$ (note the essentialness of λ being a *-homomorphism). However, by one of the corollaries, this is a positive element of $\mathcal{M}_n(\mathfrak{B})$ which completes the proof. □

Remarks Before we move on we note the following result. Note how the proof used at the end of the previous corollary and the most recent lemma are essential.

Proposition *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and let $(\mathcal{H}_\alpha)_{\alpha\in\Lambda}$ be Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodules (where the actions of $\mathfrak{A}$ and $\mathfrak{B}$ come from $\lambda_\alpha : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\alpha)$ and $\rho_\alpha : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_\alpha)$). Let $\mathcal{H} = \oplus_\Lambda \mathcal{H}_\alpha$ be the space of all functions $\alpha \mapsto h_\alpha \in H_\alpha$ such that $\sum_\Lambda \langle h_\alpha, h_\alpha\rangle_\alpha$ converges in $\mathfrak{B}$. Define $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathfrak{B}$ by $\langle (h_\alpha)_\Lambda, (k_\alpha)_\Lambda\rangle = \sum_\Lambda \langle h_\alpha, k_\alpha\rangle_\alpha$. Then $\langle \cdot, \cdot \rangle$ is a well-defined $\mathfrak{B}$-valued inner product and $\mathcal{H}$ is a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule whose actions $\lambda : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$ and $\rho : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ act component wise (i.e. $\lambda(a)(h_\alpha)_\Lambda = (\lambda_\alpha(a)h_\alpha)_\Lambda$).*

Proof: First we need to show that the $\mathfrak{B}$-valued inner product is well-defined. By the GNS construction, we can assume that $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. Let $(h_\alpha)_\Lambda, (k_\alpha)_\Lambda \in \mathcal{H}$ be arbitrary. We recall that if $P, Q, A \in \mathcal{B}(\mathcal{K})$ are such that

$$\begin{bmatrix} P & A \\ A^* & Q \end{bmatrix} \geq 0$$

then $|\langle \eta, A\xi\rangle| \leq |\langle \eta, P\eta\rangle|^{\frac{1}{2}} |\langle \xi, Q\xi\rangle|^{\frac{1}{2}}$ for all $\xi, \eta \in \mathcal{K}$. However, since each $\langle \cdot, \cdot \rangle_\alpha$ is a $\mathfrak{B}$-valued inner product, for a right Hilbert $\mathfrak{B}$-module, by a previous corollary,

$$\begin{bmatrix} \langle h_\alpha, h_\alpha\rangle_\alpha & \langle h_\alpha, k_\alpha\rangle_\alpha \\ \langle k_\alpha, h_\alpha\rangle_\alpha & \langle k_\alpha, k_\alpha\rangle_\alpha \end{bmatrix} \geq 0$$

for all α. Thus, if $J \subseteq \Lambda$ is any finite set, we obtain that

$$\begin{bmatrix} \sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha & \sum_{\alpha\in J}\langle h_\alpha, k_\alpha\rangle_\alpha \\ \sum_{\alpha\in J}\langle k_\alpha, h_\alpha\rangle_\alpha & \sum_{\alpha\in J}\langle k_\alpha, k_\alpha\rangle_\alpha \end{bmatrix} \geq 0$$

so

$$\begin{aligned} \left|\left\langle \eta, \sum_{\alpha\in J}\langle h_\alpha, k_\alpha\rangle_\alpha \xi\right\rangle\right| &\leq \left|\left\langle \eta, \sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha \eta\right\rangle\right|^{\frac{1}{2}} \left|\left\langle \xi, \sum_{\alpha\in J}\langle k_\alpha, k_\alpha\rangle_\alpha \xi\right\rangle\right|^{\frac{1}{2}} \\ &\leq \left\|\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{\alpha\in J}\langle k_\alpha, k_\alpha\rangle_\alpha\right\|_{\mathfrak{B}}^{\frac{1}{2}} \|\xi\|\,\|\eta\| \end{aligned}$$

for all $\xi, \eta \in \mathcal{K}$ and $J \subseteq \Lambda$ finite. Therefore $\left\|\sum_{\alpha\in J}\langle h_\alpha, k_\alpha\rangle_\alpha\right\|_{\mathfrak{B}} \leq \left\|\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right\|_{\mathfrak{B}}^{\frac{1}{2}} \left\|\sum_{\alpha\in J}\langle k_\alpha, k_\alpha\rangle_\alpha\right\|_{\mathfrak{B}}^{\frac{1}{2}}$. Thus, if we order all finite subsets J of Λ by inclusion, $\left(\sum_{\alpha\in J}\langle h_\alpha, k_\alpha\rangle_\alpha\right)_{J\subseteq\Lambda}$ becomes a Cauchy net by the above inequality. Since $\mathfrak{B}$ is complete, this sum converges so $\sum_{\alpha\in\Lambda}\langle h_\alpha, k_\alpha\rangle_\alpha \in \mathfrak{B}$ is well-defined. It is now easy to verify that this is a $\mathfrak{B}$-inner product as each $\langle\cdot,\cdot\rangle_\alpha$ is.

Next we need to show that $\mathcal{H}$ is complete. To see this, let $(x_n) \in \mathcal{H}$ be a Cauchy sequence. Write $x_n = (h_{\alpha,n})_\Lambda$ for all n. Then for all $\alpha \in \Lambda$ and $n, m \in \mathbb{N}$, $0 \leq \langle h_{\alpha,n}-h_{\alpha,m}, h_{\alpha,n}-h_{\alpha,m}\rangle_\alpha \leq \langle x_n - x_m, x_n - x_m\rangle_{\mathcal{H}}$ as an infinite sum of positive elements in a C*-algebra is positive. Whence $(h_{\alpha,n})_n$ is a Cauchy sequence in $\mathcal{H}_\alpha$. Since $\mathcal{H}_\alpha$ is complete, there exists $h_\alpha \in \mathcal{H}_\alpha$ such that $h_{\alpha,n} \to h_\alpha$ as $n \to \infty$.

Let $x = (h_\alpha)_\Lambda$. Then for all $\xi \in \mathcal{K}$ with $\|\xi\| \leq 1$,

$$\begin{aligned} \sum_{\alpha\in\Lambda}\langle \xi, \langle h_\alpha, h_\alpha\rangle_\alpha \xi\rangle_{\mathcal{K}} &= \sum_{\alpha\in\Lambda}\lim_{n\to\infty}\langle \xi, \langle h_{\alpha,n}, h_{\alpha,n}\rangle_\alpha \xi\rangle_{\mathcal{K}} \\ &= \lim_{n\to\infty}\sum_{\alpha\in\Lambda}\langle \xi, \langle h_{\alpha,n}, h_{\alpha,n}\rangle_\alpha \xi\rangle_{\mathcal{K}} \\ &= \lim_{n\to\infty}\langle \xi, \langle x_n, x_n\rangle \xi\rangle_{\mathcal{K}} \\ &\leq \limsup_{n\to\infty}\|\langle x_n, x_n\rangle\| \\ &= \limsup_{n\to\infty}\|x_n\|_{\mathcal{H}}^2 \end{aligned}$$

(where we can exchange the sums and limits since we are adding positive numbers). Since $(x_n)_n \in \mathcal{H}$ is a Cauchy sequence, $\|x_n\|_{\mathcal{H}}^2$ is finite. Using the fact that

$$\langle \eta, T\xi\rangle = \frac{1}{4}\left(\langle \eta+\xi, T(\eta+\xi)\rangle - \langle \eta-\xi, T(\eta-\xi)\rangle + i\langle \eta+i\xi, T(\eta+i\xi)\rangle - i\langle \eta-i\xi, T(\eta-i\xi)\rangle\right)$$

for all $T \in \mathcal{B}(\mathcal{K})$ and $\xi, \eta \in \mathcal{K}$, we obtain that $\left\|\sum_{\alpha\in\Lambda}\langle h_\alpha, h_\alpha\rangle_\alpha\right\| \leq \limsup_{n\to\infty}\|x_n\|_{\mathcal{H}}^2 < \infty$ so $\sum_{\alpha\in\Lambda}\langle h_\alpha, h_\alpha\rangle_\alpha$ defines an operator in $\mathcal{B}(\mathcal{K})$ (this does not mean the sum converges in norm). Next, by repeating the same arguments with $x - x_m$ for a fixed m, we obtain that $\|x - x_m\|_{\mathcal{H}} \leq \limsup_{n\to\infty}\|x_n - x_m\|_{\mathcal{H}}$. Since $\limsup_{n\to\infty}\|x_n - x_m\|_{\mathcal{H}} \to 0$ as $m \to \infty$, we obtain that $\|x - x_m\|_{\mathcal{H}} \to 0$ as $m \to \infty$. Lastly, we must show that $x \in \mathcal{H}$. Fix $\epsilon > 0$ and choose m such that $\|x - x_m\|_{\mathcal{H}} < \epsilon$. Then, if $J \subseteq \Lambda$ is finite, then $\left\|\sum_{\alpha\in J}\langle h_\alpha - h_{\alpha,m}, h_\alpha - h_{\alpha,m}\rangle_\alpha\right\|_{\mathfrak{B}} \leq \|x - x_m\|_{\mathcal{H}}^2 \leq \epsilon^2$. Since $\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha$ makes sense, we obtain $\left\|\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right\|_{\mathfrak{B}} \leq \epsilon^2 + \left\|\sum_{\alpha\in J}\langle h_{\alpha,m}, h_{\alpha,m}\rangle_\alpha\right\|_{\mathfrak{B}}$ by using the triangle inequality on the norm obtained by repeating the above with J replacing Λ. Order all finite subsets of Λ by reverse inclusion. Since $\sum_{\alpha\in\Lambda}\langle h_{\alpha,m}, h_{\alpha,m}\rangle_\alpha$ converges, there exists a finite subset J_ϵ so that if $J \subseteq \Lambda$ is finite with $J_\epsilon \cap J = \emptyset$, then $\left\|\sum_{\alpha\in J}\langle h_{\alpha,m}, h_{\alpha,m}\rangle_\alpha\right\|_{\mathfrak{B}} < \epsilon$. Therefore, if $J \subseteq \Lambda$ is finite with $J_\epsilon \cap J = \emptyset$ then $\left\|\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right\|_{\mathfrak{B}} < \epsilon^2 + \epsilon$. Hence $\left(\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right)_{J\subseteq\Lambda}$ is a Cauchy sequence in $\mathfrak{B}$ and thus converges. Hence $x \in \mathcal{H}$ so $\mathcal{H}$ is a Hilbert $\mathfrak{B}$-space.

To see that applying ρ_α coordinate-wise makes $\mathcal{H}$ a right Hilbert $\mathfrak{B}$-module, we first need to check that if $(h_\alpha)_\Lambda \in \mathcal{H}$ and $B \in \mathfrak{B}$, then $(\rho_\alpha(B)h_\alpha)_\Lambda \in \mathcal{H}$. To see this, we notice $\sum_\Lambda \langle h_\alpha, h_\alpha\rangle_\alpha$ converges so $\sum_\Lambda \langle \rho_\alpha(B)h_\alpha, \rho_\alpha(B)h_\alpha\rangle_\alpha = \sum_\Lambda B^*\langle h_\alpha, h_\alpha\rangle_\alpha B = B^*\left(\sum_\Lambda \langle h_\alpha, h_\alpha\rangle_\alpha\right) B$ converges. Moreover, it is clear that this coordinate-wise action has all of the necessary properties. Hence $\mathcal{H}$ a right Hilbert $\mathfrak{B}$-module.

Lastly, we need to show that applying λ_α coordinate-wise makes $\mathcal{H}$ a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. First we need to check that if $(h_\alpha)_\Lambda \in \mathcal{H}$ and $A \in \mathfrak{A}$, then $(\lambda_\alpha(A)h_\alpha)_\Lambda \in \mathcal{H}$. First, we recall from a previous corollary that $0 \leq \langle \lambda_\alpha(A)h_\alpha, \lambda_\alpha(A)h_\alpha\rangle_\alpha \leq \|A\|_{\mathfrak{A}}^2 \langle h_\alpha, h_\alpha\rangle_\alpha$ for all α. Hence for all $J \subseteq \Lambda$ finite,

$$0 \leq \sum_{\alpha\in J} \langle \lambda_\alpha(A)h_\alpha, \lambda_\alpha(A)h_\alpha\rangle_\alpha \leq \|A\|^2 \sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha$$

Hence

$$0 \leq \left\| \sum_{\alpha\in J} \langle \lambda_\alpha(A)h_\alpha, \lambda_\alpha(A)h_\alpha\rangle_\alpha \right\| \leq \|A\|^2 \left\| \sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha \right\|$$

Since $\sum_{\alpha\in\Lambda}\langle h_\alpha, h_\alpha\rangle_\alpha$ converges, $\left(\sum_{\alpha\in J}\langle h_\alpha, h_\alpha\rangle_\alpha\right)_{J\subseteq\Lambda}$ is a Cauchy net. Hence $\left(\sum_{\alpha\in J}\langle \lambda_\alpha(A)h_\alpha, \lambda_\alpha(A)h_\alpha\rangle_\alpha\right)_{J\subseteq\Lambda}$ is a Cauchy net in $\mathfrak{B}$ and thus converges as $\mathfrak{B}$ is complete. Hence the $\mathfrak{A}$ action is well-defined. It is trivial to verify all other necessary properties so $\mathcal{H}$ a Hilbert $\mathfrak{A}$-$\mathfrak{B}$-bimodule. □

One important operation on Hilbert spaces is the ability to take tensor products. In our context, we need to be a little more careful.

Construction Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be C*-algebras. Let $\mathcal{H}$ be a $\mathfrak{A}$-$\mathfrak{B}$-bimodule (where the actions of $\mathfrak{A}$ and $\mathfrak{B}$ come from $\lambda_\mathcal{H} : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H})$ and $\rho_\mathcal{H} : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$) and let $\mathcal{K}$ be a $\mathfrak{B}$-$\mathfrak{C}$-bimodule (where the actions of $\mathfrak{B}$ and $\mathfrak{C}$ come from $\lambda_\mathcal{K} : \mathfrak{B} \to \mathcal{B}_a(\mathcal{K})$ and $\rho_\mathcal{K} : \mathfrak{C} \to \mathcal{B}(\mathcal{K})$). Let $\mathcal{H} \odot \mathcal{K}$ be the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$. Define a sesquilinear form $\langle\cdot,\cdot\rangle : \mathcal{H}\odot\mathcal{K} \times \mathcal{H}\odot\mathcal{K} \to \mathfrak{C}$ by $\langle h_1\otimes k_1, h_2\otimes k_2\rangle = \langle k_1, \lambda_\mathcal{K}(\langle h_1, h_2\rangle_\mathcal{H})k_2\rangle_\mathcal{K}$. We notice that $\langle\cdot,\cdot\rangle$ will be well-defined since $\langle\cdot,\cdot\rangle_\mathcal{H}$ and $\langle\cdot,\cdot\rangle_\mathcal{K}$ are linear in the second component, conjugate linear in the first component, and $\lambda_\mathcal{K}$ is linear.

We claim that $\langle\cdot,\cdot\rangle$ is a positive $\mathfrak{C}$-valued sesquilinear form. To see this, we already know that $\langle\cdot,\cdot\rangle$ is sesquilinear. To see that it is positive, suppose $u = \sum_{i=1}^n h_i \otimes k_i \in \mathcal{H}\odot\mathcal{K}$. Then

$$\begin{aligned}\langle u, u\rangle &= \left\langle \sum_{i=1}^n h_i\otimes k_i, \sum_{j=1}^n h_j\otimes k_j\right\rangle \\ &= \sum_{i,j=1}^n \langle k_i, \lambda_\mathcal{K}(\langle h_i, h_j\rangle_\mathcal{H})k_j\rangle_\mathcal{K} \\ &= \langle k_1\oplus\cdots\oplus k_n, (\lambda_\mathcal{K})_n([\langle h_i, h_j\rangle_\mathcal{H}])(k_1\oplus\cdots\oplus k_n)\rangle_{\mathcal{K}^{\oplus n}}\end{aligned}$$

where $\mathcal{K}^{\oplus n}$ is the direct sum of n copies of $\mathcal{K}$. By one of the above lemmas, $[\langle h_i, h_j\rangle_\mathcal{H}] \in \mathcal{M}_n(\mathfrak{B})$ is positive. We claim that $(\lambda_\mathcal{K})_n$ allows us to place a $\mathcal{M}_n(\mathfrak{B})$-$\mathfrak{C}$-module structure on $\mathcal{K}^{\oplus n}$; that is $\langle\xi, (\lambda_\mathcal{K})_n(B^*)\eta\rangle_{\mathcal{K}^{\oplus n}} = \langle(\lambda_\mathcal{K})_n(B)\xi, \eta\rangle_{\mathcal{K}^{\oplus n}}$ for all $\xi, \eta \in \mathcal{K}^{\oplus n}$ and for all $B \in \mathcal{M}_n(\mathfrak{B})$. Note that we can do this whenever the direct summands are the same Hilbert bimodule. To see that this is true, let $\xi = \xi_1\oplus\cdots\oplus\xi_n, \eta = \eta_1\oplus\cdots\oplus\eta_n \in \mathcal{K}^{\oplus n}$

and $B = [b_{i,j}] \in \mathcal{M}_n(\mathfrak{B})$. Then

$$\begin{aligned}
\langle \xi, (\lambda_\mathcal{K})_n(B^*)\eta\rangle_{\mathcal{K}^{\oplus n}} &= \langle \xi, [\lambda_\mathcal{K}(b^*_{j,i})]\eta\rangle_{\mathcal{K}^{\oplus n}} \\
&= \sum_{i,j=1}^n \langle \xi_i, \lambda_\mathcal{K}(b^*_{j,i})\eta_j\rangle_\mathcal{K} \\
&= \sum_{i,j=1}^n \langle \lambda_\mathcal{K}(b_{j,i})\xi_i, \eta_j\rangle_\mathcal{K} \\
&= \sum_{i,j=1}^n \langle \lambda_\mathcal{K}(b_{i,j})\xi_j, \eta_i\rangle_\mathcal{K} \\
&= \langle (\lambda_\mathcal{K})_n(B)\xi, \eta\rangle_{\mathcal{K}^{\oplus n}}
\end{aligned}$$

as $\lambda_\mathcal{K}$ is a *-homomorphism on $\mathcal{K}$ in the module sense. Since $[\langle h_i, h_j\rangle_\mathcal{H}] \in \mathcal{M}_n(\mathfrak{B})$ is positive, $[\langle h_i, h_j\rangle_\mathcal{H}] = N^*N$ for some $N \in \mathcal{M}_n(\mathfrak{B})$. It is easy to verify that $(\lambda_\mathcal{K})_n(N^*N) = (\lambda_\mathcal{K})_n(N^*)(\lambda_\mathcal{K})_n(N)$ since λ is a homomorphism and how matrix multiplication works. Therefore

$$\langle u, u\rangle = \langle k_1 \oplus \cdots \oplus k_n, (\lambda_\mathcal{K})_n(N^*)(\lambda_\mathcal{K})_n(N)(k_1 \oplus \cdots \oplus k_n)\rangle_{\mathcal{K}^{\oplus n}} = \langle (\lambda_\mathcal{K})_n(N)(k_1 \oplus \cdots \oplus k_n), (\lambda_\mathcal{K})_n(N)(k_1 \oplus \cdots \oplus k_n)\rangle_{\mathcal{K}^{\oplus n}} \geq 0$$

Since $u \in \mathcal{H} \odot \mathcal{K}$ was arbitrary, $\langle \cdot, \cdot\rangle$ is a positive $\mathfrak{C}$-valued sesquilinear form. Moreover this shows us that $\lambda_\mathcal{K}$ is completely positive in the sense that $\langle k_1 \oplus \cdots \oplus k_n, (\lambda_\mathcal{K})_n(B)(k_1 \oplus \cdots \oplus k_n)\rangle_{\mathcal{K}^{\oplus n}} \geq 0$ for any B positive in $\mathcal{M}_n(\mathfrak{B})$. This clearly also holds for $\mathcal{H}$.

Let $\mathcal{N} = \{u \in \mathcal{H} \odot \mathcal{K} \mid \langle u, u\rangle = 0\}$. By the partial Cauchy Schwarz inequality for positive sesquilinear forms, we obtain that $\mathcal{N} = \{u \in \mathcal{H} \odot \mathcal{K} \mid \langle u, v\rangle = 0 \text{ for all } v \in \mathcal{H} \odot \mathcal{K}\}$ so that $\mathcal{N}$ is a subspace. It is useful for later reasons to know that for all $h \in \mathcal{H}$, $k \in \mathcal{K}$, and $b \in \mathfrak{B}$ that $u = \rho_\mathcal{H}(b)h \otimes k - h \otimes \lambda_\mathcal{K}(b)k \in \mathcal{N}$. To see this, we notice that

$$\begin{aligned}
\langle u, u\rangle &= \langle \rho_\mathcal{H}(b)h \otimes k, \rho_\mathcal{H}(b)h \otimes k\rangle - \langle \rho_\mathcal{H}(b)h \otimes k, h \otimes \lambda_\mathcal{K}(b)k\rangle \\
&\quad - \langle h \otimes \lambda_\mathcal{K}(b)k, \rho_\mathcal{H}(b)h \otimes k\rangle + \langle h \otimes \lambda_\mathcal{K}(b)k, h \otimes \lambda_\mathcal{K}(b)k\rangle \\
&= \langle k, \lambda_\mathcal{K}(\langle \rho_\mathcal{H}(b)h, \rho_\mathcal{H}(b)h\rangle_\mathcal{H})k\rangle_\mathcal{K} - \langle k, \lambda_\mathcal{K}(\langle \rho_\mathcal{H}(b)h, h\rangle_\mathcal{H})\lambda_\mathcal{K}(b)k\rangle_\mathcal{K} \\
&\quad - \langle \lambda_\mathcal{K}(b)k, \lambda_\mathcal{K}(\langle h, \rho_\mathcal{H}(b)h\rangle_\mathcal{H})k\rangle_\mathcal{K} + \langle \lambda_\mathcal{K}(b)k, \lambda_\mathcal{K}(\langle h, h\rangle_\mathcal{H})\lambda_\mathcal{K}(b)k\rangle_\mathcal{K} \\
&= \langle k, \lambda_\mathcal{K}(b^*\langle h, h\rangle_\mathcal{H} b)k\rangle_\mathcal{K} - \langle k, \lambda_\mathcal{K}(b^*\langle h, h\rangle_\mathcal{H})\lambda_\mathcal{K}(b)k\rangle_\mathcal{K} \\
&\quad - \langle \lambda_\mathcal{K}(b)k, \lambda_\mathcal{K}(\langle h, h\rangle_\mathcal{H} b)k\rangle_\mathcal{K} + \langle \lambda_\mathcal{K}(b)k, \lambda_\mathcal{K}(\langle h, h\rangle_\mathcal{H})\lambda_\mathcal{K}(b)k\rangle_\mathcal{K} \\
&= 0
\end{aligned}$$

as $\lambda_\mathcal{K}$ is a homomorphism.

Now take $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ and complete it with respect to the norm $\|u\| = \sqrt{\|\langle u, u\rangle\|_\mathfrak{C}}$ induced $\mathfrak{C}$-valued inner product to obtain a Hilbert $\mathfrak{C}$-space $\mathcal{H} \otimes_\mathfrak{B} \mathcal{K}$. Our next goal is to show that $\mathcal{H} \otimes_\mathfrak{B} \mathcal{K}$ is a $\mathfrak{C}$-bimodule. For each $c \in \mathfrak{C}$ define $\rho(c) \in \mathcal{L}(\mathcal{H} \odot \mathcal{K})$ by $\rho(c)(h \otimes k) = h \otimes \rho_\mathcal{K}(c)k$ (which is well-defined since it is a linear map induced by a bilinear map on $\mathcal{H} \times \mathcal{K}$). To show that we can define $\rho(c)$ on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ we notice for all

$u = \sum_{i=1}^n h_i \otimes k_i \in \mathcal{H} \odot \mathcal{K}$ that

$$\begin{aligned} 0 \leq \langle \rho(c)u, \rho(c)u \rangle &= \sum_{i,j=1}^n \langle \rho(c)(h_i \otimes k_i), \rho(c)(h_j \otimes k_j) \rangle \\ &= \sum_{i,j=1}^n \langle h_i \otimes \rho_{\mathcal{K}}(c)k_i, h_j \otimes \rho_{\mathcal{K}}(c)k_j \rangle \\ &= \sum_{i,j=1}^n \langle \rho_{\mathcal{K}}(c)k_i, \lambda_{\mathcal{K}}(\langle h_i, h_j \rangle_{\mathcal{H}})\rho_{\mathcal{K}}(c)k_j \rangle_{\mathcal{K}} \\ &= \sum_{i,j=1}^n c^* \langle k_i, \lambda_{\mathcal{K}}(\langle h_i, h_j \rangle_{\mathcal{H}})k_j \rangle_{\mathcal{K}} c \\ &= c^* \langle u, u \rangle c \end{aligned}$$

Thus, if $u \in \mathcal{N}$, $\rho(c)u \in \mathcal{N}$. Hence we can redefine $\rho(c)$ on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$. Moreover, the above computation shows $\|\rho(c)u\|^2 = \|c^*\langle u, u \rangle c\|_{\mathfrak{C}} \leq \|c\|^2 \|u\|^2$. Hence $\rho(c)$ is continuous on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ and thus extends to $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$. Define $\rho : \mathfrak{C} \to \mathcal{L}(\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K})$ by $c \mapsto \rho(c)$. It is clear by the definition of $\rho(c)$ on $\mathcal{H} \odot \mathcal{K}$ that ρ is linear and anti-multiplicative as $\rho_{\mathcal{K}}$ was. Thus these properties extend to $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ and thus to $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ by density and the boundedness of ρ. Hence $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ is a $\mathfrak{C}$-module.

We desire to make $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ a $\mathfrak{A}$-$\mathfrak{C}$-bimodule. To do this, for each $a \in \mathfrak{A}$ we define $\lambda(a) \in \mathcal{L}(\mathcal{H} \odot \mathcal{K})$ by $\lambda(a)(h \otimes k) = \lambda_{\mathcal{H}}(a)h \otimes k$ (which is well-defined since it is a linear map induced by a bilinear map on $\mathcal{H} \times \mathcal{K}$). To show that we can define $\lambda(a)$ on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ we notice for all $u = \sum_{i=1}^n h_i \otimes k_i \in \mathcal{H} \odot \mathcal{K}$ that

$$\begin{aligned} 0 \leq \langle \lambda(a)u, \lambda(a)u \rangle &= \sum_{i,j=1}^n \langle \lambda(a)(h_i \otimes k_i), \lambda(a)(h_j \otimes k_j) \rangle \\ &= \sum_{i,j=1}^n \langle \lambda_{\mathcal{H}}(a)h_i \otimes k_i, \lambda_{\mathcal{H}}(a)h_j \otimes k_j \rangle \\ &= \sum_{i,j=1}^n \langle k_i, \lambda_{\mathcal{K}}(\langle \lambda_{\mathcal{H}}(a)h_i, \lambda_{\mathcal{H}}(a)h_j \rangle_{\mathcal{H}})k_j \rangle_{\mathcal{K}} \\ &= \langle k_1 \oplus \cdots \oplus k_n, (\lambda_{\mathcal{K}})_n([\langle \lambda_{\mathcal{H}}(a)h_i, \lambda_{\mathcal{H}}(a)h_j \rangle_{\mathcal{H}}])(k_1 \oplus \cdots \oplus k_n) \rangle_{\mathcal{K}^{\oplus n}} \end{aligned}$$

However, we saw earlier that $[\langle \lambda_{\mathcal{H}}(a)h_i, \lambda_{\mathcal{H}}(a)h_j \rangle_{\mathcal{H}}] \leq \|a\|^2 [\langle h_i, h_j \rangle_{\mathcal{H}}]$ in $\mathcal{M}_n(\mathfrak{B})$ and that $\lambda_{\mathcal{K}}$ was completely positive. Therefore we obtain that

$$0 \leq \langle \lambda(a)u, \lambda(a)u \rangle \leq \|a\|^2 \langle k_1 \oplus \cdots \oplus k_n, (\lambda_{\mathcal{K}})_n([\langle h_i, h_j \rangle_{\mathcal{H}}])(k_1 \oplus \cdots \oplus k_n) \rangle_{\mathcal{K}^{\oplus n}} = \|a\|^2 \langle u, u \rangle$$

Thus, if $u \in \mathcal{N}$, $\lambda(a)u \in \mathcal{N}$. Hence we can redefine $\lambda(a)$ on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$. Moreover, the above computation shows $\|\lambda(a)u\|^2 = \|a\|^2 \|\langle u, u \rangle\|_{\mathfrak{C}} \leq \|a\|^2 \|u\|^2$. Hence $\lambda(a)$ is continuous on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ and thus extends to $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$. Define $\lambda : \mathfrak{A} \to \mathcal{L}(\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K})$ by $a \mapsto \lambda(a)$. It is clear by the definition of $\lambda(a)$ on $\mathcal{H} \odot \mathcal{K}$ that λ is a homomorphism as $\lambda_{\mathcal{H}}$ was. Lastly, we notice for all $h \otimes k, h' \otimes k' \in \mathcal{H} \odot \mathcal{K}$ that

$$\begin{aligned} \langle h' \otimes k', \lambda(a)(h \otimes k) \rangle &= \langle h' \otimes k', \lambda_{\mathcal{H}}(a)h \otimes k \rangle \\ &= \langle k', \langle h', \lambda_{\mathcal{H}}(a)h \rangle_{\mathcal{H}} k \rangle_{\mathcal{K}} \\ &= \langle k', \langle \lambda_{\mathcal{H}}(a^*)h', h \rangle_{\mathcal{H}} k \rangle_{\mathcal{K}} \\ &= \langle \lambda_{\mathcal{H}}(a^*)h' \otimes k', h \otimes k \rangle \\ &= \langle \lambda(a^*)(h' \otimes k'), h \otimes k \rangle \end{aligned}$$

Thus, by linearity, we see that $\lambda(a)^* = \lambda(a^*)$ on $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$. Thus, as these properties extend to $\mathcal{H} \odot \mathcal{K}/\mathcal{N}$ and thus to $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ by density and boundedness of λ, $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ is a $\mathfrak{A}$-$\mathfrak{C}$-module.

Remarks In an abuse of notation, we write $h \otimes k$ instead of $h \otimes k + \mathcal{N}$. Thus the linear span of these tensors is dense in $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$. Also, due to our knowledge of the kernel, we know that $\rho_{\mathcal{H}}(b)h \otimes k = h \otimes \lambda_{\mathcal{K}}(b)k$. We have not proven that these are the only relations (i.e. it might be true that $h \otimes k = 0$ for some $h \neq 0$ and $k \neq 0$). Moreover if $\mathfrak{A}$ is unital, it is easy to see that $\lambda(1_{\mathfrak{A}})$ is the identity map on $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ if $\lambda_{\mathcal{H}}(1_{\mathfrak{A}})$ was the identity map on $\mathcal{H}$.

Example Let $\mathcal{H} = \mathfrak{B}$, $\langle \cdot, \cdot \rangle : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{B}$ by $\langle A, B \rangle = A^*B$, $\rho : \mathfrak{B} \to \mathcal{L}(\mathcal{H})$ by $\rho(B)A = AB$ for all $A \in \mathcal{H}$ and $B \in \mathfrak{B}$, and $\lambda : \mathfrak{B} \to \mathcal{B}_a(\mathcal{H})$ by $\lambda(B)A = BA$ for all $A \in \mathcal{H}$ and $B \in \mathfrak{B}$. Consider $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{H}$. Then for all $A \otimes B \in \mathcal{H} \otimes_{\mathfrak{B}} \mathcal{H}$, $A \otimes B = A \otimes \lambda(B)1_{\mathfrak{B}} = \rho(B)A \otimes 1_{\mathfrak{B}} = AB \otimes 1_{\mathfrak{B}}$. Moreover, for all $C \in \mathfrak{B}$, $\langle C \otimes 1, C \otimes 1 \rangle = \langle C, \lambda(\langle 1, 1 \rangle)C \rangle = \langle C, C \rangle = C^*C$. Thus $C \otimes 1 = 0$ if and only if $C^*C = 0$ if and only if $C = 0$. Also, since $\rho(B)(C \otimes 1) = C \otimes B = CB \otimes 1$ and $\lambda(B)(C \otimes 1) = BC \otimes 1$, $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{H} \simeq \mathcal{H}$ as Hilbert $\mathfrak{B}$-$\mathfrak{B}$-bimodules.

Similarly, if $\mathfrak{B}$ is a C*-algebra, $\mathcal{H} = \mathfrak{B}^n$ with the canonical Hilbert $\mathfrak{B}$-$\mathfrak{B}$-bimodule structure, and $\mathcal{K} = \mathfrak{B}^m$ with the canonical Hilbert $\mathfrak{B}$-$\mathfrak{B}$-bimodule structure, then $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K} \simeq \mathfrak{B}^{nm}$ as Hilbert $\mathfrak{B}$-$\mathfrak{B}$-bimodules (i.e. show that if $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ are the vectors in $\mathfrak{B}^n$ and $\mathfrak{B}^m$ with one entry $1_{\mathfrak{B}}$ and the rest zeros, then every element of $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ is of the form $\sum_{i=1}^n \sum_{j=1}^m (B_{i,j}e_i) \otimes f_j$. Map this sum to an nm tuple whose entries are the $B_{i,j}$ and show this map preserves the inner product and the right and left $\mathfrak{B}$-actions).

Remarks We have already seen a relation between completely positive maps between C*-algebra and Hilbert bimodules. One question is, "How do Hilbert bimodules created by completely positive maps factor through this tensor product?"

Theorem *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be unital C*-algebras and $\varphi : \mathfrak{A} \to \mathfrak{B}$, $\psi : \mathfrak{B} \to \mathfrak{C}$ be completely positive maps. Let $\mathcal{H}$ be a $\mathfrak{A}$-$\mathfrak{B}$-bimodule with $\varphi(A) = \langle \xi, \lambda_{\mathcal{H}}(A)\xi \rangle$ for all $A \in \mathfrak{A}$ with $\xi \in \mathcal{H}$ fixed and let $\mathcal{K}$ be a $\mathfrak{B}$-$\mathfrak{C}$-bimodule with $\psi(A) = \langle \eta, \lambda(B)_{\mathcal{K}}\eta \rangle$ for all $B \in \mathfrak{A}$ with $\eta \in \mathcal{K}$ fixed. If $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ is as above, then $\psi(\varphi(a)) = \langle \xi \otimes \eta, \lambda_{\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}}(a)(\xi \otimes \eta) \rangle$. In particular, if $\psi \circ \varphi \neq 0$, $\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}$ is not trivial.*

Proof: This is a simple brute force computation:

$$\begin{aligned}
\psi(\varphi(a)) &= \psi(\langle \xi, \lambda_{\mathcal{H}}(a)\xi \rangle_{\mathcal{H}}) \\
&= \langle \eta, \lambda_{\mathcal{K}}(\langle \xi, \lambda_{\mathcal{H}}(a)\xi \rangle_{\mathcal{H}})\eta \rangle_{\mathcal{K}} \\
&= \langle \xi \otimes \eta, \lambda_{\mathcal{H}}(a)\xi \otimes \eta \rangle_{\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}} \\
&= \langle \xi \otimes \eta, \lambda_{\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}}(a)(\xi \otimes \eta) \rangle_{\mathcal{H} \otimes_{\mathfrak{B}} \mathcal{K}}
\end{aligned}$$

as desired. □