Partial Isometries

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Abstract

The purpose of this document is to define and develop the basic properties of partial isometries. Partial isometries are useful tools in the theory of C*-algebra and von Neumann algebras as they allow for the construction of an equivalence relation on the set of projections. The reader of these notes need only a basic knowledge of the bounded linear maps on a Hilbert space. Note that all inner products in this document are linear in the first variable.

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We begin with the definition of an isometry.

Definition. A bounded linear operator $V \in \mathcal{B}(\mathcal{H})$ is said to be an isometry if $\|V\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$.

It is useful to note the following properties of isometries.

Lemma. *Let $V \in \mathcal{B}(\mathcal{H})$ be an isometry. Then $\langle V\xi, V\eta\rangle = \langle \xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$.*

Proof: Fix $\xi, \eta \in \mathcal{H}$. Notice

$$\begin{aligned}
\|\xi\|^2 + 2Re(\langle \xi, \eta\rangle) + \|\eta\|^2 &= \langle \xi + \eta, \xi + \eta\rangle \\
&= \|\xi + \eta\|^2 \\
&= \|V(\xi + \eta)\|^2 \\
&= \langle V(\xi + \eta), V(\xi + \eta)\rangle \\
&= \|V\xi\|^2 + 2Re(\langle V\xi, V\eta\rangle) + \|V\eta\|^2 \\
&= \|\xi\|^2 + 2Re(\langle V\xi, V\eta\rangle) + \|\eta\|^2 .
\end{aligned}$$

Hence $Re(\langle \xi, \eta\rangle) = Re(\langle V\xi, V\eta\rangle)$. By repeating the above with η replaced with $i\eta$, we obtain that

$$Im(\langle \xi, \eta\rangle) = Re(-i\langle \xi, \eta\rangle) = Re(-i\langle V\xi, V\eta\rangle) = Im(\langle V\xi, V\eta\rangle).$$

Hence $\langle V\xi, V\eta\rangle = \langle \xi, \eta\rangle$ as desired. □

Proposition. *An operator $V \in \mathcal{B}(\mathcal{H})$ is an isometry if and only if $V^*V = I_{\mathcal{H}}$.*

Proof: Suppose $V \in \mathcal{B}(\mathcal{H})$ is an isometry. Then $\langle \xi, \eta\rangle = \langle V\xi, V\eta\rangle = \langle V^*V\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$. Hence $\langle (I_{\mathcal{H}} - V^*V)\xi, \eta\rangle = 0$ for all $\xi, \eta \in \mathcal{H}$. Therefore $V^*V = I_{\mathcal{H}}$.

Suppose $V \in \mathcal{B}(\mathcal{H})$ is such that $V^*V = I_{\mathcal{H}}$. Then for all $\xi \in \mathcal{H}$

$$\|V\xi\|^2 = \langle V\xi, V\xi\rangle = \langle V^*V\xi, \xi\rangle = \langle \xi, \xi\rangle = \|\xi\|^2 .$$

Hence $\|V\xi\| = \|\xi\|$ for all $\xi \in \mathcal{H}$ so V is an isometry. □

Based on the above proposition and the GNS construction, we make the following definition.

Definition. Let $\mathfrak{A}$ be a unital C*-algebra. An operator $V \in \mathfrak{A}$ is said to be an isometry if $V^*V = I_{\mathfrak{A}}$.

With the basic theory of isometries complete, we turn our attention to the theory of partial isometries.

Definition. A bounded linear operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a partial isometry if $V|_{ker(V)^\perp}$ is an isometry; that is, for every $\xi \in ker(V)^\perp$, $\|V\xi\| = \|\xi\|$.

The following theorem contains all essential basic properties about a given partial isometry.

Theorem. *Let $V \in \mathcal{B}(\mathcal{H})$. The following are equivalent:*

1. *V is a partial isometry.*
2. *V^* is a partial isometry.*
3. *VV^* is a projection.*
4. *V^*V is a projection.*
5. *$V^*VV^* = V^*$*
6. *$VV^*V = V$*

Moreover, the range of V is closed, VV^* is the projection onto $ran(V)$, and V^*V is the projection onto $ker(V)^\perp$.

PROOF: 1) implies 5): Suppose V is a partial isometry. Fix $\xi \in \mathcal{H}$ and consider $\langle V^*VV^*\xi, \eta\rangle$ and $\langle V^*\xi, \eta\rangle$ for $\eta \in \mathcal{H}$. If $\eta \in ker(V)$ then

$$\langle V^*VV^*\xi, \eta\rangle = \langle VV^*\xi, V\eta\rangle = 0 = \langle \xi, V\eta\rangle = \langle V^*\xi, \eta\rangle.$$

However, since V is an isometry on $ker(V)^\perp = \overline{ran(V^*)}$ and thus preserves the inner product of two elements of $\overline{ran(V^*)}$ (see above), if $\eta \in ker(V)^\perp = \overline{ran(V^*)}$ then

$$\langle V^*VV^*\xi, \eta\rangle = \langle V(V^*\xi), V\eta\rangle = \langle V^*\xi, \eta\rangle.$$

Since $ker(V) \oplus \ker(V)^\perp = \mathcal{H}$, we obtain that $\langle V^*VV^*\xi, \eta\rangle = \langle V^*\xi, \eta\rangle$ for all $\xi, \eta \in \mathcal{H}$ so $V^*VV^* = V^*$.

5) if and only if 6): Notice $V^*VV^* = V^*$ if and only if $V = (V^*)^* = (V^*VV^*)^* = VV^*V$.

5) implies 3) and 4): Notice VV^* is self-adjoint and $VV^*VV^* = V(V^*VV^*) = VV^*$ by our assumptions of 5). Thus VV^* is a projection. Similarly V^*V is self-adjoint and $V^*VV^*V = (V^*VV^*)V = V^*V$ so V^*V is a projection.

3) implies 1): Suppose VV^* is a projection and let $\xi \in ker(V)^\perp = \overline{ran(V^*)}$. Then there exists a sequence $(\xi_n)_{n\geq 1} \in \mathcal{H}$ such that $\lim_{n\to\infty} V^*\xi_n = \xi$. Notice

$$\begin{aligned}
\|V\xi\|^2 &= \lim_{n\to\infty} \|VV^*\xi_n\|^2 \\
&= \lim_{n\to\infty} \langle VV^*\xi_n, VV^*\xi_n\rangle \\
&= \lim_{n\to\infty} \langle (VV^*)^2\xi_n, \xi_n\rangle \\
&= \lim_{n\to\infty} \langle VV^*\xi_n, \xi_n\rangle \\
&= \lim_{n\to\infty} \|V^*\xi_n\|^2 = \|\xi\|^2.
\end{aligned}$$

Thus, as $\|V\xi\| = \|\xi\|$ for all $\xi \in ker(V)^\perp$, V is a partial isometry.

Rest of the Proof: In the above, we have shown that 1), 3), 5), and 6) are equivalent. By applying these equivalences to V^* instead of V, we obtain that 2), 4), 6), and 5) are equivalent. Whence all of the above are equivalent.

Suppose V satisfies the above six equivalences. To see that $ran(V)$ is closed, suppose $\xi \in \overline{ran(V)}$. Then there exists a sequence of vectors $(\xi_n)_{n\geq 1} \in \mathcal{H}$ such that $\xi = \lim_{n\to\infty} V\xi_n$. Then

$$V(V^*\xi) = \lim_{n\to\infty} V(V^*(V\xi_n)) = \lim_{n\to\infty} V\xi_n = \xi.$$

Hence $\xi \in ran(V)$ so $ran(V)$ is closed.

Next we desired to show that VV^* is the projection onto the range of V. To begin, suppose that $\xi \in ran(V)$. Then $\xi = V\eta$ for some $\eta \in \mathcal{H}$ so

$$VV^*\xi = VV^*V\eta = V\eta = \xi.$$

However, if $\xi \in (ran(V))^\perp = ker(V^*)$, clearly $VV^*\xi = 0$. Thus VV^* is clearly the orthogonal projection onto $ran(V)$.

To see that V^*V is the projection onto $ker(V)^\perp = \overline{ran(V^*)}$, we notice that V^* is a partial isometry and thus $ran(V^*)$ is closed by the above proof. Whence $ker(V)^\perp = ran(V^*)$. Since V^*V is the projection onto $ran(V^*)$ by the above paragraph, the proof is complete. □

Based on the above theorem, we make the following definition for C*-algebras and trivially obtain the subsequent result by the GNS construction.

Definition. Let $\mathfrak{A}$ be a C*-algebra. An operator $V \in \mathfrak{A}$ is said to be a partial isometry if V^*V is a projection.

Corollary. *Let $\mathfrak{A}$ be a C*-algebra. If $V \in \mathfrak{A}$ is a partial isometry then VV^* is also a projection. Hence $V^* \in \mathfrak{A}$ is also a partial isometry. Moreover, if $P := V^*V$ and $Q := VV^*$, then $VP = V$ and $QV = V$.*