# An Introduction to Multiplier Algebras

Paul Skoufranis

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**Abstract**

The purpose of this document is to develop some of the basic theory of the multiplier algebra of a C*-algebra. Three characterizations of the multiplier algebra will be given. Several examples and results pertaining to the multiplier algebra will be developed based on these descriptions. The final section will be devoted to developing additional results and applications of the multiplier algebra.

Most of these notes are developed from the three references contained in the bibliography along with the author's personal knowledge.

A reader of these notes should be familiar with the basics of C*-algebra theory. In particular, a reader should be familiar with unitizations of C*-algebras, the Continuous Functional Calculus for Normal Operators, states, representations of C*-algebra, C*-bounded approximate identities, and ideals.

For these notes, $\mathcal{H}$ will denote a Hilbert space, $\mathcal{H}_{\mathfrak{B}}$ will denote a right $\mathfrak{B}$-Hilbert module, $\tilde{\mathfrak{A}}$ will denote the unitization of a C*-algebra $\mathfrak{A}$, and $\mathcal{B}(\mathfrak{X})$ will denote the space of bounded linear maps on a Banach space $\mathfrak{X}$. An ideal of a C*-algebra will mean a closed two-sided ideal (which is then automatically self-adjoint), all inner products will be linear in the first co-ordinate, and all C*-valued inner products will be linear in the second co-ordinate.

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## Contents

# 1 Definitions

In the theory of C$^*$-algebras it is very useful that a C$^*$-algebra has a unit. However many canonical and interesting examples of C$^*$-algebras are unitless. There are several techniques and ideas in order to deal with this technicality. One technique is to consider the unitization $\tilde{\mathfrak{A}}$ of the C$^*$-algebra $\mathfrak{A}$. This is a canonical way of adding a unit to $\mathfrak{A}$ such that $\mathfrak{A}$ sits as an ideal in $\tilde{\mathfrak{A}}$ and such that $\tilde{\mathfrak{A}}/\mathfrak{A} \simeq \mathbb{C}$. Another useful technique is to consider C$^*$-bounded approximate identities. Such bounded approximate identities behave as asymptotic units and thus enable many of the analytic properties of a unit.

There is an additional notation of unitization known as the multiplier algebra of a C$^*$-algebra. The multiplier algebra of a C$^*$-algebra $\mathfrak{A}$ is a universal unital C$^*$-algebra that contains $\mathfrak{A}$ as an ideal and has the property that for every other C$^*$-algebra that contains $\mathfrak{A}$ as an ideal there is a $^*$-homomorphism from the C$^*$-algebra into the multiplier algebra that is the identity on $\mathfrak{A}$. As such, the multiplier algebra of a C$^*$-algebra can be viewed as the largest unital C$^*$-algebra containing $\mathfrak{A}$ as an ideal.

To begin our study of multiplier algebras, it is useful to describe certain special ideals of C$^*$-algebras.

**Definition 1.1.** Let $\mathfrak{A}$ be a C$^*$-algebra. An ideal $\mathcal{I}$ of $\mathfrak{A}$ is said to be essential if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for every non-zero ideal $\mathcal{J}$ of $\mathfrak{A}$.

It is trivial that every C$^*$-algebra is an essential ideal of itself. There are several more interesting canonical examples of essential ideals.

**Example 1.2.** Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, let $\mathcal{B}(\mathcal{H})$ be the bounded linear maps on $\mathcal{H}$, and let $\mathfrak{K}$ be the set of compact operators on $\mathcal{H}$. Then it is well-known that $\mathfrak{K}$ is the only non-trivial ideal of $\mathcal{B}(\mathcal{H})$ and thus $\mathfrak{K}$ is an essential ideal in $\mathcal{B}(\mathcal{H})$.

**Example 1.3.** Let $\mathfrak{A} := C[0,1]$ (the set of continuous functions on $[0,1]$). It is well-known that every ideal of $\mathfrak{A}$ is of the form $\mathcal{I}_X := \{f \in \mathfrak{A} \mid f|_X = 0\}$ for some closed subsets $X$ of $[0,1]$. Therefore $\mathcal{I}_X$ is not an essential ideal if and only if there exists a closed subset $Y \subseteq [0,1]$ with $Y \neq [0,1]$ such that $\mathcal{I}_X \cap \mathcal{I}_Y = \{0\}$. Since $\mathcal{I}_X \cap \mathcal{I}_Y = \{0\}$ if and only if $f|_X = 0$ and $f|_Y = 0$ implies $f = 0$ for all $f \in \mathfrak{A}$, it is clear that this later condition occurs if and only if $X \cup Y = [0,1]$.

Since $Y$ must be a proper closed subset of $[0,1]$, the condition that $X \cup Y = [0,1]$ occurs if and only if $X$ has interior. Indeed if $X$ has empty interior then $X \cup Y = [0,1]$ implies that $Y$ contains the complement of $X$ in $[0,1]$. Since $Y$ is closed, this implies that $Y$ must be all of $[0,1]$ and which is a contradiction. However, if $X$ has interior and $Y$ is the closure of the complement in $[0,1]$ of the interior of $X$ then it is clear that $Y \neq [0,1]$ and $X \cup Y = [0,1]$. Thus $\mathcal{I}_X$ is an essential ideal of $\mathfrak{A}$ if and only if $X$ has no interior. In particular, every maximal ideal of $\mathfrak{A}$ is essential.

In the above example, an 'essential' idea was used to determine whether or not an ideal was essential. This idea is encapsulated in Proposition 1.5. For convenience to the reader, we note the following lemma.

**Lemma 1.4.** *Let $\mathfrak{A}$ be a C$^*$-algebra and let $\mathcal{I}$ and $\mathcal{J}$ be ideals of $\mathfrak{A}$. Then $\mathcal{I} \cap \mathcal{J} = \{0\}$ if and only if $\mathcal{I} \cdot \mathcal{J} := \{AB \mid A \in \mathcal{I}, B \in \mathcal{J}\} = \{0\}$.*

*Proof.* Suppose $\mathcal{I} \cap \mathcal{J} = \{0\}$. Then for all $A \in \mathcal{I}$ and $B \in \mathcal{J}$ we note that $AB \in \mathcal{I}$ and $AB \in \mathcal{J}$ as $\mathcal{I}$ and $\mathcal{J}$ are ideals. Therefore $AB \in \mathcal{I} \cap \mathcal{J} = \{0\}$ so $AB = 0$. Hence $\mathcal{I} \cdot \mathcal{J} = \{0\}$.

For the other direction, suppose $\mathcal{I} \cdot \mathcal{J} = \{0\}$ and let $T \in \mathcal{I} \cap \mathcal{J}$ be non-zero. Then $T \in \mathcal{I}$ and $T \in \mathcal{J}$. Since $\mathcal{I}$ is closed under adjoints, $T^* \in \mathcal{I}$. Hence $T^*T \in \mathcal{I} \cdot \mathcal{J} = \{0\}$. Thus $T^*T = 0$ so $T = 0$. Thus $\mathcal{I} \cap \mathcal{J} = \{0\}$ as desired. □

**Proposition 1.5.** *Let $\mathfrak{A}$ be a C$^*$-algebra and let $\mathcal{I}$ be an ideal of $\mathfrak{A}$. Then the following are equivalent:*

1. *$\mathcal{I}$ is an essential ideal of $\mathfrak{A}$.*
2. *The only element $A \in \mathfrak{A}$ such that $AB = 0$ for all $B \in \mathcal{I}$ is the zero operator.*
3. *The only element $A \in \mathfrak{A}$ such that $BA = 0$ for all $B \in \mathcal{I}$ is the zero operator.*

*Proof.* It is clear that (2) and (3) are equivalent by taking adjoints as every ideal of $\mathfrak{A}$ is closed under adjoints.

To prove that (1) and (2) are equivalent, first let us assume that $\mathcal{I}$ is an essential ideal of $\mathfrak{A}$. Suppose that $A \in \mathfrak{A}$ is such that $AB = 0$ for all $B \in \mathcal{I}$. Let $\mathcal{J}$ be the closed ideal of $\mathfrak{A}$ generated by $A$; that is

$$\mathcal{J} := \overline{\{C_1 A C_2 \mid C_1, C_2 \in \tilde{\mathfrak{A}}\}}$$

(it is clear that the expression on the right is a closed ideal). Note $A \in \mathcal{J}$. However, for all $C_1, C_2 \in \mathfrak{A}$ and for all $B_0 \in \mathcal{I}$

$$(C_1 A C_2) B_0 = C_1 A (C_2 B_0) = C_1 0 = 0$$

as $C_2 B \in \mathcal{I}$ and $AD = 0$ for all $D \in \mathcal{I}$. Hence $TB = 0$ for all $T \in \mathcal{J}$ and $B \in \mathcal{I}$. Hence $\mathcal{J} \cdot \mathcal{I} = \{0\}$ so Lemma 1.4 implies that $\mathcal{I} \cap \mathcal{J} = \{0\}$. Therefore $\mathcal{J} = \{0\}$ as $\mathcal{I}$ was assumed to be an essential ideal of $\mathfrak{A}$. Therefore, since $A \in \mathcal{J}$, $A = 0$. Hence (1) implies (2).

To see that (2) implies (1), suppose $\mathcal{I}$ is not an essential ideal of $\mathfrak{A}$. Then there exists a non-zero ideal $\mathcal{J}$ of $\mathfrak{A}$ such that $\mathcal{I} \cap \mathcal{J} = \{0\}$. Let $A \in \mathcal{J}$ be any non-zero operator (which exists as $\mathcal{J}$ is non-zero). However, for all $B \in \mathcal{I}$, $AB \in \mathcal{I}$ and $AB \in \mathcal{J}$ as $B \in \mathcal{I}$, $A \in \mathcal{J}$, and $\mathcal{I}$ and $\mathcal{J}$ are ideals. Hence $AB \in \mathcal{I} \cap \mathcal{J} = \{0\}$ so $AB = 0$ for all $B \in \mathcal{I}$. ☐

For those familiar with Hilbert C*-bimodules, the above says that if we consider the right Hilbert $\mathfrak{A}$-module $\mathfrak{A}$ with C*-valued inner product $\langle A, B \rangle = A^* B$ for all $A, B \in \mathfrak{A}$, then $\mathcal{I}$ is an essential ideal of $\mathfrak{A}$ if and only if $\mathcal{I}^\perp := \{A \in \mathfrak{A} \mid \langle A, B \rangle = 0 \text{ for all } B \in \mathcal{I}\} = \{0\}$. Thus it is common to say that $\mathcal{I}$ is an essential ideal of $\mathfrak{A}$ if and only if its orthogonal complement (or annihilator) is zero. For an exposition of Hilbert C*-bimodules, see http://www.math.ucla.edu/∼pskoufra/OANotes-HilbertC-Bimodules.pdf.

As a simple corollary, we obtain the following.

**Corollary 1.6.** *Let $\mathfrak{A}$ be a non-unital C*-algebra and let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$. Then $\mathfrak{A}$ is an essential ideal of $\tilde{\mathfrak{A}}$.*

*Proof.* It is clear that $\mathfrak{A}$ is an ideal in $\tilde{\mathfrak{A}}$. Suppose $\lambda I_{\mathfrak{A}} + A \in \tilde{\mathfrak{A}}$ is such that $(\lambda I_{\mathfrak{A}} + A)B = 0$ for all $B \in \mathfrak{A}$. If $\lambda = 0$, then by selecting $B = A^*$, we obtain that $AA^* = 0$ and thus $A = 0$ by the C*-identity. Otherwise Then $\lambda B + AB = 0$ for all $B \in \mathfrak{A}$ so $(-\lambda^{-1}A)B = B$ for all $B \in \mathfrak{A}$. As $-\lambda^{-1}A \in \mathfrak{A}$, $-\lambda^{-1}A$ is a left identity of $\mathfrak{A}$. By taking adjoints of the above equation, we obtain that $(-\lambda^{-1}A)^*$ is a right identity of $\mathfrak{A}$. Since any left and right identities in an algebra must be equal by elementary algebra, we obtain that $-\lambda^{-1}A$ is a two sided identity in $\mathfrak{A}$ and thus $\mathfrak{A}$ is unital. As this contradicts the assumptions on $\mathfrak{A}$, $\mathfrak{A}$ is an essential ideal of $\tilde{\mathfrak{A}}$ by Proposition 1.5. ☐

Another interesting corollary of Proposition 1.5 is the following.

**Corollary 1.7.** *Let $\mathfrak{A}$ be a unital C*-algebra and let $\mathfrak{B}$ be a unital C*-algebra containing $\mathfrak{A}$ as an essential ideal. Then $\mathfrak{B} = \mathfrak{A}$.*

*Proof.* Let $I_{\mathfrak{A}}$ and $I_{\mathfrak{B}}$ be the identities of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then for all $A \in \mathfrak{A}$

$$(I_{\mathfrak{B}} - I_{\mathfrak{A}})A = I_{\mathfrak{B}}A - A = 0$$

as $I_{\mathfrak{B}}$ is an identity for $\mathfrak{B}$ which contains $\mathfrak{A}$. Therefore, since $\mathfrak{A}$ is an essential ideal of $\mathfrak{B}$, Proposition 1.5 implies that $I_{\mathfrak{B}} = I_{\mathfrak{A}} \in \mathfrak{A}$. However, as $\mathfrak{A}$ is an ideal of $\mathfrak{B}$, we obtain that $\mathfrak{B} \subseteq \mathfrak{A}$ as desired. ☐

With the above examples as motivation, we shall finally define the multiplier algebra of a C*-algebra.

**Definition 1.8.** Let $\mathfrak{A}$ be a C*-algebra. The multiplier algebra of $\mathfrak{A}$, denoted $\mathcal{M}(\mathfrak{A})$, is the universal C*-algebra with the property that $\mathcal{M}(\mathfrak{A})$ contains $\mathfrak{A}$ as an essential ideal and for any C*-algebra $\mathfrak{B}$ containing $\mathfrak{A}$ as an essential ideal there exists a unique *-homomorphism $\pi : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$.

It is a priori not apparent that the multiplier algebra of a C*-algebra exists. However, it is clear that if such an object exists, it is unique up to isomorphisms by the universal properties. Indeed suppose $\mathfrak{C}$ is a C*-algebra containing $\mathfrak{A}$ as an essential ideal with the property that for any C*-algebra $\mathfrak{B}$ containing $\mathfrak{A}$ as an essential ideal there exists a unique $^*$-homomorphism $\pi : \mathfrak{B} \to \mathfrak{C}$ that is the identity on $\mathfrak{A}$. Since $\mathfrak{C}$ and $\mathcal{M}(\mathfrak{A})$ contain $\mathfrak{A}$ as an essential ideal, the universal property implies that the only $^*$-automorphism on each of $\mathfrak{C}$ and $\mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$ is the identity $^*$-homomorphism. Furthermore the universal property implies that there are unique $^*$-homomorphisms $\pi_1 : \mathfrak{C} \to \mathcal{M}(\mathfrak{A})$ and $\pi_2 : \mathcal{M}(\mathfrak{A}) \to \mathfrak{C}$ that are the identity on $\mathfrak{A}$. Hence $\pi_1 \circ \pi_2 : \mathcal{M}(\mathfrak{A}) \to \mathcal{M}(\mathfrak{A})$ and $\pi_2 \circ \pi_1 : \mathfrak{C} \to \mathfrak{C}$ are $^*$-homomorphisms that are the identity on $\mathfrak{A}$ and thus must be the identity $^*$-homomorphisms. Hence $\pi_1$ is a $^*$-isomorphism from $\mathfrak{C}$ to $\mathcal{M}(\mathfrak{A})$.

Assuming the existence of the multiplier algebra of a C*-algebra, Corollary 1.7 immediately provides information about the multiplier algebra of a unital C*-algebra.

**Lemma 1.9.** *Let $\mathfrak{A}$ be a unital C*-algebra. If $\mathcal{M}(\mathfrak{A})$ is unital then $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$.*

*Proof.* By the definition of the multiplier algebra, $\mathfrak{A}$ is an essential ideal of $\mathcal{M}(\mathfrak{A})$. If $\mathcal{M}(\mathfrak{A})$ is unital then Corollary 1.7 implies that $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$. □

In the next three sections, we shall demonstrate various proofs and constructions that demonstrate the multiplier algebra of a C*-algebra exists. Using these three constructions, we shall demonstrate multiple properties of the multiplier algebra and most properties of the multiplier algebra can usually be deduced by using one of these constructions and elementary arguments. For example, we will see that the multiplier algebra is always unital and thus $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$ for any unital C*-algebra $\mathfrak{A}$.

# 2 Construction of Multiplier Algebra by Multipliers

In this section we will demonstrate the existence of the multiplier algebra of an arbitrary C$^*$-algebra via multipliers. As in the construction of the unitization of a non-unital C$^*$-algebra $\mathfrak{A}$, we can let an element $A \in \mathfrak{A}$ act on $\mathfrak{A}$ by left multiplication, $L_A$, or by right multiplication, $R_A$. These operators are called the multiplier operators. It is clear that for all $A, B, C \in \mathfrak{A}$ that $AL_C(B) = R_C(A)B$. It is pairs of bounded linear operators on $\mathfrak{A}$ with this interconnecting property that will enable the construction of the multiplier algebra. This construction relies only on elementary C$^*$-algebra results and no sophisticated results on representations of C$^*$-algebras (that will be necessary in Section 3).

We begin with the definition of the set of all 'nice' pairs of bounded linear maps on a C$^*$-algebra $\mathfrak{A}$.

**Definition 2.1.** Let $\mathfrak{A}$ be a C$^*$-algebra. A pair $(L, R)$ where $L, R \in \mathcal{B}(\mathfrak{A})$ is said to be a double centralizer of $\mathfrak{A}$ if $AL(B) = R(A)B$ for all $A, B \in \mathfrak{A}$. We will denote the set of all double centralizers by $\mathcal{DC}(\mathfrak{A})$.

Our first goal is to equip $\mathcal{DC}(\mathfrak{A})$ with a $^*$-algebra structure. For the involution we must place an additional structure on $\mathcal{B}(\mathfrak{A})$.

**Lemma 2.2.** *Let $\mathfrak{A}$ be a C$^*$-algebra. For an operator $T \in \mathcal{B}(\mathfrak{A})$ we define $T^\sharp \in \mathcal{B}(\mathfrak{A})$ by $T^\sharp(A) = T(A^*)^*$ for all $A \in \mathfrak{A}$. Then $^\sharp$ is an isometric, multiplicative, conjugate linear, idempotent on $\mathcal{B}(\mathfrak{A})$.*

*Proof.* It is clear under this definition that $T^\sharp$ is linear with $\left\|T^\sharp\right\| = \|T\|$ as $\|A^*\| = \|A\|$ for all $A \in \mathfrak{A}$. Thus $^\sharp$ is a well-defined isometric operation on $\mathcal{B}(\mathfrak{A})$. For $S, T \in \mathcal{B}(\mathfrak{A})$ and $\lambda \in \mathbb{C}$ we notice that

$$(\lambda T + S)^\sharp(A) = (\lambda T + S)(A^*)^* = \overline{\lambda} T(A^*)^* + S(A^*)^* = \left(\overline{\lambda} T^\sharp + S^\sharp\right)(A)$$

and

$$\left(T^\sharp\right)^\sharp(A) = \left(T^\sharp\right)(A^*)^* = (T((A^*)^*)^*)^* = T(A)$$

and

$$(T \circ S)^\sharp(A) = T(S(A^*))^* = T((S(A^*)^*)^*)^* = T(S^\sharp(A)^*)^* = T^\sharp(S^\sharp(A)) = (T^\sharp \circ S^\sharp)(A)$$

for all $A \in \mathfrak{A}$. Hence $(\lambda T + S)^\sharp = \overline{\lambda} T^\sharp + S^\sharp$, $(T^\sharp)^\sharp = T$, and $(T \circ S)^\sharp = T^\sharp \circ S^\sharp$ for all $T, S \in \mathcal{B}(\mathfrak{A})$. Hence the result is complete ☐

With the above operation, it is easy to turn $\mathcal{DC}(\mathfrak{A})$ into a unital $^*$-algebra.

**Lemma 2.3.** *Let $\mathfrak{A}$ be a C$^*$-algebra. For $\lambda \in \mathbb{C}$, $(L, R), (L_1, R_1), (L_2, R_2) \in \mathcal{DC}(\mathfrak{A})$, the operations*

$$\lambda(L, R) := (\lambda L, \lambda R) \qquad (L_1, R_1) + (L_2, R_2) := (L_1 + L_2, R_1 + R_2)$$
$$(L_1, R_1)(L_2, R_2) := (L_1 \circ L_2, R_2 \circ R_1) \qquad (L, R)^* := (R^\sharp, L^\sharp)$$

*are well-defined and turn $\mathcal{DC}(\mathfrak{A})$ into a unital $^*$-algebra where the zero vector is $(0, 0)$ where $0 \in \mathcal{B}(\mathfrak{A})$ and the unit is $(Id, Id)$ where $Id \in \mathcal{B}(\mathfrak{A})$ is the identity map.*

*Proof.* Let $\lambda \in \mathbb{C}$ and $(L, R), (L_1, R_1), (L_2, R_2) \in \mathcal{DC}(\mathfrak{A})$. It is then elementary to verify that $\lambda L, \lambda R, L_1 + L_2, R_1 + R_2, L_1 \circ L_2, R_2 \circ R_1 \in \mathcal{B}(\mathfrak{A})$. Furthermore it is elementary to verify that $(\lambda L, \lambda R), (L_1 + L_2, R_1 + R_2) \in \mathcal{DC}(\mathfrak{A})$. To see that $(L_1 \circ L_2, R_2 \circ R_1) \in \mathcal{DC}(\mathfrak{A})$, we note for all $A, B \in \mathfrak{A}$ that

$$A(L_1(L_2(B))) = R_1(A)L_2(B) = R_2(R_1(A))B$$

by the definition of a double centralizer. Hence $(L_1 \circ L_2, R_2 \circ R_1) \in \mathcal{DC}(\mathfrak{A})$. Thus addition, scalar multiplication, and multiplication are well-defined on $\mathcal{DC}(\mathfrak{A})$. Furthermore it is elementary to verify that $(0, 0)$ is a zero vector and these operations turn $\mathcal{DC}(\mathfrak{A})$ into an algebra. Notice if $Id \in \mathcal{B}(\mathfrak{A})$ is the identity operator then

$$A(Id)(B) = AB = (Id)(A)B$$

for all $A, B \in \mathfrak{A}$ so $(Id, Id) \in \mathcal{DC}(\mathfrak{A})$. It is clear that $(Id, Id)$ is the multiplicative unit in $\mathcal{DC}(\mathfrak{A})$.

Thus to show that $\mathcal{DC}(\mathfrak{A})$ is a unital $^*$-algebra, it suffices to check that we have a well-defined involution. To see this, we notice if $(L, R) \in \mathcal{DC}(\mathfrak{A})$ then

$$AR^\sharp(B) = A(R(B^*))^* = (R(B^*)A^*)^* = (B^*L(A^*))^* = L(A^*)^*B = L^\sharp(A)B$$

for all $A, B \in \mathfrak{A}$. Hence $(L, R) \in \mathcal{DC}(\mathfrak{A})$ implies that $(R^\sharp, L^\sharp) \in \mathcal{DC}(\mathfrak{A})$. Hence $(L, R)^* := (R^\sharp, L^\sharp)$ is a well-defined operation.

It remains to verify that the operation $(L, R)^* := (R^\sharp, L^\sharp)$ is an involution. It is trivial to verify that this operation is conjugate linear by Lemma 2.2. Moreover it is clear that

$$((L, R)^*)^* = (R^\sharp, L^\sharp)^* = ((L^\sharp)^\sharp, (R^\sharp)^\sharp) = (L, R)$$

by Lemma 2.2. Finally we notice that

$$\begin{aligned} ((L_1, R_1)(L_2, R_2))^* &= (L_1 \circ L_2, R_2 \circ R_1)^* \\ &= ((R_2 \circ R_1)^\sharp, (L_1 \circ L_2)^\sharp) \\ &= (R_2^\sharp \circ R_1^\sharp, L_1^\sharp \circ L_2^\sharp) \\ &= (R_2^\sharp, L_2^\sharp)(R_1^\sharp, L_1^\sharp) = (L_2, R_2)^*(L_1, R_1)^* \end{aligned}$$

by Lemma 2.2. Hence $(L, R)^* := (R^\sharp, L^\sharp)$ is an involution and thus $\mathcal{DC}(\mathfrak{A})$ is a $^*$-algebra. □

With the above $^*$-algebra structure, it is easy to turn $\mathcal{DC}(\mathfrak{A})$ into a C$^*$-algebra.

**Lemma 2.4.** *Let $\mathfrak{A}$ be a C$^*$-algebra. If $(L, R) \in \mathcal{DC}(\mathfrak{A})$ then $\|L\| = \|R\|$. Moreover if we equip $\mathcal{DC}(\mathfrak{A})$ with the norm $\|(L, R)\| := \|L\|$ for all $(L, R) \in \mathcal{DC}(\mathfrak{A})$ then $\mathcal{DC}(\mathfrak{A})$ is a unital C$^*$-algebra.*

*Proof.* Let $(L, R) \in \mathcal{DC}(\mathfrak{A})$. To see that $\|L\| = \|R\|$ we recall that if $A \in \mathfrak{A}$ and $\mathfrak{A}_1$ is the unit ball of $\mathfrak{A}$ then

$$\|A\| = \sup_{B \in \mathfrak{A}_1} \|AB\| = \sup_{B \in \mathfrak{A}_1} \|BA\|$$

(these equalities come from the C$^*$-equation and we omit the proof as these equations are used in the proof that the unitization of a C$^*$-algebra is in fact a C$^*$-algebra). Therefore

$$\begin{aligned} \|L\| &= \sup_{B \in \mathfrak{A}_1} \|L(B)\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|AL(B)\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|R(A)B\| \\ &= \sup_{A \in \mathfrak{A}_1} \|R(A)\| \\ &= \|R\| \end{aligned}$$

as desired.

The fact that $\|(L, R)\| := \|L\|$ is a norm on $\mathcal{DC}(\mathfrak{A})$ follows trivially as $L \mapsto \|L\|$ is a norm on $\mathcal{B}(\mathfrak{A})$ and by the operations on $\mathcal{DC}(\mathfrak{A})$ given in Lemma 2.3. To see that this norm is submultiplicative we notice that

$$\|(L_1, R_1)(L_2, R_2)\| = \|(L_1 \circ L_2, R_2 \circ R_1)\| = \|L_1 \circ L_2\| \leq \|L_1\| \|L_2\| = \|(L_1, R_1)\| \|(L_2, R_2)\|$$

for all $(L_1, R_1), (L_2, R_2) \in \mathcal{DC}(\mathfrak{A})$. Thus the norm is submultiplicative. To verify that the norm satisfies the C$^*$-equation we notice that

$$\begin{aligned} \|(L, R)^*(L, R)\| = \left\|(R^\sharp, L^\sharp)(L, R)\right\| = \left\|(R^\sharp \circ L, R \circ L^\sharp)\right\| &= \left\|R^\sharp \circ L\right\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \left\|AR^\sharp(L(B))\right\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|A(R(L(B)^*)^*)\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|(R(L(B)^*)A^*)^*\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|R(L(B)^*)A^*\| \\ &= \sup_{A, B \in \mathfrak{A}_1} \|L(B)^*L(A^*)\| . \end{aligned}$$

However

$$\sup_{A,B\in\mathfrak{A}_1} \|L(B)^*L(A^*)\| \leq \sup_{A,B\in\mathfrak{A}_1} \|L(B)^*\| \, \|L(A^*)\| = \sup_{A,B\in\mathfrak{A}_1} \|L(B)\| \, \|L(A)\| = \|L\|^2$$

and

$$\sup_{A,B\in\mathfrak{A}_1} \|L(B)^*L(A^*)\| \geq \sup_{A\in\mathfrak{A}_1} \|L(A^*)^*L(A^*)\| = \sup_{A\in\mathfrak{A}_1} \|L(A)^*L(A)\| = \sup_{A\in\mathfrak{A}_1} \|L(A)\|^2 = \|L\|^2 .$$

Therefore $\|(L,R)^*(L,R)\| = \|L\|^2 = \|(L,R)\|^2$. Therefore $\|(L,R)\| := \|L\|$ is a C*-norm.

To complete the proof that $\mathcal{DC}(\mathfrak{A})$ is a C*-algebra, it suffices to demonstrate that $\mathcal{DC}(\mathfrak{A})$ is complete with respect to this norm. Let $((L_n, R_n))_{n\geq 1}$ be a Cauchy sequence in $\mathcal{DC}(\mathfrak{A})$. Then $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ are Cauchy sequences in $\mathcal{B}(\mathfrak{A})$ as $\|(L_n,R_n)-(L_m,R_m)\| = \|L_n - L_m\| = \|R_n - R_m\|$ and by the addition on $\mathcal{DC}(\mathfrak{A})$. Therefore, as $\mathcal{B}(\mathfrak{A})$ is complete, there exists $L, R \in \mathcal{B}(\mathfrak{A})$ such that $\lim_{n\to\infty} L_n = L$ and $\lim_{n\to\infty} R_n = R$. However for all $A, B \in \mathfrak{A}$

$$AL(B) = \lim_{n\to\infty} AL_n(B) = \lim_{n\to\infty} R_n(A)B = \lim_{n\to\infty} R(A)B$$

as norm convergence implies pointwise convergence and since multiplication by a fixed operator in $\mathfrak{A}$ is a continuous operation as $\mathfrak{A}$ is a C*-algebra. Hence $(L,R) \in \mathcal{DC}(\mathfrak{A})$. Since $\lim_{n\to\infty} \|(L_n,R_n)-(L,R)\| = \lim_{n\to\infty} \|L_n - L\| = 0$, $((L_n,R_n))_{n\geq 1}$ converges to $(L,R) \in \mathcal{DC}(\mathfrak{A})$. Hence $\mathcal{DC}(\mathfrak{A})$ is complete and thus a unital C*-algebra. ☐

For $\mathcal{DC}(\mathfrak{A})$ to be the multiplier algebra of $\mathfrak{A}$, it is necessary to demonstrate that $\mathfrak{A}$ sits as an essential ideal in $\mathcal{DC}(\mathfrak{A})$. The injective of $\mathfrak{A}$ into $\mathcal{DC}(\mathfrak{A})$ is easy to describe but to show that $\mathfrak{A}$ is an ideal in $\mathcal{DC}(\mathfrak{A})$ we will need some additional knowledge about double centralizers. This knowledge gives some additional light to why we call these operators multipliers.

**Lemma 2.5.** *Let $\mathfrak{A}$ be a C*-algebra and let $(L,R) \in \mathcal{DC}(\mathfrak{A})$. Then $L(AB) = L(A)B$ and $R(AB) = AR(B)$ for all $A, B \in \mathfrak{A}$.*

*Proof.* Fix $A, B \in \mathfrak{A}$. Then for all $C \in \mathfrak{A}$

$$CL(AB) = R(C)(AB) = (R(C)A)B = (CL(A))B = CL(A)B$$

and

$$R(AB)C = (AB)L(C) = A(BL(C)) = AR(B)C.$$

In particular, if we choose $C = L(AB)^* - (L(A)B)^*$ the first equation gives $CC^* = 0$ so $C^* = L(AB) - L(A)B = 0$ by the C*-identity. Similarly if we choose $C = R(AB)^* - (AR(B))^*$ the second equation gives $C^*C = 0$ so $C^* = R(AB) - AR(B) = 0$ by the C*-identity. Thus the result is complete. ☐

With the above lemma showing that $\mathfrak{A}$ embeds as an essential ideal in $\mathcal{DC}(\mathfrak{A})$ is simple. For later use, we prove a more general result.

**Lemma 2.6.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras with $\mathfrak{A}$ an ideal in $\mathfrak{B}$. For each $B \in \mathfrak{B}$ define $L_B, R_B \in \mathcal{B}(\mathfrak{A})$ by $L_B(A) = BA$ and $R_B(A) = AB$ for all $A \in \mathfrak{A}$. Then $(L_B, R_B) \in \mathcal{DC}(\mathfrak{A})$. Moreover the map $\pi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{DC}(\mathfrak{A})$ defined by $\pi_{\mathfrak{B}}(B) = (L_B, R_B)$ is a ${}^*$-homomorphism. Furthermore $\pi_{\mathfrak{A}}$ is injective and $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an essential ideal in $\mathcal{DC}(\mathfrak{A})$.*

*Proof.* Since $\mathfrak{A}$ is an ideal of $\mathfrak{B}$, it is clear that $AB, BA \in \mathfrak{A}$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Thus it is clear that $L_B, R_B \in \mathcal{B}(\mathfrak{A})$ for all $B \in \mathfrak{B}$. To see that $(L_B, R_B) \in \mathcal{DC}(\mathfrak{A})$ we notice that

$$AL_B(C) = A(BC) = (AB)C = R_B(A)C$$

for all $A, C \in \mathfrak{A}$. Hence $(L_B, R_B) \in \mathcal{DC}(\mathfrak{A})$.

It is clear that the map $\pi_{\mathfrak{B}}$ is well-defined. To see that $\pi_{\mathfrak{B}}$ is a $^*$-homomorphism, we notice that

$$\begin{aligned} L_{\lambda A_1+A_2} &= \lambda L_{A_1} + L_{A_2} \qquad & R_{\lambda A_1+A_2} &= \lambda R_{A_1} + R_{A_2} \\ L_{A_1A_2} &= L_{A_1} \circ L_{A_2} \qquad & R_{A_1A_2} &= R_{A_2} \circ R_{A_1} \end{aligned}$$

for all $B_1, B_2 \in \mathfrak{B}$ and $\lambda \in \mathbb{C}$. Therefore, by the definition of the operations of $\mathcal{DC}(\mathfrak{A})$ given in Lemma 2.3, it is clear that $\pi_{\mathfrak{B}}$ is a homomorphism. To see that $\pi_{\mathfrak{B}}$ preserves adjoints, we notice that

$$L_B^\sharp(A) = L_B(A^*)^* = (BA^*)^* = AB^* = R_{B^*}(A)$$

and

$$R_B^\sharp(A) = R_B(A^*)^* = (A^*B)^* = B^*A = L_{B^*}(A)$$

for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Hence $L_B^\sharp = R_{B^*}$ and $R_B^\sharp = L_{B^*}$ so

$$\pi_{\mathfrak{B}}(B^*) = (L_{B^*}, R_{B^*}) = (R_B^\sharp, L_B^\sharp) = (L_B, R_B)^* = \pi_{\mathfrak{B}}(B)^*$$

for all $B \in \mathfrak{B}$. Hence $\pi_{\mathfrak{B}}$ is a $^*$-homomorphism.

To see that $\pi_{\mathfrak{A}}$ is injective, we notice if $\pi_{\mathfrak{A}}(A)$ is zero then $R_A = 0$. Therefore $A^*A = R_A(A^*) = 0$ so $A = 0$ by the C$^*$-identity. Hence $\pi_{\mathfrak{A}}$ is injective.

To see that $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an ideal in $\mathcal{DC}(\mathfrak{A})$, it suffices to show that $(L, R)(L_A, R_A), (L_A, R_A)(L, R) \in \pi_{\mathfrak{A}}(\mathfrak{A})$ for all $A \in \mathfrak{A}$ and $(L, R) \in \mathcal{DC}(\mathfrak{A})$ as $\pi_{\mathfrak{A}}(\mathfrak{A})$ is already closed in $\mathcal{DC}(\mathfrak{A})$ being the image of a $^*$-homomorphism of a C$^*$-algebra. Moreover, since $\pi_{\mathfrak{A}}(\mathfrak{A})$ is closed under adjoints, it suffices to check that $(L_A, R_A)(L, R) = (L_A \circ L, R \circ R_A) \in \pi_{\mathfrak{A}}(\mathfrak{A})$ for all $A \in \mathfrak{A}$ and $(L, R) \in \mathcal{DC}(\mathfrak{A})$ (i.e. it suffices to check that $\pi_{\mathfrak{A}}(\mathfrak{A})$ is a right ideal). However we notice

$$(L_A \circ L)(B) = L_A(L(B)) = AL(B) = R(A)B = L_{R(A)}(B)$$

and

$$(R \circ R_A)(B) = R(R_A(B)) = R(BA) = BR(A) = R_{R(A)}(B)$$

(by Lemma 2.5) for all $A, B \in \mathfrak{A}$. Hence $(L_A, R_A)(L, R) = (L_{R(A)}, R_{R(A)}) \in \pi_{\mathfrak{A}}(\mathfrak{A})$ as desired. Hence $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an ideal in $\mathcal{DC}(\mathfrak{A})$.

To show that $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an essential ideal in $\mathfrak{A}$, it suffices by Proposition 1.5 to show that if $(L, R) \in \mathcal{DC}(\mathfrak{A})$ is such that $(L_A, R_A)(L, R) = 0$ for all $A \in \mathfrak{A}$ then $(L, R) = 0$. However, by the above computation, $(L_A, R_A)(L, R) = (L_{R(A)}, R_{R(A)}) = \pi_{\mathfrak{A}}(R(A))$. Therefore, if $(L_A, R_A)(L, R) = 0$ for all $A \in \mathfrak{A}$ then $R(A) = 0$ for all $A \in \mathfrak{A}$ as $\pi_{\mathfrak{A}}$ is injective. Therefore $R = 0$ so $\|L\| = \|(L, R)\| = \|R\| = 0$ by Lemma 2.4. Hence $(L, R) = 0$ and thus $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an essential ideal in $\mathfrak{A}$. □

With the above technical lemma complete, the proof that $\mathcal{DC}(\mathfrak{A})$ is the multiplier algebra reduces to verifying that $\mathcal{DC}(\mathfrak{A})$ has the universal property.

**Theorem 2.7.** *Let $\mathfrak{A}$ be a C$^*$-algebra. The double centralizer of $\mathfrak{A}$, $\mathcal{DC}(\mathfrak{A})$, is the multiplier algebra of $\mathfrak{A}$ when we view $\mathfrak{A} \subseteq \mathcal{DC}(\mathfrak{A})$ via $\pi_{\mathfrak{A}}$ from Lemma 2.6.*

*Proof.* By Lemma 2.4 and Lemma 2.6 $\mathcal{DC}(\mathfrak{A})$ is a unital C$^*$-algebra that contains $\mathfrak{A}$ as an essential ideal via $\pi_{\mathfrak{A}}$. To verify that $\mathcal{DC}(\mathfrak{A})$ has the universal property of the multiplier algebra, let $\mathfrak{B}$ be a C$^*$-algebra that contains $\mathfrak{A}$ as an essential ideal. By Lemma 2.6, there exists a $^*$-homomorphism $\pi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{DC}(\mathfrak{A})$. It is clear that $\pi_{\mathfrak{B}}|_{\mathfrak{A}} = \pi_{\mathfrak{A}}$ by construction. Thus, to complete the proof, it suffices to show that $\pi_{\mathfrak{B}}$ is the unique $^*$-homomorphism that gives $\pi_{\mathfrak{A}}$ when restricted to $\mathfrak{A}$.

Suppose $\sigma : \mathfrak{B} \to \mathcal{DC}(\mathfrak{A})$ is such that $\sigma|_{\mathfrak{A}} = \pi_{\mathfrak{A}}$. Since $\mathfrak{A}$ is an ideal of $\mathfrak{B}$, we notice that

$$\pi_{\mathfrak{B}}(B)\pi_{\mathfrak{A}}(A) = \pi_{\mathfrak{B}}(B)\pi_{\mathfrak{B}}(A) = \pi_{\mathfrak{B}}(BA) = \pi_{\mathfrak{A}}(BA) = \sigma(BA) = \sigma(B)\sigma(A) = \sigma(B)\pi_{\mathfrak{A}}(A)$$

for all $B \in \mathfrak{B}$ and $A \in \mathfrak{A}$. Hence

$$(\pi_{\mathfrak{B}}(B) - \sigma(B))\pi_{\mathfrak{A}}(A) = 0$$

for all $B \in \mathfrak{B}$ and $A \in \mathfrak{A}$. Since $\pi_{\mathfrak{A}}(\mathfrak{A})$ is an essential ideal in $\mathcal{DC}(\mathfrak{A})$, Proposition 1.5 implies that $\pi_{\mathfrak{B}}(B) - \sigma(B) = 0$ for all $B \in \mathfrak{B}$. Hence $\pi_{\mathfrak{B}} = \sigma$ as desired. □

The proof given above of the existence of the multiplier algebra has many corollaries which provide more information about the structure of the multiplier algebra.

**Corollary 2.8.** *Let $\mathfrak{A}$ be a C*-algebra. Then $\mathcal{M}(\mathfrak{A})$ is unital.*

*Proof.* Lemma 2.4 showed that $\mathcal{DC}(\mathfrak{A})$ is unital. ☐

**Corollary 2.9.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras such that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$. Then there exists a unique *-homomorphism $\pi : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$. If $\mathfrak{B}$ is unital then $\pi$ is unital. Furthermore if $\mathfrak{A}^\perp := \{B \in \mathfrak{B} \mid BA = 0 \text{ for all } A \in \mathfrak{A}\}$ then $ker(\pi) = \mathfrak{A}^\perp$. Finally $\pi$ is injective if and only if $\mathfrak{A}$ is an essential ideal of $\mathfrak{B}$.*

*Proof.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras such that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$. The assumption that $\mathfrak{A}$ was an essential ideal in $\mathfrak{B}$ was not used in the proof of the universal property of $\mathcal{M}(\mathfrak{A})$ in Theorem 2.7. Therefore there exists a unique *-homomorphism $\pi : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$ where $\pi = \pi|_{\mathfrak{B}}$ is from Lemma 2.6. If $I_{\mathfrak{B}}$ is a unit of $\mathfrak{B}$, it is clear that $L_{I_{\mathfrak{B}}} = Id_{\mathfrak{A}} = R_{I_{\mathfrak{B}}}$ so $\pi_{\mathfrak{B}}$ is unital.

If $B \in \mathfrak{A}^\perp$ then $L_B = 0$. Hence $\pi_{\mathfrak{B}}(B) = (L_B, R_B)$ must be zero as $\|L_B\| = \|R_B\| = 0$ by Lemma 2.4. Thus $\mathfrak{A}^\perp \subseteq ker(\pi)$. For the other inclusion, suppose that $B \in ker(\pi)$. Hence $(L_B, R_B) = \pi_{\mathfrak{B}}(B) = (0, 0)$. Thus $L_B = 0$ so $BA = L_B(A) = 0$ for all $A \in \mathfrak{A}$. Hence $B \in \mathfrak{A}^\perp$. Thus $ker(\pi) = \mathfrak{A}^\perp$.

Proposition 1.5 implies that $\mathfrak{A}$ is an essential ideal of $\mathfrak{B}$ if and only if $\mathfrak{A}^\perp = \{0\}$ which is equivalent to the fact that $\pi|_{\mathfrak{B}}$ is injective by the previous paragraph. ☐

**Corollary 2.10.** *If $\mathfrak{A}$ is unital then $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$.*

*Proof.* We shall provide two proofs of this fact. For the first, we note from Corollary 2.8 that $\mathcal{M}(\mathfrak{A})$ is unital. Hence $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$ by Lemma 1.9.

For the second proof, suppose $(L, R) \in \mathcal{DC}(\mathfrak{A})$. Let $A := L(I_{\mathfrak{A}}) \in \mathfrak{A}$. Then for all $B \in \mathfrak{A}$

$$L(B) = L(I_{\mathfrak{A}}B) = L(I_{\mathfrak{A}})B = AB = L_A(B)$$

by Lemma 2.5. Moreover

$$R(B) = R(B)I_{\mathfrak{A}} = BL(I_{\mathfrak{A}}) = BA = R_A(B)$$

by the definition of a double centralizer. Hence $(L, R) = (L_A, R_A) \in \pi_{\mathfrak{A}}(\mathfrak{A})$. Therefore, since $(L, R) \in \mathcal{DC}(\mathfrak{A})$, $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$. ☐

With the above important corollaries complete, we turn our attention to a method of concretely realizing the double centralizer of a C*-algebra. As seen in Section 1, there seems to be a connection between multiplier algebras and right Hilbert C*-modules. We will investigate this connection in greater detail in Section 4. To begin this discussion and obtain a preliminary result, we note the following.

**Remarks 2.11.** Let $\mathfrak{A}$ be a C*-algebra. Then $\mathfrak{A}$ can be viewed as a right Hilbert $\mathfrak{A}$-module with inner product $\langle A_1, A_2\rangle = A_1^* A_2$ for all $A_1, A_2 \in \mathfrak{A}$ and right action $\rho : \mathfrak{A} \to \mathcal{B}(\mathfrak{A})$ by $\rho(A)A' = A'A$ for all $A, A' \in \mathfrak{A}$. Recall that $\mathcal{B}_a(\mathfrak{A})$ is the set of all bounded linear maps $T$ on $\mathfrak{A}$ such that there exists a bounded linear map $T^*$ such that $\langle A_1, T(A_2)\rangle = \langle T^*(A_1), A_2\rangle$ for all $A_1, A_2 \in \mathfrak{A}$. It is not difficult to verify that $\mathcal{B}_a(\mathfrak{A})$ is a unital C*-algebra. Furthermore, due to the structure of the inner product selected on $\mathfrak{A}$, a bounded linear map $T_1$ on $\mathfrak{A}$ is in $\mathcal{B}_a(\mathfrak{A})$ if and only if there exists a $T_2 \in \mathcal{B}(\mathfrak{A})$ such that $A_1^* T_1(A_2) = (T_2(A_1))^* A_2$ for all $A_1, A_2 \in \mathfrak{A}$ if and only if $A_1 T_1(A_2) = (T_2)^\sharp(A_1)A_2$ for all $A_1, A_2 \in \mathfrak{A}$ if and only if $(T_1, T_2^\sharp) \in \mathcal{DC}(\mathfrak{A})$. Hence there exists a bijective map $\Psi : \mathcal{B}_a(\mathfrak{A}) \to \mathcal{DC}(\mathfrak{A})$ defined by $\Psi(T) = (T, (T^*)^\sharp)$ for all $T \in \mathcal{B}_a(\mathfrak{A})$.

Thus, using the above remarks, the following theorem gives a concrete description of the multiplier algebra via the adjointable linear maps on a right Hilbert C*-module.

**Corollary 2.12.** *Let $\mathfrak{A}$ be a C*-algebra and equip $\mathfrak{A}$ with the right Hilbert $\mathfrak{A}$-module structure as in Remarks 2.11. Then the map $\Psi$ from Remarks 2.11 is a *-isomorphism so $\mathcal{M}(\mathfrak{A}) = \mathcal{B}_a(\mathfrak{A})$.*

*Proof.* Since $\Psi$ is bijective by Remarks 2.11, it suffices to show that $\Psi$ is a $^*$-isomorphism by Theorem 2.7. Since both $^*$ and $^\sharp$ are conjugate linear, clearly $\Psi$ is linear. Since $^*$ is antimultiplicative, since $^\sharp$ is multiplicative, and since the multiplication in the double centralizer is antimultiplicative in the second component, clearly $\Psi$ is multiplicative. To see that $\Psi$ is $^*$-preserving, we notice that

$$\Psi(T^*) = (T^*, ((T^*)^*)^\sharp) = (T^*, T^\sharp) = ((T^\sharp)^\sharp, (T^*)^\sharp)^* = (T, (T^*)^\sharp)^* = \Psi(T)^*$$

for all $T \in \mathcal{B}_a(\mathfrak{A})$. Hence $\Psi$ is a $^*$-isomorphism. □

Although there are many other technical details and interesting results that may be obtained from the double centralizer version of the multiplier algebra, we will complete this section with the following result. We remind the reader that there are many other results and properties of multiplier algebras that may be obtained using one of the three descriptions of the multiplier algebra that we will develop. We shall investigate some of these properties in Section 5.

**Proposition 2.13.** *Let $\{\mathfrak{A}_i\}_{i\in I}$ be a set of $C^*$-algebras. If $\bigoplus_{i\in I}\mathfrak{A}_i$ denotes the direct sum of $\mathfrak{A}_i$ (the closure with respect to the supremum norm of all functions indexed by $I$ with values at $i$ in $\mathfrak{A}_i$ with finite support) then*

$$\mathcal{M}\left(\bigoplus_{i\in I}\mathfrak{A}_i\right) \simeq \prod_{i\in I}\mathcal{M}(\mathfrak{A}_i).$$

*where $\prod_{i\in I}\mathcal{M}(\mathfrak{A}_i)$ is the product of $\mathcal{M}(\mathfrak{A}_i)$ (all bounded functions indexed by $I$ with values at $i$ in $\mathcal{M}(\mathfrak{A}_i)$).*

*Proof.* Consider the map $\Psi : \prod_{i\in I}\mathcal{DC}(\mathfrak{A}_i) \to \mathcal{DC}(\bigoplus_{i\in I}\mathfrak{A}_i)$ defined by

$$\Psi(((L_i, R_i))_{i\in I}) = (\psi_{(L_i)_{i\in I}}, \psi_{(R_i)_{i\in I}})$$

where

$$\psi_{(T_i)_{i\in I}}((A_i)_{i\in I}) = (T_i(A_i))_{i\in I}$$

for any bounded set $\{T_i\}_{i\in I} \subseteq \mathcal{B}(\mathfrak{A})$. It is clear that if $((L_i, R_i))_{i\in I} \in \prod_{i\in I}\mathcal{DC}(\mathfrak{A}_i)$ then $\psi_{(L_i)_{i\in I}}, \psi_{(R_i)_{i\in I}} \in \mathcal{B}(\bigoplus_{i\in I}\mathfrak{A}_i)$. Moreover, if $(A_i)_{i\in I}, (B_i)_{i\in I} \in \bigoplus_{i\in I}\mathfrak{A}_i$ then

$$\begin{aligned}
(A_i)_{i\in I}\psi_{(L_i)_{i\in I}}((B_i)_{i\in I}) &= (A_i)_{i\in I}(L_i(B_i))_{i\in I} \\
&= (A_iL_i(B_i))_{i\in I} \\
&= (R_i(A_i)B_i)_{i\in I} \\
&= (R_i(A_i))_{i\in I}(B_i)_{i\in I} \\
&= \psi_{(R_i)_{i\in I}}((A_i)_{i\in I})(B_i)_{i\in I}.
\end{aligned}$$

Hence $\Psi$ is a well-defined map from $\prod_{i\in I}\mathcal{DC}(\mathfrak{A}_i)$ to $\mathcal{DC}(\bigoplus_{i\in I}\mathfrak{A}_i)$. Since $\psi_\cdot$ is linear and multiplicative in its subscript, it is clear that $\Psi$ is a homomorphism. Furthermore, since $\psi_{\bullet^*} = \psi_\bullet^*$, it is clear that $\Psi$ is a $^*$-homomorphism.

To see that $\Psi$ is injective, suppose that $\Psi(((L_i, R_i))_{i\in I}) = 0$. Therefore $\psi_{(L_i)_{i\in I}} = 0$ so $L_i(A) = 0$ for all $A \in \mathfrak{A}_i$ and all $i \in I$. Hence $L_i = 0$ for all $i \in I$. Since $\|R_i\| = \|L_i\|$ for all $i \in I$, we clearly obtain that $\Psi$ is injective.

To see that $\Psi$ is surjective, let $(L, R) \in \mathcal{DC}(\bigoplus_{i\in I}\mathfrak{A}_i)$. Therefore $L, R \in \mathcal{B}(\bigoplus_{i\in I}\mathfrak{A}_i)$. We claim that $L$ and $R$ must map each $\mathfrak{A}_i$ to itself; that is, if $(A_i)_{i\in I} \in \bigoplus_{i\in I}\mathfrak{A}_i$ is such that $A_j = 0$ unless $j = i_0$ for some $i_0 \in I$ and $L((A_i)_{i\in I}) = (D_i)_{i\in I}$ then $D_j = 0$ unless $j = i_0$. To see this, suppose $(A_i)_{i\in I} \in \bigoplus_{i\in I}\mathfrak{A}_i$ is such that $A_j = 0$ unless $j = i_0$ for some $i_0 \in I$. Let $(D_i)_{i\in I} := L((A_i)_{i\in I})$. Let $(B_i)_{i\in I} \in \bigoplus_{i\in I}\mathfrak{A}_i$ be arbitrary and define $C_j := 0_{\mathfrak{A}_j}$ if $j \neq i_0$ and $C_{i_0} := B_{i_0}$. Then, by Lemma 2.5,

$$L((A_i)_{i\in I})(B_i)_{i\in I} = L((A_iB_i)_{i\in I}) = L((A_iC_i)_{i\in I}) = L((A_i)_{i\in I})(C_i)_{i\in I}.$$

Hence $D_jB_j = D_jC_j = 0$ for all $j \neq i_0$. Since $(B_i)_{i\in I} \in \bigoplus_{i\in I}\mathfrak{A}_i$ was arbitrary, $D_j = 0$ if $j \neq i_0$. Hence the result is complete for $L$. The proof of the result for $R$ is identical.

For each $k \in I$ we define $L_k \in \mathcal{B}(\mathfrak{A}_k)$ such that if $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{A}_i$ is a sequence where $A_i = 0$ unless $i = k$ and $L((A_i)_{i\in I}) = (B_i)_{i\in I}$ where $B_i = 0$ if $i \neq k$ then $L(A_k) = B_k$. Since $L$ is linear, it is trivial to verify that $L_k$ is well-defined and linear. Similarly we define $R_k \in \mathcal{B}(\mathfrak{A}_k)$. It is clear that $L_k$, $R_k$ are well-defined elements of $\mathcal{B}(\mathfrak{A}_k)$ and uniformly bounded over $k \in I$ by $\|L\|$ and $\|R\|$ respectively. Moreover, since functions with finite support are dense in $\bigoplus_{i\in I} \mathfrak{A}_i$, it is easy to see that $L = \psi_{(L_i)_{i\in I}}$ and $R = \psi_{(R_i)_{i\in I}}$. Thus it suffices to verify that $(L_i, R_i) \in \mathcal{DC}(\mathfrak{A}_i)$ for all $i \in I$; that is $AL_i(B) = R_i(A)B$ for all $A, B \in \mathfrak{A}_i$ and for all $i \in I$. However this follows trivially from the definition of $L_i$ and $R_i$ and the fact that $(L, R) \in \mathcal{DC}(\bigoplus_{i\in I} \mathfrak{A}_i)$. Hence $\Psi$ is surjective and thus a $^*$-isomorphism. $\square$

## 3 Construction of Multiplier Algebra by Representations

In this section we will demonstrate the existence of the multiplier algebra of an arbitrary C*-algebra by an application of the representation theory of C*-algebras. In particular we will demonstrate that given a 'nice' representation of a given C*-algebra we can construct a set with a certain algebraic property that will be the multiplier algebra. Although Section 2 required slightly less machinery and had a more explicit description of the multiplier algebra, the results of this section are, in general, easier to deal with.

We begin with the definition of the 'nice' representations of a given C*-algebra that we will deal with.

**Definition 3.1.** Let $\mathfrak{A}$ be a C*-algebra. A representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is said to be non-degenerate if $\pi(\mathfrak{A})\mathcal{H} := \{\pi(A)\xi \mid A \in \mathfrak{A}, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

A representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is said to have trivial null-space if $\pi(A)\xi = 0$ for all $\xi \in \mathcal{H}$ implies that $A = 0$.

If $\mathfrak{A}$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$, we say that $\mathfrak{A}$ is non-degenerate (has trivial null-space) if and only if the identity representation is non-degenerate (has trivial null-space).

It is clear from the GNS construction that every GNS representation is non-degenerate. Moreover it is clear that the direct sum of representations with trivial null-spaces have trivially null-space. It turns out that the concepts of non-degenerate representations and representations with trivial null-spaces coincide and thus every C*-algebra has a faithful non-degenerate representation. It is this later fact that will be necessary for the rest of the section. As it is not clear from the definition that the direct sum of non-degenerate representations is non-degenerate, we will prove the following results which the author feels should be explicitly stated in more textbooks on C*-algebras.

**Proposition 3.2.** *Let $\mathfrak{A}$ be a C*-algebra, let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a representation of $\mathfrak{A}$, and let $\mathcal{K} := \overline{\pi(\mathfrak{A})\mathcal{H}}$. Then $\mathcal{K}$ is a closed subspace of $\mathcal{H}$.*

*Furthermore $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity of $\mathfrak{A}$ then $\pi(E_\lambda)$ converges in the SOT to $P_\mathcal{K}$ (the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$).*

*Let $\pi' : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ be defined by $\pi'(A) = \pi(A)|_\mathcal{K}$ for all $A \in \mathfrak{A}$. Then $\pi'$ is a non-degenerate representation of $\mathfrak{A}$. In fact, $\pi(A)\xi = 0$ for all $\xi \in \mathcal{K}^\perp$. Hence $\|\pi'(A)\| = \|\pi(A)\|$ for all $A \in \mathfrak{A}$. Moreover $\pi'$ is faithful whenever $\pi$ is faithful. Lastly $\pi$ is non-degenerate if and only if $\pi(\mathfrak{A})$ has trivial null-space.*

*Proof.* First we will demonstrate that if $\xi \in \mathcal{K}$ and $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity of $\mathfrak{A}$ (note that at least one exists) then $\lim_\Lambda \pi(E_\lambda)\xi = \xi$. Let $\xi \in \mathcal{K} = \overline{\pi(\mathfrak{A})\mathcal{H}}$ and let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity of $\mathfrak{A}$. By the definition of $\pi(\mathfrak{A})\mathcal{H}$ there exists $A_n \in \mathfrak{A}$ and $\eta_n \in \mathcal{H}$ such that $\xi = \lim_n \pi(A_n)\eta_n$. Let $\epsilon > 0$. Since $\xi = \lim_n \pi(A_n)\eta_n$, there exists an $N \in \mathbb{N}$ such that $\|\xi - \pi(A_N)\eta_N\| \leq \frac{\epsilon}{3}$. Moreover, since $(E_\lambda)_\Lambda$ is a bounded approximate identity for $\mathfrak{A}$, there exists a $\lambda' \in \Lambda$ such that for all $\lambda \geq \lambda'$, $\|E_\lambda A_N - A_N\| \leq \frac{\epsilon}{3(\|\eta_N\|+1)}$. Hence, since $\pi$ is a contraction, $\|\pi(E_\lambda A_N) - \pi(A_N)\| \leq \frac{\epsilon}{3(\|\eta_N\|+1)}$. Hence for all $\lambda \geq \lambda'$

$$\begin{aligned} \|\xi - \pi(E_\lambda)\xi\| &\leq \|\xi - \pi(A_N)\eta_N\| + \|\pi(A_N)\eta_N - \pi(E_\lambda A_N)\eta_N\| + \|\pi(E_\lambda A_N)\eta_N - \pi(E_\lambda)\xi\| \\ &\leq \frac{\epsilon}{3} + \|\pi(A_N) - \pi(E_\lambda A_N)\| \, \|\eta_N\| + \|\pi(E_\lambda)\| \, \|\pi(A_N)\eta_N - \xi\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3(\|\eta_N\|+1)} \|h_N\| + 1\frac{\epsilon}{3} \leq \epsilon \end{aligned}$$

Hence $\lim_\Lambda \pi(E_\lambda)\xi = \xi$ as claimed.

To see that $\mathcal{K}$ is a Hilbert space, it suffices to verify that $\mathcal{K}$ is a linear subspace of $\mathcal{H}$. If $\xi_1, \xi_2 \in \mathcal{K}$ and $\alpha \in \mathbb{C}$ then $\lim_\Lambda \pi(E_\lambda)(\alpha\xi_1 + \xi_2) = \alpha\xi_1 + \xi_2$ so that $\alpha\xi_1 + \xi_2 \in \mathcal{K}$ by the definition of $\mathcal{K}$. Hence $\mathcal{K}$ is a Hilbert space.

It is clear that $\pi'$ is a well-defined representation of $\mathfrak{A}$ as $\mathcal{K}$ is $\pi(\mathfrak{A})$-invariant (and thus reducing) subspace. Moreover, since $\lim_\Lambda \pi'(E_\lambda)\xi = \lim_\Lambda \pi(E_\lambda)\xi = \xi$ for all $\xi \in \mathcal{K}$, $\xi \in \overline{\pi'(\mathfrak{A})\mathcal{K}}$ for all $\xi \in \mathcal{K}$. Hence $\pi'$ is a non-degenerate representation of $\mathfrak{A}$.

Next suppose $\eta \in \mathcal{K}^{\perp}$. Then for all $\xi \in \mathcal{K}$

$$\langle \pi(A)\eta, \xi \rangle = \langle \eta, \pi(A^*)\xi \rangle = 0$$

since $\eta \in \mathcal{K}^{\perp}$ and $\pi(A^*)\xi \in \mathcal{K}$. Similarly, if $\xi \in \mathcal{K}^{\perp}$ then $\langle \pi(A)\eta, \xi \rangle = 0$. Hence $\pi(A)\eta = 0$ as desired. Consequently $\|\pi(A)\| = \|\pi'(A)\|$ as $\pi'(A) = \pi(A)|_{\mathcal{K}}$. Furthermore if $\pi$ is faithful and $\pi'(A_1) = \pi'(A_2)$ then $\pi(A_1)|_{\mathcal{K}} = \pi(A_2)|_{\mathcal{K}}$ and, since $\pi(A_1)|_{\mathcal{K}^{\perp}} = \pi(A_2)|_{\mathcal{K}^{\perp}} = 0$, $\pi(A_1) = \pi(A_2)$ so $A_1 = A_2$. Hence $\pi'$ is faithful.

To see that $(\pi(E_\lambda))_\Lambda$ converges in the SOT to $P_{\mathcal{K}}$, let $\xi \in \mathcal{H}$ be arbitrary and write $\xi = \xi_{\mathcal{K}} + \eta$ where $\xi \in \mathcal{K}$ and $\eta \in \mathcal{K}^{\perp}$. Then

$$\lim_{\Lambda} \pi(E_\lambda)\xi = \lim_{\Lambda} \pi(E_\lambda)\xi_{\mathcal{K}} + 0 = \xi_{\mathcal{K}} = P_{\mathcal{K}}\xi$$

as $\pi(A)\eta = 0$ for all $A \in \mathfrak{A}$. As $\xi \in \mathcal{H}$ was arbitrary, the claim is complete

Lastly suppose $\pi$ is non-degenerate so that $\mathcal{H} = \mathcal{K} = \overline{\pi(\mathfrak{A})\mathcal{H}}$. Suppose $\pi(\mathfrak{A})\xi = \{0\}$. Then if $(E_\lambda)_\lambda$ is any C*-bounded approximate identity of $\mathfrak{A}$, $(\pi(E_\lambda))_\Lambda$ converges in the SOT to $I_{\mathcal{H}}$ and thus

$$\xi = \lim_{\Lambda} \pi(E_\lambda)\xi = \lim_{\Lambda} 0 = 0.$$

Hence $\pi(\mathfrak{A})$ has trivial null-space. Now suppose $\pi(\mathfrak{A})$ has trivial null-space. If $\mathcal{H} \neq \overline{\pi(\mathfrak{A})\mathcal{H}}$ then there exists a $\xi \in \mathcal{H}$ such that $\xi \in \overline{\pi(\mathfrak{A})\mathcal{H}}^{\perp}$. However, from earlier work, this implies that $\pi(A)\xi = 0$ for all $A \in \mathfrak{A}$ which contradicts the fact that $\pi(\mathfrak{A})$ had trivial null-space. ☐

The relation between non-degenerate representations and the multiplier algebra is any non-degenerate representation of a ideal can be extended to a representation on the entire C*-algebra. Thus, if the ideal is essential and the representation is faithful, the extension will also be faithful. This is very close to what is necessary in the multiplier algebra. There is an additional property of the extension that will be desirable and thus we make the following definition and postpone the desired result.

**Definition 3.3.** Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. The idealizer of $\mathcal{A}$, denoted $\mathcal{ID}(\mathcal{A})$, is the set

$$\mathcal{ID}(\mathcal{A}) := \{T \in \mathcal{B}(\mathcal{H}) \mid T\mathcal{A} \subseteq \mathcal{A}, \mathcal{A}T \subseteq \mathcal{A}\}$$

(where $T\mathcal{A} := \{S \in \mathcal{B}(\mathcal{H}) \mid S = TA \text{ for some } A \in \mathcal{A}\}$).

It is fairly clear that if $\mathcal{A}$ is a *-algebra then $\mathcal{ID}(\mathcal{A})$ is also a *-algebra that contains $\mathcal{A}$ as an ideal (see the proof below). In particular, $\mathcal{ID}(\mathcal{A})$ is the largest algebra that contains $\mathcal{A}$ as an ideal. The idealizer of a C*-algebra is a nice subset of $\mathcal{B}(\mathcal{H})$.

**Lemma 3.4.** *Let $\mathfrak{A}$ be a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Then $\mathcal{ID}(\mathfrak{A})$ is a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$ that contains $\mathfrak{A}$ as an ideal. If $\mathfrak{A}$ is non-degenerate then $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$.*

*Proof.* Since $\mathfrak{A}$ is closed under addition, scalar multiplication, multiplication, and adjoints, it is clear that $\mathcal{ID}(\mathfrak{A})$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$. Moreover, since $\mathfrak{A}$ is closed, it is trivial to verify that $\mathcal{ID}(\mathfrak{A})$ is closed. Hence $\mathcal{ID}(\mathfrak{A})$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Furthermore it is clear by the definition of the idealizer that $\mathfrak{A}$ is an ideal in $\mathcal{ID}(\mathfrak{A})$ and $I_{\mathcal{H}} \in \mathcal{ID}(\mathfrak{A})$.

Suppose $\mathfrak{A}$ is non-degenerate. To show that $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$, suppose that $T \in \mathcal{ID}(\mathfrak{A})$ is such that $TA = 0$ for all $A \in \mathfrak{A}$. Therefore $TA\xi = 0$ for all $A \in \mathfrak{A}$ and $\xi \in \mathcal{H}$. Since $\mathfrak{A}$ is a non-degenerate subalgebra of $\mathcal{B}(\mathcal{H})$, the set $\{A\xi \mid A \in \mathfrak{A}, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Hence $T\eta = 0$ for all $\eta \in \mathcal{H}$ so $T = 0$. Therefore Proposition 1.5 implies that $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$. ☐

The above shows how when given a C*-algebra $\mathfrak{A}$ we can construct a C*-algebra containing $\mathfrak{A}$ as an essential ideal. In particular, the above is how we will construct the multiplier algebra of $\mathfrak{A}$. In order to prove that the construction is the multiplier algebra, we will need to be able to extend representations from ideals. In particular, the following result looks very similar to the conditions necessary and conclusions obtained in Section 2 for the multiplier algebra.

**Lemma 3.5.** *Let $\mathcal{I}$ be an ideal of a C*-algebra $\mathfrak{A}$ and let $\pi : \mathcal{I} \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Then there exists a unique $^*$-homomorphism $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ extending $\pi$ (that must then be non-degenerate). If $\mathfrak{A}$ is unital then $\tilde{\pi}$ is unital. Furthermore $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$. Finally, if in addition $\pi$ is faithful and $\mathcal{I}^\perp := \{A \in \mathfrak{A} \mid AB = 0 \text{ for all } B \in \mathcal{I}\}$, then $ker(\tilde{\pi}) = \mathcal{I}^\perp$. Thus if $\pi$ is faithful then $\mathcal{I}$ is an essential ideal of $\mathfrak{A}$ if and only if $\tilde{\pi}$ is faithful.*

*Proof.* Let $\mathcal{I}$ be an ideal of a C*-algebra $\mathfrak{A}$ and let $\pi : \mathcal{I} \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Fix a C*-bounded approximate identity $(E_\lambda)_\Lambda$ for $\mathcal{I}$. For each $A \in \mathfrak{A}$, we claim that the operator $\tilde{\pi}_0(A)$ defined on $\pi(\mathcal{I})\mathcal{H}$ by

$$\tilde{\pi}_0(A)(\pi(B)\xi) = \pi(AB)\xi$$

for all $B \in \mathcal{I}$ and $\xi \in \mathcal{H}$ is well-defined. To see this, we notice that $AB \in \mathcal{I}$ for all $B \in \mathcal{I}$ as $\mathcal{I}$ is an ideal so $\pi(AB)$ makes sense. To see that $\tilde{\pi}_0(A)$ is well-defined, suppose $B_1, B_2 \in \mathcal{I}$ and $\xi_1, \xi_2 \in \mathcal{H}$ are such that $\pi(B_1)\xi_1 = \pi(B_2)\xi_2$. Then

$$\begin{aligned} \pi(AB_1)\xi_1 &= \lim_\Lambda \pi(AE_\lambda B_1)\xi_1 \\ &= \lim_\Lambda \pi(AE_\lambda)\pi(B_1)\xi_1 \\ &= \lim_\Lambda \pi(AE_\lambda)\pi(B_2)\xi_2 \\ &= \lim_\Lambda \pi(AE_\lambda B_2)\xi_2 \\ &= \pi(AB_2)\xi_2. \end{aligned}$$

Hence $\tilde{\pi}_0(A)$ is well-defined. Moreover we notice that

$$\begin{aligned} \|\tilde{\pi}_0(A)(\pi(B_1)\xi) - \tilde{\pi}_0(A)(\pi(B_2)\xi)\| &= \|\pi(AB_1 - AB_2)\xi\| \\ &= \lim_\Lambda \|\pi(AE_\lambda(B_1 - B_2))\xi\| \\ &= \lim_\Lambda \|\pi(AE_\lambda)\pi(B_1 - B_2)\xi\| \\ &\leq \lim_\Lambda \|\pi(AE_\lambda)\| \, \|\pi(B_1 - B_2)\xi\| \\ &\leq \limsup_\Lambda \|AE_\lambda\| \, \|\pi(B_1 - B_2)\xi\| \\ &\leq \|A\| \, \|\pi(B_1 - B_2)\xi\|. \end{aligned}$$

Therefore, since $\lim_\Lambda \pi(E_\lambda)\xi = \xi$ for all $\xi \in \mathcal{H}$ by Proposition 3.2, the above inequalities implies that $(\tilde{\pi}_0(A)\pi(E_\lambda)\xi)_\Lambda$ is a Cauchy sequence. Hence we define $\tilde{\pi}(A)$ to be the function on $\mathcal{H}$ defined by

$$\tilde{\pi}(A)\xi := \lim_\Lambda \tilde{\pi}_0(A)\pi(E_\lambda)\xi = \lim_\Lambda \pi(AE_\lambda)\xi$$

for all $\xi \in \mathcal{H}$. It is then clear that $\tilde{\pi}(A)$ is a linear map and is bounded since $\|\pi(AE_\lambda)\xi\| \leq \|\pi(AE_\lambda)\| \, \|\xi\| \leq \|A\| \, \|\xi\|$ for all $\xi \in \mathcal{H}$ and $\lambda \in \Lambda$. Hence $\tilde{\pi}(A) \in \mathcal{B}(\mathcal{H})$. Furthermore it is clear $A \mapsto \tilde{\pi}(A)$ is linear as $\pi$ is linear. Hence $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a linear map. In addition, if $A \in \mathcal{I}$ then

$$\tilde{\pi}(A)\xi = \lim_\Lambda \pi(AE_\lambda)\xi = \pi(A)\xi$$

for all $\xi \in \mathcal{H}$ so $\tilde{\pi}$ extends $\pi$.

To show that $\tilde{\pi}$ is multiplicative and preserves adjoints, we first notice that if $A \in \mathfrak{A}$, $B \in \mathcal{I}$, and $\xi \in \mathcal{H}$ then

$$\tilde{\pi}(A)(\pi(B)\xi) = \lim_\Lambda \pi(AE_\lambda)\pi(B)\xi = \lim_\Lambda \pi(AE_\lambda B)\xi = \pi(AB)\xi.$$

Thus $\tilde{\pi}(A)$ extends $\tilde{\pi}_0(A)$ for all $A \in \mathfrak{A}$. Therefore, if $A, B \in \mathfrak{A}$

$$\tilde{\pi}(AB)\xi = \lim_\Lambda \pi(ABE_\lambda)\xi = \lim_\Lambda \tilde{\pi}(A)(\pi(BE_\lambda)\xi) = \tilde{\pi}(A)\tilde{\pi}(B)\xi$$

for all $\xi \in \mathcal{H}$. Hence $\tilde{\pi}$ is multiplicative.

To see that $\tilde{\pi}$ preserves adjoints, we notice for all $A \in \mathfrak{A}$ and $\xi, \eta \in \mathcal{H}$ that

$$\begin{aligned} \langle \tilde{\pi}(A^*)\xi, \eta \rangle &= \lim_\Lambda \langle \pi(A^* E_\lambda)\xi, \pi(E_\lambda)\eta \rangle \\ &= \lim_\Lambda \langle \xi, \pi(E_\lambda A)\pi(E_\lambda)\eta \rangle \\ &= \lim_\Lambda \langle \xi, \pi(E_\lambda A E_\lambda)\eta \rangle \\ &= \lim_\Lambda \langle \xi, \pi(E_\lambda)\pi(AE_\lambda)\eta \rangle \\ &= \lim_\Lambda \langle \pi(E_\lambda)\xi, \pi(AE_\lambda)\eta \rangle \\ &= \langle \xi, \tilde{\pi}(A)\eta \rangle. \end{aligned}$$

Hence $\tilde{\pi}(A^*) = \tilde{\pi}(A)^*$ for all $A \in \mathfrak{A}$. Hence $\tilde{\pi}$ is a $^*$-homomorphism.

Suppose $\sigma : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ was an extension of $\pi$. Then

$$\sigma(A)\pi(B)\xi = \sigma(A)\sigma(B)\xi = \sigma(AB)\xi = \pi(AB)\xi$$

for all $A \in \mathfrak{A}$, $B \in \mathcal{I}$, and $\xi \in \mathcal{H}$. Since $\pi(\mathcal{I})\mathcal{H}$ is dense in $\mathcal{H}$ as $\pi$ is non-degenerate, we obtain that $\sigma(A) = \tilde{\pi}(A)$ for all $A \in \mathfrak{A}$. Hence $\tilde{\pi}$ is the unique extension of $\pi$.

Suppose $\mathfrak{A}$ is unital. Then

$$\tilde{\pi}(I_{\mathfrak{A}})\xi = \lim_{\Lambda} \pi(I_{\mathfrak{A}}E_\lambda)\xi = \lim_{\Lambda} \pi(E_\lambda)\xi = \xi$$

for all $\xi \in \mathcal{H}$. Hence $\tilde{\pi}$ is unital if $\mathfrak{A}$ is unital.

To see that $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$, we notice that if $A \in \mathfrak{A}$ and $B \in \mathcal{I}$ then

$$\tilde{\pi}(A)\pi(B) = \tilde{\pi}(A)\tilde{\pi}(B) = \tilde{\pi}(AB) = \pi(AB) \in \pi(\mathcal{I})$$

and

$$\pi(B)\tilde{\pi}(A) = \tilde{\pi}(B)\tilde{\pi}(A) = \tilde{\pi}(BA) = \pi(BA) \in \pi(\mathcal{I}).$$

Hence $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$ by the definition of the idealizer.

Finally, suppose in addition that $\pi$ is faithful. If $A \in \mathcal{I}^\perp$ then $AB = 0$ for all $B \in \mathcal{I}$. Therefore

$$\tilde{\pi}(A)\xi = \lim_{\Lambda} \pi(AE_\lambda)\xi = 0$$

for all $\xi \in \mathcal{H}$. Hence $B \in ker(\tilde{\pi})$ so $ker(\tilde{\pi}) \subseteq \mathcal{I}$. To see the other inclusion, suppose that $\tilde{\pi}(A) = 0$ for some $A \in \mathfrak{A}$. Then for all $B \in \mathcal{I}$ and $\xi \in \mathcal{H}$

$$0 = \tilde{\pi}(A)(\pi(B)\xi) = \pi(AB)\xi.$$

Therefore $\pi(AB) = 0$ for all $B \in \mathcal{I}$. Since $\pi$ is faithful, this implies that $AB = 0$ for all $B \in \mathcal{I}$ and thus $A \in \mathcal{I}^\perp$ by definition. Hence $ker(\tilde{\pi}) = \mathcal{I}^\perp$. Finally Proposition 1.5 implies that $\mathcal{I}$ is an essential ideal if and only if $\mathcal{I}^\perp = \{0\}$ which, by the above proof, occurs if and only if $\tilde{\pi}$ is faithful. □

With the above results in hand, it is simple to construct the multiplier algebra of a C$^*$-algebra

**Theorem 3.6.** *Let $\mathfrak{A}$ be a C$^*$-algebra and let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a faithful, non-degenerate representation of $\mathfrak{A}$ (which exists by the GNS construction and Proposition 3.2). Then $\mathcal{ID}(\pi(\mathfrak{A}))$ is the multiplier algebra of $\mathfrak{A}$ where we view $\mathfrak{A} \subseteq \mathcal{ID}(\pi(\mathfrak{A}))$ via $\pi$.*

*Proof.* By Lemma 3.4 $\mathcal{ID}(\pi(\mathfrak{A}))$ is a C$^*$-algebra that contains $\pi(\mathfrak{A})$ as an essential ideal. Since $\pi$ is faithful $\pi(\mathfrak{A}) \simeq \mathfrak{A}$ so $\mathcal{ID}(\pi(\mathfrak{A}))$ contains $\mathfrak{A}$ as an essential ideal via $\pi$.

Suppose that $\mathfrak{B}$ is another C$^*$-algebra that contains $\mathfrak{A}$ as an essential ideal. By Lemma 3.5 there exists a unique representation $\tilde{\pi} : \mathfrak{B} \to \mathcal{ID}(\pi(\mathfrak{A}))$ that extends $\pi$. Hence $\mathcal{ID}(\pi(\mathfrak{A}))$ is the multiplier algebra of $\mathfrak{A}$ by definition. □

The proof given above of the existence of the multiplier algebra has the same corollaries as shown in Section 2 with different proofs.

**Corollary 3.7.** *Let $\mathfrak{A}$ be a C$^*$-algebra. Then $\mathcal{M}(\mathfrak{A})$ is unital.*

*Proof.* Lemma 3.4 showed that $\mathcal{ID}(\pi(\mathfrak{A}))$ is unital for any faithful, non-degenerate representation $\pi$ of $\mathfrak{A}$. □

**Corollary 3.8.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C$^*$-algebras such that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$. Then there exists a unique $^*$-homomorphism $\pi : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$. If $\mathfrak{B}$ is unital then $\pi$ is unital. Furthermore if $\mathfrak{A}^\perp := \{B \in \mathfrak{B} \mid BA = 0$ for all $A \in \mathfrak{A}\}$ then $ker(\pi) = \mathfrak{A}^\perp$. Finally $\pi$ is injective if and only if $\mathfrak{A}$ is an essential ideal of $\mathfrak{B}$.*

*Proof.* The result follows trivial from Lemma 3.5 (where we did all of the additional work to obtain this result). ☐

**Corollary 3.9.** *If $\mathfrak{A}$ is unital then $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$.*

*Proof.* We shall provide two proofs of this fact. For the first, we note from Corollary 3.7 that $\mathcal{M}(\mathfrak{A})$ is unital. Hence $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$ by Lemma 1.9.

For the second proof, suppose $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a faithful, non-degenerate representation. Since $I_{\mathfrak{A}}$ is a C$^*$-bounded approximate identity of $\mathfrak{A}$, we obtain from Proposition 3.2 that $\pi(I_{\mathfrak{A}})\xi = \xi$ for all $\xi \in \mathcal{H}$ so $I_{\mathcal{H}} \in \pi(\mathfrak{A})$. However, from the definition of the idealizer, it is clear that $I_{\mathcal{H}} \in \pi(\mathfrak{A})$ implies that $\mathcal{ID}(\pi(\mathfrak{A})) \subseteq \mathfrak{A}$. Hence $\mathcal{ID}(\pi(\mathfrak{A})) = \mathfrak{A}$ as desired. ☐

To complete this section we will note one useful result of Theorem 3.6 that Theorem 2.7 did not give and another whose proof is significantly simplified.

**Corollary 3.10.** *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and let $\mathfrak{K}$ be the set of compact operators on $\mathcal{H}$. Then $\mathcal{M}(\mathfrak{K}) = \mathcal{B}(\mathcal{H})$.*

*Proof.* It is clear that the identity representation of $\mathfrak{K}$ is a non-degenerate, faithful representation of $\mathfrak{K}$. Since $\mathfrak{K}$ is an ideal of $\mathcal{B}(\mathcal{H})$, it is clear that $\mathcal{ID}(\mathfrak{K}) = \mathcal{B}(\mathcal{H})$. Hence the result follows from Theorem 3.6. ☐

**Proposition 3.11.** *Let $\{\mathfrak{A}_i\}_{i\in I}$ be a set of C$^*$-algebras. If $\bigoplus_{i\in I} \mathfrak{A}_i$ denotes the direct sum of $\mathfrak{A}_i$ (the closure with respect to the supremum norm of all functions indexed by $I$ with values at $i$ in $\mathfrak{A}_i$ with finite support) then*

$$\mathcal{M}\left(\bigoplus_{i\in I} \mathfrak{A}_i\right) \simeq \prod_{i\in I} \mathcal{M}(\mathfrak{A}_i) \qquad \text{and} \qquad \mathcal{M}\left(\prod_{i\in I} \mathfrak{A}_i\right) \simeq \prod_{i\in I} \mathcal{M}(\mathfrak{A}_i)$$

*where $\prod_{i\in I} \mathcal{M}(\mathfrak{A}_i)$ is the product of $\mathcal{M}(\mathfrak{A}_i)$ (all bounded functions indexed by $I$ with values at $i$ in $\mathcal{M}(\mathfrak{A}_i)$).*

*Proof.* We shall only prove the first result as the second follows verbatim by replacing $\bigoplus_{i\in I} \mathfrak{A}_i$ with $\prod_{i\in I} \mathfrak{A}_i$. For each $i \in I$ let $\pi_i : \mathfrak{A}_i \to \mathcal{B}(\mathcal{H}_i)$ be a faithful, non-degenerate representation. Then $\bigoplus_{i\in I} \pi_i$ is a faithful, non-degenerate representation of $\bigoplus_{i\in I} \mathfrak{A}_i$. Therefore

$$\mathcal{M}\left(\bigoplus_{i\in I} \mathfrak{A}_i\right) = \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right) \subseteq \mathcal{B}\left(\bigoplus_{i\in I} \mathcal{H}_i\right).$$

Consider the map $\Psi : \prod_{i\in I} \mathcal{ID}(\pi_i(\mathfrak{A}_i)) \to \mathcal{B}\left(\bigoplus_{i\in I} \mathcal{H}_i\right)$ defined by

$$\Psi((T_i)_{i\in I}) = \bigoplus_{i\in I} T_i$$

for all $(T_i)_{i\in I} \in \prod_{i\in I} \mathcal{ID}(\pi_i(\mathfrak{A}_i))$. It is clear that $\Psi((T_i)_{i\in I})$ is indeed a bounded linear operator and it is clear that $\Psi$ is injective. To verify that $\Psi((T_i)_{i\in I}) \in \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right)$, we notice for all $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{A}_i$ that $T_i\pi_i(A_i) \in \pi_i(\mathfrak{A}_i)$ and $\pi_i(A_i)T_i \in \pi_i(\mathfrak{A}_i)$ for all $i \in I$. Moreover, since for every $\epsilon > 0$ there are only finitely many $A_i$ with $\|A_i\| > \epsilon$, there are only finitely many $i$ such that $\|T_i\pi_i(A_i)\| > \|(T_i)_{i\in I}\| \epsilon$ and $\|\pi_i(A_i)T_i\| > \|(T_i)_{i\in I}\| \epsilon$. Hence $\left(\bigoplus_{i\in I} T_i\right)\left(\bigoplus_{i\in I} \pi_i(A_i)\right) \in \bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)$ and $\left(\bigoplus_{i\in I} \pi_i(A_i)\right)\left(\bigoplus_{i\in I} T_i\right) \in \bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)$. Hence $\Psi((T_i)_{i\in I}) \in \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right)$ as desired.

To see that $\Psi$ is surjective, let $T \in \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right)$. For each pair $i, j \in I$ let $T_{i,j} : \mathcal{H}_j \to \mathcal{H}_i$ be the bounded operator defined by restricting the domain of $T$ to $\mathcal{H}_j$ and the range of $T$ to $\mathcal{H}_i$. We claim that $T_{i,j} = 0$ unless $i = j$. To see this, for each $k \in I$ consider all operators $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{A}_i$ such that $A_i = 0$ unless $i = k$. Since $T \in \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right)$, we obtain that $T\left(\bigoplus_{i\in I} \pi_i(A_i)\right) \in \bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)$. Thus $T_{i,k}\pi_k(A_k) = 0$ whenever $i \neq k$ and $A_k \in \mathfrak{A}_k$. Since $\pi_k$ is non-degenerate, $\overline{\pi_k(\mathfrak{A}_k)\mathcal{H}_k} = \mathcal{H}_k$ by Proposition 3.2 so the equation $T_{i,k}\pi_k(A_k) = 0$ for all $A_k \in \mathfrak{A}_k$ implies that $T_{i,k} = 0$ whenever $i \neq k$. Thus the claim is complete.

Next we claim that $T_{i,i} \in \mathcal{ID}(\pi_i(\mathfrak{A}_i))$ for all $i \in I$. To see this, for each $k \in I$ consider all operators $(A_i)_{i\in I} \in \bigoplus_{i\in I} \mathfrak{A}_i$ such that $A_i = 0$ unless $i = k$. Since $T \in \mathcal{ID}\left(\bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)\right)$, we obtain that $T\left(\bigoplus_{i\in I} \pi_i(A_i)\right) \in \bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)$ and $\left(\bigoplus_{i\in I} \pi_i(A_i)\right) T \in \bigoplus_{i\in I} \pi_i(\mathfrak{A}_i)$. These equations clearly imply that $T_{k,k}\pi_k(A_k) \in \pi_k(\mathfrak{A}_k)$ and $\pi_k(A_k)T_{k,k} \in \pi_k(\mathfrak{A}_k)$ for all $A_k \in \mathfrak{A}_k$. Hence $T_{k,k} \in \mathcal{ID}(\pi_k(\mathfrak{A}_k))$ for all $k \in I$ as desired. Since $\|T_{i,i}\| \leq \|T\|$ for all $i \in I$, it is clear that $(T_i)_{i\in I} \in \prod_{i\in I} \mathcal{ID}(\pi_i(\mathfrak{A}_i))$ and $\Psi((T_i)_{i\in I}) = T$. Hence $\Psi$ is surjective. $\square$

# 4 Construction of Multiplier Algebra by Bimodules

In this section we will demonstrate the existence of the multiplier algebra of an arbitrary C$^*$-algebra through the existence of faithful, non-degenerate representations on right Hilbert C$^*$-modules. Although the representation theory and C$^*$-theory used in Section 3 is simpler for the average reader, the results of this section follow along the same lines and the realization of the multiplier algebra in this section has many theoretical applications. In particular, this section is a generalization of Section 3 (that is, the results of this section include those of Section 3) and draws an important connection between multiplier algebras and right Hilbert C$^*$-modules. Those that do not desire to discuss the representation theory of C$^*$-algebras on right Hilbert C$^*$-modules may simply read the examples, read the proof of Theorem 4.24, and then derive the main additional results of this section from Examples 4.18, 4.19, and 4.20. A reader that is unfamiliar with right Hilbert C$^*$-modules is referred to http://www.math.ucla.edu/∼pskoufra/OANotes-HilbertC-Bimodules.pdf.

The results of this section will follow those of Section 3 by replacing Hilbert spaces with right Hilbert C$^*$-modules. Thus we begin with the analogue of a non-degenerate representation.

**Definition 4.1.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be C$^*$-algebras. A representation $\pi : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ is said to be non-degenerate if $\pi(\mathfrak{A})\mathcal{H}_\mathfrak{B} := \{\pi(A)\xi \mid A \in \mathfrak{A}, \xi \in \mathcal{H}_\mathfrak{B}\}$ is dense in $\mathcal{H}_\mathfrak{B}$.

If $\mathfrak{A}$ is a C$^*$-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$, we say that $\mathfrak{A}$ is non-degenerate if and only if the identity representation is non-degenerate.

The theory of representations of C$^*$-algebras on right Hilbert C$^*$-modules is more complex than the theory of representations on Hilbert spaces. We do have the following analogue of Proposition 3.2.

**Proposition 4.2.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C$^*$-algebras, let $\pi : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ be a representation of $\mathfrak{A}$, and let $\mathcal{K} := \overline{\pi(\mathfrak{A})\mathcal{H}_\mathfrak{B}}$. Then $\mathcal{K}$ is a closed subspace of $\mathcal{H}_\mathfrak{B}$.*

*Furthermore $(E_\lambda)_\Lambda$ is a C$^*$-bounded approximate identity of $\mathfrak{A}$ then $\lim_\Lambda \pi(E_\lambda)\xi = \xi$ for all $\xi \in \mathcal{K}$. In particular, if $\mathcal{K} = \mathcal{H}_\mathfrak{B}$ then $(\pi(E_\lambda))_\Lambda$ converges to $I_{\mathcal{H}_\mathfrak{B}}$ in the SOT.*

*Proof.* First we prove the second claim. Let $\xi \in \mathcal{K} = \overline{\pi(\mathfrak{A})\mathcal{H}_\mathfrak{B}}$ and let $(E_\lambda)_\Lambda$ be a C$^*$-bounded approximate identity of $\mathfrak{A}$. By the definition of $\pi(\mathfrak{A})\mathcal{H}_\mathfrak{B}$ there exists $A_n \in \mathfrak{A}$ and $\eta_n \in \mathcal{H}_\mathfrak{B}$ such that $\xi = \lim_n \pi(A_n)\eta_n$. Let $\epsilon > 0$. Since $\xi = \lim_n \pi(A_n)\eta_n$, there exists an $N \in \mathbb{N}$ such that $\|\xi - \pi(A_N)\eta_N\| \leq \frac{\epsilon}{3}$. Moreover, since $(E_\lambda)_\Lambda$ is a bounded approximate identity for $\mathfrak{A}$, there exists a $\lambda' \in \Lambda$ such that for all $\lambda \geq \lambda'$, $\|E_\lambda A_N - A_N\| \leq \frac{\epsilon}{3(\|\eta_N\|+1)}$. Hence, since $\pi$ is a contraction, $\|\pi(E_\lambda A_N) - \pi(A_N)\| \leq \frac{\epsilon}{3(\|\eta_N\|+1)}$. Hence for all $\lambda \geq \lambda'$

$$\begin{aligned}
\|\xi - \pi(E_\lambda)\xi\| &\leq \|\xi - \pi(A_N)\eta_N\| + \|\pi(A_N)\eta_N - \pi(E_\lambda A_N)\eta_N\| + \|\pi(E_\lambda A_N)\eta_N - \pi(E_\lambda)\xi\| \\
&\leq \frac{\epsilon}{3} + \|\pi(A_N) - \pi(E_\lambda A_N)\| \, \|\eta_N\| + \|\pi(E_\lambda)\| \, \|\pi(A_N)\eta_N - \xi\| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3(\|\eta_N\| + 1)} \|h_N\| + 1\frac{\epsilon}{3} \leq \epsilon
\end{aligned}$$

Hence $\lim_\Lambda \pi(E_\lambda)\xi = \xi$ as claimed.

To see that $\mathcal{K}$ is a closed subspace, it suffices to verify that $\mathcal{K}$ is a linear subspace of $\mathcal{H}_\mathfrak{B}$. If $\xi_1, \xi_2 \in \mathcal{K}$ and $\alpha \in \mathbb{C}$ then $\lim_\Lambda \pi(E_\lambda)(\alpha\xi_1 + \xi_2) = \alpha\xi_1 + \xi_2$ so that $\alpha\xi_1 + \xi_2 \in \mathcal{K}$ by the definition of $\mathcal{K}$. Hence $\mathcal{K}$ is a closed subspace. □

Although the GNS representations along with Proposition 3.2 provide a plethora of non-degenerate representations, the most important non-degenerate representations of a C$^*$-algebra on a right Hilbert C$^*$-module for this section are the following.

**Example 4.3.** Let $\mathfrak{A}$ be a C$^*$-algebra. Recall from Remarks 2.11 that $\mathfrak{A}$ can be viewed as a right Hilbert $\mathfrak{A}$-module. Define $\pi : \mathfrak{A} \to \mathcal{B}_a(\mathfrak{A})$ by $\pi(A)A' = AA'$ for all $A, A' \in \mathfrak{A}$. It is trivial to verify that $\pi$ is a well-defined map that maps into $\mathcal{B}_a(\mathfrak{A})$ and is a $^*$-homomorphism. This C$^*$-valued inner product, this right action, and this adjointable action gives $\mathfrak{A}$ a canonical Hilbert $\mathfrak{A}$-$\mathfrak{A}$-bimodule structure.

We claim that $\pi$ is faithful. Indeed if $A \in \mathfrak{A}$ is such that $\pi(A)B = 0$ for all $B \in \mathfrak{A}$ then $AA^* = 0$ so $A = 0$. Hence $\pi$ is faithful.

We claim that $\pi$ is a non-degenerate representation. To see this, we notice if $A \in \mathfrak{A}$ is positive then $A = \pi(A^{\frac{1}{2}})A^{\frac{1}{2}} \in \overline{\pi(\mathfrak{A})\mathfrak{A}}$. Since $\overline{\pi(\mathfrak{A})\mathfrak{A}}$ contains every positive elements of $\mathfrak{A}$, the span of the positive elements of $\mathfrak{A}$ is $\mathfrak{A}$, and $\overline{\pi(\mathfrak{A})\mathfrak{A}}$ is a subspace of $\mathfrak{A}$ by Proposition 4.2, $\overline{\pi(\mathfrak{A})\mathfrak{A}} = \mathfrak{A}$. Hence $\pi$ is a non-degenerate representation.

There are many more canonical examples of non-degenerate representations. We shall postpone these examples until we have demonstrated the relation between the multiplier algebra of a C*-algebra and non-degenerate representations of C*-algebras on right Hilbert C*-modules.

Following the ideas of Section 3, the idealizer of a faithful, non-degenerate representation is the correct construct for the multiplier algebra.

**Definition 4.4.** Let $\mathcal{H}_\mathfrak{B}$ be a right Hilbert $\mathfrak{B}$-module and let $\mathcal{A} \subseteq \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$. The idealizer of $\mathcal{A}$, denoted $\mathcal{ID}(\mathcal{A})$, is the set

$$\mathcal{ID}(\mathcal{A}) := \{T \in \mathcal{B}_a(\mathcal{H}_\mathfrak{B}) \mid T\mathcal{A} \subseteq \mathcal{A}, \mathcal{A}T \subseteq \mathcal{A}\}$$

(where $T\mathcal{A} := \{S \in \mathcal{B}_a(\mathcal{H}_\mathfrak{B}) \mid S = TA \text{ for some } A \in \mathcal{A}\}$).

The following is the generalization of Lemma 3.4.

**Lemma 4.5.** *Let $\mathfrak{B}$ be a C*-algebra and let $\mathfrak{A}$ be a C*-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$. Then $\mathcal{ID}(\mathfrak{A})$ is a unital C*-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ that contains $\mathfrak{A}$ as an ideal. If $\mathfrak{A}$ is non-degenerate then $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$.*

*Proof.* Since $\mathfrak{A}$ is closed under addition, scalar multiplication, multiplication, and adjoints, it is clear that $\mathcal{ID}(\mathfrak{A})$ is a $^*$-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$. Moreover, since $\mathfrak{A}$ is closed, it is trivial to verify that $\mathcal{ID}(\mathfrak{A})$ is closed. Hence $\mathcal{ID}(\mathfrak{A})$ is a C*-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$. Furthermore it is clear by the definition of the idealizer that $\mathfrak{A}$ is an ideal in $\mathcal{ID}(\mathfrak{A})$ and $I_{\mathcal{H}_\mathfrak{B}} \in \mathcal{ID}(\mathfrak{A})$.

Suppose $\mathfrak{A}$ is non-degenerate. To show that $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$ suppose that $T \in \mathcal{ID}(\mathfrak{A})$ is such that $TA = 0$ for all $A \in \mathfrak{A}$. Therefore $TA\xi = 0$ for all $A \in \mathfrak{A}$ and $\xi \in \mathcal{H}_\mathfrak{B}$. Since $\mathfrak{A}$ is a non-degenerate subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$, the set $\{A\xi \mid A \in \mathfrak{A}, \xi \in \mathcal{H}_\mathfrak{B}\}$ is dense in $\mathcal{H}_\mathfrak{B}$. Hence $T\eta = 0$ for all $\eta \in \mathcal{H}_\mathfrak{B}$ so $T = 0$. Therefore Proposition 1.5 implies that $\mathfrak{A}$ is an essential ideal of $\mathcal{ID}(\mathfrak{A})$. ☐

As in Section 3 it is necessary to be able to extend representations of ideals of C*-algebras on right Hilbert C*-modules to the entire C*-algebra in 'nice' ways.

**Lemma 4.6.** *Let $\mathfrak{B}$ be a C*-algebra, let $\mathcal{I}$ be an ideal of a C*-algebra $\mathfrak{A}$, and let $\pi : \mathcal{I} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ be a non-degenerate representation. Then there exists a unique $^*$-homomorphism $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ extending $\pi$ (that must then be non-degenerate). If $\mathfrak{A}$ is unital then $\tilde{\pi}$ is unital. Furthermore $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$. If in addition $\pi$ is faithful and $\mathcal{I}^\perp := \{A \in \mathfrak{A} \mid AB = 0 \text{ for all } B \in \mathcal{I}\}$, then $ker(\tilde{\pi}) = \mathcal{I}^\perp$. Therefore if $\pi$ is faithful the $\mathcal{I}$ is an essential ideal of $\mathfrak{A}$ if and only if $\tilde{\pi}$ is faithful.*

*Proof.* Let $\mathcal{I}$ be an ideal of a C*-algebra $\mathfrak{A}$ and let $\pi : \mathcal{I} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ be a non-degenerate representation. Fix a C*-bounded approximate identity $(E_\lambda)_\Lambda$ for $\mathcal{I}$. For each $A \in \mathfrak{A}$, we claim that the operator $\tilde{\pi}_0(A)$ defined on $\pi(\mathcal{I})\mathcal{H}_\mathfrak{B}$ by

$$\tilde{\pi}_0(A)(\pi(B)\xi) = \pi(AB)\xi$$

for all $B \in \mathcal{I}$ and $\xi \in \mathcal{H}_\mathfrak{B}$ is well-defined. To see this, we notice that $AB \in \mathcal{I}$ for all $B \in \mathcal{I}$ as $\mathcal{I}$ is an ideal so $\pi(AB)$ makes sense. To see that $\tilde{\pi}_0(A)$ is well-defined, suppose $B_1, B_2 \in \mathcal{I}$ and $\xi_1, \xi_2 \in \mathcal{H}_\mathfrak{B}$ are such that $\pi(B_1)\xi_1 = \pi(B_2)\xi_2$. Then

$$\begin{aligned} \pi(AB_1)\xi_1 &= \lim_\Lambda \pi(AE_\lambda B_1)\xi_1 \\ &= \lim_\Lambda \pi(AE_\lambda)\pi(B_1)\xi_1 \\ &= \lim_\Lambda \pi(AE_\lambda)\pi(B_2)\xi_2 \\ &= \lim_\Lambda \pi(AE_\lambda B_2)\xi_2 \\ &= \pi(AB_2)\xi_2. \end{aligned}$$

Hence $\tilde{\pi}_0(A)$ is well-defined. Moreover we notice that

$$\begin{aligned}
\|\tilde{\pi}_0(A)(\pi(B_1)\xi) - \tilde{\pi}_0(A)(\pi(B_2)\xi)\| &= \|\pi(AB_1 - AB_2)\xi\| \\
&= \lim_\Lambda \|\pi(AE_\lambda(B_1 - B_2))\xi\| \\
&= \lim_\Lambda \|\pi(AE_\lambda)\pi(B_1 - B_2)\xi\| \\
&\leq \lim_\Lambda \|\pi(AE_\lambda)\| \, \|\pi(B_1 - B_2)\xi\| \\
&\leq \limsup_\Lambda \|AE_\lambda\| \, \|\pi(B_1 - B_2)\xi\| \\
&\leq \|A\| \, \|\pi(B_1 - B_2)\xi\| \, .
\end{aligned}$$

Therefore, since $\lim_\Lambda \pi(E_\lambda)\xi = \xi$ for all $\xi \in \mathcal{H}_{\mathfrak{B}}$ by Proposition 4.2, the above inequalities implies that $(\tilde{\pi}_0(A)\pi(E_\lambda)\xi)_\Lambda$ is a Cauchy sequence. Hence we define $\tilde{\pi}(A)$ to be the function on $\mathcal{H}_{\mathfrak{B}}$ defined by

$$\tilde{\pi}(A)\xi := \lim_\Lambda \tilde{\pi}_0(A)\pi(E_\lambda)\xi = \lim_\Lambda \pi(AE_\lambda)\xi$$

for all $\xi \in \mathcal{H}_{\mathfrak{B}}$. It is then clear that $\tilde{\pi}(A)$ is a linear map and is bounded since $\|\pi(AE_\lambda)\xi\| \leq \|\pi(AE_\lambda)\| \, \|\xi\| \leq \|A\| \, \|\xi\|$ for all $\xi \in \mathcal{H}_{\mathfrak{B}}$ and $\lambda \in \Lambda$. Hence $\tilde{\pi}(A) \in \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$. Furthermore it is clear that $A \mapsto \tilde{\pi}(A)$ is linear as $\pi$ is linear. Hence $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_{\mathfrak{B}})$ is a linear map. In addition if $A \in \mathcal{I}$ then

$$\tilde{\pi}(A)\xi = \lim_\Lambda \pi(AE_\lambda)\xi = \pi(A)\xi$$

for all $\xi \in \mathcal{H}_{\mathfrak{B}}$ so $\tilde{\pi}$ extends $\pi$.

To show that $\tilde{\pi}$ is multiplicative and preserves adjoints we first notice that if $A \in \mathfrak{A}$, $B \in \mathcal{I}$, and $\xi \in \mathcal{H}_{\mathfrak{B}}$ then

$$\tilde{\pi}(A)(\pi(B)\xi) = \lim_\Lambda \pi(AE_\lambda)\pi(B)\xi = \lim_\Lambda \pi(AE_\lambda B)\xi = \pi(AB)\xi.$$

Thus $\tilde{\pi}(A)$ extending $\tilde{\pi}_0(A)$ for all $A \in \mathfrak{A}$. Therefore, if $A, B \in \mathfrak{A}$

$$\tilde{\pi}(AB)\xi = \lim_\Lambda \pi(ABE_\lambda)\xi = \lim_\Lambda \tilde{\pi}(A)(\pi(BE_\lambda)\xi) = \tilde{\pi}(A)\tilde{\pi}(B)\xi$$

for all $\xi \in \mathcal{H}_{\mathfrak{B}}$. Hence $\tilde{\pi}$ is multiplicative.

To see that $\tilde{\pi}(A) \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ for all $A \in \mathfrak{A}$ and that $\tilde{\pi}$ preserves adjoints we notice for all $A \in \mathfrak{A}$ and $\xi, \eta \in \mathcal{H}_{\mathfrak{B}}$ that

$$\begin{aligned}
\langle \tilde{\pi}(A^*)\xi, \eta\rangle &= \lim_\Lambda \langle \pi(A^*E_\lambda)\xi, \pi(E_\lambda)\eta\rangle \\
&= \lim_\Lambda \langle \xi, \pi(E_\lambda A)\pi(E_\lambda)\eta\rangle \\
&= \lim_\Lambda \langle \xi, \pi(E_\lambda AE_\lambda)\eta\rangle \\
&= \lim_\Lambda \langle \xi, \pi(E_\lambda)\pi(AE_\lambda)\eta\rangle \\
&= \lim_\Lambda \langle \pi(E_\lambda)\xi, \pi(AE_\lambda)\eta\rangle \\
&= \langle \xi, \tilde{\pi}(A)\eta\rangle.
\end{aligned}$$

Hence $\tilde{\pi}(A^*) = \tilde{\pi}(A)^*$ for all $A \in \mathfrak{A}$. Hence $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ is a $^*$-homomorphism.

Suppose $\sigma : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ was an extension of $\pi$. Then

$$\sigma(A)\pi(B)\xi = \sigma(A)\sigma(B)\xi = \sigma(AB)\xi = \pi(AB)\xi$$

for all $A \in \mathfrak{A}$, $B \in \mathcal{I}$, and $\xi \in \mathcal{H}_{\mathfrak{B}}$. Since $\pi(\mathcal{I})\mathcal{H}_{\mathfrak{B}}$ is dense in $\mathcal{H}_{\mathfrak{B}}$ as $\pi$ is non-degenerate, we obtain that $\sigma(A) = \tilde{\pi}(A)$ for all $A \in \mathfrak{A}$. Hence $\tilde{\pi}$ is the unique extension of $\pi$.

Suppose $\mathfrak{A}$ is unital. Then

$$\tilde{\pi}(I_{\mathfrak{A}})\xi = \lim_\Lambda \pi(I_{\mathfrak{A}}E_\lambda)\xi = \lim_\Lambda \pi(E_\lambda)\xi = \xi$$

for all $\xi \in \mathcal{H}_{\mathfrak{B}}$. Hence $\tilde{\pi}$ is unital if $\mathfrak{A}$ is unital.

To see that $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$, we notice that if $A \in \mathfrak{A}$ and $B \in \mathcal{I}$ then

$$\tilde{\pi}(A)\pi(B) = \tilde{\pi}(A)\tilde{\pi}(B) = \tilde{\pi}(AB) = \pi(AB) \in \pi(\mathcal{I})$$

and

$$\pi(B)\tilde{\pi}(A) = \tilde{\pi}(B)\tilde{\pi}(A) = \tilde{\pi}(BA) = \pi(BA) \in \pi(\mathcal{I}).$$

Hence $\tilde{\pi}(\mathfrak{A}) \subseteq \mathcal{ID}(\pi(\mathcal{I}))$ by the definition of the idealizer.

Finally, suppose in addition that $\pi$ is faithful. If $A \in \mathcal{I}^\perp$ then $AB = 0$ for all $B \in \mathcal{I}$. Therefore

$$\tilde{\pi}(A)\xi = \lim_{\Lambda} \pi(AE_\lambda)\xi = 0$$

for all $\xi \in \mathcal{H}_\mathfrak{B}$. Hence $B \in ker(\tilde{\pi})$ so $ker(\tilde{\pi}) \subseteq \mathcal{I}$. To see the other inclusion, suppose that $\tilde{\pi}(A) = 0$ for some $A \in \mathfrak{A}$. Then for all $B \in \mathcal{I}$ and $\xi \in \mathcal{H}_\mathfrak{B}$

$$0 = \tilde{\pi}(A)(\pi(B)\xi) = \pi(AB)\xi.$$

Therefore $\pi(AB) = 0$ for all $B \in \mathcal{I}$. Since $\pi$ is faithful, this implies that $AB = 0$ for all $B \in \mathcal{I}$ and thus $A \in \mathcal{I}^\perp$ by definition. Hence $ker(\tilde{\pi}) = \mathcal{I}^\perp$. Furthermore $\mathcal{I}$ is an essential ideal if and onyl if $\mathcal{I}^\perp = \{0\}$ by Proposition 1.5 which is equivalent to $\tilde{\pi}$ being faithful by the above proof. □

With the above results in hand it is simple to construct the multiplier algebra of a C*-algebra as we did in Section 3.

**Theorem 4.7.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and let $\pi : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ be a faithful, non-degenerate representation of $\mathfrak{A}$ (which exists by Example 4.3). Then $\mathcal{ID}(\pi(\mathfrak{A}))$ is the multiplier algebra of $\mathfrak{A}$ where we view $\mathfrak{A} \subseteq \mathcal{ID}(\pi(\mathfrak{A}))$ via $\pi$.*

*Proof.* By Lemma 4.5 $\mathcal{ID}(\pi(\mathfrak{A}))$ is a C*-algebra that contains $\pi(\mathfrak{A})$ as an essential ideal. Since $\pi$ is faithful $\pi(\mathfrak{A}) \simeq \mathfrak{A}$ so $\mathcal{ID}(\pi(\mathfrak{A}))$ contains $\mathfrak{A}$ as an essential ideal via $\pi$.

Suppose that $\mathfrak{C}$ is another C*-algebra that contains $\mathfrak{A}$ as an essential ideal. By Lemma 4.6 there exists a unique representation $\tilde{\pi} : \mathfrak{C} \to \mathcal{ID}(\pi(\mathfrak{A}))$ that extends $\pi$. Hence $\mathcal{ID}(\pi(\mathfrak{A}))$ is the multiplier algebra of $\mathfrak{A}$ by definition. □

The proof given above of the existence of the multiplier algebra has the same corollaries as shown in Section 3 with identical proofs.

**Corollary 4.8.** *Let $\mathfrak{A}$ be a C*-algebra. Then $\mathcal{M}(\mathfrak{A})$ is unital.*

*Proof.* Lemma 4.5 showed that $\mathcal{ID}(\pi(\mathfrak{A}))$ is unital for any faithful, non-degenerate representation $\pi$ of $\mathfrak{A}$. □

**Corollary 4.9.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras such that $\mathfrak{A}$ is an ideal in $\mathfrak{B}$. Then there exists a unique *-homomorphism $\pi : \mathfrak{B} \to \mathcal{M}(\mathfrak{A})$ that is the identity on $\mathfrak{A}$. If $\mathfrak{B}$ is unital then $\pi$ is unital. Furthermore if $\mathfrak{A}^\perp := \{B \in \mathfrak{B} \mid BA = 0 \text{ for all } A \in \mathfrak{A}\}$ then $ker(\pi) = \mathfrak{A}^\perp$. Finally $\pi$ is injective if and only if $\mathfrak{A}$ is an essential ideal of $\mathfrak{B}$.*

*Proof.* The result follows trivial from Lemma 4.6 (where we did all of the additional work to obtain this result). □

**Corollary 4.10.** *If $\mathfrak{A}$ is unital then $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$.*

*Proof.* We shall provide two proofs of this fact. For the first, we note from Corollary 4.8 that $\mathcal{M}(\mathfrak{A})$ is unital. Hence $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$ by Lemma 1.9.

For the second proof, suppose $\pi : \mathfrak{A} \to \mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ is a faithful, non-degenerate representation on a right Hilbert $\mathfrak{B}$-module. Since $I_\mathfrak{A}$ is a C*-bounded approximate identity of $\mathfrak{A}$, we obtain from Proposition 4.2 that $\pi(I_\mathfrak{A})\xi = \xi$ for all $\xi \in \mathcal{H}_\mathfrak{B}$ so $I_{\mathcal{H}_\mathfrak{B}} \in \pi(\mathfrak{A})$. However, from the definition of the idealizer, it is clear that $I_{\mathcal{H}_\mathfrak{B}} \in \pi(\mathfrak{A})$ implies that $\mathcal{ID}(\pi(\mathfrak{A})) \subseteq \mathfrak{A}$. Hence $\mathcal{ID}(\pi(\mathfrak{A})) = \mathfrak{A}$ as desired. □

The benefit of Theorem 4.7 is that the multiplier algebra of a C*-algebra can be realized as 'nice' operators on a right Hilbert C*-module. Using these ideas, we obtain a third proof of the above result and an alternate proof of Corollary 2.12.

**Corollary 4.11.** *Let $\mathfrak{A}$ be a C*-algebra and equip $\mathfrak{A}$ with the Hilbert $\mathfrak{A}$-$\mathfrak{A}$-bimodule structure from Example 4.3. Then $\mathcal{B}_a(\mathfrak{A}) = \mathcal{M}(\mathfrak{A})$.*

*Proof.* By Theorem 4.7 it suffices to show that $\mathfrak{B}_a(\mathfrak{A}) = \mathcal{ID}(\pi(\mathfrak{A}))$. Clearly $\mathcal{ID}(\pi(\mathfrak{A})) \subseteq \mathfrak{B}_a(\mathfrak{A})$. Let $T \in \mathfrak{B}_a(\mathfrak{A})$ be arbitrary and fix $A \in \mathfrak{A}$. Then $T(A) \in \mathfrak{A}$. Furthermore for all $B \in \mathfrak{A}$

$$(T\pi(A))B = T(AB) = T(\rho(B)A) = \rho(B)(T(A)) = T(A)B = \pi(T(A))B$$

as adjointable maps commute with the right action. Hence $T\pi(A) = \pi(T(A)) \in \pi(\mathfrak{A})$ for all $T \in \mathfrak{B}_a(\mathfrak{A})$ and $A \in \mathfrak{A}$. Furthermore

$$\pi(A)T = (T^*\pi(A^*))^* = (\pi(T^*(A^*)))^* = \pi((T^*(A^*))^*) \in \pi(\mathfrak{A})$$

for all $T \in \mathfrak{B}_a(\mathfrak{A})$ and $A \in \mathfrak{A}$. Hence $T \in \mathcal{ID}(\pi(\mathfrak{A}))$ so $\mathfrak{B}_a(\mathfrak{A}) = \mathcal{ID}(\pi(\mathfrak{A}))$ as desired. □

Thus Example 4.3 provides a concrete realization of the multiplier algebra of a C*-algebra. Through other additional canonical right Hilbert C*-modules, the multiplier algebras of other C*-algebras (such as matrix algebras of a C*-algebra) may be realized.

**Example 4.12.** Let $\mathfrak{A}$ be a C*-algebra and let $n \in \mathbb{N}$. Then $\mathfrak{A}^n := \{(A_1, \ldots, A_n) \mid A_j \in \mathfrak{A}\}$ can be viewed as a right Hilbert $\mathfrak{A}$-module with inner product $\langle (A_1, \ldots, A_n), (B_1, \ldots, B_n)\rangle = \sum_{j=1}^n A_j^* B_j$ for all $A_j, B_j \in \mathfrak{A}$ and right action $\rho : \mathfrak{A} \to \mathcal{B}(\mathfrak{A}^n)$ by $\rho(A)(A_1, \ldots, A_n) = (A_1 A, \ldots, A_n A)$ for all $A, A_j \in \mathfrak{A}$. Define $\pi : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{B}_a(\mathfrak{A}^n)$ by

$$\pi([A_{i,j}])(A_1, \ldots, A_n) = \left(\sum_{j=1}^n A_{1,j}A_j, \ldots, \sum_{j=1}^n A_{n,j}A_j\right)$$

for all $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ and $(A_1, \ldots, A_n) \in \mathfrak{A}^n$. It is trivial to verify that $\pi$ is a well-defined map that maps into $\mathcal{B}_a(\mathfrak{A}^n)$ and is a *-homomorphism. This C*-valued inner product, this right action, and this adjointable action gives $\mathfrak{A}^n$ a canonical Hilbert $\mathcal{M}_n(\mathfrak{A})$-$\mathfrak{A}$-bimodule structure.

We claim that $\pi$ is faithful. Indeed suppose $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$ is such that $\pi([A_{i,j}]) = 0$. Fix $k \in \{1, \ldots, n\}$ and consider all elements $(A_1, \ldots, A_n) \in \mathfrak{A}^n$ such that $A_j = 0$ unless $j = k$. The above definition of $\pi$ implies that $A_{i,k}A_k = 0$ for all $A_k \in \mathfrak{A}$ and $i \in \{1, \ldots, n\}$ and thus $A_{i,k} = 0$ for all $i \in \{1, \ldots, n\}$. Thus $[A_{i,j}] = 0$ as $k$ was arbitrary. Thus $\pi$ is faithful.

We claim that $\pi$ is a non-degenerate representation. To see this, we notice if $(A_1, \ldots, A_n) \in \mathfrak{A}^n$ is such that $A_j \geq 0$ for all $j \in \{1, \ldots, n\}$ then

$$(A_1, \ldots, A_n) = \pi(diag(A_1^{\frac{1}{2}}, \ldots, A_n^{\frac{1}{2}}))(A_1^{\frac{1}{2}}, \ldots, A_n^{\frac{1}{2}}) \in \overline{\pi(\mathcal{M}_n(\mathfrak{A}))\mathfrak{A}^n}.$$

Since $\overline{\pi(\mathcal{M}_n(\mathfrak{A}))\mathfrak{A}^n}$ contains $n$-tuple with a positive element in each entry, the span of the positive elements of $\mathfrak{A}$ is $\mathfrak{A}$, and $\overline{\pi(\mathcal{M}_n(\mathfrak{A}))\mathfrak{A}^n}$ is a subspace of $\mathfrak{A}^n$ by Proposition 4.2, $\overline{\pi(\mathcal{M}_n(\mathfrak{A}))\mathfrak{A}^n} = \mathfrak{A}^n$. Hence $\pi$ is a non-degenerate representation.

**Corollary 4.13.** *Let $\mathfrak{A}$ be a C*-algebra, let $n \in \mathbb{N}$, and equip $\mathfrak{A}^n$ with the Hilbert $\mathcal{M}_n(\mathfrak{A})$-$\mathfrak{A}$-bimodule structure from Example 4.12. Then $\mathcal{B}_a(\mathfrak{A}^n) = \mathcal{M}(\mathcal{M}_n(\mathfrak{A}))$.*

*Proof.* By Theorem 4.7 it suffices to show that $\mathfrak{B}_a(\mathfrak{A}^n) = \mathcal{ID}(\pi(\mathcal{M}_n(\mathfrak{A})))$. Clearly $\mathcal{ID}(\pi(\mathcal{M}_n(\mathfrak{A}))) \subseteq \mathfrak{B}_a(\mathfrak{A}^n)$. Let $T \in \mathfrak{B}_a(\mathfrak{A}^n)$ be arbitrary and fix $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$. Let $(C_{1,j}, \ldots, C_{n,j}) := T(A_{1,j}, \ldots, A_{n,j}) \in \mathfrak{A}^n$. Then for all $(B_1, \ldots, B_n) \in \mathfrak{A}^n$

$$\begin{aligned}
(T\pi([A_{i,j}]))(B_1, \ldots, B_n) &= T\left(\textstyle\sum_{j=1}^n A_{1,j}B_j, \ldots, \sum_{j=1}^n A_{n,j}B_j\right) \\
&= \textstyle\sum_{j=1}^n T\left(A_{1,j}B_j, \ldots, A_{n,j}B_j\right) \\
&= \textstyle\sum_{j=1}^n T\left(\rho(B_j)\left(A_{1,j}, \ldots, A_{n,j}\right)\right) \\
&= \textstyle\sum_{j=1}^n \rho(B_j)T\left(A_{1,j}, \ldots, A_{n,j}\right) \\
&= \textstyle\sum_{j=1}^n \rho(B_j)(C_{1,j}, \ldots, C_{n,j}) \\
&= \textstyle\sum_{j=1}^n (C_{1,j}B_j, \ldots, C_{n,j}B_j) \\
&= \pi([C_{i,j}])(B_1, \ldots, B_n)
\end{aligned}$$

(as adjointable maps commute with the right action). Hence $T\pi([A_{i,j}]) = \pi([C_{i,j}]) \in \pi(\mathcal{M}_n(\mathfrak{A}))$ for all $T \in \mathfrak{B}_a(\mathfrak{A}^n)$ and $[A_{i,j}] \in \mathfrak{A}^n$. Furthermore

$$\pi([A_{i,j}])T = (T^*\pi([A_{i,j}]^*))^* = (\pi(T^*([A_{i,j}]^*)))^* = \pi((T^*([A_{i,j}]^*))^*) \in \pi(\mathcal{M}_n(\mathfrak{A}))$$

for all $T \in \mathfrak{B}_a(\mathfrak{A}^n)$ and $[A_{i,j}] \in \mathcal{M}_n(\mathfrak{A})$. Hence $T \in \mathcal{ID}(\pi(\mathcal{M}_n(\mathfrak{A})))$ so $\mathfrak{B}_a(\mathfrak{A}^n) = \mathcal{ID}(\pi(\mathcal{M}_n(\mathfrak{A})))$ as desired. □

It is not difficult to extend the above to infinite tuples.

**Example 4.14.** Let $\mathfrak{A}$ be a C*-algebra. Then

$$\mathfrak{A}^\infty := \left\{ (A_n)_{n\geq 1} \mid A_n \in \mathfrak{A}, \sum_{n\geq 1} A_n^* A_n \text{ converges in } \mathfrak{A} \right\}$$

can be viewed as a right Hilbert $\mathfrak{A}$-module with inner product $\langle (A_n)_{n\geq 1}, (B_n)_{n\geq 1}\rangle = \sum_{n\geq 1} A_n^* B_n$ for all $(A_n)_{n\geq 1}, (B_n)_{n\geq 1} \in \mathfrak{A}^\infty$ and right action $\rho : \mathfrak{A} \to \mathcal{B}(\mathfrak{A}^\infty)$ by $\rho(A)(A_n)_{n\geq 1} = (A_n A)_{n\geq 1}$ for all $A \in \mathfrak{A}$ and $(A_n)_{n\geq 1} \in \mathfrak{A}^\infty$. Notice for each $k \in \mathbb{N}$ that $\mathfrak{A}^k$ (as in Example 4.12) can be embedded as a right Hilbert $\mathfrak{A}$-module into $\mathfrak{A}^\infty$ by embedding into the first $k$-coordinates. Notice that $\bigcup_{k\geq 1} \mathfrak{A}^k$ is then dense in $\mathfrak{A}^\infty$.

Notice for each $k \in \mathbb{N}$ the $\mathcal{M}_k(\mathfrak{A})$-left-action on $\mathfrak{A}^k$ from Example 4.12 can be extended to a left-$\mathcal{M}_k(\mathfrak{A})$-action on $\mathfrak{A}^\infty$ by letting $\mathcal{M}_k(\mathfrak{A})$ act on the first $k$-coordinates as it does on $\mathfrak{A}^k$ and letting it act as zero on the other coordinates. Furthermore, these actions are contractive and are preserved under the canonical embeddings of $\mathcal{M}_k(\mathfrak{A})$ into $\mathcal{M}_{k+1}(\mathfrak{A})$. Therefore, if $\mathfrak{K}$ is the C*-algebra of compact operators on a separable, infinite dimensional Hilbert space and $\mathfrak{A} \otimes_{\min} \mathfrak{K}$ is the inductive limit of $\mathcal{M}_k(\mathfrak{A})$ then there exists a *-homomorphism $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{K} \to \mathcal{B}_a(\mathfrak{A}^\infty)$ obtained by extending the $\mathcal{M}_k(\mathfrak{A})$-left-action on $\mathfrak{A}^k$. Since $\bigcup_{k\geq 1} \mathfrak{A}^k$ is then dense in $\mathfrak{A}^\infty$ and the $\mathcal{M}_k(\mathfrak{A})$-left-action on $\mathfrak{A}^k$ is non-degenerate, it follows that $\pi$ is non-degenerate.

To see that $\pi$ is faithful, we note that if $(E_\lambda)_\Lambda$ and $(P_n)_{n\geq 1}$ are C*-bounded approximate identities of $\mathfrak{A}$ and $\mathfrak{K}$ respectively (where $P_n$ is the rank $n$ projection onto the first $n$-co-ordinates) then $(E_\lambda \otimes P_n)_{\Lambda\times\mathbb{N}}$ is a C*-bounded approximate identity of $\mathfrak{A} \otimes_{\min} \mathfrak{K}$ such that $(E_\lambda \otimes P_n)T(E_\lambda \otimes P_n) \in \mathfrak{A} \otimes_{\min} \mathcal{M}_n(\mathbb{C})$ for all $T \in \mathfrak{A} \otimes_{\min} \mathfrak{K}$. Suppose $T \in \mathfrak{A} \otimes_{\min} \mathfrak{K}$ is such that $\pi(T) = 0$. Then

$$T = \lim_{\Lambda\times\mathbb{N}} (E_\lambda \otimes P_n)T(E_\lambda \otimes P_n).$$

Furthermore

$$\pi((E_\lambda \otimes P_n)T(E_\lambda \otimes P_n)) = \pi(E_\lambda \otimes P_n)\pi(T)\pi(E_\lambda \otimes P_n) = 0$$

for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$. However, since $\pi$ is faithful on $\mathfrak{A} \otimes_{\min} \mathcal{M}_n(\mathbb{C})$ (being an extension of the left-$\mathcal{M}_n(\mathfrak{A})$-action), $\pi((E_\lambda \otimes P_n)T(E_\lambda \otimes P_n)) = 0$ implies $(E_\lambda \otimes P_n)T(E_\lambda \otimes P_n) = 0$ for all $\lambda \in \Lambda$ and $n \in \mathbb{N}$. Hence $T = 0$. Thus $\pi$ is faithful as desired.

**Corollary 4.15.** *Let $\mathfrak{A}$ be a C*-algebra, let $\mathfrak{K}$ denote the C*-algebra of compact operators on a complex, infinite dimensional, separable Hilbert space, and equip $\mathfrak{A}^\infty$ with the Hilbert $\mathfrak{A} \otimes_{\min} \mathfrak{K}$-$\mathfrak{A}$-bimodule structure from Example 4.14. Then $\mathcal{B}_a(\mathfrak{A}^\infty) = \mathcal{M}(\mathfrak{A} \otimes_{\min} \mathfrak{K})$.*

*Proof.* By Theorem 4.7 it suffices to show that $\mathfrak{B}_a(\mathfrak{A}^\infty) = \mathcal{ID}(\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K}))$. Clearly $\mathcal{ID}(\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})) \subseteq \mathfrak{B}_a(\mathfrak{A}^\infty)$. Let $T \in \mathfrak{B}_a(\mathfrak{A}^\infty)$. To show that $T\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K}) \subseteq \pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})$ it suffices to show if $\{E_{i,j}\}_{i,j\geq 1}$ are the canonical matrix units for $\mathfrak{K}$ and $A \in \mathfrak{A}$ then $T\pi(A \otimes E_{i,j}) \in \pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})$. Fix $A \in \mathfrak{A}$ and let $D \in \mathfrak{A}^\infty$ be the sequence with $A$ in the $i^{th}$ spot and zero elsewhere. Then $T(D) \in \mathfrak{A}^\infty$. Let $(C_n)_{n\geq 1} := T(D)$. We claim that $\sum_{n\geq 1} C_n \otimes E_{n,j}$ converges in $\mathfrak{A} \otimes_{\min} \mathfrak{K}$. To see this, we notice for any finite subset $I \subseteq \mathbb{N}$ that

$$\left\| \sum_{n\in I} C_n \otimes E_{n,j} \right\|^2 = \left\| \left( \sum_{m\in I} C_m \otimes E_{m,j} \right)^* \left( \sum_{n\in I} C_n \otimes E_{n,j} \right) \right\| = \left\| \sum_{n\in I} C_n^* C_n \otimes E_{j,j} \right\| = \left\| \sum_{n\in I} C_n^* C_n \right\|.$$

Therefore, since $\sum_{n\geq 1} C_n^* C_n$ converges in $\mathfrak{A}$ as $(C_n)_{n\geq 1} \in \mathfrak{A}^\infty$, $\sum_{n\geq 1} C_n \otimes E_{n,j}$ converges in $\mathfrak{A} \otimes_{\min} \mathfrak{K}$. Hence for all $(B_n)_{n\geq 1} \in \mathfrak{A}^\infty$

$$\begin{aligned} (T\pi(A \otimes E_{i,j}))(B_n)_{n\geq 1} &= T(\rho(B_j)D) \\ &= \rho(B_j)T(D) \\ &= \rho(B_j)(C_n)_{n\geq 1} \\ &= (C_n B_j)_{n\geq 1} \\ &= \pi(\textstyle\sum_{n\geq 1} C_n \otimes E_{n,j})(B_n)_{n\geq 1} \end{aligned}$$

(as adjointable maps commute with the right action). Hence $T\pi(\mathfrak{A}\otimes_{\min}\mathfrak{K}) \subseteq \pi(\mathfrak{A}\otimes_{\min}\mathfrak{K})$ for all $T \in \mathcal{B}_a(\mathfrak{A}^\infty)$. Since the adjoint operation is a bijection on $\mathcal{B}_a(\mathfrak{A}^\infty)$ and $\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})$, we obtain that $\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})T \subseteq \pi(\mathfrak{A} \otimes_{\min} \mathfrak{K})$ for all $T \in \mathcal{B}_a(\mathfrak{A}^\infty)$. Hence $\mathfrak{B}_a(\mathfrak{A}^\infty) = \mathcal{ID}(\pi(\mathfrak{A} \otimes_{\min} \mathfrak{K}))$ as desired. □

All of the above examples give concrete realization of the multiplier algebra as the bounded adjointable maps on a right Hilbert C$^*$-module. These examples are a specific subscase of a result we now desire to demonstrate. This result is motivated by the fact that the compact operators on a Hilbert space are a non-degenerated subset of the bounded linear maps on the same Hilbert space and thus the bounded linear maps are the multiplier algebra of the compact operators as in Corollary 3.10. Thus we desire the analogue of the compact operators on a right Hilbert C$^*$-module.

**Definition 4.16.** Let $\mathfrak{B}$ be a C$^*$-algebra and let $\mathcal{H}_\mathfrak{B}$ be a right Hilbert $\mathfrak{B}$-module. For each $\xi, \eta \in \mathcal{H}_\mathfrak{B}$ we define the operator $\theta_{\xi,\eta} \in \mathcal{B}(\mathcal{H}_\mathfrak{B})$ by

$$\theta_{\xi,\eta}(\zeta) = \rho(\langle \eta, \zeta\rangle)\xi$$

(or $\xi\langle\eta,\zeta\rangle$ if the reader is comfortable with omitting notation for the right action) for all $\zeta \in \mathcal{H}_\mathfrak{B}$ (clearly these are the analogues of the rank one operators for right Hilbert C$^*$-modules). It is trivial to verify that $\theta_{\xi,\eta}$ is linear and bounded with norm at most $\|\xi\|\,\|\eta\|$. Furthermore for all $\zeta, \omega \in \mathcal{H}_\mathfrak{B}$

$$\begin{aligned} \langle \zeta, \theta_{\eta,\xi}(\omega)\rangle &= \langle \zeta, \rho(\langle \xi, \omega\rangle)\eta\rangle \\ &= \langle \zeta, \eta\rangle\langle \xi, \omega\rangle \\ &= \langle \rho(\langle \zeta, \eta\rangle^*)\xi, \omega\rangle \\ &= \langle \rho(\langle \eta, \zeta\rangle)\xi, \omega\rangle \\ &= \langle \theta_{\xi,\eta}(\zeta), \omega\rangle \end{aligned}$$

so that $\theta_{\xi,\eta} \in \mathfrak{B}_a(\mathcal{H}_\mathfrak{B})$ with $\theta^*_{\xi,\eta} = \theta_{\eta,\xi}$. Furthermore it is clear that

$$\theta_{\lambda\xi_1+\xi_2,\eta} = \lambda\theta_{\xi_1,\eta} + \theta_{\xi_2,\eta}$$

and

$$\theta_{\xi,\lambda\eta_1+\eta_2} = \overline{\lambda}\theta_{\xi,\eta_1} + \theta_{\xi,\eta_2}$$

for all $\lambda \in \mathbb{C}$ and $\xi, \xi_1, \xi_2, \eta, \eta_1, \eta_2 \in \mathcal{H}_\mathfrak{B}$. Moreover

$$\begin{aligned} \theta_{\xi_1,\eta_1}(\theta_{\xi_2,\eta_2}(\zeta)) &= \theta_{\xi_1,\eta_1}(\rho(\langle \eta_2, \zeta\rangle\xi_2) \\ &= \rho(\langle \eta_1, \rho(\langle \eta_2, \zeta\rangle)\xi_2\rangle)\xi_1 \\ &= \rho(\langle \eta_1, \xi_2\rangle\langle \eta_2, \zeta\rangle)\xi_1 \\ &= \rho(\langle \eta_2, \zeta\rangle)\rho(\langle \eta_1, \xi_2\rangle)\xi_1 \\ &= \theta_{\rho(\langle \eta_1,\xi_2\rangle)\xi_1,\eta_2}(\zeta) \end{aligned}$$

for all $\zeta, \xi_j, \eta_j \in \mathcal{H}_\mathfrak{B}$. Hence $\theta_{\xi_1,\eta_1} \circ \theta_{\xi_2,\eta_2} = \theta_{\rho(\langle \eta_1,\xi_2\rangle)\xi_1,\eta_2}$.

Let $\mathfrak{K}(\mathcal{H}_\mathfrak{B})$ be the closure in $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ of the linear span of all $\theta_{\xi,\eta}$ where $\xi, \eta \in \mathcal{H}_\mathfrak{B}$. By the above computations, $\mathfrak{K}(\mathcal{H}_\mathfrak{B})$ is a C$^*$-subalgebra of $\mathcal{B}_a(\mathcal{H}_\mathfrak{B})$ known as the compact operators on $\mathcal{H}_\mathfrak{B}$.

To obtain some intuition behind $\mathfrak{K}(\mathcal{H}_\mathfrak{B})$ we note the following examples. Our first example shows why $\mathfrak{K}(\mathcal{H}_\mathfrak{B})$ is indeed a generalization of the compact operators on a Hilbert space.

**Example 4.17.** Let $\mathcal{H}_{\mathbb{C}}$ be a right Hilbert $\mathbb{C}$-module (that is, $\mathcal{H}_{\mathbb{C}}$ is a Hilbert space). For each $\xi, \eta \in \mathcal{H}_{\mathbb{C}}$ notice $\theta_{\xi,\eta}$ is the rank one operator sending $\eta$ to $\xi$. Hence $\mathfrak{K}(\mathcal{H}_{\mathbb{C}})$ is the closure of the span of the rank one operators in $\mathcal{B}_a(\mathcal{H}_{\mathbb{C}}) = \mathcal{B}(\mathcal{H}_{\mathbb{C}})$ and thus $\mathfrak{K}(\mathcal{H}_{\mathbb{C}})$ is precisely the compact operators on $\mathcal{H}_{\mathbb{C}}$.

**Example 4.18.** Let $\mathfrak{A}$ be a C$^*$-algebra and view $\mathfrak{A}$ as a Hilbert $\mathfrak{A}$-$\mathfrak{A}$-bimodule as in Example 4.3. If $A, B \in \mathfrak{A}$ then

$$\theta_{A,B}(C) = AB^*C = \pi(AB^*)C$$

for all $C \in \mathfrak{A}$. Therefore, since $\mathfrak{A}$ is the span of its positive elements (which are of the form $A^*A$ for some $A \in \mathfrak{A}$), the above shows that $\mathfrak{K}(\mathfrak{A}) = \pi(\mathfrak{A})$.

Suppose further that $\mathfrak{A}$ is a unital C$^*$-algebra. If $T \in \mathcal{B}_a(\mathfrak{A})$ let $A := T(I_{\mathfrak{A}}) \in \mathfrak{A}$. Then for all $B \in \mathfrak{A}$

$$T(B) = T(\rho(B)I_{\mathfrak{A}}) = \rho(B)T(I_{\mathfrak{A}}) = \rho(B)A = AB = \pi(A)B.$$

Hence $T = \pi(A)$ so $T \in \pi(\mathfrak{A})$. Hence $\mathcal{B}_a(\mathfrak{A}) = \mathfrak{K}(\mathfrak{A}) = \pi(\mathfrak{A})$ whenever $\mathfrak{A}$ is unital.

**Example 4.19.** Let $\mathfrak{A}$ be a C$^*$-algebra and view $\mathfrak{A}^n$ as a Hilbert $\mathcal{M}_n(\mathfrak{A})$-$\mathfrak{A}$-bimodule as in Example 4.12. If $(A_1, \ldots, A_n), (B_1, \ldots, B_n) \in \mathfrak{A}^n$ then

$$\begin{aligned}
\theta_{(A_1,\ldots,A_n),(B_1,\ldots,B_n)}(C_1, \ldots, C_n) &= \rho(\langle (B_1, \ldots, B_n), (C_1, \ldots, C_n)\rangle)(A_1, \ldots, A_n) \\
&= \left(A_1 \textstyle\sum_{j=1}^n B_j^* C_j, \ldots, A_n \sum_{j=1}^n B_j^* C_j\right) \\
&= \left(\textstyle\sum_{j=1}^n (A_1 B_j^*)C_j, \ldots, \sum_{j=1}^n (A_n B_j^*)C_j\right) \\
&= \pi([A_i B_j^*])(C_1, \ldots, C_n)
\end{aligned}$$

for all $(C_1, \ldots, C_n) \in \mathfrak{A}^n$. Therefore, since $\mathcal{M}_n(\mathfrak{A})$ is the span of its positive elements (all of which can be written as sums of elements of the form $[A_i A_j^*]$), the above shows that $\mathfrak{K}(\mathfrak{A}^n) = \pi(\mathcal{M}_n(\mathfrak{A}))$.

Suppose further that $\mathfrak{A}$ is a unital C$^*$-algebra. If $T \in \mathcal{B}_a(\mathfrak{A}^n)$ let $E_j \in \mathfrak{A}^n$ be the element with $I_{\mathfrak{A}}$ in the $j^{th}$ component and zeros elsewhere and let $(A_{1,j}, \ldots, A_{n,j}) := T(E_j) \in \mathfrak{A}^n$. Then for all $(B_1, \ldots, B_n) \in \mathfrak{A}^n$

$$\begin{aligned}
T(B_1, \ldots, B_n) &= \textstyle\sum_{j=1}^n T(\rho(B_j)E_j) \\
&= \textstyle\sum_{j=1}^n \rho(B_j)T(E_j) \\
&= \textstyle\sum_{j=1}^n \rho(B_j)(A_{1,j}, \ldots, A_{n,j}) \\
&= \textstyle\sum_{j=1}^n (A_{1,j}B_j, \ldots, A_{n,j}B_j) \\
&= \pi([A_{i,j}])(B_1, \ldots, B_n).
\end{aligned}$$

Hence $T = \pi([A_{i,j}])$ so $T \in \pi(\mathcal{M}_n(\mathfrak{A}))$. Hence $\mathcal{B}_a(\mathfrak{A}^n) = \mathfrak{K}(\mathfrak{A}^n) = \pi(\mathcal{M}_n(\mathfrak{A}))$ whenever $\mathfrak{A}$ is unital.

**Example 4.20.** Let $\mathfrak{A}$ be a C$^*$-algebra and view $\mathfrak{A}^\infty$ as a Hilbert $\mathfrak{A} \otimes_{\min} \mathfrak{K}$-$\mathfrak{A}$-bimodule as in Example 4.14. If $(A_n)_{n\geq 1}, (B_n)_{n\geq 1} \in \mathfrak{A}^\infty$ are such that $A_n, B_n = 0$ if $n \geq m$ for some $m \in \mathbb{N}$ then it is easy to verify that $\theta_{(A_n)_{n\geq 1},(B_n)_{n\geq 1}}$ is the operator on $\mathfrak{A}^\infty$ obtained by using the left-action of $\mathcal{M}_m(\mathfrak{A})$ on $\mathfrak{A}^\infty$ (as in Example 4.14) with the operator $[A_i B_j^*]_{i,j=1}^m$. Therefore, since $\theta_{\xi,\eta}$ is continuous in $\xi$ and $\eta$, since $\bigcup_{n\geq 1} \mathfrak{A}^n$ is dense in $\mathfrak{A}^\infty$, since $\bigcup_{n\geq 1} \mathcal{M}_n(\mathfrak{A})$ is dense in $\mathfrak{A} \otimes_{\min} \mathfrak{K}$, and since the span of elements of the form $[A_i B_j^*]_{i,j=1}^m$ is all of $\mathcal{M}_m(\mathfrak{A})$, we obtain that $\mathfrak{K}(\mathfrak{A}^\infty) = \mathfrak{A} \otimes_{\min} \mathfrak{K}$.

If $\mathfrak{A}$ is unital it need not be the case that $\mathfrak{K}(\mathfrak{A}^\infty) = \mathcal{B}_a(\mathfrak{A}^\infty)$. Indeed if $\mathfrak{A} = \mathbb{C}$, $\mathfrak{A}^\infty = \ell_2(\mathbb{N})$, $\mathfrak{K}(\mathfrak{A}^\infty)$ is the usual C$^*$-algebra of compact operators, and $\mathcal{B}_a(\mathfrak{A}^\infty)$ is the usual C$^*$-algebra of all bounded linear operators on $\ell_2(\mathbb{N})$.

One important lemma before discussing the multiplier algebra of $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is that $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is an ideal in $\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ just as the compact operators are an ideal of the bounded linear maps on a Hilbert space.

**Lemma 4.21.** *Let $\mathcal{H}_{\mathfrak{B}}$ be a right Hilbert $\mathfrak{B}$-module. Then $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is a closed ideal of $\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$.*

*Proof.* It is clear by Definition 4.16 that $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is a C$^*$-subalgebra of $\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$. Thus, to show that $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is an ideal, it suffices to show that $T\mathfrak{K}(\mathcal{H}_{\mathfrak{B}}) \subseteq \mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ and $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})T \subseteq \mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ for all $T \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$. However it suffices by linearity and density to show that $T\theta_{\xi,\eta}$ and $\theta_{\xi,\eta}T$ are compact for all $\xi, \eta \in \mathcal{H}_{\mathfrak{B}}$. To see this we notice that if $\zeta \in \mathcal{H}_{\mathfrak{B}}$ then

$$T\theta_{\xi,\eta}(\zeta) = T(\rho(\langle \eta, \zeta\rangle)\xi) = \rho(\langle \eta, \zeta\rangle)T(\xi) = \theta_{T(\xi),\eta}(\zeta)$$

and

$$\theta_{\xi,\eta}T(\zeta) = \rho(\langle \eta, T\zeta\rangle)\xi = \rho(\langle T^*\eta, \zeta\rangle)\xi = \theta_{\xi,T^*\eta}(\zeta)$$

so $T\theta_{\xi,\eta} = \theta_{T\xi,\eta}$ and $\theta_{\xi,\eta}T = \theta_{\xi,T^*\eta}$. Hence the result is complete. □

To begin our discussion of the multiplier algebra of $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ we note the following technical lemma.

**Lemma 4.22.** *Let $\mathcal{H}_{\mathfrak{B}}$ be a right Hilbert $\mathfrak{B}$-module. If $\xi \in \mathcal{H}_{\mathfrak{B}}$ then*

$$\xi = \lim_{\epsilon\to 0^+} \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi.$$

*Proof.* First note for all $\epsilon > 0$ that $\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}}$ is invertible in the unitization of $\mathfrak{B}$ (as $\langle \xi, \xi\rangle$ is positive) and thus $\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1}$ is a well-defined element of $\mathfrak{B}$. Furthermore

$$\begin{array}{rl} & \left\|\xi - \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi\right\|^2 \\ = & \langle \xi, \xi\rangle - \langle \xi, \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi\rangle - \langle \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi, \xi\rangle \\ & +\langle \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi, \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi\rangle \\ = & \langle \xi, \xi\rangle - \langle \xi, \xi\rangle^2(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1}) - ((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\langle \xi, \xi\rangle^2 \\ & -((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\langle \xi, \xi\rangle^3(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1}) \end{array}$$

which by the Continuous Functional Calculus for Normal Operators converges to zero as $\epsilon$ converges to zero from above. Thus the result follows. □

Using Theorem 4.7 and Lemma 4.21 we will prove the following result. This result along with the above examples generalizes the results of Corollary 4.11, Corollary 4.13, and Corollary 4.15.

**Corollary 4.23.** *If $\mathcal{H}_{\mathfrak{B}}$ a right Hilbert $\mathfrak{B}$-module then $\mathcal{M}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) = \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$.*

*Proof.* To conclude using Theorem 4.7 that $\mathcal{M}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) = \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ it suffices to show that $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is non-degenerate and $\mathcal{ID}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) = \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$. For the first, we recall that the right action of $\mathfrak{B}$ on $\mathcal{H}_{\mathfrak{B}}$ clearly extends to the unitization of $\mathfrak{B}$. Hence we may assume that $\mathfrak{B}$ is unital. Thus we notice if $\xi \in \mathcal{H}_{\mathfrak{B}}$ and $\epsilon > 0$ then

$$\theta_{\rho(((\langle \xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\xi}(\xi) = \rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi.$$

Hence $\xi \in \overline{\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})\mathcal{H}_{\mathfrak{B}}}$ by Lemma 4.22. Since $\xi \in \mathcal{H}_{\mathfrak{B}}$ was arbitrary, $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is non-degenerate.

To see that $\mathcal{ID}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) = \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$, we note clearly that $\mathcal{ID}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) \subseteq \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ and $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is an ideal of $\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ by Lemma 4.21. Therefore $\mathcal{ID}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})) = \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ follows from the definition of the idealizer so the result is complete. □

To complete this section we desire to provide another proof of Corollary 4.23 directly from Theorem 2.7. This proof then provides a short-cut to obtain the main results and examples of this section directly from those in Section 2 thereby avoiding the discussion of representation theory on right Hilbert C*-modules. However, although the theory of representations of right Hilbert C*-modules was difficult, the following proof is quite technical.

**Theorem 4.24.** *Let $\mathcal{H}_{\mathfrak{B}}$ be a right Hilbert $\mathfrak{B}$-module. Then the map $\psi : \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \to \mathcal{DC}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}}))$ (where the C*-algebra of double centralizers is as described in Definition 2.1) defined by $\Psi(T) = (L_T, R_T)$ where $L_T(K) = TK$ and $R_T(K) = KT$ for all $K \in \mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is a isomorphism of C*-algebras.*

*Proof.* Recall we may assume $\mathfrak{B}$ is unital as we may extend the definition of $\rho$. Since $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ is an ideal of $\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ by Lemma 4.21, the map $\Psi$ exists and is $^*$-homomorphism by Lemma 2.6. Thus it suffices to show that $\Psi$ is bijective. Recall from Corollary 2.9 that

$$ker(\Psi) = \{T \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \mid TK = 0 \text{ for all } K \in \mathfrak{K}(\mathcal{H}_{\mathfrak{B}})\}.$$

Thus to see that $ker(\Psi) = \{0\}$, suppose $T \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ is such that $TK = 0$ for all $K \in \mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$. Let $\xi \in \mathcal{H}_{\mathfrak{B}}$ be arbitrary. Then, by Lemma 4.22 and the proof of Corollary 4.23,

$$T\xi = \lim_{\epsilon \to 0^+} T(\rho(\langle \xi, \xi\rangle(\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi) = \lim_{\epsilon \to 0^+} T(\theta_{\rho(((\langle \xi,\xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi, \xi}(\xi)) = 0$$

as $T\theta_{\rho(((\langle \xi,\xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})\xi, \xi} = 0$. Hence $T = 0$ so $\Psi$ is injective.

To see that $\Psi$ is surjective, let $(L, R) \in \mathcal{DC}(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}}))$ be arbitrary. We claim that for each $\xi \in \mathcal{H}_{\mathfrak{B}}$ the limit $\lim_{\epsilon \to 0^+} \rho((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))$ exists. To see this, we first note that

$$L(\theta_{\xi,\xi})^* L(\theta_{\xi,\xi}) = \langle \theta_{\xi,\xi}, (L^*L)(\theta_{\xi,\xi})\rangle_{\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})} \leq \|L\|^2 \langle \theta_{\xi,\xi}, \theta_{\xi,\xi}\rangle_{\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})} = \|L\|^2 \theta_{\rho(\langle \xi,\xi\rangle)\xi,\xi}$$

where the C$^*$-valued inner product is the canonical C$^*$-valued inner product on $\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})$ from Remarks 2.11 (and $L \in \mathcal{B}_a(\mathfrak{K}(\mathcal{H}_{\mathfrak{B}}))$ with $L^* = R^\sharp$). Therefore if $\epsilon_1 > \epsilon_2 > 0$ then

$$\begin{aligned}
& \left\|\rho((\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi)) - \rho((\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))\right\|^2 \\
= \ & \left\|\left((\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1} - (\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right) \langle L(\theta_{\xi,\xi})(\xi), L(\theta_{\xi,\xi})(\xi)\rangle \left((\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1} - (\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right)\right\| \\
= \ & (\epsilon_2 - \epsilon_1)^2 \left\|(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\langle \xi, L(\theta_{\xi,\xi})^* L(\theta_{\xi,\xi})(\xi)\rangle(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right\| \\
\leq \ & \|L\|^2 (\epsilon_2 - \epsilon_1)^2 \left\|(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\langle \xi, \theta_{\rho(\langle \xi,\xi\rangle)\xi,\xi}(\xi)\rangle(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right\| \\
= \ & \|L\|^2 (\epsilon_2 - \epsilon_1)^2 \left\|(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\langle \xi, \rho(\langle \xi, \xi\rangle)\rho(\langle \xi, \xi\rangle)\xi\rangle(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right\| \\
= \ & \|L\|^2 (\epsilon_2 - \epsilon_1)^2 \left\|(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\langle \xi, \xi\rangle^3(\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1}(\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1}\right\|.
\end{aligned}$$

Thus if $f_{\epsilon_1,\epsilon_2}(x) := \|L\|^2 \frac{(\epsilon_2-\epsilon_1)^2 x^3}{(x+\epsilon_1)^2(x+\epsilon_2)^2}$ then

$$\left\|\rho((\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi)) - \rho((\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))\right\|^2 \leq \|f_{\epsilon_1,\epsilon_2}(\langle \xi, \xi\rangle)\|.$$

However, since $\epsilon_1 > \epsilon_2 > 0$, it is easy to see that if $g_{\epsilon_1,\epsilon_2}(x) = \|L\|^2 \frac{(\epsilon_2-\epsilon_1)^2 x}{(x+\epsilon_1)^2}$ then $f_{\epsilon_1,\epsilon_2} \leq g_{\epsilon_1,\epsilon_2}$ on $[0, \infty)$. Since the supremum of $g_{\epsilon_1,\epsilon_2}$ is easily seen to be at $x = \epsilon_1$, $g_{\epsilon_1,\epsilon_2}(\epsilon_1) = \|L\|^2 \frac{(\epsilon_2-\epsilon_1)^2\epsilon_1}{4\epsilon_1^2}$, we easily obtain that

$$\left\|\rho((\langle \xi, \xi\rangle + \epsilon_1 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi)) - \rho((\langle \xi, \xi\rangle + \epsilon_2 I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))\right\|^2 \leq \|L\|^2 \frac{(\epsilon_2 - \epsilon_1)^2 \epsilon_1}{4\epsilon_1^2} \leq \frac{1}{2}\|L\|^2 \epsilon_1.$$

Hence the limit exists as desired.

A similar computation shows that $\lim_{\epsilon \to 0^+} \rho((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})((R(\theta_{\xi,\xi}))^*(\xi))$ exists. Therefore, for each $\xi \in \mathcal{H}_{\mathfrak{B}}$ we define

$$T(\xi) := \lim_{\epsilon \to 0^+} \rho((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))$$

and

$$S(\xi) := \lim_{\epsilon \to 0^+} \rho((\langle \xi, \xi\rangle + \epsilon I_{\mathfrak{B}})^{-1})((R(\theta_{\xi,\xi}))^*(\xi)).$$

We claim that $T, S \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$. To show this, it suffices to show that

$$\langle \eta, T(\xi)\rangle = \langle S(\eta), \xi\rangle$$

for all $\xi, \eta \in \mathcal{H}_{\mathfrak{B}}$ (as this will clearly enable us to show that $T$ and $S$ are linear; the rest of the claim follows from results in http://www.math.ucla.edu/∼pskoufra/OANotes-HilbertC-Bimodules.pdf). Thus, if $\xi, \eta \in \mathcal{H}_{\mathfrak{B}}$, we note that

$$\begin{aligned}
(L(\theta_{\xi,\xi}))^*\theta_{\eta,\eta} &= \langle L(\theta_{\xi,\xi}), \theta_{\eta,\eta}\rangle_{\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})} \\
&= \langle \theta_{\xi,\xi}, R^\sharp(\theta_{\eta,\eta})\rangle_{\mathfrak{K}(\mathcal{H}_{\mathfrak{B}})} \\
&= \theta^*_{\xi,\xi}(R^\sharp(\theta_{\eta,\eta}))
\end{aligned}$$

(recall that $L^* = R^\sharp$ by Remarks 2.11) so by Lemma 4.22 we have that

$$\begin{aligned}
\langle \eta, T(\xi)\rangle &= \lim_{\epsilon\to 0^+}\langle \rho(\langle\eta,\eta\rangle(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1})\eta, \rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi))\rangle \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle\eta,\eta\rangle\langle\eta, L(\theta_{\xi,\xi})(\xi)\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle\rho(\langle\eta,\eta\rangle)\eta, L(\theta_{\xi,\xi})(\xi)\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle\theta_{\eta,\eta}(\eta), L(\theta_{\xi,\xi})(\xi)\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle(((L(\theta_{\xi,\xi}))^*)\theta_{\eta,\eta})\eta, \xi\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle(\theta_{\xi,\xi})^*(R^\sharp(\theta_{\eta,\eta}))\eta, \xi\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}(\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle(R(\theta^*_{\eta,\eta})^*)(\eta), \theta_{\xi,\xi}(\xi)\rangle\,(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1} \\
&= \lim_{\epsilon\to 0^+}\langle\rho\left((\langle\eta,\eta\rangle+\epsilon I_{\mathfrak{B}})^{-1}\right)((R(\theta_{\eta,\eta})^*)(\eta)), \rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(\theta_{\xi,\xi}))(\xi))\rangle \\
&= \langle S(\eta),\xi\rangle
\end{aligned}$$

as desired. Hence $T, S \in \mathcal{B}_a(\mathcal{H}_{\mathfrak{B}})$ and $S = T^*$.

To complete the proof it suffices to show that $\Psi(T) = (L, R)$. To see that $L_T = L$, we notice for all $\xi, \eta, \zeta \in \mathcal{H}_{\mathfrak{B}}$

$$\begin{aligned}
L_T(\theta_{\xi,\eta})(\zeta) &= T(\theta_{\xi,\eta}(\zeta)) \\
&= T(\rho(\langle\eta,\zeta\rangle)\xi) \\
&= \rho(\langle\eta,\zeta\rangle)T(\xi) && \text{as } T\in\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \\
&= \lim_{\epsilon\to 0^+}\rho(\langle\eta,\zeta\rangle)\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(L(\theta_{\xi,\xi})(\xi)) && \text{definition of } T(\xi) \\
&= \lim_{\epsilon\to 0^+}L(\theta_{\xi,\xi})\rho(\langle\eta,\zeta\rangle)\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(\xi) && \text{as } L(\theta_{\xi,\xi})\in\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \\
&= \lim_{\epsilon\to 0^+}L(\theta_{\xi,\xi})\theta_{\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\eta}(\zeta) \\
&= \lim_{\epsilon\to 0^+}L(\theta_{\xi,\xi}\circ\theta_{\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\eta})(\zeta) && \text{by Lemma 2.5} \\
&= \lim_{\epsilon\to 0^+}L(\theta_{\rho(\langle\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\xi\rangle)\xi,\eta})(\zeta) \\
&= \lim_{\epsilon\to 0^+}L(\theta_{\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1}\langle\xi,\xi\rangle)\xi,\eta})(\zeta) \\
&= L(\theta_{\xi,\eta})(\zeta)
\end{aligned}$$

where the last line follows by Lemma 4.22 and the fact that $\theta$ is continuous in each subscript. Therefore $L_T = L$. Furthermore, since $R_T^\sharp = L_{T^*}$ by (the proof of) Lemma 2.6, since $T^* = S$, and since

$$\begin{aligned}
L_{T^*}(\theta_{\xi,\eta})(\zeta) &= S(\theta_{\xi,\eta}(\zeta)) \\
&= S(\rho(\langle\eta,\zeta\rangle)\xi) && \text{as } S\in\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \\
&= \rho(\langle\eta,\zeta\rangle)S(\xi) \\
&= \lim_{\epsilon\to 0^+}\rho(\langle\eta,\zeta\rangle)\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(R(\theta_{\xi,\xi})^*(\xi)) && \text{definition of } S(\xi) \\
&= \lim_{\epsilon\to 0^+}(R(\theta_{\xi,\xi}))^*\rho(\langle\eta,\zeta\rangle)\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})(\xi) && \text{as } R(\theta_{\xi,\xi})\in\mathcal{B}_a(\mathcal{H}_{\mathfrak{B}}) \\
&= \lim_{\epsilon\to 0^+}(R(\theta_{\xi,\xi}))^*\theta_{\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\eta}(\zeta) \\
&= \lim_{\epsilon\to 0^+}(\theta_{\eta,\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi}R(\theta_{\xi,\xi}))^*(\zeta) \\
&= \lim_{\epsilon\to 0^+}(R(\theta_{\eta,\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi}\circ\theta_{\xi,\xi}))^*(\zeta) && \text{by Lemma 2.5} \\
&= \lim_{\epsilon\to 0^+}R^\sharp((\theta_{\eta,\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi}\circ\theta_{\xi,\xi})^*)(\zeta) \\
&= \lim_{\epsilon\to 0^+}R^\sharp(\theta_{\xi,\xi}\circ\theta_{\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi,\eta})(\zeta) \\
&= \lim_{\epsilon\to 0^+}R^\sharp(\theta_{\rho(\langle\xi,\rho((\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1})\xi\rangle)\xi,\eta})(\zeta) \\
&= \lim_{\epsilon\to 0^+}R^\sharp(\theta_{\rho(\langle\xi,\xi\rangle(\langle\xi,\xi\rangle+\epsilon I_{\mathfrak{B}})^{-1}\xi,\eta})(\zeta) \\
&= R^\sharp(\theta_{\xi,\eta})(\zeta)
\end{aligned}$$

(where the last line follows by Lemma 4.22 and the fact that $\theta$ is continuous in each subscript) for all $\xi, \eta, \omega \in \mathcal{H}_{\mathfrak{B}}$, we obtain that $R_T^\sharp = R^\sharp$ so $R_T = R$. Hence $\Psi(T) = (R, L)$ as desired. □

# 5 Applications and Other Interesting Results

In this section we will develop some interesting and important results pertaining to multiplier algebras of C*-algebras. These proofs will be developed from the descriptions of the multiplier algebra demonstrated in the previous three sections. As such, due to the different possible techniques, we may (or may not) present multiple proofs of each fact.

The first results we desire to discuss is the minimal tensor product of multiplier algebras. As the minimal tensor product of two C*-algebras can be derived from the images of faithful representations, we are in a prime position to apply Section 3.

**Proposition 5.1.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. Then there exists a faithful, unital *-homomorphism* $\Psi : \mathcal{M}(\mathfrak{A}) \otimes_{\min} \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{A} \otimes_{\min} \mathfrak{B})$.

*Proof.* By Theorem 3.6 there exists faithful, non-degenerate representations $\pi_{\mathfrak{A}} : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{B}(\mathcal{K})$ such that $\mathcal{M}(\mathfrak{A}) = \mathcal{ID}(\pi_{\mathfrak{A}}(\mathfrak{A}))$ and $\mathcal{M}(\mathfrak{B}) = \mathcal{ID}(\pi_{\mathfrak{B}}(\mathfrak{B}))$. Therefore, by properties of the minimal tensor product, there exists a faithful *-homomorphism $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ such that $\pi(A \otimes B) = \pi_{\mathfrak{A}}(A) \otimes \pi_{\mathfrak{B}}(B)$ for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Therefore, since $\overline{\pi_{\mathfrak{A}}(\mathfrak{A})\mathcal{H}} = \mathcal{H}$ and $\overline{\pi_{\mathfrak{B}}(\mathfrak{B})\mathcal{K}} = \mathcal{K}$, it is easy to see that $\overline{\pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})(\mathcal{H} \otimes \mathcal{K})}$ contains all tensor of the form $\xi \otimes \eta$ where $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$. Therefore, since $\overline{\pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})(\mathcal{H} \otimes \mathcal{K})}$ is a subspace of $\mathcal{H} \otimes \mathcal{K}$ by Proposition 3.2, $\pi$ is a non-degenerate representation of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. Hence $\mathcal{M}(\mathfrak{A} \otimes_{\min} \mathfrak{B}) = \mathcal{ID}(\pi(\mathfrak{A} \otimes_{\min} \mathfrak{B}))$.

Since $\mathcal{M}(\mathfrak{A})$ and $\mathcal{M}(\mathfrak{B})$ are C*-subalgebras of $\mathcal{B}(\mathcal{H})$ and $\mathcal{B}(\mathcal{K})$ respectively, we easily obtain that $\mathcal{M}(\mathfrak{A}) \otimes_{\min} \mathcal{M}(\mathfrak{B})$ is the C*-subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ obtained by taking the closure of the span of all operators of the form $T \otimes S$ where $T \in \mathcal{ID}(\pi_{\mathfrak{A}}(\mathfrak{A}))$ and $S \in \mathcal{ID}(\pi_{\mathfrak{B}}(\mathfrak{B}))$. Since for any $T \in \mathcal{ID}(\pi_{\mathfrak{A}}(\mathfrak{A}))$ and $S \in \mathcal{ID}(\pi_{\mathfrak{B}}(\mathfrak{B}))$ we have that

$$(T \otimes S)\pi(A \otimes B) = T\pi(A) \otimes S\pi(B) \in \pi_{\mathfrak{A}}(\mathfrak{A}) \odot \pi_{\mathfrak{B}}(\mathfrak{B}) \subseteq \pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})$$

and

$$\pi(A \otimes B)(T \otimes S) = \pi(A)T \otimes \pi(B)S \in \pi_{\mathfrak{A}}(\mathfrak{A}) \odot \pi_{\mathfrak{B}}(\mathfrak{B}) \subseteq \pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})$$

for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ and since the closed linear span of elements of the form $A \otimes B$ (where $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$) is dense in $\mathfrak{A} \otimes_{\min} \mathfrak{B}$, we obtain that $T \otimes S \in \mathcal{ID}(\pi_{\mathfrak{B}}(\mathfrak{B}))$. Thus the result follows. □

We note that the map in the above theorem need not be surjective. Indeed if $\mathfrak{A} = \mathfrak{B} = \mathfrak{K}$ then $\Psi$ is the canonical inclusion of $\mathcal{B}(\mathcal{H}) \otimes_{\min} \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ which is not surjective when $\mathcal{H}$ is infinite dimensional. The following is an important corollary of a case where $\Psi$ is surjective.

**Proposition 5.2.** *Let $\mathfrak{A}$ be a C*-algebra. Then $\mathcal{M}_n(\mathcal{M}(\mathfrak{A})) \simeq \mathcal{M}(\mathcal{M}_n(\mathfrak{A}))$.*

*Proof.* In the proof of Proposition 5.1, if we let $\mathfrak{B} := \mathcal{M}_n(\mathbb{C})$ then we can take $\pi_{\mathfrak{B}}$ to be the identity map. To see that $\Psi$ is surjective, suppose $T \in \mathcal{ID}(\pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})) = \mathcal{ID}((\pi_{\mathfrak{A}})_n(\mathcal{M}_n(\mathfrak{A})))$ where $(\pi_{\mathfrak{A}})_n : \mathcal{M}_n(\mathfrak{A}) \to \mathcal{B}(\mathcal{H}^{\oplus n}) \simeq \mathcal{M}_n(\mathcal{B}(\mathcal{H}))$ is the map defined by $(\pi_{\mathfrak{A}})_n([A_{i,j}]) = [\pi_{\mathfrak{A}}(A_{i,j})]$. Therefore we can write $T = [T_{i,j}]$ where $T_{i,j} \in \mathcal{B}(\mathcal{H})$. To complete the proof it suffices to show that $T_{i,j} \in \mathcal{ID}(\pi_{\mathfrak{A}}(\mathfrak{A}))$ for all $i, j \in \{1, \ldots, n\}$. To see this, we notice for each $k, \ell \in \{1, \ldots, n\}$ that if $E_{i,j} \in \mathcal{M}_n(\mathbb{C})$ are the canonical matrix units then

$$\sum_{i=1}^{n} T_{i,k}\pi(A) \otimes E_{i,\ell} = T(\pi_{\mathfrak{A}})_n(A \otimes E_{k,\ell}) \in (\pi_{\mathfrak{A}})_n(\mathcal{M}_n(\mathfrak{A}))$$

and

$$\sum_{j=1}^{n} \pi(A)T_{\ell,j} \otimes E_{k,j} = (\pi_{\mathfrak{A}})_n(A \otimes E_{k,\ell})T \in (\pi_{\mathfrak{A}})_n(\mathcal{M}_n(\mathfrak{A}))$$

for all $A \in \mathfrak{A}$. Therefore we easily conclude that $\pi(A)T_{i,j}, T_{i,j}\pi(A) \in \pi_{\mathfrak{A}}(\mathfrak{A})$ for all $A \in \mathfrak{A}$ and $i, j \in \{1, \ldots, n\}$ so that $T_{i,j} \in \mathcal{ID}(\pi_{\mathfrak{A}}(\mathfrak{A})) = \mathcal{M}(\mathfrak{A})$ as desired. □

With the above example complete we turn our attention to placing a new topology on our multiplier algebras. This topology generalizes several common topologies on C*-algebras.

**Definition 5.3.** Let $\mathfrak{A}$ be a C*-algebra. For each $A \in \mathfrak{A}$ define the seminorms $l_A$ and $r_A$ on $\mathcal{M}(\mathfrak{A})$ by

$$l_A(T) = \|AT\|_{\mathfrak{A}} \qquad \text{and} \qquad r_A(T) = \|TA\|_{\mathfrak{A}}$$

for all $T \in \mathcal{M}(\mathfrak{A})$. The strict topology on $\mathcal{M}(\mathfrak{A})$ is the locally convex topology generated by the seminorms $\{l_A, r_A\}_{A\in\mathfrak{A}}$.

**Remarks 5.4.** Note that the strict topology on $\mathcal{M}(\mathfrak{A})$ is indeed a locally convex topology since $\mathfrak{A}$ is an essential ideal in $\mathcal{M}(\mathfrak{A})$ so Lemma 1.5 implies that $\{l_A, r_A\}_{A\in\mathfrak{A}}$ is a separating family of seminorms. Moreover the strict topology on $\mathcal{M}(\mathfrak{A})$ is the topology on $\mathcal{M}(\mathfrak{A})$ such that a net $(T_\lambda)_\Lambda$ converges to an element $T \in \mathcal{M}(\mathfrak{A})$ if and only if $\lim_\Lambda T_\lambda A = TA$ and $\lim_\Lambda AT_\lambda = AT$ for all $A \in \mathfrak{A}$. If $\mathfrak{A}$ is separable it is clear that we may consider sequences instead of nets as the strict topology can easily be seen to be generated by a countable family of seminorms and thus is metrizable.

**Example 5.5.** Let $\mathfrak{A}$ be a unital C*-algebra. Then, by Corollary 2.10 or Corollary 3.9, $\mathcal{M}(\mathfrak{A}) = \mathfrak{A}$ and it is easy to see that the strict topology on $\mathcal{M}(\mathfrak{A})$ is the norm topology on $\mathfrak{A}$.

**Example 5.6.** Recall that $\mathcal{M}(\mathfrak{K}) = \mathcal{B}(\mathcal{H})$ by Corollary 3.10. It is easy to see for $T, (T_\lambda)_\Lambda \subseteq \mathcal{B}(\mathcal{H})$ that $\lim_\Lambda T_\lambda A = TA$ for all $A \in \mathfrak{K}$ if and only if $(T_\lambda)_\Lambda$ converges to $T$ in the $\sigma$-strong topology. Similarly $\lim_\Lambda AT_\lambda = AT$ for all $A \in \mathfrak{K}$ if and only if $\lim_\Lambda T_\lambda^* A = T^*A$ for all $A \in \mathfrak{K}$ if and only if $(T_\lambda^*)_\Lambda$ converges to $T^*$ in the $\sigma$-strong topology. Thus the strict topology on $\mathcal{B}(\mathcal{H})$ induced by $\mathfrak{K}$ is the $\sigma$-strong* topology.

With the above definition of the strict topology, we note the following two important results.

**Proposition 5.7.** *Let $\mathfrak{A}$ be a C*-algebra. Then every bounded net in $\mathcal{M}(\mathfrak{A})$ that is Cauchy in the strict topology converges in the strict topology. Furthermore if $\mathfrak{A}$ is separable then $\mathcal{M}(\mathfrak{A})$ is complete.*

*Proof.* By Theorem 2.7 the double centralizer, $\mathcal{DC}(\mathfrak{A})$ is equal to the multiplier algebra of $\mathfrak{A}$. Suppose $(T_\lambda)_\Lambda$ is a bounded net that is Cauchy in the strict topology on $\mathcal{M}(\mathfrak{A})$. By viewing $\mathcal{M}(\mathfrak{A})$ as $\mathcal{DC}(\mathfrak{A})$ we can write $T_\lambda = (L_\lambda, R_\lambda) \in \mathcal{DC}(\mathfrak{A})$. Recall that we view $\mathfrak{A} \subseteq \mathcal{DC}(\mathfrak{A})$ via $A \mapsto (L_A, R_A)$ where $L_A(B) = AB$ and $R_A(B) = BA$ for all $B \in \mathfrak{A}$.

Notice that $(L_\lambda, R_\lambda)(L_A, R_A) = (L_\lambda \circ L_A, R_A \circ R_\lambda)$ and $(L_A, R_A)(L_\lambda, R_\lambda) = (L_A \circ L_\lambda, R_\lambda \circ R_A)$. However

$$(L_\lambda \circ L_A)(B) = L_\lambda(AB) = L_\lambda(A)B = L_{L_\lambda(A)}(B)$$

for all $B \in \mathfrak{A}$ by Lemma 2.5 and similarly

$$(R_\lambda \circ R_A)(B) = R_\lambda(BA) = BR_\lambda(A) = R_{R_\lambda(A)}(B).$$

Hence $L_\lambda \circ L_A = L_{L_\lambda(A)}$ and $R_\lambda \circ R_A = R_{R_\lambda(A)}$.

Since $(T_\lambda)_\Lambda$ is a Cauchy net in the strict topology on $\mathcal{M}(\mathfrak{A})$ we obtain that $((L_\lambda, R_\lambda)(L_A, R_A))_\Lambda$ and $((L_A, R_A)(L_\lambda, R_\lambda) = (L_A \circ L_\lambda, R_\lambda \circ R_A))_\Lambda$ are Cauchy nets in $\mathfrak{A} \subseteq \mathcal{DC}(\mathfrak{A})$ for all $A \in \mathfrak{A}$. By the above description this implies that $(L_\lambda(A))_\Lambda$ and $(R_\lambda(A))_\Lambda$ are Cauchy nets in $\mathfrak{A}$ for all $A \in \mathfrak{A}$. Therefore, since $\mathfrak{A}$ is complete, we define

$$L(A) := \lim_\Lambda L_\lambda(A) \qquad \text{and} \qquad R(A) := \lim_\Lambda R_\lambda(A)$$

for each $A \in \mathfrak{A}$. It is then clear that $L$ and $R$ are bounded linear operators as $(L_\lambda)_\Lambda$ and $(R_\lambda)_\Lambda$ are bounded nets. Moreover we notice for all $A, B \in \mathfrak{A}$ that

$$AL(B) = \lim_\Lambda AL_\lambda(B) = \lim_\Lambda R_\lambda(A)B = BR(A)$$

so $(L,R) \in \mathcal{DC}(\mathfrak{A})$. Therefore, to complete the proof, it suffices to show that $((L_\lambda, R_\lambda))_\Lambda$ converges to $(L,R)$ in the strict topology. However Lemma 2.4 implies

$$\begin{aligned}\lim_\Lambda \|((L_\lambda, R_\lambda) - (L,R))(L_A, R_A)\| &= \lim_\Lambda \|(L_\lambda - L)\circ L_A\| \\ &= \lim_\Lambda \left\|L_{L_\lambda(A) - L(A)}\right\| \\ &= \lim_\Lambda \|L_\lambda(A) - L(A)\| = 0\end{aligned}$$

and

$$\begin{aligned}\lim_\Lambda \|(L_A, R_A)((L_\lambda, R_\lambda) - (L,R))\| &= \lim_\Lambda \|R_A \circ (R_\lambda - R)\| \\ &= \lim_\Lambda \left\|R_{R_\lambda(A) - R(A)}\right\| \\ &= \lim_\Lambda \|R_\lambda(A) - R(A)\| = 0\end{aligned}$$

for all $A \in \mathfrak{A}$. Hence the proof of the first claim is complete.

To see the second claim, we recall that since $\mathfrak{A}$ is separable it suffices to consider sequence in the strict topology on $\mathcal{M}(\mathfrak{A})$. Moreover, if $(T_n := (L_n, R_n))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}(\mathfrak{A})$ with regards to the strict topology then it is easy to see that $(L_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ are pointwise bounded as $(L_n \circ L_A = L_{L_n(A)})_{n\geq 1}$ and $(R_n \circ R_A = R_{R_n(A)})_{n\geq 1}$ are Cauchy sequence in $\mathfrak{A}$ for all $A \in \mathfrak{A}$. Hence the Uniform Boundedness Principle implies that $(T_n)_{n\geq 1}$ is uniformly bounded and thus the proof is proceeds as above. ☐

**Proposition 5.8.** *Let $\mathfrak{A}$ be a separable C*-algebra, let $(T_n)_{n\geq 1}$ be a bounded sequence of self-adjoint elements in $\mathcal{M}(\mathfrak{A})$, and let $\mathcal{S}$ be a total subset of $\mathfrak{A}$ (that is, $\overline{span(\mathcal{S})} = \mathfrak{A}$). Then $(T_n)_{n\geq 1}$ converge strictly in $\mathcal{M}(\mathfrak{A})$ if and only if $(T_nA)_{n\geq 1}$ is a norm Cauchy sequence in $\mathfrak{A}$ for all $A \in \mathcal{S}$.*

*Proof.* It is clear that if $(T_n)_{n\geq 1}$ converge strictly in $\mathcal{M}(\mathfrak{A})$ then $(T_nA)_{n\geq 1}$ is a norm Cauchy sequence in $\mathfrak{A}$ for all $A \in \mathcal{S}$.

Suppose that $(T_nA)_{n\geq 1}$ is a norm Cauchy sequence in $\mathfrak{A}$ for all $A \in \mathcal{S}$ where $\mathcal{S}$ is a total subset of $\mathfrak{A}$. Since the span of $\mathcal{S}$ is dense in $\mathfrak{A}$ and $(T_n)_{n\geq 1}$ is bounded, it is elementary to see that $(T_nA)_{n\geq 1}$ is Cauchy in $\mathfrak{A}$ for all $A \in \mathfrak{A}$. Furthermore, since each $T_n$ is self-adjoint, we easily obtain that $(AT_n)_{n\geq 1}$ is Cauchy in $\mathfrak{A}$ for all $A \in \mathfrak{A}$. Hence $(T_n)_{n\geq 1}$ is a Cauchy sequence in the strict topology on $\mathcal{M}(\mathfrak{A})$ and thus converges by Proposition 5.7. ☐

One use of strict convergence is the ability to sum an infinite number of elements in the multiplier algebra. Proposition 5.8 implies that if we can check that a sum of self-adjoint elements in the multiplier algebra converges when tested against an total subset of the C*-algebra, then the sum converges in the multiplier algebra.

To complete this section, we desire to discuss $\sigma$-unital C*-algebras and relations to multiplier algebras. One of the reasons for this is that multiplier algebras occur canonically in KK-Theory for C*-algebras and adding the condition that the C*-algebra under investigation is $\sigma$-unital aids in the theory. We begin with the following definition that is essential to developing the idea of a $\sigma$-unital C*-algebra.

**Definition 5.9.** Let $\mathfrak{A}$ be a C*-algebra. A positive element $A \in \mathfrak{A}$ is said to be strictly positive if $\varphi(A) > 0$ for every state $\varphi$ on $\mathfrak{A}$.

**Example 5.10.** If $\mathfrak{A}$ is a unital C*-algebra the identity in $\mathfrak{A}$ is clearly a strictly positive element.

**Example 5.11.** Let $f \in C_0(0,1)$ be a strictly positive function; that is $f(x) > 0$ for all $x \in (0,1)$. Then, by the Riesz Representation Theorem characterizing the states on $C_0(0,1)$, $f$ is a strictly positive element of $C_0(0,1)$. This is the reason for Definition 5.9.

We first desire to develop a proposition that easily enables us to determine when a C*-algebra has a strictly positive element in some cases. In addition, if we know a C*-algebra has a strictly positive element the proposition will say something nice about C*-bounded approximate identities. We begin with a technical lemma.

**Lemma 5.12.** *Let $\mathfrak{A}$ be a C*-algebra, let $A \in \mathfrak{A}$ be a strictly positive element, and let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Then $\overline{\pi(A)\mathcal{H}} = \mathcal{H}$.*

*Proof.* This lemma can be obtained as a result of Proposition 5.17 but as the proof of said proposition is more complicated than needed for this result, we shall prove this result directly.

Suppose $\overline{\pi(A)\mathcal{H}} \neq \mathcal{H}$. Let $\xi \in \left(\overline{\pi(A)\mathcal{H}}\right)^{\perp}$ be any unit vector and consider the state $\varphi$ on $\mathfrak{A}$ defined by $\varphi(T) = \langle \pi(T)\xi, \xi\rangle$ for all $T \in \mathfrak{A}$ ($\varphi$ is indeed a state as $\pi$ is non-degenerate; see Proposition 3.2). Then, as $A$ is strictly positive

$$0 < \varphi(A) = \langle \pi(A)\xi, \xi\rangle = 0$$

as $\xi \in \left(\overline{\pi(A)\mathcal{H}}\right)^{\perp}$. Hence we have a contradiction so the result is complete. □

**Proposition 5.13.** *Let $\mathfrak{A}$ be a C*-algebra. Then $\mathfrak{A}$ has a strictly positive element if and only if $\mathfrak{A}$ has a countable C*-bounded approximate identity. In particular, if $A \in \mathfrak{A}$ is a strictly positive element of norm one then $(A^{\frac{1}{n}})_{n\geq 1}$ is a C*-bounded approximate identity of $\mathfrak{A}$.*

*Proof.* The following proof is based on [AK]. Suppose $\mathfrak{A}$ has a countable C*-bounded approximate identity. Let $\{E_n\}_{n\geq 1}$ be the elements in the C*-bounded approximate identity (the C*-bounded approximate identity need not be a sequence). Let $A := \sum_{n\geq 1} \frac{1}{2^n} E_n$ which is clearly a well-defined positive element of $\mathfrak{A}$. We claim $A$ is strictly positive. To see this, suppose $\varphi$ is a state on $\mathfrak{A}$. As the norm of $\varphi$ is the limit of $\varphi$ applied to any C*-bounded approximate identity there exists some $m \in \mathbb{N}$ such that $\varphi(E_m) > 0$. Hence

$$\varphi(A) \geq \frac{1}{2^m}\varphi(E_m) > 0.$$

Hence $A$ is strictly positive.

Suppose $A$ is a strictly positive element of $\mathfrak{A}$ of norm one. For each $n \in \mathbb{N}$ define $E_n := A^{\frac{1}{n}}$. By the Continuous Functional Calculus for Normal Operators it is clear that $(E_n)_{n\geq 1}$ is bounded by one and is an increasing sequence of element of $\mathfrak{A}$. To show that $(E_n)_{n\geq 1}$ is a C*-bounded approximate identity it suffices to show that $\lim_{n\to\infty} \|T - TE_n\| = 0$ for all positive elements $T \in \mathfrak{A}$ (by taking adjoints and linear combinations of positive elements). Thus fix a positive element $T \in \mathfrak{A}$. Notice that

$$\begin{aligned} \|T - TE_n\|^2 = \left\|T(I_{\tilde{\mathfrak{A}}} - E_n)\right\|^2 &\leq \left\|T^{\frac{1}{2}}\right\|^2 \left\|T^{\frac{1}{2}}(I_{\tilde{\mathfrak{A}}} - E_n)^{\frac{1}{2}}\right\|^2 \left\|(I_{\tilde{\mathfrak{A}}} - E_n)^{\frac{1}{2}}\right\|^2 \\ &\leq \|T\| \left\|T^{\frac{1}{2}}(I_{\tilde{\mathfrak{A}}} - E_n)T^{\frac{1}{2}}\right\| \\ &= \|T\| \left\|T - T^{\frac{1}{2}}E_nT^{\frac{1}{2}}\right\|. \end{aligned}$$

Hence it suffices to show that $\lim_{n\to\infty} \left\|T - T^{\frac{1}{2}}E_nT^{\frac{1}{2}}\right\| = 0$.

For each $n \in \mathbb{N}$ let $S_n := T - T^{\frac{1}{2}}E_nT^{\frac{1}{2}} \geq 0$. Since $(T^{\frac{1}{2}}E_nT^{\frac{1}{2}})_{n\geq 1}$ is clearly an increasing sequence of positive operators, it is clear that $(S_n)_{n\geq 1}$ is a decreasing sequence of positive operators. We claim that $\lim_{n\to\infty} \|S_n\| = 0$. To begin, let $\varphi$ be any positive linear functional on $\mathfrak{A}$. By the GNS construction there exists a Hilbert space $\mathcal{H}_\varphi$, a non-degenerate representation $\pi_\varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\varphi)$, and a vector $\xi \in \mathcal{H}_\varphi$ such that $\varphi(B) = \langle \pi(B)\xi, \xi\rangle$ for all $B \in \mathfrak{A}$. It is clear by the Borel Functional Calculus for Normal Operators that $\pi(E_n) = \pi(A)^{\frac{1}{n}}$ converges in the weak operator topology to the range projection of $A$ as $n$ tends to infinity. Therefore Lemma 5.12 implies that $(\pi(E_n))_{n\geq 1}$ converges to $I_{\mathcal{H}_\varphi}$ in the strong operator topology. Hence

$$\lim_{n\to\infty} \varphi(S_n) = \lim_{n\to\infty} \langle \pi(T)\xi, \xi\rangle - \langle \pi(E_n)\pi(T^{\frac{1}{2}})\xi, \pi(T^{\frac{1}{2}})\xi\rangle = \langle \pi(T)\xi, \xi\rangle - \langle \pi(T^{\frac{1}{2}})\xi, \pi(T^{\frac{1}{2}})\xi\rangle = 0.$$

Hence $\lim_{n\to\infty} \varphi(S_n) = 0$ for all positive linear functional $\varphi$ on $\mathfrak{A}$. However, $(S_n)_{n\geq 1}$ defines a sequence of decreasing weak*-continuous linear functionals on the weak*-compact set of all positive linear functionals of norm at most one on $\mathfrak{A}$. Hence Dini's Theorem implies that $(\varphi(S_n))_{n\geq 1}$ converges uniformly to zero on all positive linear functionals on $\mathfrak{A}$ with norm at most one. Hence it trivially follows that $(S_n)_{n\geq 1}$ converges to zero on $\mathfrak{A}$ as desired. □

With the above proof complete we easily obtain that certain C*-algebras have strictly positive operators.

**Example 5.14.** As the compact operators on a separable Hilbert space clearly have a countable C*-bounded approximate identity, the compact operators have a strictly positive operator.

**Example 5.15.** It is well-known that every separable C*-algebra has a countable C*-bounded approximate identity. Hence every separable C*-algebra has a strictly positive element.

We desire a term to classify when a C*-algebra has a strictly positive operator.

**Definition 5.16.** A C*-algebra $\mathfrak{A}$ is said to be $\sigma$-unital if $\mathfrak{A}$ contains at least one strictly positive element.

The reason for the term $\sigma$-unital is that Proposition 5.13 implies a C*-algebra having a strictly positive element is the same as the C*-algebra having a countable approximate unit. By the above results, it is clear that unital C*-algebras and separable C*-algebras are $\sigma$-unital.

To complete these notes, we desire to show that the multiplier algebra is well-behaved with respect to inclusion for C*-subalgebras that contain strictly positive elements. To see this, we require the following proposition.

**Proposition 5.17.** *Let $\mathfrak{A}$ be a C*-algebra and let $A \in \mathfrak{A}$ be positive. Then $A$ is a strictly positive operator if and only if $A\mathfrak{A}$ (or $\mathfrak{A}A$) is dense in $\mathfrak{A}$.*

*Proof.* It is clear that by taking adjoints the statements $A\mathfrak{A}$ is dense in $\mathfrak{A}$ and $\mathfrak{A}A$ is dense in $\mathfrak{A}$ are equivalent. Suppose $A \in \mathfrak{A}$ is a positive element that is not strictly positive. Therefore there exists a state $\varphi$ on $\mathfrak{A}$ such that $\varphi(A) = 0$. Therefore if $C \in \mathfrak{A}$ then

$$|\varphi(AC)|^{\frac{1}{2}} = |\varphi(A^{\frac{1}{2}}(A^{\frac{1}{2}}C))|^{\frac{1}{2}} \leq \varphi(A)\varphi(C^*AC) = 0$$

by the Cauchy Schwarz Inequality for positive linear functionals. Hence $\varphi(A\mathfrak{A}) = \{0\}$. If $A\mathfrak{A}$ were dense in $\mathfrak{A}$, the previous equation would imply that $\varphi = 0$ which contradicts the fact that $\varphi$ was a state. Hence if $A \in \mathfrak{A}$ is a positive element that is not strictly positive then $A\mathfrak{A}$ is not dense in $\mathfrak{A}$.

The remainder of the proof is based on [Di] and [AK]. Suppose $A\mathfrak{A}$ is not dense in $\mathfrak{A}$. Then it is easy to see that there exists a positive element $B \in \mathfrak{A}$ that is not in $\overline{A\mathfrak{A}}$ (as every element in $\mathfrak{A}$ is a linear combination of four positive elements). Let $\epsilon > 0$ be arbitrary and let $\mathcal{S}_\epsilon$ be the set of all positive linear functionals on $\mathfrak{A}$ with norm at most one such that $f(B) \geq \epsilon$. Clearly $\mathcal{S}_\epsilon$ is compact in the weak*-topology.

Suppose there exists a $\varphi \in \mathcal{S}_\epsilon$ such that $\varphi(\overline{A\mathfrak{A}}) = \{0\}$. Since $\varphi(B) \geq \epsilon$, $\varphi \neq 0$ so a multiple of $\varphi$ is a state on $\mathfrak{A}$. However, by the Continuous Functional Calculus for Normal Operators, there exists a sequence of functions $f_n \in C(\sigma(A))$ such that $\lim_{n\to\infty} Af_n(A) = A$. Hence $\varphi(A) = 0$ so $A$ is not a strictly positive operator. Thus we may assume that each element in $\mathcal{S}_\epsilon$ does not vanish on $\overline{A\mathfrak{A}}$.

For each $\varphi \in \mathcal{S}_\epsilon$ choose an element $A_\varphi \in \overline{A\mathfrak{A}}$ such that $\varphi(A_\varphi) \neq 0$. Therefore, by the Cauchy Schwarz Inequality for positive linear functionals, $0 < |\varphi(A_\varphi)|^2 \leq \|\varphi\| \, \varphi(A_\varphi A_\varphi^*)$. Hence there exists a weak*-neighbourhood $U_\varphi$ of $\varphi$ in $\mathcal{S}_\epsilon$ such that $\psi(A_\varphi A_\varphi^*) > 0$ for all $\psi \in U_\varphi$. Therefore, since $\mathcal{S}_\epsilon$ is weak*-compact, there exists $A_1, \ldots, A_n \in \overline{A\mathfrak{A}}$ such that

$$\varphi(A_1A_1 + \cdots + A_nA_n^*) > 0$$

for all $\varphi \in \mathcal{S}_\epsilon$.

Let $M_\epsilon := \inf\{\varphi(A_1A_1 + \cdots + A_nA_n^*) \mid \varphi \in \mathcal{S}_\epsilon\}$. Since $\mathcal{S}_\epsilon$ is weak*-compact, it is clear that $M_\epsilon > 0$. Let $T_\epsilon := \frac{1}{M_\epsilon}(A_1A_1 + \cdots + A_nA_n^*)$ which is a positive element in $\overline{A\mathfrak{A}}$ (as $\overline{A\mathfrak{A}}$ is a right ideal). Hence

$$\varphi(T_\epsilon + \epsilon I_{\tilde{\mathfrak{A}}} - B) \geq 1 + \epsilon \|\varphi\| - 1 = \epsilon > 0$$

for all $\varphi \in \mathcal{S}_\epsilon$. Moreover, if $\varphi$ is a state on $\mathfrak{A}$ that is not in $\mathcal{S}_\epsilon$ then

$$\varphi(T_\epsilon + \epsilon I_{\tilde{\mathfrak{A}}} - B) \geq 0 + \epsilon - \epsilon \geq 0.$$

Hence $\varphi(T_\epsilon + \epsilon I_{\tilde{\mathfrak{A}}} - B) \geq 0$ for all states $\varphi$ on $\mathfrak{A}$ and hence $B \leq T_\epsilon + \epsilon I_{\tilde{\mathfrak{A}}}$.

Notice that $B^{\frac{1}{2}}\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1}T_\epsilon^{\frac{1}{2}}$ is a well-defined element in $\mathfrak{A}$ such that

$$\begin{aligned}
& \left\| B^{\frac{1}{2}} - B^{\frac{1}{2}}\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1}T_\epsilon^{\frac{1}{2}} \right\|^2 \\
= & \left\| \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}} - T_\epsilon^{\frac{1}{2}}\right)\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} B \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1}\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}} - T_\epsilon^{\frac{1}{2}}\right) \right\| \\
= & \ \epsilon^2 \left\| \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} B \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} \right\| \\
\leq & \ \epsilon^2 \left\| \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} (T_\epsilon + \epsilon I_{\tilde{\mathfrak{A}}}) \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} \right\| \\
\leq & \ \epsilon
\end{aligned}$$

as $f(x) = \epsilon^2 \frac{x^2+\epsilon}{(x+\epsilon)^2}$ obtains its maximum at $x = 0$. Thus

$$B = \lim_{\epsilon \to 0^+} T_\epsilon^{\frac{1}{2}}\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} B \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} T_\epsilon^{\frac{1}{2}}.$$

However, $T_\epsilon^{\frac{1}{2}} \in \overline{A\mathfrak{A}}$ as $T_\epsilon \in \overline{A\mathfrak{A}}$, $T_\epsilon^{\frac{1}{2}}$ is a limit of polynomials in $T_\epsilon$ that vanish at zero, and $\overline{A\mathfrak{A}}$ is a closed right ideal. Furthermore, since $\left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} B \left(T_\epsilon^{\frac{1}{2}}+\epsilon I_{\tilde{\mathfrak{A}}}\right)^{-1} T_\epsilon^{\frac{1}{2}} \in \mathfrak{A}$, we obtain that $B \in \overline{A\mathfrak{A}}$ which is a contradiction. Hence if $A\mathfrak{A}$ is not dense in $\mathfrak{A}$, $A$ is not a strictly positive element. □

**Theorem 5.18.** *Let $\mathfrak{A}$ be a C*-algebra and let $\mathfrak{B}$ be a C*-subalgebra of $\mathfrak{A}$. Suppose there exists a strictly positive element $A$ of $\mathfrak{A}$ such that $A \in \mathfrak{B}$. Then there exists a unital, injective $^*$-homomorphism $\pi : \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{A})$.*

*Proof.* Let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a faithful, non-degenerate representation. Therefore Theorem 3.6 implies that $\mathcal{ID}(\pi(\mathfrak{A})) = \mathcal{M}(\mathfrak{A})$.

We claim that $\pi|_{\mathfrak{B}}$ is a faithful, non-degenerate representation of $\mathfrak{B}$. Clearly $\pi|_{\mathfrak{B}}$ is faithful as $\pi$ is faithful. To see that $\pi|_{\mathfrak{B}}$ is non-degenerate, we notice that $\overline{\pi(A)\mathcal{H}}$ is dense in $\mathcal{H}$ by Lemma 5.12. Hence $\pi|_{\mathfrak{B}}$ is a faithful, non-degenerate representation of $\mathfrak{B}$. Therefore Theorem 3.6 implies that $\mathcal{M}(\mathfrak{B}) = \mathcal{ID}(\pi(\mathfrak{B}))$.

To complete the proof it suffices to show that if $T \in \mathcal{ID}(\pi(\mathfrak{B}))$ then $T \in \mathcal{ID}(\pi(\mathfrak{A}))$. Suppose $T \in \mathcal{ID}(\pi(\mathfrak{B}))$. Hence $T\pi(A) \in \pi(\mathfrak{B})$ so $T\pi(A\mathfrak{A}) \subseteq \pi(\mathfrak{B})\pi(\mathfrak{A}) \subseteq \pi(\mathfrak{A})$. Therefore, as $A$ is a strictly positive element of $\mathfrak{A}$, Proposition 5.17 implies that $A\mathfrak{A}$ is dense in $\mathfrak{A}$ so $T\pi(\mathfrak{A}) \subseteq \pi(\mathfrak{A})$. Similarly $\pi(\mathfrak{A})T \subseteq \pi(\mathfrak{A})$ so $T \in \mathcal{ID}(\pi(\mathfrak{A})$ as desired. □

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