Separable Exact C*-Algebras Embed Into the Cuntz Algebra

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July 8, 2016

Abstract

Do you want to read these notes? You sure you want to know? The proofs contained in this document are not for the faint of heart. If somebody said it was a happy little tale, if somebody told you it was just your average straightforward proof not a technicality in sight... somebody lied.

The purpose of these notes is to prove the embedding theorem of Kirchberg and Phillips, Theorem 11.11, that states every unital, separable, exact C*-algebra has a unital embedding into the Cuntz algebra $\mathcal{O}_2$. These notes are based are based on several references that will be acknowledged at the beginning of each chapter. These notes are meant to be as self-contained as possible except for some well-known results in theory of C*-algebras. In particular, the author assumes the reader has a basic knowledge of the following ideas:

1. Basic C*-Algebra Theory (including C*-norms, invertible elements, normal operators, self-adjoint operators, positive operators, continuous functional calculus, abelian C*-algebras, finite dimensional C*-algebras, polar decomposition, ideals, quotients, pure states, representations, irreducible representations, GNS, continuity of *-homomorphisms, compact operators, C*-bounded approximate identities, quasicentral C*-bounded approximate identities)
2. Basic von Neumann Algebra Theory (WOT-convergence, SOT-convergence, von Neumann's Double Commutant Theory, Borel functional calculus, partial isometries, Murray von Neumann equivalence of projections, polar decomposition, commutants, the Strong Kadison Transitivity Theorem)
3. Completely Positive Maps (definitions, operator systems, completely bounded norms, Stinespring's Theorem, Arveson's Extension Theorem, injectivity, conditional expectations, point-norm topology, bounded-weak topology)
4. Tensor Products of C*-Algebras (minimal and maximal tensor products, theory of states and representations on tensor products)
5. Nuclear C*-Algebras (tensor product and completely positive map definition, examples of nuclear C*-algebras)
6. Exact C*-Algebras (completely positive map and tensor product definition, examples of exact C*-algebras)
7. Inductive Limits of C*-Algebras (including AF C*-algebras)
8. Cross Products of C*-Algebras (definitions of reduced and full cross products, cross product of a nuclear C*-algebra by $\mathbb{Z}$ is nuclear, reduced and full cross products by $\mathbb{Z}$ are the same)
9. Quasidiagonal C*-Algebras (definition given, cones of C*-algebras are quasidiagonal)

Two excellent references that cover most of these topics are [Da] and [BO].

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Contents

1 Basic Properties of the Cuntz Algebras

In this chapter we will develop some basic properties of the Cuntz algebras. To be more specific, we will show that the Cuntz algebras are simple and nuclear. In fact, in our proof that the Cuntz algebras are simple we will prove a stronger result which, in the next chapter, will imply that the Cuntz algebras are purely infinite.

The results for this chapter were developed from the excellent book [Da] (if you are reading these notes, you should definitely invest in this book) and from the original paper [Cu2]. Note that Lemma V.4.5 in [Da] has a small problem at the end as Lemma V.4.4 does not apply directly. In these notes, we modify Lemma V.4.4 to correct this mistake.

We begin with the definition of the Cuntz algebras.

Definition 1.1. For a natural number $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal C*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ such that $\sum_{i=1}^n S_i S_i^* = I$. The Cuntz algebra $\mathcal{O}_\infty$ is the universal C*-algebra generated by an infinite collection of isometries $\{S_i\}_{i=1}^\infty$ such that $\sum_{i=1}^n S_i S_i^* \leq I$ for all $n \in \mathbb{N}$.

Remarks 1.2. The statement "the Cuntz algebra $\mathcal{O}_n$ is the universal C*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ such that $\sum_{i=1}^n S_i S_i^* = I$" means that if $\mathfrak{A}$ is any C*-algebra with n isometries $\{T_i^{\mathfrak{A}}\}_{i=1}^n \subseteq \mathfrak{A}$ such that $\sum_{i=1}^n T_i^{\mathfrak{A}} (T_i^{\mathfrak{A}})^* = I_{\mathfrak{A}}$ (such C*-algebra exists by considering the specific isometries in $\mathcal{B}(\mathcal{H})$) then there exists a C*-homomorphism $\pi : \mathcal{O}_n \to \mathfrak{A}$ such that $\pi(S_i) = T_i^{\mathfrak{A}}$. We note that such a universal C*-algebra exists by taking a direct sum of all such C*-algebras and the fact that $\left\|\oplus_{\mathfrak{A}} T_i^{\mathfrak{A}}\right\| = 1$ for all i so the norm of any element in *-$alg\{\oplus_{\mathfrak{A}} T_1^{\mathfrak{A}}, \ldots, \oplus_{\mathfrak{A}} T_n^{\mathfrak{A}}\}$ is finite. The same remarks apply for $\mathcal{O}_\infty$.

Remarks 1.3. Clearly $\mathcal{O}_\infty$ and each $\mathcal{O}_n$ are separable being the closure of a *-algebra generated by a countable number of operators. Using the fact that the S_i's in $\mathcal{O}_n$ are isometries and $\sum_{i=1}^n S_i S_i^* = I$ imply that $S_i S_i^*$ are projections and thus $S_i^* S_j = 0$ if $i \neq j$ (and the same for $\mathcal{O}_\infty$). Thus $S_i^* S_j = \delta_{i,j} I$.

To discuss the Cuntz algebras, it is useful to develops some notation.

Notation 1.4. For a word $\mu = (i_1, i_2, \ldots, i_m)$ with $i_j \in \{1, 2, \ldots, n\}$ (or $\mathbb{N}$ for $\mathcal{O}_\infty$), we define

$$S_\mu := S_{i_1} S_{i_2} \cdots S_{i_m}.$$

Let $|\mu|$ denote the length of the word μ.

With this notation in hand, we make the following observations using Remarks 1.3.

Lemma 1.5. *Let μ and ν be words in $\{1, \ldots, n\}$ (or $\mathbb{N}$) such that $S_\mu^* S_\nu \neq 0$. Then*

1. *If $|\mu| = |\nu|$ then $\mu = \nu$ and $S_\mu^* S_\nu = I$.*
2. *If $|\mu| > |\nu|$ then there exists a word μ' such that $\mu = \nu\mu'$ (as words) and $S_\mu^* S_\nu = S_{\mu'}^*$.*
3. *If $|\mu| < |\nu|$ then there exists a word ν' such that $\nu = \mu\nu'$ (as words) and $S_\mu^* S_\nu = S_{\nu'}$.*

As a simple corollary, we have the following.

Corollary 1.6. *For $n \geq 2$ or $n = \infty$, every element in *-$alg\{S_i\}_{i=1}^n$ can be written as a linear combination of elements of the form $S_\mu S_\nu^*$ where μ and ν are words with letters in $\{1, \ldots, n\}$.*

To prove the desired properties of the Cuntz algebras, we will need a specific C*-subalgebra which will be of vital importance.

Notation 1.7. For each $n \geq 2$ or $n = \infty$ and for each $k \in \mathbb{N}$, let

$$\mathfrak{F}_k^n := span\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k, \mu \text{ and } \nu \text{ are words with letters in } \{1, \ldots, n\}\}$$

Let $\mathfrak{F}^n = \overline{\bigcup_{k \geq 1} \mathfrak{F}_k^n}$. Notice that $\mathfrak{F}^n, \mathfrak{F}_k^n \subseteq \mathcal{O}_m$ for all $m \geq n$.

Lemma 1.8. *For $n \geq 2$, $\mathfrak{F}_k^n \simeq \mathcal{M}_{n^k}(\mathbb{C})$ and $\mathfrak{F}^n$ is the UHF algebra with supernatural number n^∞. Moreover $\mathfrak{F}_k^\infty \simeq \mathfrak{K}$ and $\mathfrak{F}^\infty$ is an AF C^*-algebra.*

Proof. To see that $\mathfrak{F}_k^n \simeq \mathcal{M}_{n^k}(\mathbb{C})$, we simply note that the set

$$\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k, \mu \text{ and } \nu \text{ are words with letters in } \{1, \ldots, n\}\}$$

is a set of matrix units for $\mathfrak{F}_k^n$ by Lemma 1.5 with precisely n^k elements. To see that $\mathfrak{F}^n$ is the UHF algebra with supernatural number n^∞, we need to analyze the embeddings of $\mathfrak{F}_k^n$ into $\mathfrak{F}_{k+1}^n$.

To see that $\mathfrak{F}_k^n$ embeds into $\mathfrak{F}_{k+1}^n$ with the 'correct' embedding, we notice for any word μ and ν with letters in $\{1, \ldots, n\}$ and $|\mu| = |\nu| = k$ that

$$S_\mu S_\nu^* = S_\mu \left(\sum_{i=1}^n S_i S_i^* \right) S_\nu^* = \sum_{i=1}^n S_{\mu i} S_{\nu i}^*$$

Therefore, by grouping the matrix units of $\mathfrak{F}_{k+1}^n$ in the appropriate way, we obtain that $\mathfrak{F}^n$ is the UHF algebra with supernatural number n^∞.

The proof that $\mathfrak{F}_k^\infty \simeq \mathfrak{K}$ is identical. To see that $\mathfrak{F}^\infty$ is AF, we note since $\mathfrak{F}_k^n \subseteq \mathfrak{F}_{k+1}^n \subseteq \mathfrak{F}_{k+1}^{n+1}$ for all n that $\mathfrak{F}^\infty = \overline{\bigcup_{n \geq 1} \mathfrak{F}_n^n}$ and thus the result follows. $\square$

Next we note that there exists a very important map from $\mathcal{O}_n$ to $\mathfrak{F}^n$ for all $n \geq 2$ and $n = \infty$.

Theorem 1.9. *There exists a faithful conditional expectation $\Phi_n : \mathcal{O}_n \to \mathfrak{F}^n$ for all $n \geq 2$ or $n = \infty$. That is, $\Phi_n : \mathcal{O}_n \to \mathfrak{F}^n$ is a unital, (completely) positive map such that $\Phi_n(T) = T$ for all $T \in \mathfrak{F}^n$.*

Proof. Fix $n \geq 2$ or $n = \infty$. For each $\lambda \in \mathbb{T}$, we notice that $\{\lambda S_i\}_{i=1}^n$ are also a set of isometries that satisfy the universal property of the Cuntz algebras. Therefore there must exists an *-automorphism ρ_λ of $\mathcal{O}_n$ such that $\rho_\lambda(S_i) = \lambda S_i$. Hence $\rho_\lambda(S_i^*) = \lambda^{-1} S_i$ and $\rho_\lambda(S_\mu S_\nu^*) = \lambda^{|\mu| - |\nu|} S_\mu S_\nu^*$. Thus the map from $\mathbb{T}$ to $\mathcal{O}_n$ defined by $\lambda \mapsto \rho_\lambda(T)$ is continuous for all $T \in {}^*\text{-}alg\{S_i\}_{i=1}^n$. Therefore, since ${}^*\text{-}alg\{S_i\}_{i=1}^n$ is dense in $\mathcal{O}_n$ and $\|\rho_\lambda\| = 1$ for all $\lambda \in \mathbb{T}$, the map $\mathbb{T}$ to $\mathcal{O}_n$ defined by $\lambda \mapsto \rho_\lambda(T)$ is continuous for all $T \in \mathcal{O}_n$.

Define $\Phi_n : \mathcal{O}_n \to \mathcal{O}_n$ by

$$\Phi_n(T) = \int_{\mathbb{T}} \rho_\lambda(T) d\lambda$$

which exists by continuity. We notice for all words μ and ν with letters in $\{1, 2, \ldots, n\}$ that

$$\Phi_n(S_\mu S_\nu^*) = \int_{\mathbb{T}} \lambda^{|\mu| - |\nu|} S_\mu S_\nu^* d\lambda = \begin{cases} 0 & \text{if } |\mu| \neq |\nu| \\ S_\mu S_\nu^* & \text{if } |\mu| = |\nu| \end{cases}$$

Hence it is easy to see that Φ_n maps into $\mathfrak{F}^n$. Moreover, if $T \in \mathfrak{F}_k^n$ then $\Phi_n(T) = T$. Hence, by extending by continuity, $\Phi_n|_{\mathfrak{F}^n} = Id_{\mathfrak{F}^n}$. In addition, since each ρ_λ is a *-homomorphism and the integration of positive (or matrices of positive) operators is positive, Φ_n is a conditional expectation onto $\mathfrak{F}^n$.

To see that Φ_n is faithful, let $T \in \mathcal{O}_n$ be positive with $T \neq 0$. Therefore there exists a state φ on $\mathcal{O}_n$ such that $\varphi(T) > 0$. Since $\rho_1(T) = T$, $\rho_\lambda(T) \geq 0$ for all λ, and $\lambda \mapsto \rho_\lambda(T)$ is continuous, the function $\lambda \mapsto \varphi(\rho_\lambda(T))$ is a continuous function from $\mathbb{T}$ to $[0, \infty)$ that is strictly positive at 1. Hence standard integration theory implies

$$\phi(\Phi_n(T)) = \int_{\mathbb{T}} \phi(\rho_\lambda(T)) d\lambda > 0$$

so $\Phi_n(T) \neq 0$. Hence Φ_n is faithful. $\square$

To prove that $\mathcal{O}_n$ is simple, the above conditional expectation will need to be examined further. To begin, we need a technical lemma.

Lemma 1.10. *Let $n \geq 2$ or $n = \infty$. Let μ and ν be words in $\{1, 2, \ldots, n\}$ such that $|\mu| \neq |\nu|$. Let $m \geq \max\{|\mu|, |\nu|\}$ and let $S_\gamma = S_1^m S_2$. Then $S_\gamma^*(S_\mu^* S_\nu) S_\gamma = 0$.*

Proof. Since $|\mu| \neq |\nu|$, Lemma 1.5 implies that if $S_\mu^* S_\nu \neq 0$, then either $S_\mu^* S_\nu = S_{\mu'}^*$ where μ' is a word of length at least one and at most m or $S_\mu^* S_\nu = S_{\nu'}$ where ν' is a word of length at least one and at most m. In the first case, $(S_\mu^* S_\nu)S_\gamma = S_{\mu'}^* S_\gamma$ is non-zero only if $S_{\mu'} = S_1^{|\mu'|}$ as $|\mu'| \leq m$. However, if $S_{\mu'} = S_1^{|\mu'|}$ then

$$S_\gamma^*(S_\mu^* S_\nu)S_\gamma = S_\gamma^*(S_1^*)^{|\mu'|}S_\gamma = S_2^*(S_1^*)^m S_1^{m-|\mu'|}S_2 = 0$$

as $S_1^* S_2 = 0$.

In the second case, $S_\gamma^*(S_\mu^* S_\nu) = S_\gamma^* S_{\nu'}$ is non-zero only if $S_{\nu'} = S_1^{|\nu'|}$ as $|\nu'| \leq m$. However, if $S_{\nu'} = S_1^{|\nu'|}$ then

$$S_\gamma^*(S_\mu^* S_\nu)S_\gamma = S_\gamma^*(S_1)^{|\nu'|}S_\gamma = S_2^*(S_1^*)^{m-|\nu'|}S_1^m S_2 = 0$$

as $S_2^* S_1 = 0$. □

Theorem 1.11. *Let $n \geq 2$. For each $m \in \mathbb{N}$ there exists an isometry $W_{n,m} \in \mathcal{O}_n$ that commutes with $\mathfrak{F}_m^n$ such that $\Phi_n(T) = W_{n,m}^* T W_{n,m} \in \mathfrak{F}_m^n$ for all*

$$T \in span\{S_\mu S_\nu^* \mid |\mu|, |\nu| \leq m, \mu \text{ and } \nu \text{ are words with letters in } \{1, \ldots, n\}\}.$$

Proof. Let $S_\gamma = S_1^m S_2$ and let $W_{n,m} = \sum_{|\delta|=m} S_\delta S_\gamma S_\delta^*$. We claim that $W_{n,m}$ is an isometry. To see this, we notice that

$$W_{n,m}^* W_{n,m} = \sum_{|\epsilon|=|\delta|=m} S_\epsilon S_\gamma^* S_\epsilon^* S_\delta S_\gamma S_\delta^* = \sum_{|\delta|=m} S_\delta S_\gamma^* S_\gamma S_\delta^* = \sum_{|\delta|=m} S_\delta S_\delta^* = I$$

where $\sum_{|\delta|=m} S_\delta S_\delta^* = I$ comes from the fact that $\sum_{i=1}^n S_i S_i^* = I$, by dividing the sum into all S_δ that start with the same $m-1$ letters, using the identity to decrease the length of the words, and repeating.

To see that $W_{n,m}$ commutes with $\mathfrak{F}_m^n$ (and to begin to obtain the other equality), we notice that if μ is a word of length m then

$$W_{n,m} S_\mu = S_\mu S_\gamma \qquad \text{and} \qquad S_\mu^* W_{n,m} = S_\gamma S_\mu^*$$

Therefore, if $S_\mu S_\nu^*$ is one of the matrix units for $\mathfrak{F}_m^n$ (so $|\mu| = |\nu| = m$) then

$$W_{n,m} S_\mu S_\nu^* = S_\mu S_\gamma S_\nu^* = S_\mu S_\nu^* W_{n,m}$$

Hence $W_{n,m}$ must commute with $\mathfrak{F}_m^n$. Moreover, from the above computation, the fact that $W_{n,m}$ is an isometry, and our knowledge of Φ_n from Theorem 1.8, we obtain that $W_{n,m}^* S_\mu S_\nu W_{n,m} = S_\mu S_\nu = \Phi_n(S_\mu S_\nu^*)$.

Next notice that if μ and ν are words with letters in $\{1, \ldots, n\}$ of length at most m with $|\mu| \neq |\nu|$, then

$$W_{n,m}^* S_\mu S_\nu^* W_{n,m} = \sum_{|\epsilon|=|\delta|=m} S_\delta S_\gamma^* S_\delta^* S_\mu S_\nu^* S_\epsilon S_\gamma S_\epsilon^* = 0 = \Phi_n(S_\mu S_\nu^*)$$

as if $S_\delta^* S_\mu S_\nu^* S_\epsilon$ is non-zero, it can be written as $S_{\mu'}^* S_\nu$ with $|\mu'| = m - |\mu| \neq m - |\nu| = |\nu'|$ and so $S_\gamma^* S_\delta^* S_\mu S_\nu^* S_\epsilon S_\gamma = S_\gamma^* S_{\mu'}^* S_{\nu'} S_\gamma = 0$ by Lemma 1.9. Hence the result follows. □

Using the above proof, it is easy to prove the following for $\mathcal{O}_\infty$.

Theorem 1.12. *Let $n \geq 2$. For each $m \in \mathbb{N}$ there exists an isometry $W'_{n,m} \in \mathcal{O}_\infty$ such that $\Phi_\infty(T) = (W'_{n,m})^* T W'_{n,m} \in \mathfrak{F}_m^n \subseteq \mathcal{O}_\infty$ for all*

$$T \in span\{S_\mu S_\nu^* \mid |\mu|, |\nu| \leq m, \mu \text{ and } \nu \text{ are words with letters in } \{1, \ldots, n\}\}$$

Using the above isometries and some clever tricks, we are finally able to prove the following.

Theorem 1.13. *Let $n \geq 2$. If $X \in \mathcal{O}_n$ is non-zero then there exists $A, B \in \mathcal{O}_n$ such that $AXB = I$.*

Proof. Since $X \neq 0$, $X^*X \neq 0$ and thus $\Phi_n(X^*X) \neq 0$ as Φ_n is faithful. Hence we may assume without loss of generality that $\|\Phi_n(X^*X)\| = 1$. By density, we can choose Y in the algebraic span of elements of the form $S_\mu S_\nu^*$ such that $\|X^*X - Y\| < \frac{1}{4}$. By considering the real part of Y, we may assume that Y is self-adjoint. Thus $\|\Phi_n(X^*X) - \Phi_n(Y)\| \leq \frac{1}{4}$ so $\|\Phi_n(Y)\| \geq \frac{3}{4}$.

Since Y is in the algebraic span of elements of the form $S_\mu S_\nu^*$, there exists an $m \in \mathbb{N}$ such that Y is a linear combination of elements of the form $S_\mu S_\nu^*$ where $|\mu|, |\nu| \leq m$. Therefore, by Theorem 1.10, there exists an isometry $W_{n,m}$ such that $\Phi_n(Y) = W_{n,m}^* Y W_{n,m} \in \mathfrak{F}_m^n$. Since $\|\Phi_n(Y)\| \geq \frac{3}{4}$ and $\Phi_n(Y)$ is a self-adjoint element of a matrix algebra, there exists a rank one projection $P \in \mathfrak{F}_m^n$ such that

$$P\Phi_n(Y) = \Phi_n(Y)P = \|\Phi_n(Y)\| P \geq \frac{3}{4}P$$

Moreover, since P and $S_1^m(S_1^*)^m$ are both rank one projections in $\mathfrak{F}_m^n$, there exists an isometry $U \in \mathfrak{F}_m^n$ such that $UPU^* = S_1^m(S_1^*)^m$.

Finally, let

$$Z := \frac{1}{\|\Phi_n(Y)\|^{\frac{1}{2}}}(S_1^*)^m UPW_{n,m}^* \in \mathcal{O}_n.$$

Then

$$\|Z\| \leq \frac{1}{\|\Phi_n(Y)\|^{\frac{1}{2}}} \|S_1^*\|^m \|U\| \|P\| \left\|W_{n,m}^*\right\| \leq \frac{2}{\sqrt{3}}$$

(as S_1, U, and $W_{n,m}$ are isometries and P is a projection) and

$$ZYZ^* = \frac{1}{\|\Phi_n(Y)\|}(S_1^*)^m UPW_{n,m}^* Y W_{n,m} PU^* S_1^m = (S_1^*)^m UPU^* S_1^m = (S_1^*)^m S_1^m (S_1^*)^m S_1^m = I.$$

Hence

$$\|I - ZX^*XZ^*\| = \|Z(Y - X^*X)Z^*\| \leq \|Z\|^2 \|Y - X^*X\| \leq \frac{4}{3}\frac{1}{4} = \frac{1}{3}$$

so ZX^*XZ^* is a self-adjoint, invertible operator.

Let $B = Z^*(ZX^*XZ^*)^{-\frac{1}{2}}$. Then

$$(B^*X^*)XB = (ZX^*XZ^*)^{-\frac{1}{2}} ZX^*XZ^*(ZX^*XZ^*)^{-\frac{1}{2}} = I$$

as desired. ☐

If we follow the above proof with $n = \infty$, we notice at the step where Y is chosen that we can bound the number of letters used in the words in the algebraic expression for Y as Y is a finite sum of operators of the form $S_\mu S_\nu^*$. Therefore, by applying Theorem 1.11, we see that the remainder of the proof follows (with $W_{n,m}$ replaced with $W'_{n,m}$). Hence we obtain the following.

Theorem 1.14. *If $X \in \mathcal{O}_\infty$ is non-zero, then there exists $A, B \in \mathcal{O}_\infty$ such that $AXB = I$.*

Using the above theorems, we easily obtain the following result.

Theorem 1.15. *$\mathcal{O}_\infty$ and $\mathcal{O}_n$ are simple for all $n \geq 2$. Moreover, if $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ are isometries such that $\sum_{i=1}^n T_iT_i^* = I$, then $C^*(T_1, \ldots, T_n) \simeq \mathcal{O}_n$. In addition, if $\{T_i\}_{i=1}^\infty \in \mathcal{B}(\mathcal{H})$ are isometries such that $\sum_{i=1}^n T_iT_i^* \leq I$ for all $n \in \mathbb{N}$, then $C^*(\{T_i\}_{i=1}^\infty) \simeq \mathcal{O}_\infty$.*

Proof. The proof that the C*-algebras are simple is trivial.

If $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ are isometries such that $\sum_{i=1}^n T_iT_i^* = I$, then, by the universal property of the Cuntz algebra, there exists a *-homomorphism $\pi : \mathcal{O}_n \to C^*(T_1, \ldots, T_n)$ such that $\pi(S_i) = T_i$. Clearly this implies that π is surjective. Since $\mathcal{O}_n$ is simple and π is not the zero map, π must be injective.

The $\mathcal{O}_\infty$ proof is similar. ☐

With the above result in hand, we can prove that if $\mathfrak{A}$ is a C*-algebra generated by n isometries, then $\mathfrak{A}$ is either $\mathcal{O}_n$ or a quotient of $\mathfrak{A}$ is isomorphic to $\mathcal{O}_n$.

Lemma 1.16. *Let $\mathfrak{A}$ be a C*-algebra generated by n isometries $S_1, S_2, \ldots, S_n$ such that $\sum_{i=1}^n S_i S_i^* = P < I$. Then the ideal $\langle I - P\rangle$ generated by $I - P$ is isomorphic to the compact operators and $\mathfrak{A}/\mathfrak{K} \simeq \mathcal{O}_n$.*

Proof. Since P contains the range of each S_i, $(I-P)S_i = 0 = S_i^*(I-P)$ for all i. Therefore, since it is trivial to see that Lemma 1.5 applies to $\mathfrak{A}$, we obtain that $\langle I-P\rangle$ has

$$span\{S_\mu(I-P)S_\nu^* \mid |\mu| < \infty, |\nu| < \infty\}$$

as a dense subset. Moreover, it is trivial to verify that

$$(S_\mu(I-P)S_\nu^*)(S_{\mu'}(I-P)S_{\nu'}^*) = \begin{cases} 0 & \text{if } \nu \neq \mu' \\ S_\mu(I-P)S_{\nu'}^* & \text{if } \nu = \mu' \end{cases}$$

and thus $\{S_\mu(I-P)S_\nu^*\}$ forms an infinite collection of matrix units whose span is dense in $\langle I-P\rangle$. Hence $\langle I-P\rangle \simeq \mathfrak{K}$ as claimed.

To see that $\mathfrak{A}/\langle I-P\rangle \simeq \mathcal{O}_n$, we notice that if $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ is the canonical quotient map, then $\pi(S_i)$ are isometries in $\mathfrak{A}/\mathfrak{K}$ such that

$$\sum_{i=1}^n \pi(S_i)\pi(S_i)^* = \pi(P) = \pi(P) + \pi(I-P) = \pi(I)$$

which is the unit of $\mathfrak{A}/\mathfrak{K}$. Hence, as $\mathfrak{A}/\mathfrak{K}$ is generated by $\pi(S_i)$, we obtain that $\mathfrak{A}/\mathfrak{K} \simeq \mathcal{O}_n$ as claimed. □

Remarks 1.17. Notice that the above result implies that $\mathcal{O}_m$ contains a C*-subalgebra $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{K} \simeq \mathcal{O}_n$ for all $m > n \geq 2$. Similarly, for all $n \geq 2$, $\mathcal{O}_\infty$ contains a C*-subalgebra $\mathfrak{A}$ such that $\mathfrak{A}/\mathfrak{K} \simeq \mathcal{O}_n$ for all $n \geq 2$.

Our next goal is to show that each $\mathcal{O}_n$ and $\mathcal{O}_\infty$ are nuclear C*-algebras. The idea behind the proof is to construct a C*-algebra $\mathfrak{B}$ that is the reduced cross product of a nuclear C*-algebra $\mathfrak{A}$ by the integers and show that $\mathcal{O}_n$ is isomorphic to a compression of this cross product C*-algebra. We remark that the reduced cross product of a nuclear C*-algebra by the integers is nuclear (see Chapter 4 of [BO] for this proof and the construction of the reduced cross product. The idea of the proof of nuclearity is to compress $\mathfrak{B}$ by projections corresponding to finite subsets of $\mathbb{Z}$. This operation is a completely positive map into $\mathfrak{A} \otimes_{\min} \mathcal{M}_n(\mathbb{C})$ where n is the number of elements of the finite subset of $\mathbb{Z}$. Then a completely positive map is constructed from $\mathfrak{A} \otimes_{\min} \mathcal{M}_n(\mathbb{C})$ to $\mathfrak{B}$ that asymptotically does the right thing as long as Følner sets are taken for the finite subsets of $\mathbb{Z}$. Then $\mathfrak{B}$ is nuclear as each $\mathfrak{A} \otimes_{\min} \mathcal{M}_n(\mathbb{C})$ is nuclear. This also can be used to show that the reduced cross product is the same as the full cross product) and the compression of a nuclear C*-algebra is nuclear (as if $\mathfrak{C} \subseteq \mathfrak{D}$ are nuclear and there is a conditional expectation of $\mathfrak{D}$ onto $\mathfrak{C}$, then $\mathfrak{C}$ must be nuclear by elementary arguments). To begin this proof, we start with a fixed $n \geq 2$ as we will deal with $\mathcal{O}_\infty$ separately.

Notation 1.18. For all $j \in \mathbb{Z}$ let $\mathfrak{A}_j = \otimes_{i=j}^\infty \mathcal{M}_n(\mathbb{C})$ (where this means the closure of all operators of the form $A_j \otimes \cdots \otimes A_m \otimes I \otimes I \otimes \cdots$ with respect to the infinite tensor norm). Then $\mathfrak{A}_j \simeq \mathfrak{F}^n$ for all j.

Construction 1.19. With the notation as above, we have a canonical sequence of embeddings

$$\cdots \hookrightarrow \mathfrak{A}_3 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_1 \hookrightarrow \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_{-1} \hookrightarrow \mathfrak{A}_{-2} \hookrightarrow \cdots$$

where the inclusion $\mathfrak{A}_j \hookrightarrow \mathfrak{A}_{j-1}$ is given by $X \mapsto E_{1,1} \otimes X \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}_j \simeq \mathfrak{A}_{j-1}$ (where $\{E_{i,j}\}$ are the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$). Let $\mathfrak{B}$ be the C*-algebra that is the direct limit of this chain. Hence $\mathfrak{B}$ is an inductive limit of AF C*-algebras and thus $\mathfrak{B}$ is AF. In fact $\mathfrak{B} \simeq \mathfrak{K} \otimes_{\min} \mathfrak{F}^n$ (to see this, we notice that the embeddings do not change the $\mathfrak{A}_0 \simeq \mathfrak{F}^n$ term and $\mathfrak{K} = \lim_\to \mathcal{M}_{n^k}(\mathbb{C})$ with the embeddings $\mathcal{M}_{n^k}(\mathbb{C}) \hookrightarrow \mathcal{M}_{n^{k+1}}(\mathbb{C})$ by $T \mapsto T \oplus 0_n \oplus \cdots \oplus 0_n$). Therefore, since $\mathfrak{B}$ is AF, $\mathfrak{B}$ is nuclear.

Since each $\mathfrak{A}_j$ is isomorphic, there is a canonical automorphism of $\mathfrak{B}$, which we will denote Ψ, given by shifting the sequence to the left. Notice if $T \in \mathfrak{A}_j$ then $\Psi(T) \in \mathfrak{A}_{j+1}$ is the operator $T \in \mathfrak{A}_{j+1}$ which is the operator $E_{1,1} \otimes T$ in $\mathfrak{A}_j$.

Let $\mathfrak{C} = \mathfrak{B} \rtimes_\Psi \mathbb{Z}$. Thus $\mathfrak{C}$ is a nuclear C*-algebra by the above discussion. Let $U \in \mathfrak{C}$ be the unitary implementing Ψ (that is $\Psi(X) = UXU^*$ for all $X \in \mathfrak{B}$). Notice that $\mathfrak{C}$ is the closure of all operator of the form

$$A = \sum_{i=-N}^{N} T_i U^i$$

where $T_i \in \mathfrak{B}$ and $N \in \mathbb{N}$. By letting $\tilde{T}_i = U^{-i} T_i U^i$ (for $i < 0$), we obtain that $\mathfrak{C}$ is the closure of all operator of the form

$$A = \sum_{i<0} U^i \tilde{T}_i + T_0 + \sum_{i>0} T_i U^i$$

where $\tilde{T}_i \in \mathfrak{B}$.

Let $P \in \mathfrak{A}_0$ be the unit. Therefore $P \in \mathfrak{C}$ is a projection. Notice that

$$UPU^* = \Psi(P) = E_{1,1} \otimes P \in \mathfrak{A}_0.$$

Hence $UPU^* = P(UPU^*)$ (as P is the unit for $\mathfrak{A}_0$) and thus $UP = PUP$ as U^* is invertible. Therefore it is easy to see that

$$PT_i U^i P = (PT_i P)(UP)^i \text{ for } i > 0 \qquad \text{and} \qquad PU^i \tilde{T}_i P = (UP)^* P\tilde{T}_i P \text{ for } i < 0.$$

Let $V = UP$. Thus

$$PAP = \sum_{i<0} V^i P\tilde{T}_i P + PT_0 P + \sum_{i>0} PT_i P V^i.$$

Let $\mathfrak{E} = P\mathfrak{C}P$. Thus the above computations show that $\mathfrak{E}$ is generated by $P\mathfrak{B}P = \mathfrak{A}_0$ (think about it!) with V. Moreover $\mathfrak{E}$ is nuclear being the compression of a nuclear C*-algebra. Our goal is to show that $\mathfrak{E} \simeq \mathcal{O}_n$. To show this, it suffices by Theorem 1.14 to construct n isometries in $\mathfrak{E}$ that generate $\mathfrak{E}$ with the desired properties.

Theorem 1.20. *With n and $\mathfrak{E}$ as above, $\mathfrak{E} \simeq \mathcal{O}_n$ so $\mathcal{O}_n$ is nuclear when $n \geq 2$.*

Proof. Let $S_i = (E_{i,1} \otimes P)V$ (where $E_{i,1} \otimes P \in \mathfrak{A}_0$) for $i \in \{1, \ldots, n\}$. It suffices to show that each S_i is an isometry, $\sum_{i=1}^n S_i S_i^* = P$, and $\mathfrak{E} = C^*(S_1, \ldots, S_n)$. To begin, we notice that

$$S_i^* S_i = PU^*(E_{1,1} \otimes P)UP = P\Psi^{-1}(E_{1,1} \otimes P)P = PPP = P$$

(where any elements and tensors are viewed in $\mathfrak{A}_0$). Hence each S_i is an isometry. Moreover

$$S_j S_i^* = (E_{j,1} \otimes P)UPPU^*(E_{1,i} \otimes P) = (E_{j,1} \otimes P)\Psi(P)(E_{1,i} \otimes P) = (E_{j,1} \otimes P)(E_{1,1} \otimes P)(E_{1,i} \otimes P) = E_{i,j} \otimes P.$$

Thus

$$\sum_{i=1}^n S_i S_i^* = \sum_{i=1}^n E_{i,i} \otimes P = I \otimes P = P.$$

Thus it remains only to show that $C^*(S_1, \ldots, S_n) = \mathfrak{E}$. Since $\mathfrak{A}_0$ and V generate $\mathfrak{E}$, it suffices to show that $\mathfrak{A}_0 \cup \{V\} \subseteq C^*(S_1, \ldots, S_n)$.

To see that $\mathfrak{A}_0 \subseteq C^*(S_1, \ldots, S_n)$, we notice that $\mathfrak{A}_0 = \otimes_{i=0}^\infty \mathcal{M}_n(\mathbb{C}) = \mathcal{M}_n(\mathbb{C})^{\otimes k} \otimes \mathfrak{A}_k$. Thus, a little thought shows that

$$span \left\{ \bigcup_{k>0} \{E_{j_1,i_1} \otimes \cdots \otimes E_{j_k,i_k} \otimes P \mid P \text{ the unit of } \mathfrak{A}_k\} \right\}$$

is dense in $\mathfrak{A}_0$. To show that the above span is in $C^*(S_1, \ldots, S_n)$, we recall that $S_i S_j^* = E_{i,j} \otimes P$ and

$$\begin{aligned} S_k(E_{i,j} \otimes P)S_\ell^* &= (E_{k,1} \otimes P)(UP(E_{i,j} \otimes P)PU^*)(E_{1,\ell} \otimes P) \\ &= (E_{k,1} \otimes P)(U(E_{i,j} \otimes P)U^*)(E_{1,\ell} \otimes P) \\ &= (E_{k,1} \otimes P)(E_{1,1} \otimes (E_{i,j} \otimes P))(E_{1,\ell} \otimes P) \\ &= E_{k,\ell} \otimes (E_{i,j} \otimes P) = E_{k,\ell} \otimes E_{i,j} \otimes P \end{aligned}$$

Thus, by repeating the above arguments, we see that if $\mu = (j_1, \ldots, j_k)$ and $\nu = (i_1, \ldots, i_k)$ then

$$S_\mu S_\nu^* = E_{j_1,i_1} \otimes \cdots \otimes E_{j_k,i_k} \otimes P$$

and thus $\mathfrak{A}_0 \in C^*(S_1, \ldots, S_n)$.

Finally, to see that $V \in C^*(S_1, \ldots, S_n)$, we notice that

$$VV^* = UPU^* \in \mathfrak{A}_0.$$

Thus

$$V = UP = UP(P)P = UPU^*(E_{1,1} \otimes P)UP = VV^*(S_1) \in \mathfrak{A}_0 \cdot S_1 \in C^*(S_1, \ldots, S_n)$$

as desired.

Thus $\mathfrak{E} = C^*(S_1, \ldots, S_n) \simeq \mathcal{O}_n$ so $\mathcal{O}_n$ is nuclear. □

To prove that $\mathcal{O}_\infty$ is also nuclear, we will only sketch the differences that need to be taken and the proof will follow by similar arguments to those shown above.

Theorem 1.21. *$\mathcal{O}_\infty$ is nuclear.*

Proof. For each $j \in \mathbb{N} \cup \{0\}$ let $\mathfrak{A}_j = S_1^j \mathfrak{F}^\infty (S_1^*)^j \subseteq \mathcal{O}_\infty$. Then it is clear that $\mathfrak{A}_j \simeq \mathfrak{A}_0 = \mathfrak{F}^\infty$ for all $j \geq 0$ (by the *-homomorphism $T \mapsto (S_1^*)^j T(S_1)^j$). Moreover it is not difficult to see (but perhaps slightly annoying to write down) that $\mathfrak{A}_{j-1} \simeq \mathbb{C}I + (\mathfrak{K} \otimes_{\min} \mathfrak{A}_j)$ where the $\mathbb{C}I$ comes from $S_1^{j-1} I (S_1^*)^{j-1} \in \mathfrak{A}_{j-1}$ and

$$S_1^{j-1}(S_{i_1} S_{i_2} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_2}^* S_{j_1}^*)(S_1^*)^{j-1}$$

corresponds to the operator

$$E_{i_1,j_1} \otimes (S_1^j (S_{i_2} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_2}^*)(S_1^*)^j)$$

in $\mathfrak{K} \otimes_{\min} \mathfrak{A}_j$.

Next we extend our notation by letting $\mathfrak{A}_{j-1} = \mathbb{C}I + (\mathfrak{K} \otimes_{\min} \mathfrak{A}_j)$ for all $j \in \mathbb{Z}$. Then we can consider the sequence of C*-algebras

$$\cdots \hookrightarrow \mathfrak{A}_3 \hookrightarrow \mathfrak{A}_2 \hookrightarrow \mathfrak{A}_1 \hookrightarrow \mathfrak{A}_0 \hookrightarrow \mathfrak{A}_{-1} \hookrightarrow \mathfrak{A}_{-2} \hookrightarrow \cdots$$

where the inclusion $\mathfrak{A}_j \hookrightarrow \mathfrak{A}_{j-1}$ is given by $X \mapsto E_{1,1} \otimes X \in \mathfrak{K} \otimes_{\min} \mathfrak{A}_j \subseteq \mathfrak{A}_{j-1}$ (where $\{E_{i,j}\}_{i,j=1}^\infty$ are matrix units for $\mathfrak{K}$). Let $\mathfrak{B}$ be the C*-algebra that is the direct limit of this chain. Since each $\mathfrak{A}_j$ is AF, it is clear that $\mathfrak{B}$ is AF and thus nuclear. Since each $\mathfrak{A}_j$ is isomorphic, let Ψ be the automorphism of $\mathfrak{B}$ given by shifting to the left. The remainder of the proof follows as in the $\mathcal{O}_n$ case. □

To conclude this section, we desire to draw a relation between the various Cuntz algebras and show that the matrix algebras of certain Cuntz algebras are Cuntz algebras. We will show that certain Cuntz algebras embed into others and that $\mathcal{O}_\infty$ embeds into each $\mathcal{O}_n$.

Theorem 1.22. *$\mathcal{O}_{k(n-1)+1}$ can be unitarily embedded into $\mathcal{O}_n$ for all $k \geq 1$. Moreover $\mathcal{O}_\infty$ can be embedded in $\mathcal{O}_n$ for all $n \geq 2$.*

Proof. Fix $n \geq 2$ and $k \geq 1$. If $k = 1$ then $k(n-1)+1 = n$ so $\mathcal{O}_{k(n-1)+1}$ sits inside $\mathcal{O}_n$. Otherwise suppose $k \geq 2$. Let $\{S_1, \ldots, S_n\}$ be the generators for $\mathcal{O}_n$. Let

$$X := \{S_n^\ell S_m\}_{1 \leq m \leq n-1, 0 \leq \ell \leq k-1} \cup \{S_n^k\}.$$

Thus $|X| = k(n-1)+1$. Notice $(S_n^\ell S_m)^*(S_n^\ell S_m) = I$ and $(S_n^k)^*(S_n^k) = I$ for all ℓ, m in our ranges. Moreover

$$\begin{aligned}
S_n^k(S_n^*)^k + \sum_{m=1}^{n-1}\sum_{\ell=0}^{k-1} S_n^\ell S_m S_m^* (S_n^*)^\ell &= S_n^k(S_n^*)^k + \sum_{m=1}^{n-1} S_n^{k-1} S_m S_m^* (S_n^*)^{k-1} + \sum_{m=1}^{n-2}\sum_{\ell=0}^{k-1} S_n^\ell S_m S_m^* (S_n^*)^\ell \\
&= S_n^{k-1}\left(\sum_{m=1}^{n} S_m S_m^*\right)(S_n^*)^{k-1} + \sum_{m=1}^{n-1}\sum_{\ell=0}^{k-2} S_n^\ell S_m S_m^* (S_n^*)^\ell \\
&= S_n^{k-1}(S_n^*)^{k-1} + \sum_{m=1}^{n-1}\sum_{\ell=0}^{k-2} S_n^\ell S_m S_m^* (S_n^*)^\ell \\
&\vdots \\
&= S_n^2(S_n^*)^2 + \sum_{m=1}^{n-1}\sum_{\ell=0}^{1} S_n^\ell S_m S_m^* (S_n^*)^\ell \\
&= S_n^2(S_n^*)^2 + \sum_{m=1}^{n-1} S_n S_m S_m^* S_n^* + \sum_{m=1}^{n-1} S_m S_m^* \\
&= S_n\left(\sum_{m=1}^{n} S_m S_m^*\right) S_n^* + \sum_{m=1}^{n-1} S_m S_m^* \\
&= S_n S_n^* + \sum_{m=1}^{n-1} S_m S_m^* = I.
\end{aligned}$$

Whence X generates a copy of $\mathcal{O}_{k(n-1)+1}$ inside $\mathcal{O}_n$ as desired.

Let S_1 and S_2 be two of the generators for $\mathcal{O}_n$. Let $X = \{S_1^\ell S_2\}_{\ell \geq 0}$. Notice $(S_1^\ell S_2)^*(S_1^\ell S_2) = I$ for all $\ell \geq 0$. Moreover $(S_1^\ell S_2)^*(S_1^k S_2) = 0$ if $\ell \neq k$. Therefore $\{(S_1^\ell S_2)(S_1^\ell S_2)^*\}_{\ell \geq 0}$ are projections with orthogonal ranges (as $S_1^\ell S_2$ is an isometry) so $\sum_{\ell=0}^n (S_1^\ell S_2)(S_1^\ell S_2)^* \leq I$ for all $n \geq 0$. Whence X generates a copy of $\mathcal{O}_\infty$ inside $\mathcal{O}_n$ as desired. ☐

The following result is our first result that shows the matrix algebras of some Cuntz algebra is a Cuntz algebra.

Proposition 1.23. *If k divides n then $\mathcal{M}_k(\mathcal{O}_n)$ is isomorphic to $\mathcal{O}_n$.*

Proof. Suppose k divides n ($n \geq 2$) and that $\mathcal{O}_n$ is generated by $S_1, \ldots, S_n$. Let $\{E_{i,j}\}$ be the canonical matrix units for $\mathcal{M}_k(\mathbb{C}) \subseteq \mathcal{M}_k(\mathcal{O}_n)$. For $0 \leq j < \frac{n}{k}$ and $1 \leq i \leq k$, consider the operator $T_{i,j} = \sum_{\ell=1}^k S_{kj+\ell} E_{i,\ell}$. We notice $\{T_{i,j}\}_{0 \leq j < \frac{n}{k}, 1 \leq i \leq k}$ has $k\left(\frac{n}{k}\right) = n$ elements such that

$$\begin{aligned}
T_{i,j}^* T_{i,j} &= \left(\sum_{m=1}^k S_{kj+m} E_{i,m}\right)^* \left(\sum_{\ell=1}^k S_{kj+\ell} E_{i,\ell}\right) \\
&= \sum_{m,\ell=1}^k S_{kj+m}^* S_{kj+\ell} E_{m,\ell} \\
&= \sum_{\ell=1}^k E_{\ell,\ell} = I_k
\end{aligned}$$

and

$$T_{i,j}T_{i',j'}^* = \left(\sum_{m=1}^{k} S_{kj+m}E_{i,m}\right)\left(\sum_{\ell=1}^{k} S_{kj'+\ell}E_{i',\ell}\right)^* = \sum_{\ell=1}^{k} S_{kj+\ell}S_{kj'+\ell}^*E_{i,i'}.$$

Therefore

$$\sum_{i=1}^{k}\sum_{0\leq j<\frac{n}{k}} T_{i,j}T_{i,j}^* = \sum_{i=1}^{k}\sum_{0\leq j<\frac{n}{k}}\sum_{\ell=1}^{k} S_{kj+\ell}S_{kj+\ell}^*E_{i,i} = \sum_{i=1}^{k}\sum_{q=1}^{n} S_qS_q^*E_{i,i} = \sum_{i=1}^{k} E_{i,i} = I_k.$$

Whence $\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k}$ generates a copy of $\mathcal{O}_n$ inside $\mathcal{M}_k(\mathcal{O}_n)$. We claim that $C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k}) = \mathcal{M}_k(\mathcal{O}_n)$. To see this, we notice that

$$\sum_{0\leq j<\frac{n}{k}} T_{i,j}T_{i',j} = \sum_{0\leq j<\frac{n}{k}}\sum_{\ell=1}^{k} S_{kj+\ell}S_{kj+\ell}^*E_{i,i'} = \sum_{q=1}^{n} S_qS_q^*E_{i,i'}. = E_{i,i'}$$

Whence $\{E_{i,j}\} \subseteq C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k})$. Since $S_{kj+\ell}E_{1,1} = T_{1,j}E_{\ell,1}$, we have

$$S_qE_{1,1} \in C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k}))$$

for all $1 \leq q \leq n$. Whence

$$\mathcal{O}_nE_{1,1} \subseteq C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k})).$$

Using the fact that

$$E_{i,j} \in C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k}))$$

for all i, j, $\mathcal{M}_k(\mathcal{O}_n) = C^*(\{T_{i,j}\}_{0\leq j<\frac{n}{k},1\leq i\leq k}))$ as desired. Whence $\mathcal{M}_k(\mathcal{O}_n) \simeq \mathcal{O}_n$ for all k that divides n. ☐

Finally, we will demonstrate that $\mathcal{O}_2$ has a very interesting property (that the author cannot recall what it is called).

Theorem 1.24. *For all $n \geq 1$ $\mathcal{M}_n(\mathcal{O}_2)$ is isomorphic to $\mathcal{O}_2$.*

Proof. If $n = 1$ we are trivially done and if $n = 2$ we are done by Theorem 1.20. Thus suppose $n = k + 1$ where $k \geq 2$. Let $\mathcal{O}_2$ be generated by the isometries S_1 and S_2. Let

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & \dots & \dots & 0 \\ I & 0 & 0 & \dots & \dots & 0 \\ 0 & I & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & I & 0 & 0 \\ 0 & \dots & 0 & 0 & S_1 & S_2 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} S_1 & S_2S_1 & S_2^2S_1 & \dots & S_2^{k-1}S_1 & S_2^k \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\{E_{i,j}\}_{i,j=1}^{k+1}$ be the canonical matrix units. It is trivial to verify that $T_1^*T_1 = I_{k+1} = T_2^*T_2$, $T_1T_1^* = \sum_{j=2}^{k+1} E_{j,j}$, and $T_2T_2^* = E_{1,1}$. Hence $T_1T_1^* + T_2T_2^* = I$. Whence, if $\mathcal{M}_n(\mathcal{O}_2)$ is generated by T_1 and T_2, then $\mathcal{M}_n(\mathcal{O}_2) = C^*(T_1, T_2) \simeq \mathcal{O}_2$ by Theorem 1.14.

We notice that $E_{1,1} = T_2T_2^* \in C^*(T_1, T_2)$. Moreover $E_{q+1,1} = T_1E_{q,1}$ for all $1 \leq q \leq k-1$. Hence $\{E_{i,1}\}_{i=1}^{k} \subseteq C^*(T_1, T_2)$. Since $C^*(T_1, T_2)$ is self-adjoint, we obtain $\{E_{i,j}\}_{i,j=1}^{k} \subseteq C^*(T_1, T_2)$.

Fix $1 \leq q \leq k$. Notice

$$T_2E_{q,1} = S_2^{q-1}S_1E_{1,1} \in C^*(T_1, T_2).$$

Since $C^*(T_1, T_2)$ is self-adjoint,

$$S_1^*(S_2^*)^{q-1}E_{1,1} \in C^*(T_1, T_2).$$

However $T_1E_{k,1} = S_1E_{k+1,1} \in C^*(T_1, T_2)$. Therefore

$$(S_1E_{k+1,1})(S_1^*(S_2^*)^{q-1}E_{1,1}) = S_1S_1^*(S_2^*)^{q-1}E_{k+1,1} \in C^*(T_1, T_2)$$

for $1 \leq q \leq k$. However

$$E_{k+1,k+1} = T_1T_1^* - \sum_{j=2}^{k} E_{j,j} \in C^*(T_1, T_2).$$

Whence

$$E_{k+1,k+1}T_1E_{k+1,k+1} = S_2E_{k+1,k+1} \in C^*(T_1, T_2).$$

Therefore

$$S_2^{q-1}S_1S_1^*(S_2^*)^{q-1}E_{k+1,1} \in C^*(T_1, T_2)$$

for all $1 \leq q \leq k$.

Next notice that

$$\left(T_1 - \sum_{i=1}^{k-1} E_{i+1,i}\right)T_2^* = (S_1S_1^*(S_2^*)^{k-1} + S_2S_2^*(S_2^*)^{k-1})E_{k+1,1} = (S_2^*)^{k-1}E_{k+1,1} \in C^*(T_1, T_2).$$

Therefore

$$(S_2E_{k+1,k+1})^{k-1}((S_2^*)^{k-1}E_{k+1,1}) = (S_2)^{k-1}(S_2^*)^{k-1}E_{k+1,1} \in C^*(T_1, T_2).$$

Since

$$(S_2)^{k-1}(S_2^*)^{k-1} + \sum_{q=1}^{k-1} S_2^{q-1}S_1S_1^*(S_2^*)^{q-1} = \cdots = S_2S_2^* + S_1S_1^* = I$$

and

$$S_2^{q-1}S_1S_1^*(S_2^*)^{q-1}E_{k+1,1} \in C^*(T_1, T_2)$$

for all $1 \leq q \leq k$, $E_{k+1,1} \in C^*(T_1, T_2)$ Whence $C^*(T_1, T_2)$ contains all the matrix units.

Next we notice that $S_1E_{1,1} \in C^*(T_1, T_2)$ from above and

$$E_{1,k+1}(S_2E_{k+1,k+1})E_{k+1,1} = S_2E_{1,1} \in C^*(T_1, T_2).$$

Therefore $\mathcal{O}_nE_{1,1} \subseteq C^*(T_1, T_2)$ and since the matrix units are in $C^*(T_1, T_2)$, we obtain that $\mathcal{M}_n(\mathcal{O}_2) = C^*(T_1, T_2)$ as desired. $\square$

2 Purely Infinite C*-Algebras

In this chapter we will further our knowledge of simple C*-algebras. In particular, we will be interested in simple C*-algebras that have certain types of projections. We will then narrow our focus to the 'purely infinite' C*-algebras and we will use Theorem 1.12 to conclude that the Cuntz algebras are purely infinite. Purely infinite C*-algebras are of major interest to us and will be studied further in the next chapter.

The results for this chapter were developed from the excellent book [Da] (if you are reading these notes, you should definitely invest in this book).

We begin with several definitions pertaining to projections in a C*-algebra.

Definition 2.1. Let $\mathfrak{A}$ be a C*-algebra. A projection $P \in \mathfrak{A}$ is said to be infinite if there exists a non-zero proper subprojection $Q \in \mathfrak{A}$ of P such that $Q \sim P$ (that is, there exists a partial isometry $V \in \mathfrak{A}$ such that $P = V^*V$ and $Q := VV^* < P$). We say that P is properly infinite if there exists non-zero projections Q_1 and Q_2 in $\mathfrak{A}$ such that $P \sim Q_1 \sim Q_2$ and $Q_1 + Q_2 \leq P$ (note this last condition automatically implies that Q_1 and Q_2 are orthogonal).

A C*-algebra $\mathfrak{A}$ is said to be infinite if it contains an infinite projection and is said to be properly infinite if it contains a properly infinite projection.

Our first result is that if a C*-algebra is simple and infinite then it is properly infinite. Before we prove this, we have a simple technical lemma.

Lemma 2.2. *Let $\mathfrak{A}$ be a simple (not necessarily unital) C*-algebra. If $Q \in \mathfrak{A}$ is a projection and $P \in \mathfrak{A}$ is any non-zero positive operator then there exists elements $Z_i \in \mathfrak{A}$ such that $Q = \sum_{i=1}^n Z_i P Z_i^*$.*

Proof. Without loss of generality, $\|P\| = 1$. Since $\mathfrak{A}$ is simple, Q is in the closure of the algebraic ideal generated by P. Therefore there exists $\{X_i\}_{i=1}^n, \{Y_i\}_{i=1}^n \subseteq \mathfrak{A}$ such that $\|Q - \sum_{i=1}^n X_i P Y_i\| < \frac{1}{2}$. Hence

$$\left\| 2Q - \sum_{i=1}^n X_i P Y_i - \sum_{i=1}^n Y_i^* P X_i^* \right\| < 1.$$

Thus

$$2Q \leq I + \sum_{i=1}^n X_i P Y_i + \sum_{i=1}^n Y_i^* P X_i^*$$

in the unitization of $\mathfrak{A}$ so

$$2Q \leq Q + \sum_{i=1}^n Q X_i P Y_i Q + \sum_{i=1}^n Q Y_i^* P X_i^* Q.$$

However, since $(X_i - Y_i^*)P(X_i^* - Y_i) \geq 0$, $X_i P Y_i + Y_i^* P X_i^* \leq X_i P X_i^* + Y_i^* P Y_i^*$ so

$$\begin{aligned} Q &\leq \sum_{i=1}^n Q X_i P Y_i Q + \sum_{i=1}^n Q Y_i^* P X_i^* Q \\ &\leq \sum_{i=1}^n Q X_i P X_i^* Q + \sum_{i=1}^n Q Y_i^* P Y_i Q \\ &\leq \left(\sum_{i=1}^n \|X_i\|^2 + \|Y_i\|^2 \right) Q. \end{aligned}$$

Let

$$c := \sum_{i=1}^n \|X_i\|^2 + \|Y_i\|^2 \qquad \text{and} \qquad A := \sum_{i=1}^n Q X_i P X_i^* Q + \sum_{i=1}^n Q Y_i^* P Y_i Q.$$

Hence $Q \leq A \leq cQ$. By viewing Q as a projection in $\mathcal{B}(\mathcal{H})$ and using the fact that $Q \leq A \leq cQ$, we see that A commutes with Q, $(I-Q)A = A(I-Q) = 0$, and $\sigma(A) \subseteq \{0\} \cup [1, c]$. Define $f \in C(\{0\} \cup [1, c])$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^{-\frac{1}{2}} & \text{if } x \in [1, c] \end{cases}.$$

Hence $f(A)$ is well-defined and, by considering the decomposition of $\mathcal{B}(\mathcal{H})$ given by Q,

$$Q = f(A)Af(A) = \sum_{i=1}^{n} f(A)QX_iPX_i^*Qf(A) + \sum_{i=1}^{n} f(A)QY_i^*PY_iQf(A).$$

Therefore Q can be written as $Q = \sum_{i=1}^{m} Z_iPZ_i^*$ with $Z_i \in \mathfrak{A}$. $\square$

Lemma 2.3. *Let $\mathfrak{A}$ be a simple, infinite C*-algebra. Then for every infinite projection $Q \in \mathfrak{A}$ there exists partial isometries $\{V_n\}_{n=1}^{\infty} \subseteq \mathfrak{A}$ such that $Q = V_n^*V_n$ for all $n \in \mathbb{N}$ and $\sum_{k=1}^{n} V_kV_k^* < Q$ for all $n \in \mathbb{N}$. Hence Q is a properly infinite projection.*

Proof. Let $Q \in \mathfrak{A}$ be an infinite projection. Let $V \in \mathfrak{A}$ be a non-zero partial isometry such that $P := VV^* < V^*V = Q$. Since $\mathfrak{A}$ is simple and $Q - P > 0$, Lemma 2.2 implies that there exists $X_i \in \mathfrak{A}$ such that $\sum_{i=1}^{n} X_i^*(Q-P)X_i = Q$. Let

$$T_1 := \sum_{i=1}^{n} V^{i-1}(Q-P)X_i.$$

Since V is a partial isometry in $\mathfrak{A}$ with $(Q-P)V = 0 = V^*(Q-P)$, $V^i(Q-P)$ have pairwise orthogonal ranges for all i (as $(V^i(Q-P))^*(V^j(Q-P)) = (Q-P)V^{j-i}(Q-P) = 0$ for all $j > i$). Moreover, each $V^{i-1}(Q-P)$ is a partial isometry as $(V^i(Q-P))^*(V^i(Q-P)) = (Q-P)$. Therefore

$$T_1^*T_1 = \sum_{i,j=1}^{n} X_i^*(Q-P)(V^*)^{i-1}V^{j-1}(Q-P)X_i = \sum_{i=1}^{n} X_i^*(Q-P)X_i = Q.$$

Hence T_1 must be a partial isometry so $T_1T_1^*$ is a projection. Since the range of T_1 is clearly contained in the span of the ranges of $V^{i-1}(Q-P)$ and each $V^{i-1}(Q-P)$ is a partial isometry, we obtain that

$$T_1T_1^* \leq \sum_{i=1}^{n} V^{i-1}(Q-P)^2(V^*)^{i-1} = \sum_{i=1}^{n} V^{i-1}(Q-VV^*)(V^*)^{i-1} = Q - V^n(V^*)^n.$$

For each $i \geq 2$ let $T_i = V^{n(i-1)}T_1 \in \mathfrak{A}$. Then clearly each T_i is an isometry with $T_i^*T_i = Q$ and

$$T_iT_i^* = V^{n(i-1)}T_1T_1^*(V^*)^{n(i-1)} \leq V^{n(i-1)}(V^*)^{n(i-1)} - V^{n(i)}(V^*)^{n(i)}$$

for all $i \geq 2$. Hence $\sum_{i=1}^{k} T_iT_i^* = Q - V^{k(i)}(V^*)^{k(i)} < Q$ for all $k \geq 1$ as desired. $\square$

Applying the above result and Theorem 1.15, we trivially obtain the following.

Corollary 2.4. *Let $\mathfrak{A}$ be a simple, infinite C*-algebra. Then $\mathfrak{A}$ contains $\mathcal{O}_\infty$ as a C*-subalgebra.*

Moreover, combining this result with Remarks 1.16, we obtain the following.

Corollary 2.5. *If $\mathfrak{A}$ is a simple, infinite C*-algebra then $\mathcal{O}_n$ is a quotient of a C*-subalgebra of $\mathfrak{A}$ for all $n \geq 2$.*

Proof. By Corollary 2.4 $\mathcal{O}_\infty \subseteq \mathfrak{A}$. By Remarks 1.16 $\mathcal{O}_\infty$ contains a C*-subalgebra $\mathfrak{A}$ such that $\mathcal{O}_n$ is a quotient of $\mathfrak{A}$. $\square$

Our final result pertaining to simple C*-algebras and infinite projections is the following.

Proposition 2.6. *Let $\mathfrak{A}$ be a simple C*-algebra. Suppose that P and Q are projections in $\mathfrak{A}$ and P is infinite. Then Q is equivalent to a subprojection of P.*

Proof. By Lemma 2.2 there exists elements $\{Z_i\}_{i=1}^m \subseteq \mathfrak{A}$ such that $Q = \sum_{i=1}^m Z_i^* P Z_i$ and, by Lemma 2.3, there exists partial isometries $\{V_i\}_{i=1}^m \subseteq \mathfrak{A}$ such that $V_i^* V_i = P$ and $\sum_{i=1}^m V_i V_i^* < P$. Let $V := \sum_{i=1}^m Z_i P V_i^*$. Then

$$VV^* = \sum_{i,j=1}^m Z_i P V_i^* V_j P Z_j^* = \sum_{i=1}^m Z_i P Z_i^* = Q.$$

Hence V is a partial isometry so V^*V is a projection. Moreover, since $\sum_{i=1}^m V_i V_i^* < P$, $V_i^* P = V_i^*$ and $PV_i = V_i$ so

$$PV^*VP = \sum_{i,j=1}^m PV_i P Z_i^* Z_j P V_j^* P = V^*V$$

and thus $Q = VV^* \sim V^*V \leq P$ as claimed. ☐

Before we move onto purely infinite C*-algebra, we first need to develop a little theory about hereditary C*-subalgebras.

Definition 2.7. Let $\mathfrak{A}$ be a C*-algebra. A non-zero subset $\mathfrak{B}$ of $\mathfrak{A}$ is said to be hereditary if whenever $0 \leq A \leq B$ with $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ then $A \in \mathfrak{B}$.

To prove some results about hereditary subalgebras, we need the following technical lemma.

Lemma 2.8. *Let $\mathfrak{A}$ be a C*-algebra. Suppose $A, B \in \mathfrak{A}$ are such that $0 \leq A^*A \leq B$. Then there exists an element $C \in \mathfrak{A}$ such that $A = CB^{\frac{1}{4}}$.*

Proof. Let $\tilde{\mathfrak{A}}$ be the unitization of $\mathfrak{A}$. For each $n \in \mathbb{N}$, let $C_n := A\left(B + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{1}{4}}$ which lies in $\mathfrak{A}$ as $\mathfrak{A}$ is an ideal in $\tilde{\mathfrak{A}}$. We claim that C_n is a Cauchy sequence in $\mathfrak{A}$. To see this, for each $n, m \in \mathbb{N}$ let

$$D_{n,m} := \left(B + \frac{1}{n}\right)^{-\frac{1}{2}} - \left(B + \frac{1}{m}\right)^{-\frac{1}{2}} \in \tilde{\mathfrak{A}}$$

and let $f_n \in C([0, \|B\|])$ be the continuous functions defined $f_n(x) = x^{\frac{3}{4}}\left(x + \frac{1}{n}\right)^{-\frac{1}{2}}$. It is clear that $(f_n)_{n\geq 1}$ is a Cauchy sequence in the uniform norm on $C([0, \|B\|])$ and thus $(f_n(B))_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$. Moreover we notice that

$$\begin{aligned}
\|C_n - C_m\|^2 &= \|(C_n - C_m)^*(C_n - C_m)\| \\
&= \left\| B^{\frac{1}{4}} D_{n,m} A^*A D_{n,m} B^{\frac{1}{4}} \right\| \\
&\leq \left\| B^{\frac{1}{4}} D_{n,m} B D_{n,m} B^{\frac{1}{4}} \right\| \\
&= \left\| B^{\frac{3}{4}} D_{n,m} D_{n,m} B^{\frac{3}{4}} \right\| \\
&= \|f_n(B) - f_m(B)\|^2
\end{aligned}$$

and thus $(C_n)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$.

Let $C := \lim_{n\to\infty} C_n \in \mathfrak{A}$. Then

$$\begin{aligned}
\left\| A - CB^{\frac{1}{4}} \right\|^2 &= \lim_{n\to\infty} \left\| (A - C_n B^{\frac{1}{4}})^*(A - C_n B^{\frac{1}{4}}) \right\| \\
&= \lim_{n\to\infty} \left\| \left(I_{\mathfrak{A}} - \left(B + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{1}{2}} \right)^* A^*A \left(I_{\mathfrak{A}} - \left(B + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{1}{2}} \right) \right\| \\
&\leq \limsup_{n\to\infty} \left\| \left(I_{\mathfrak{A}} - \left(B + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{1}{2}} \right)^* B \left(I_{\mathfrak{A}} - \left(B + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{1}{2}} \right) \right\| \\
&= \limsup_{n\to\infty} \|g_n(B)\|
\end{aligned}$$

where $g_n(x) = \left(1 - \sqrt{\frac{x}{x+\frac{1}{n}}}\right)^2 x$. Since $0 \leq \sqrt{\frac{x}{x+\frac{1}{n}}} \leq 1$, we have

$$1 - 2\sqrt{\frac{x}{x+\frac{1}{n}}} + \left(\sqrt{\frac{x}{x+\frac{1}{n}}}\right)^2 \leq 1 - \left(\sqrt{\frac{x}{x+\frac{1}{n}}}\right)^2$$

so

$$0 \leq g_n(x) \leq \left(1 - \left(\sqrt{\frac{x}{x+\frac{1}{n}}}\right)^2\right) x = \frac{\frac{1}{n}}{x+\frac{1}{n}} x \leq \frac{1}{n}$$

for all $x \geq 0$. Hence $\|g_n(B)\| \leq \frac{1}{n}$ so $\left\|A - CB^{\frac{1}{4}}\right\|^2 \leq \limsup_{n\to\infty} \frac{1}{n} = 0$. Hence $A = CB^{\frac{1}{4}}$ as desired. $\square$

Note that the above result can be used to show that closed ideals in a C*-algebra (which are automatically C*-subalgebras) are hereditary C*-subalgebras.

Lemma 2.9. *Let $\mathfrak{A}$ be a C*-algebra and $A \in \mathfrak{A}$ be positive. Then $\overline{A\mathfrak{A}A}$ is the smallest hereditary C*-subalgebra containing A. Moreover every separable hereditary subalgebra has this form.*

Proof. Let A be a positive element of $\mathfrak{A}$. Clearly $\overline{A\mathfrak{A}A}$ is a C*-subalgebra of $\mathfrak{A}$. For every $\epsilon > 0$ define

$$f_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^2}x & \text{if } x \in [0,\epsilon] \\ \frac{1}{x} & \text{if } x > \epsilon \end{cases}.$$

Then f_ϵ is continuous on $\sigma(A)$ (and $f_\epsilon(0) = 0$) so $f_\epsilon(A) \in \mathfrak{A}$. Since $f_\epsilon(x)x^2 = x$ if $x > \epsilon$ and $0 \leq f_\epsilon(x)x^2 \leq x$ if $x \in [0,\epsilon]$, $\left\|f_\epsilon(x)x^2 - x\right\|_{\sigma(A)} \to 0$ as $\epsilon \to 0$. Hence $Af_\epsilon(A)A \to A$ as $\epsilon \to 0$. Thus $A \in \overline{A\mathfrak{A}A}$.

Next we claim that $\overline{A\mathfrak{A}A}$ is contained in any hereditary C*-subalgebra of $\mathfrak{A}$ containing A. To see this, suppose $\mathfrak{B}$ is a hereditary C*-subalgebra of $\mathfrak{A}$ containing $C^*(A)$. If $C \in \mathfrak{A}$ is positive then $0 \leq ACA \leq \|C\|^2 A^2$. Since $\|C\|^2 A^2 \in C^*(A) \subseteq \mathfrak{B}$ and $\mathfrak{B}$ is hereditary, $ACA \in \mathfrak{B}$. As this holds for all positive $C \in \mathfrak{A}$ and every element of $\mathfrak{A}$ is the linear combination of four positive elements, $A\mathfrak{A}A \subseteq \mathfrak{B}$. Hence $\overline{A\mathfrak{A}A} \subseteq \mathfrak{B}$.

To see that $\overline{A\mathfrak{A}A}$ is a hereditary C*-subalgebra of $\mathfrak{A}$, suppose $B \in \mathfrak{A}$ and $C \in \overline{A\mathfrak{A}A}$ are such that $0 \leq B \leq C$. Thus $0 \leq B^{\frac{1}{2}}B^{\frac{1}{2}} \leq C$. By Lemma 2.8, there exists a $D \in \mathfrak{A}$ so that $B^{\frac{1}{2}} = DC^{\frac{1}{4}}$. Whence $B = C^{\frac{1}{4}}D^*DC^{\frac{1}{4}}$. Since $C \in \overline{A\mathfrak{A}A}$, $C^{\frac{1}{4}} \in C^*(C) \subseteq \overline{A\mathfrak{A}A}$. Thus there exists $A_n \in \mathfrak{A}$ so that $C^{\frac{1}{4}} = \lim_{n\to\infty} AA_nA$. Thus $B = \lim_{n\to\infty} A(A_nAD^*DAA_n)A \in \overline{A\mathfrak{A}A}$. Whence $\overline{A\mathfrak{A}A}$ is hereditary.

Lastly, we desire to show that every separable hereditary C*-subalgebra of $\mathfrak{A}$ has the form $\overline{A\mathfrak{A}A}$ for some $A \in \mathfrak{A}$ positive. Suppose $\mathfrak{B}$ be a C*-subalgebra of $\mathfrak{A}$ that is separable and hereditary. Since $\mathfrak{B}$ is separable, there is a countable set $\{A_n\}_{n\geq 1}$ of positive elements of $\mathfrak{B}$ of norm at most 1 so that $\mathfrak{B} = C^*(\{A_n\}_{n\geq 1})$. Let $A := \sum_{n\geq 1} \frac{1}{2^n}A_n \in \mathfrak{B}$. Consider $\mathfrak{C} := \overline{A\mathfrak{A}A}$ which is a hereditary C*-subalgebra of $\mathfrak{A}$ since $A \geq 0$. Since $A \in \mathfrak{C}$ by above, $0 \leq \frac{1}{2^n}A_n \leq A$ for all n, and $\mathfrak{C}$ is hereditary, $A_n \in \mathfrak{C}$ for all n so $\mathfrak{B} = C^*(\{A_n\}_{n\geq 1}) \subseteq \mathfrak{C}$. However, if $X \in \mathfrak{A}$ is positive, $0 \leq AXA \leq \|X\| A^2$ and $\|X\| A^2 \in \mathfrak{B}$. Whence $A(\mathfrak{A}_+)A \subseteq \mathfrak{B}$ so $\overline{A\mathfrak{A}A} \subseteq \mathfrak{B}$ and thus $\mathfrak{C} \subseteq \mathfrak{B}$. Hence $\mathfrak{B} = \mathfrak{C}$ as desired. $\square$

Now we are finally ready to define one of the main objects of study in these notes.

Definition 2.10. A C*-algebra $\mathfrak{A}$ is said to be purely infinite if every hereditary C*-subalgebra is an infinite C*-algebra.

It turns out that the Cuntz algebras are our first examples of purely infinite C*-algebras. The easiest way to show this is the following theorem.

Theorem 2.11. *Let $\mathfrak{A}$ be a unital, simple C*-algebra that is not isomorphic to $\mathbb{C}$. Then the following are equivalent.*

1. *$\mathfrak{A}$ is purely infinite.*

2. *For all $A \in \mathfrak{A} \setminus \{0\}$, there exists $X, Y \in \mathfrak{A}$ such that $XAY = I$.*

3. *For all positive $A \in \mathfrak{A} \setminus \{0\}$ and $\epsilon > 0$, there exists an $X \in \mathfrak{A}$ such that $XAX^* = I$ and $\|X\| < \|A\|^{-\frac{1}{2}} + \epsilon$.*

Proof. Suppose (3) holds and let $X \in \mathfrak{A} \setminus \{0\}$. Then $X^*X > 0$ so there exists a $Y \in \mathfrak{A}$ such that $I = Y(X^*X)Y^* = (YX^*)X(Y^*)$. Therefore (3) implies (2).

Suppose that (2) holds. We desire to show that $\mathfrak{A}$ is purely infinite. Let $\mathfrak{B}$ be a hereditary C*-subalgebra of $\mathfrak{A}$ and let $B \in \mathfrak{B}$ be a non-zero positive element that is not invertible (such an element always exists unless $\mathfrak{B} = \mathbb{C}$ and thus, since $\mathfrak{B}$ is hereditary, we would have $\mathfrak{A} = \mathbb{C}$). By (2) there exists $X, Y \in \mathfrak{A} \setminus \{0\}$ such that $XB^{\frac{1}{2}}Y = I$. Therefore

$$I = Y^*B^{\frac{1}{2}}X^*XB^{\frac{1}{2}}Y \leq \|X\|^2 Y^*BY$$

Therefore, if $Z_0 := \|X\| (Y^*BY)^{-\frac{1}{2}}$, $I = (Z_0Y^*)B(YZ_0)$. Hence there exists a $Z \in \mathfrak{A}$ such that $ZBZ^* = I$ (thus proving (3) without the norm estimates).

Let $V := B^{\frac{1}{2}}Z^*$. Therefore $V^*V = I$. Moreover $P := VV^* = B^{\frac{1}{2}}Z^*ZB^{\frac{1}{2}} \in \mathfrak{B}$. Thus, as $P \leq \|Z\|^2 B$ and B is not invertible, $P \neq I$ so V is a proper isometry. Hence $V(I-P)V^*$ is a non-zero projection. Let $W := VP$. Then

$$W = B^{\frac{1}{2}}Z^*B^{\frac{1}{2}}Z^*ZB^{\frac{1}{2}} \in B^{\frac{1}{2}}\mathfrak{A}B^{\frac{1}{2}} \subseteq \mathfrak{B}$$

by Lemma 2.9. Moreover $W^*W = PV^*VP = P$ and $WW^* = VPV^*$. However, since VPV^* and $V(I-P)V^*$ are orthogonal projections with $VPV^* + V(I-P)V^* = VV^* = P$ and $V(I-P)V^*$ is non-zero, VPV^* must be a proper subprojection of P in $\mathfrak{B}$ that is equivalent to P in $\mathfrak{B}$. Hence $\mathfrak{B}$ is infinite and, as $\mathfrak{B}$ was an arbitrary hereditary C*-subalgebra of $\mathfrak{A}$, $\mathfrak{A}$ is purely infinite.

Lastly, suppose that (1) holds and let $A \in \mathfrak{A}$ be a positive operator of norm 1. For each $0 < \epsilon < \frac{1}{2}$, define the function

$$f_\epsilon(x) = \begin{cases} 0 & \text{if } x < 1-\epsilon \\ 1 - \frac{1}{\epsilon}(1-x) & \text{if } x \in [1-\epsilon, 1] \end{cases}.$$

Let $\mathfrak{B}_\epsilon := \overline{f_\epsilon(A)\mathfrak{A}f_\epsilon(A)}$ which is a hereditary C*-subalgebra of $\mathfrak{A}$ by Lemma 2.9. Therefore, since $\mathfrak{A}$ is purely infinite, there exists an infinite projection $P_\epsilon \in \mathfrak{B}_\epsilon$. By considering spectral projections in $\mathcal{B}(\mathcal{H})$ and by considering the definition of $\mathfrak{B}_\epsilon$, we clearly have that $P_\epsilon \leq E_A([1-\epsilon, 1])$. Therefore $P_\epsilon AP_\epsilon \geq (1-\epsilon)P_\epsilon$. Since $\mathfrak{A}$ is simple, Proposition 2.6 implies that the identity I of $\mathfrak{A}$ is equivalent to a subprojection of P_ϵ. Hence there exists a proper isometry V_ϵ such that $V_\epsilon V_\epsilon^* \leq P_\epsilon$. Therefore $V_\epsilon^*P_\epsilon = V_\epsilon^*$ and $P_\epsilon V_\epsilon = V_\epsilon$.

Let

$$B_\epsilon := V_\epsilon^*AV_\epsilon = (V_\epsilon^*P_\epsilon)A(P_\epsilon V_\epsilon) \geq (1-\epsilon)V_\epsilon^*P_\epsilon V_\epsilon = (1-\epsilon)V_\epsilon^*V_\epsilon = (1-\epsilon)I.$$

Therefore B_ϵ is invertible and

$$(B_\epsilon^{-\frac{1}{2}}V_\epsilon^*)A(V_\epsilon B^{-\frac{1}{2}}) = I.$$

Finally, we notice that

$$\left\|V_\epsilon B_\epsilon^{-\frac{1}{2}}\right\| \leq \left\|B_\epsilon^{-\frac{1}{2}}\right\| \leq (1-\epsilon)^{-\frac{1}{2}} < 1 + \epsilon$$

(as $B_\epsilon \geq (1-\epsilon)I$ and $0 < \epsilon < \frac{1}{2}$) which completes the proof. □

Thus, by Theorems 1.12 and 1.13, we have the following.

Corollary 2.12. *$\mathcal{O}_\infty$ and $\mathcal{O}_n$ are purely infinite for all $n \geq 2$.*

To conclude this section, we make the following observation (thus explaining the term 'purely infinite').

Lemma 2.13. *Let $\mathfrak{A}$ be a unital, purely infinite C*-algebra. If $P \in \mathfrak{A}$ is a non-zero projection then $P\mathfrak{A}P$ is a unital, purely infinite C*-algebra. Hence all projections in a purely infinite C*-algebra are infinite. Moreover, if $\mathfrak{A}$ is simple, $P\mathfrak{A}P$ is simple.*

Proof. Clearly $P\mathfrak{A}P$ is a unital C*-algebra. To see that $P\mathfrak{A}P$ is purely infinite, suppose $\mathfrak{B}$ is a hereditary C*-subalgebra of $P\mathfrak{A}P$. Clearly $\mathfrak{B}$ is a C*-subalgebra of $\mathfrak{A}$. We claim that $\mathfrak{B}$ is a hereditary C*-subalgebra of $\mathfrak{A}$. To see this, suppose that $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$ are such that $0 \leq A \leq B$. Since $B \in P\mathfrak{A}P$ and $P\mathfrak{A}P$ is a hereditary C*-subalgebra of $\mathfrak{A}$ by Lemma 2.9, $A \in P\mathfrak{A}P$. Therefore, since $\mathfrak{B}$ is a hereditary C*-subalgebra of $P\mathfrak{A}P$, $A \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a hereditary C*-subalgebra of $\mathfrak{A}$. Since $\mathfrak{A}$ is purely infinite, $\mathfrak{B}$ is an infinite C*-algebra. Hence $P\mathfrak{A}P$ is purely infinite. Therefore $P\mathfrak{A}P$ contains an infinite projection and thus P is infinite.

Lastly, suppose that $\mathfrak{A}$ is simple. Suppose $\mathfrak{J}$ is a non-zero ideal in $P\mathfrak{A}P$. Therefore, there exists a non-zero positive operator $A \in \mathfrak{J}$. By Lemma 2.2 there exists $X_j, Y_j \in \mathfrak{A}$ such that $I = \sum_{j=1}^{n} X_j A Y_j$. Hence, as $A \in P\mathfrak{A}P$, $P = \sum_{j=1}^{n} (PX_jP)A(PY_jP) \in \mathfrak{J}$. Hence $\mathfrak{J} = P\mathfrak{A}P$ as desired. □

Using the above lemma along with Lemma 2.3 and Proposition 2.6, we have the following important result.

Theorem 2.14. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let P and Q be projections in $\mathfrak{A}$ with P non-zero. Then there exists a projection $Q' \in \mathfrak{A}$ such that $Q \sim Q'$ and $Q' < P$.*

Proof. Since P is non-zero, Lemma 2.13 implies that P is an infinite projection. Therefore, since $\mathfrak{A}$ is unital and simple, Lemma 2.3 implies that P is properly infinite. Therefore, there exists a non-zero projection $P' \in \mathfrak{A}$ such that $P \sim P'$ and $P' < P$. However Lemma 2.13 also implies P' is an infinite projection so Proposition 2.6 implies that there exists a projection $Q' \in \mathfrak{A}$ such that $Q \sim Q'$ and $Q' \leq P' < P$. □

3 Tensor Products of Purely Infinite C*-Algebras

In this chapter, we will study the minimal tensor product of two unital, simple, purely infinite C*-algebras. The main goal of this chapter is to prove that the three properties listed in the previous sentence are preserved under taking minimal tensor products.

Most of the results for this chapter were developed from the book [Ro2] and the additional papers referenced there. The portion of the chapter on excising states is from the paper [AAP].

It is clear that the minimal tensor product of two unital C*-algebras is again a unital C*-algebra. It is also well-known that the minimal tensor product of two simple C*-algebras is again a simple C*-algebra. As many proofs involving tensor products of C*-algebras are incorrect, we include a proof here. We begin with the following observation.

Lemma 3.1. *Let $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathfrak{C}$ be a *-homomorphism such that $\pi|_{\mathfrak{A}\odot\mathfrak{B}}$ is injective. Then π is injective.*

Proof. Let $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathfrak{C}$ be a *-homomorphism which is injective when restricted to $\mathfrak{A} \odot \mathfrak{B}$. Let α be the C*-norm on $\pi(\mathfrak{A} \odot \mathfrak{B}) \simeq \mathfrak{A} \odot \mathfrak{B}$ induced by $\mathfrak{C}$. Thus, as $\mathfrak{A} \otimes_\alpha \mathfrak{B} \subseteq \mathfrak{C}$ is the smallest C*-algebra generated by $\pi(\mathfrak{A} \odot \mathfrak{B})$, we have that $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathfrak{A} \otimes_\alpha \mathfrak{B}$ is a continuous *-homomorphism. Since every *-homomorphism of a C*-algebra is contractive, $\alpha(t) \leq \|t\|_{\min}$ for all $t \in \mathfrak{A} \odot \mathfrak{B}$ and thus $\alpha = \|\cdot\|_{\min}$ (as $\|\cdot\|_{\min}$ is the smallest C*-norm on $\mathfrak{A} \odot \mathfrak{B}$). Whence π is an isometry that is the identity on a dense subset and thus π is injective on $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. ☐

To proceed with the proof that the minimal tensor product of two simple C*-algebras is simple, we will need the technical Lemma 3.3. To prove said lemma, we will need to make some common definitions.

Definition 3.2. Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a C*-algebra. The commutant of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$, denoted $\mathfrak{A}'$, is the set

$$\mathfrak{A}' := \{T \in \mathcal{B}(\mathcal{H}) \mid AT = TA \text{ for all } A \in \mathfrak{M}\}.$$

The double commutant of $\mathfrak{A}$, denoted $\mathfrak{M}''$, is the set $\mathfrak{A}'' := (\mathfrak{A}')'$.

We say that a von Neumann algebra $\mathfrak{M}$ is a factor if $\mathfrak{M} \cap \mathfrak{M}' = \{\mathbb{C}I_\mathcal{H}\}$.

The following proof is based on Proposition 4.20 of [Ta].

Lemma 3.3. *Let $\mathfrak{M} \subseteq \mathcal{B}(\mathcal{H})$ be a factor and let $\pi : \mathfrak{M}\odot\mathfrak{M}' \to \mathcal{B}(\mathcal{H})$ be the product map (i.e. $\pi(T\otimes S) = TS$ which will be well-defined since $\mathfrak{M}$ and $\mathfrak{M}'$ commute). If $\pi\left(\sum_{i=1}^n A_i \otimes B_i\right) = 0$ for some $(A_i)_{i=1}^n \subseteq \mathfrak{M}$ and $(B_i)_{i=1}^n \subseteq \mathfrak{M}'$, then $\sum_{i=1}^n A_i \otimes B_i = 0$.*

Proof. Suppose that $\pi\left(\sum_{i=1}^n A_i \otimes B_i\right) = 0$ so $\sum_{i=1}^n A_iB_i = 0$. Let $\mathcal{H}_n$ be any finite dimensional Hilbert space with orthonormal basis $\{e_1, \ldots, e_n\}$ and let $\mathcal{K} := \mathcal{H} \otimes \mathcal{H}_n$. Let

$$\mathcal{K}_0 := \overline{span\left\{\sum_{i=1}^n BB_i\xi \otimes e_i \;\middle|\; B \in \mathfrak{M}', \xi \in \mathcal{H}\right\}}.$$

Notice that

$$\begin{aligned}\left\langle \sum_{i=1}^n BB_i\xi \otimes e_i, \sum_{j=1}^n A_j^*\eta \otimes e_j \right\rangle &= \sum_{i=1}^n \langle BB_i\xi, A_i^*\eta\rangle \\ &= \sum_{i=1}^n \langle A_iBB_i\xi, \eta\rangle \\ &= \left\langle B\left(\sum_{i=1}^n A_iB_i\right)\xi, \eta\right\rangle = 0\end{aligned}$$

for all $\eta, \xi \in \mathcal{H}$ and $B \in \mathfrak{M}'$. Thus $\sum_{j=1}^n A_j^*\eta \otimes e_j$ is orthogonal to $\mathcal{K}_0$ for all $\eta \in \mathcal{H}$.

Let P be the projection of $\mathcal{K}$ onto $\mathcal{K}_0$. We can view P as an $n \times n$ matrix $[P_{i,j}]$ where $P_{i,j} \in \mathcal{B}(\mathcal{H})$ since $\mathcal{H}_n$ is finite dimensional. Notice for any $B \in \mathfrak{M}'$ and $\xi \in \mathcal{H}$ that

$$\sum_{j=1}^{n} BB_j\xi \otimes e_j = [P_{i,j}] \sum_{j=1}^{n} BB_j\xi \otimes e_j = \sum_{i,j=1}^{n} P_{i,j}BB_j\xi \otimes e_j.$$

Therefore we clearly obtain that $BB_j = \sum_{i=1}^{n} P_{i,j}BB_j$ for all $B \in \mathfrak{M}'$ (specifically $B = I$ will be useful) and all $1 \leq j \leq n$.

Since we have seen that $\sum_{j=1}^{n} A_j^*\eta \otimes e_j$ is orthogonal to $\mathcal{K}_0$ for all $\eta \in \mathcal{H}$, we have that

$$0 = P\sum_{j=1}^{n} A_j^*\eta \otimes e_j = \sum_{i,j=1}^{n} P_{i,j}A_j^*\eta \otimes e_j$$

so $\sum_{j=1}^{n} P_{i,j}A_j^* = 0$. By taking adjoints (and noting that $P = P^*$ so $P_{i,j}^* = P_{j,i}$), we obtain that $\sum_{j=1}^{n} A_jP_{j,i} = 0$.

Notice that $(B \otimes I)\mathcal{K}_0 \subseteq \mathcal{K}_0$ for all $B \in \mathfrak{M}'$ and for any $A \in \mathfrak{M}$ and $B \in \mathfrak{M}'$

$$(A \otimes I)\left(\sum_{i=1}^{n} BB_i\xi \otimes e_i\right) = \sum_{i=1}^{n} ABB_i\xi \otimes e_i = \sum_{i=1}^{n} BB_i(A\xi) \otimes e_i \in \mathcal{K}_0.$$

Whence $\mathcal{K}_0$ is invariant under $\mathfrak{M} \otimes \mathbb{C}I$ and $\mathfrak{M}' \otimes \mathbb{C}I$. Thus $\mathcal{K}_0$ is invariant under $(\mathfrak{M} \cup \mathfrak{M}') \otimes \mathbb{C}I$ and by taking SOT-limits, it is invariant under $(\mathfrak{M} \cup \mathfrak{M}')'' \otimes \mathbb{C}I$. However $(\mathfrak{M} \cup \mathfrak{M}')' = \mathbb{C}I$ since $\mathfrak{M}$ is a factor so $\mathcal{K}_0$ is invariant under $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I$. Since $\mathcal{K}_0$ is fixed by $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I$, P must commute with $\mathcal{B}(\mathcal{H}) \otimes \mathbb{C}I$ and thus $P \in \mathbb{C}I \otimes \mathcal{B}(\mathcal{H}_n)$. This implies that each $P_{i,j}$ is a scalar multiple of $I_{\mathcal{H}}$. But then

$$\begin{aligned}
\sum_{i=1}^{n} A_i \otimes B_i &= \sum_{i=1}^{n} A_i \otimes \left(\sum_{j=1}^{n} P_{i,j}B_j\right) \\
&= \sum_{i,j=1}^{n} A_i \otimes P_{i,j}B_j \\
&= \sum_{i,j=1}^{n} P_{i,j}A_i \otimes B_j \\
&= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} P_{i,j}A_i\right) \otimes B_j = 0
\end{aligned}$$

as desired. $\square$

Proposition 3.4. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Then $\mathfrak{A} \otimes_\alpha \mathfrak{B}$ is simple if and only if $\alpha = \|\cdot\|_{\min}$ and both $\mathfrak{A}$ and $\mathfrak{B}$ are simple.*

Proof. First suppose that $\alpha \neq \|\cdot\|_{\min}$. Then the *-homomorphism $\pi : \mathfrak{A} \otimes_\alpha \mathfrak{B} \to \mathfrak{A} \otimes_{\min} \mathfrak{B}$ is such that $0 \subsetneq ker(\pi) \subsetneq \mathfrak{A} \otimes_\alpha \mathfrak{B}$. Thus $\mathfrak{A} \otimes_\alpha \mathfrak{B}$ is not simple. Similarly, if $\alpha = \|\cdot\|_{\min}$ and $\mathfrak{A}$ is not simple, then there exists a non-zero ideal $\mathfrak{J} \neq \mathfrak{A}$. Whence $\mathfrak{J} \odot \mathfrak{B}$ is an algebraic ideal of $\mathfrak{A} \odot \mathfrak{B}$. Thus $\mathfrak{J} \otimes_{\min} \mathfrak{B}$ is a non-zero ideal of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. To see that $\mathfrak{J} \otimes_{\min} \mathfrak{B} \neq \mathfrak{A} \otimes_{\min} \mathfrak{B}$ we note that there exists a state φ on $\mathfrak{A}$ such that $\varphi(\mathfrak{J}) = 0$ (consider the quotient map). Then, if ψ is any state on $\mathfrak{B}$, $\varphi \times \psi$ extends to a state on $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. However, $(\varphi \times \psi)(\mathfrak{J} \odot \mathfrak{B}) = \{0\}$ so $(\varphi \times \psi)(\mathfrak{J} \otimes_{\min} \mathfrak{B}) = \{0\}$ by continuity. Since $\varphi \times \psi \neq 0$ on $\mathfrak{A} \otimes_{\min} \mathfrak{B}$, $\mathfrak{J} \otimes_{\min} \mathfrak{B} \neq \mathfrak{A} \otimes_{\min} \mathfrak{B}$ so $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is not simple. Similarly, if $\mathfrak{B}$ is not simple, $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is not simple.

Suppose that $\alpha = \|\cdot\|_{\min}$ and both $\mathfrak{A}$ and $\mathfrak{B}$ are simple. If $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ was not simple, by considering the quotient map and an irreducible representation of the quotient algebra there would exists a non-faithful irreducible representation of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. Thus it suffices to check that every irreducible representation of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is faithful. Let $\pi : \mathfrak{A} \otimes_{\min} \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ be an irreducible representation. Then there exists non-degenerate *-homomorphism $\pi_1 : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and $\pi_2 : \mathfrak{B} \to \mathcal{B}(\mathcal{H})$ such that $\pi = \pi_1 \times \pi_2$ (π_j must be non-degenerate or else π would have an invariant subspace). Since π is irreducible, $\pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})' = \mathbb{C}I$. However, by density and continuity, $\mathbb{C}I = \pi(\mathfrak{A} \otimes_{\min} \mathfrak{B})' = \pi_1(\mathfrak{A})' \cap \pi_2(\mathfrak{B})'$. Since $\pi_1(\mathfrak{A}) \subseteq \pi_2(\mathfrak{B})'$, $\pi_1(\mathfrak{A})''$ is a factor. Moreover $\pi_2(\mathfrak{B}) \subseteq \pi_1(\mathfrak{A})'$.

By Lemma 3.1, to show that π is injective, it suffices to show that $\pi|_{\mathfrak{A}\odot\mathfrak{B}}$ is injective. However, if $\pi\left(\sum_{i=1}^n A_i \otimes B_i\right) = 0$, $\sum_{i=1}^n \pi_1(A_i)\pi_2(B_i) = 0$. By Lemma 3.3,

$$0 = \sum_{i=1}^n \pi_1(A_i) \odot \pi_2(B_i) = (\pi_1 \otimes \pi_2)\left(\sum_{i=1}^n A_i \odot B_i\right).$$

Since $\mathfrak{A}$ and $\mathfrak{B}$ are simple (and thus have no closed ideals) π_1 and π_2 must be injective. Whence $\pi_1 \otimes \pi_2$ is injective and thus $0 = (\pi_1 \otimes \pi_2)\left(\sum_{i=1}^n A_i \odot B_i\right)$ implies $\sum_{i=1}^n A_i \odot B_i = 0$. Whence $\pi|_{\mathfrak{A}\odot\mathfrak{B}}$ is injective so $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is simple. ☐

It remains to show that the minimal tensor product of purely infinite C*-algebras is again purely infinite. The proof of this fact will come from a simple application of Kirchberg's Slice Lemma. However, to prove Kirchberg's Slice Lemma (and for later results), we will need to discuss excising states in a C*-algebra.

Definition 3.5. Let $\mathfrak{A}$ be a C*-algebra and let φ be a state on $\mathfrak{A}$. A net $(A_\lambda)_\Lambda$ of positive elements of $\mathfrak{A}$ with norm one is said to excise φ if $\lim_\Lambda \left\|\varphi(A)A_\lambda^2 - A_\lambda A A_\alpha\right\| = 0$ for all $A \in \mathfrak{A}$.

Example 3.6. Let X be a compact Hausdorff space, let $x \in X$ be fixed, and define $\varphi : C(X) \to \mathbb{C}$ by $\varphi(f) = f(x)$. Let $(U_\lambda)_\Lambda$ be a neighbourhood basis of the point x. By Urysohn's Lemma there exists a net $(f_\alpha)_\Lambda$ of positive, norm one elements of $C(X)$ such that $f_\lambda(x) = 1$ and $f_\lambda|_{U_\lambda^c} = 0$. By standard continuous function arguments, it is easy to see that $\lim_\Lambda \left\|f(x)f_\lambda^2 - f_\lambda f f_\lambda\right\|_\infty = 0$ for all $f \in C(X)$. Hence φ excised by $(f_\lambda)_\Lambda$.

Notice in the above example that φ was a pure state. This leads us to the following result.

Proposition 3.7. *Let $\mathfrak{A}$ be a unital C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a pure state. Then φ can be excised by a decreasing net $(A_\lambda)_\Lambda$ such that $\varphi(A_\lambda) = 1$ for all λ.*

Proof. Let $\mathcal{L} := \{B \in \mathfrak{A} \mid \varphi(B^*B) = 0\}$ and let $\mathfrak{N} := \mathcal{L} \cap \mathcal{L}^*$. First we claim that $\mathfrak{N}$ is a C*-algebra. To see this, we recall from the GNS construction that $\mathcal{L}$ is a closed left ideal in $\mathfrak{A}$. Therefore it is clear that $\mathfrak{N}$ is a closed, self-adjoint linear space. To see that $\mathfrak{N}$ is an algebra, we notice that if $A, B \in \mathfrak{N}$, then $AB \in \mathcal{L}$ since $B \in \mathcal{L}$ and $\mathcal{L}$ is a left ideal and $AB \in \mathcal{L}^*$ since $A \in \mathcal{L}^*$ and $\mathcal{L}^*$ is a right ideal. Hence $\mathfrak{N}$ is a C*-algebra (in fact, it can be shown to be hereditary).

Let $(E_\lambda)_\Lambda$ is any C*-bounded approximate identity for the C*-algebra $\mathfrak{N}$. For each $\lambda \in \Lambda$, define

$$A_\lambda := I_\mathfrak{A} - E_\lambda.$$

Clearly $(A_\lambda)_\Lambda$ is a decreasing net of positive operators that is majorized by $I_\mathfrak{A}$ and

$$\varphi(A_\lambda) = \varphi(I_\mathfrak{A}) - \varphi(E_\lambda) = 1$$

since $E_\lambda^{\frac{1}{2}} \in \mathfrak{N} \subseteq \mathcal{L}$ so the definition of $\mathcal{L}$ implies $\varphi(E_\lambda) = 0$. Hence $\|A_\lambda\| = 1$ for all $\lambda \in \Lambda$.

Next we claim that $ker(\varphi) = \mathcal{L} + \mathcal{L}^*$. To see this, we notice that if $A \in \mathcal{L}$ then

$$0 \leq |\varphi(A)| \leq \varphi(A^*A)\varphi(I_\mathfrak{A}^* I_\mathfrak{A}) = 0$$

by Cauchy Schwarz inequality for positive sesquilinear forms. Hence $\mathcal{L} \subseteq ker(\varphi)$. Similarly, if $A \in \mathcal{L}^*$, $\varphi(A^*) = 0$ so $\varphi(A) = 0$ as φ is positive. Hence $\mathcal{L} + \mathcal{L}^* \subseteq ker(\varphi)$. To see the other inclusion, suppose

$B \in ker(\varphi)$. Since φ is positive and thus $\varphi(B^*) = \overline{\varphi(B)} = 0$, $Re(B), Im(B) \in ker(\varphi)$. Hence it suffices to show that if $B \in ker(\varphi)$ and B is self-adjoint then $B \in \mathcal{L} + \mathcal{L}^*$.

Let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ and $\xi \in \mathcal{H}$ be the cyclic representation of φ given by the GNS construction. Hence $\varphi(A) = \langle \pi(A)\xi, \xi\rangle$ for all $A \in \mathfrak{A}$. Since φ is a pure state, π is irreducible. Let $B \in ker(\varphi)$ be self-adjoint. Therefore $\varphi(B) = \langle \pi(B)\xi, \xi\rangle = 0$. Hence $\pi(B)\xi$ and ξ are orthogonal vectors. By the Strong Kadison Transitivity Theorem, there exists a self-adjoint $A \in \mathfrak{A}$ such that $\pi(A)(\pi(B)\xi) = 0$ and $\pi(A)\xi = \xi$. Therefore

$$\varphi((AB)^*(AB)) = \langle \pi(A)\pi(B)\xi, \pi(A)\pi(B)\xi\rangle = 0$$

and

$$\varphi((B - AB)(B - AB)^*) = \langle \pi(B)\xi - \pi(B)\pi(A)\xi, \pi(B)\xi - \pi(B)\pi(A)\xi\rangle = 0$$

as all the operators under consideration are self-adjoint. Hence $AB \in \mathcal{L}$ and $B - AB \in \mathcal{L}^*$ so $B \in \mathcal{L} + \mathcal{L}^*$. Hence $ker(\varphi) = \mathcal{L} + \mathcal{L}^*$ as desired.

Finally we notice that if $A \in \mathfrak{A}$ then $A - \varphi(A)I_{\mathfrak{A}} \in ker(\varphi)$. Hence there exists $T, S \in \mathcal{L}$ such that $A - \varphi(A)I_{\mathfrak{A}} = T + S^*$. Notice $T^*T, S^*S \in \mathcal{L}$ since $\mathcal{L}$ is a left ideal and $T, S \in \mathcal{L}$. Since T^*T and S^*S are self-adjoint, $T^*T, S^*S \in \mathfrak{N}$. Hence

$$\begin{aligned}
\lim_\Lambda \left\|A_\lambda A A_\lambda - \varphi(A)A_\lambda^2\right\| &= \lim_\Lambda \|A_\lambda(A - \varphi(A)I_{\mathfrak{A}})A_\lambda\| \\
&= \lim_\Lambda \|A_\lambda(T + S^*)A_\lambda\| \\
&\leq \limsup_\Lambda \|T(I_{\mathfrak{A}} - E_\lambda)\| + \|(I_{\mathfrak{A}} - E_\lambda)S^*\| \\
&\leq \limsup_\Lambda \|T(I_{\mathfrak{A}} - E_\lambda)\| + \|S(I_{\mathfrak{A}} - E_\lambda)\| \\
&\leq \limsup_\Lambda \|T^*T(I_{\mathfrak{A}} - E_\lambda)\|^{\frac{1}{2}} + \|S^*S(I_{\mathfrak{A}} - E_\lambda)\|^{\frac{1}{2}} = 0
\end{aligned}$$

as $(E_\lambda)_\Lambda$ is a C*-bounded approximate identity of $\mathfrak{N}$. Therefore, since $A \in \mathfrak{A}$ is arbitrary, $(A_\lambda)_\Lambda$ excises φ. ☐

It turns out to be easy to extend our knowledge of state that can be excise by taking weak*-limits.

Proposition 3.8. *Let $\mathfrak{A}$ be a unital C*-algebra and let $\varphi : \mathfrak{A} \to \mathbb{C}$ be a state that is a weak*-limit of pure states. Then φ can be excised in $\mathfrak{A}$.*

Proof. For each finite subset $A_1, \ldots, A_n$ and each $\epsilon > 0$ there exists a pure state ψ on $\mathfrak{A}$ such that $|\psi(A_i) - \varphi(A_i)| < \epsilon$ for all $i \in \{1, \ldots, n\}$. By Proposition 3.7, there exists a positive element $B \in \mathfrak{A}$ with $\|B\| = 1$ such that $\left\|\psi(A_i)B^2 - BA_iB\right\| < \epsilon$ for all $i \in \{1, \ldots, n\}$. Hence

$$\left\|\varphi(A_i)B^2 - BA_iB\right\| \leq |\varphi(A_i) - \psi(A_i)| \, \|B\|^2 + \left\|\psi(A_i)B^2 - BA_iB\right\| < 2\epsilon$$

for all $i \in \{1, \ldots, n\}$. Therefore, since the above works for every $\epsilon > 0$ and every finite subset of $\mathfrak{A}$, φ can be excised on $\mathfrak{A}$. ☐

Now that we have developed which states we can excise, we can finally prove Kirchberg's Slice Lemma. Before we begin, we shall prove a small technical lemma.

Lemma 3.9. *Let $\mathfrak{A}$ be a unital C*-algebra, let $A, B \in \mathfrak{A}_+$, and fix $\epsilon > \|A - B\|$. Then there exists a contraction $D \in \mathfrak{A}$ such that $DBD^* = (A - \epsilon I)_+$ (where $(A - \epsilon I)_+$ is the positive part of $A - \epsilon I$).*

Proof. For each $r > 1$ define $g_r : [0, \infty) \to [0, \infty)$ by $g_r(x) = \min\{x, x^r\}$. It is clear that $g_r(B) \to B$ as $r \to 1$. Since $\|A - B\| < \epsilon$, there exists an $r_0 > 1$ such that $\|A - g_{r_0}(B)\| < \epsilon$. Let $B_0 := g_{r_0}(B)$ and let $0 \leq \epsilon_1 := \|A - g_{r_0}(B)\| < \epsilon$. Thus $A - \epsilon_1 \leq B_0$. By the definition of g_{r_0}, we see that $B_0 \leq B$ and $B_0 \leq B^{r_0}$. Since $\epsilon_1 < \epsilon$, by considering the Continuous Functional Calculus (and assuming $\epsilon \leq 1$) we can find a contraction $E \in C^*(A)$ such that $E(A - \epsilon_1 I)E = (A - \epsilon I)_+$. Hence $(A - \epsilon I)_+ \leq EB_0E$.

Let $X := B_0^{\frac{1}{2}} E \in \mathfrak{A}$ and view $\mathfrak{A}$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Therefore there exists a partial isometry $V \in \mathcal{B}(\mathcal{H})$ such that $X = V|X|$. It is well-known that V is the WOT-limit of $X\left(X^*X + \frac{1}{n}I\right)^{-\frac{1}{2}}$ and thus $V \in \mathfrak{A}''$. Let $Y := V((A-\epsilon I)_+)^{\frac{1}{2}} \in \mathfrak{A}''$. We claim that $Y \in \mathfrak{A}$. To see this, we notice that Y is the WOT-limit of $T_n := X\left(X^*X + \frac{1}{n}I\right)^{-\frac{1}{2}}((A-\epsilon I)_+)^{\frac{1}{2}} \in \mathfrak{A}$. Hence it suffices to show that $(T_n)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$. However, since $(A-\epsilon I)_+ \leq EB_0E = X^*X$, we see that if

$$Z_{n,m} := \left(\left(X^*X + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(X^*X + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)$$

then

$$\begin{aligned}
&\|T_n - T_m\|^2 \\
&= \|(T_n - T_m)(T_n^* - T_m^*)\| \\
&= \|XZ_{n,m}(A-\epsilon I)_+ Z_{n,m}X^*\| \\
&\leq \left\|X\left(\left(X^*X + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(X^*X + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)X^*X\left(\left(X^*X + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(X^*X + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)X^*\right\| \\
&= \left\|XX^*\left(XX^* + \frac{1}{n}I\right)^{-\frac{1}{2}} - XX^*\left(XX^* + \frac{1}{m}I\right)^{-\frac{1}{2}}\right\|^2.
\end{aligned}$$

However, since $f_n(x) = \frac{x}{\sqrt{x+\frac{1}{n}}}$ converges to $\sqrt{x}$ uniformly on $\sigma(XX^*)$, we see that $(T_n)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$. Hence $Y \in \mathfrak{A}$ as desired.

Next we notice that

$$Y^*Y = ((A-\epsilon I)_+)^{\frac{1}{2}}V^*V((A-\epsilon I)_+)^{\frac{1}{2}} = (A-\epsilon I)_+$$

since $(A-\epsilon I)_+ \leq EB_0E = X^*X$ and V^*V is the projection onto $ker(|X|)^\perp = \overline{ran(X^*X)}$. Moreover, we see that

$$YY^* = V(A-\epsilon I)_+V^* \leq VX^*XV^* = V|X|(V|X|)^* = XX^* = B_0^{\frac{1}{2}}E^2B_0^{\frac{1}{2}} \leq B_0 \leq B^{r_0}.$$

For each $n \in \mathbb{N}$ let $D_n := Y^*\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{r_0-1}{2}} \in \mathfrak{A}$ (and is well-defined as $r_0 > 1$). We claim that $(D_n)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{A}$. To see this, we notice that

$$\begin{aligned}
&\|D_n - D_m\|^2 \\
&= \|(D_n - D_m)^*(D_n - D_m)\| \\
&= \left\|B^{\frac{r_0-1}{2}}\left(\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(B^{r_0} + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)YY^*\left(\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(B^{r_0} + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)B^{\frac{r_0-1}{2}}\right\| \\
&\leq \left\|B^{\frac{r_0-1}{2}}\left(\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(B^{r_0} + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)B^{r_0}\left(\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} - \left(B^{r_0} + \frac{1}{m}I\right)^{-\frac{1}{2}}\right)B^{\frac{r_0-1}{2}}\right\| \\
&= \left\|B^{r_0-\frac{1}{2}}\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} - B^{r_0-\frac{1}{2}}\left(B^{r_0} + \frac{1}{m}I\right)^{-\frac{1}{2}}\right\|^2.
\end{aligned}$$

Therefore, since $h_n(x) = \frac{x^{r_0-\frac{1}{2}}}{\sqrt{x^{r_0-\frac{1}{2}}}}$ converges to $\sqrt{x^{r_0-\frac{1}{2}}}$ uniformly on $\sigma(B)$, we see that $(D_n)_{n\geq 1}$ is Cauchy in $\mathfrak{A}$.

Let $D := \lim_{n\to\infty} D_n$. Then

$$DB^{\frac{1}{2}} = \lim_{n\to\infty} Y^*\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{r_0}{2}} = Y^*$$

since $\left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{r_0}{2}}$ converges in the WOT to the projection P onto $ker(B^{r_0})^{\perp}$, $YY^* \leq B^{r_0}$ implies $Y^*P = Y^*$, and the norm limit exists and thus must be the same as the WOT-limit. Hence

$$DBD^* = Y^*Y = (A - \epsilon I)_+$$

Finally, to see that D is a contraction, we notice (since $YY^* \leq B_0 \leq B$)

$$\begin{aligned} D_n^* D_n &= B^{\frac{r_0-1}{2}} \left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} YY^* \left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{r_0-1}{2}} \\ &\leq B^{\frac{r_0-1}{2}} \left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B \left(B^{r_0} + \frac{1}{n}I\right)^{-\frac{1}{2}} B^{\frac{r_0-1}{2}} \\ &= B^{r_0} \left(B^{r_0} + \frac{1}{n}I\right)^{-1} \end{aligned}$$

and thus $\|D_n^* D_n\| \leq 1$ for all n. Hence $\|D\| \leq 1$ as desired. □

Lemma 3.10 (Kirchberg's Slice Lemma). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C*-algebras, and let $\mathfrak{D}$ be a hereditary C*-subalgebra of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. Then there exists a non-zero element $Z \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$ such that $ZZ^* \in \mathfrak{D}$ and $Z^*Z = A \otimes B$ for some $A \in \mathfrak{A}_+$ and $B \in \mathfrak{B}_+$.*

Proof. Let $T \in \mathfrak{D}$ be a non-zero positive element. Since the elementary tensors of the pure states of $\mathfrak{A}$ and $\mathfrak{B}$ separate points in $\mathfrak{A} \otimes_{\min} \mathfrak{B}$, there exists pure states $\varphi \in \mathfrak{A}^*$ and $\psi \in \mathfrak{B}^*$ such that $(\varphi \otimes \psi)(T) \neq 0$. Let $B_1 := (\varphi \otimes Id_{\mathfrak{B}})(T) \in \mathfrak{B}_+$ (as $\varphi \otimes Id$ is a positive map). Therefore $\psi(B_1) = (\varphi \otimes \psi)(T) \neq 0$ so B_1 is a non-zero element of $\mathfrak{B}$. By scaling T, we may assume without loss of generality that $\|B_1\| = 1$.

Since $T \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$ there exists $X_i \in \mathfrak{A}$ and $Y_i \in \mathfrak{B}$ such that $\|Y_i\| = 1$ for all i and

$$\left\| T - \sum_{i=1}^{n} X_i \otimes Y_i \right\|_{\min} < \frac{1}{12}.$$

Hence

$$\left\| B_1 - \sum_{i=1}^{n} \varphi(X_i) Y_i \right\| < \frac{1}{12}.$$

By Proposition 3.7, there exists a positive element $A_1 \in \mathfrak{A}$ with $\|A_1\| = 1$ such that

$$\left\| A_1^{\frac{1}{2}} X_i A_1^{\frac{1}{2}} - \varphi(X_i) A_1 \right\| < \frac{1}{12n}$$

for all $i \in \{1, \ldots, n\}$. Hence

$$\begin{aligned} &\left\| (A_1^{\frac{1}{2}} \otimes I) T (A_1^{\frac{1}{2}} \otimes I) - A_1 \otimes B_1 \right\| \\ &\leq \frac{1}{12} + \left\| (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) \left(\sum_{i=1}^{n} X_i \otimes Y_i \right) (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) - A_1 \otimes B_1 \right\| \\ &< \frac{1}{12} + \frac{n}{12n} + \left\| \left(\sum_{i=1}^{n} (\varphi(X_i) A_1) \otimes Y_i \right) - A_1 \otimes B_1 \right\| \\ &\leq \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \left\| \left(\sum_{i=1}^{n} (\varphi(X_i) A_1) \otimes Y_i \right) - A_1 \otimes \left(\sum_{i=1}^{n} \varphi(X_i) Y_i \right) \right\| = \frac{1}{4}. \end{aligned}$$

Hence, by Lemma 3.9 (as everything is positive), there exists an $R \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$ such that

$$R^* (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) T (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) R = \left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes \mathfrak{B}} \right)_+.$$

Fix δ such that $\frac{1}{2} < \delta < 1$ and let $A := (A_1 - \delta I_{\mathfrak{A}})_+ \in \mathfrak{A}_+ \setminus \{0\}$ and $B := (B_1 - \delta I_{\mathfrak{B}})_+ \in \mathfrak{B}_+ \setminus \{0\}$. We claim that there exists an element $S \in C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$ such that

$$S^* \left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+ S = A \otimes B.$$

To see this, we notice that

$$\left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+, A \otimes B \in C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$$

and $C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$ is an abelian C*-algebra. However, if ϕ is a multiplicative linear functional on $C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$, then it is easy to see that $\phi = \phi_1 \otimes \phi_2$ where ϕ_1 is a multiplicative linear functional on $C^*(A_1, I_{\mathfrak{A}})$ and ϕ_2 is a multiplicative linear functional on $C^*(B_1, I_{\mathfrak{B}})$. If $\phi(A \otimes B) \neq 0$, then $\phi_1(A) \neq 0$ and $\phi_2(B) \neq 0$. Therefore, there must exists $\lambda_1, \lambda_2 > \delta$ such that $\phi_1(A_1) = \lambda_1$ and $\phi_2(B_1) = \lambda_2$ (as multiplicative linear functionals on abelian C*-algebras are precisely the pure state and thus evaluations at a point). Hence

$$\phi \left(\left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+ \right) \geq \lambda_1 \lambda_2 - \frac{1}{4} > \delta^2 - \frac{1}{4}.$$

Therefore, as $\frac{1}{2} < \delta < 1$, the above implies that

$$\overline{\{\phi \in \Delta \mid \phi(A \otimes B) \neq 0\}} \subseteq \left\{ \phi \in \Delta \mid \phi \left(\left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+ \right) \neq 0 \right\}$$

where Δ is the maximal ideal space of $C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$. Therefore, by considering $C(\Delta) \simeq C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$, we obtain that there exists a positive element $S \in C^*(A_1, I_{\mathfrak{A}}) \otimes_{\min} C^*(B_1, I_{\mathfrak{B}})$ such that

$$S^* \left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+ S = A \otimes B.$$

Let $Z := T^{\frac{1}{2}} (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) RS$. Then

$$Z^* Z = S^* R^* (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) T (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) RS = S^* \left((A_1 \otimes B_1) - \frac{1}{4} I_{\mathfrak{A} \otimes_{\min} \mathfrak{B}} \right)_+ S = A \otimes B$$

and

$$0 \leq ZZ^* = T^{\frac{1}{2}} (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) RSS^* R^* (A_1^{\frac{1}{2}} \otimes I_{\mathfrak{B}}) T^{\frac{1}{2}} \leq \|RSS^* R\| T$$

so $ZZ^* \in \mathfrak{D}$ as $\mathfrak{D}$ is hereditary. □

Theorem 3.11. *Let $\mathfrak{A}$ be a unital, purely infinite C*-algebra and let $\mathfrak{B}$ be a unital C*-algebra such that every hereditary C*-subalgebra contains a non-zero projection. Then $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is purely infinite. Therefore $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ and $\mathcal{M}_n(\mathfrak{A})$ are purely infinite if $\mathfrak{A}$ and $\mathfrak{B}$ are unital, purely infinite C*-algebras.*

Proof. Let $\mathfrak{D}$ be a hereditary C*-subalgebra of $\mathfrak{A} \otimes_{\min} \mathfrak{B}$. By Kirchberg's Slice Lemma, there exists a non-zero element $Z \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$ such that $ZZ^* \in \mathfrak{D}$ and $Z^* Z = A \otimes B$ where $A \in \mathfrak{A}_+ \setminus \{0\}$ and $B \in \mathfrak{B}_+ \setminus \{0\}$.

Notice that $\overline{(ZZ^*)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(ZZ^*)} \subseteq \mathfrak{D}$ as $\mathfrak{D}$ is hereditary. Define

$$\pi : \overline{(Z^* Z)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(Z^* Z)} \to \overline{(ZZ^*)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(ZZ^*)}$$

by $\pi(T) = VTV^*$ where $V \in (\mathfrak{A} \otimes_{\min} \mathfrak{B})'' \subseteq \mathcal{B}(\mathcal{H})$ is the partial isometry such that $Z = V|Z|$ (it is not yet clear π has the correct codomain). We claim that π is a well-defined isomorphism. To see this we notice that

$VZ^*ZV^* = ZZ^*$ and that π is clearly a *-homomorphism on these spaces as VV^* is the projection onto the range of Z and V^*V is the projection onto the range of Z^*. Finally, if $T \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$ is positive,

$$V((Z^*Z)T(Z^*Z))V^* = Z|Z|T|Z|Z^* \in \mathfrak{A} \otimes_{\min} \mathfrak{B}$$

and

$$V((Z^*Z)T(Z^*Z))V^* = Z|Z|T|Z|Z^* \leq \|T\| \, ZZ^*ZZ^*$$

so $V((Z^*Z)T(Z^*Z))V^* \in \overline{(ZZ^*)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(ZZ^*)}$ as $\overline{(ZZ^*)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(ZZ^*)}$ is hereditary. Hence π does indeed map $\overline{(Z^*Z)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(Z^*Z)}$ to $\overline{(ZZ^*)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(ZZ^*)}$. Since $\pi^{-1}(T) = V^*TV$ will also be a *-homomorphism, we obtain that π is an isomorphism.

Therefore, to show that $\mathfrak{D}$ has an infinite projection, it suffices to show that $\overline{(Z^*Z)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(Z^*Z)}$ has an infinite projection. However

$$\overline{A\mathfrak{A}A} \otimes_{\min} \overline{B\mathfrak{B}B} \subseteq \overline{(A \otimes B)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(A \otimes B)} = \overline{(Z^*Z)(\mathfrak{A} \otimes_{\min} \mathfrak{B})(Z^*Z)}$$

Therefore, since $\overline{A\mathfrak{A}A}$ and $\overline{B\mathfrak{B}B}$ are hereditary C*-subalgebra of $\mathfrak{A}$ and $\mathfrak{B}$ respectively, since $\mathfrak{A}$ is purely infinite, and since every hereditary C*-subalgebra of $\mathfrak{B}$ contains a non-zero projection, it is easy to see that $\mathfrak{D}$ has an infinite projection. Hence $\mathfrak{A} \otimes_{\min} \mathfrak{B}$ is purely infinite. $\square$

4 K-Theory for Purely Infinite C*-Algebras

In this chapter we will develop the K-theory for unital, simple, purely infinite C*-algebras. Luckily, as the K-theory for said algebras is straightforward, a reader without knowledge of K-theory may survive this chapter. For a unital, simple, purely infinite C*-algebras $\mathfrak{A}$, we will define two abelian groups, $K_0(\mathfrak{A})$ and $K_1(\mathfrak{A})$, using the projections and unitaries in $\mathfrak{A}$. As the theory of projections in unital, simple, purely infinite C*-algebras is 'nice', the development of $K_0(\mathfrak{A})$ can be done in a simpler fashion than usual. The development of $K_1(\mathfrak{A})$ must be done in the usual fashion and then shown to be 'nice' for unital, simple, purely infinite C*-algebras.

The ideas of this chapter were developed from the original paper [Cu1]. There is a significant amount of information in this paper that we will not use and the interested reader should take the time to go through this paper.

We will begin with the construction of $K_0(\mathfrak{A})$ for a unital, simple, purely infinite C*-algebra $\mathfrak{A}$.

Construction 4.1. Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. For each non-zero projection $P \in \mathfrak{A}$ let $[P]_0$ denote the equivalent class of P (see Definition 2.1 for the equivalent relation).

Recall that $I_{\mathfrak{A}}$ is a properly infinite projection in $\mathfrak{A}$ by Lemma 2.3 and thus we can write $I_{\mathfrak{A}} = P_0 + Q_0$ where P_0 and Q_0 are orthogonal infinite projections. Therefore, if P and Q are non-zero projections in $\mathfrak{A}$, Proposition 2.6 implies that there exists projections $P' \sim P$ and $Q' \sim Q$ such that $P' \leq P_0$ and $Q' \leq Q_0$ (and thus $P'Q' = 0$).

If P and Q are non-zero projections in $\mathfrak{A}$, we define

$$[P]_0 + [Q]_0 = [P' + Q']_0$$

where P' and Q' are any non-zero projections in $\mathfrak{A}$ such that $P' \sim P$, $Q' \sim Q$, and $P'Q' = 0$ (so $P' + Q'$ is a non-zero projection). The above paragraph shows that such P' and Q' exist. Moreover, if $P'' \sim P$, $Q'' \sim Q$, and $P''Q'' = 0$, it is not difficult to show that $P' + Q' \sim P'' + Q''$ as, if $V^*V = P'$, $VV^* = P''$, $W^*W = Q'$, and $WW^* = Q''$, orthogonality implies $(V + W)^*(V + W) = P' + Q'$ and $(V + W)(V + W)^* = P'' + Q''$. Hence this is a well-defined operator on the non-zero projections.

Let

$$K_0(\mathfrak{A}) := \{[P]_0 \mid P \text{ a non-zero projection in } \mathfrak{A}\}$$

equipped with the additive operator given above. Clearly

$$[P]_0 + [Q]_0 = [P' + Q']_0 = [Q' + P']_0 = [Q]_0 + [P]_0$$

for all non-zero projections $P, Q \in \mathfrak{A}$ so $K_0(\mathfrak{A})$ is an abelian semigroup.

Before continuing, we point out the following technical yet common lemmas.

Lemma 4.2. *Let $\mathfrak{A}$ be a C*-algebra and suppose P, Q, and R are projections in $\mathfrak{A}$ such that $Q, R \leq P$, $Q \sim R$, and $QR = 0$. Then $P - Q \sim P - R$.*

Proof. Let $V \in \mathfrak{A}$ be the partial isometry such that $V^*V = Q$ and $VV^* = R$. Since $QR = 0 = RQ$, $QV = 0$ (as the range of V is the range of R) and $RV^* = 0$ (as the range of V^* is the range of Q). Hence $V^*Q = 0$ and $VR = 0$.

Let $W := P - Q - R + V$. Then

$$\begin{aligned} W^*W &= (P - Q - R + V^*)(P - Q - R + V) \\ &= (P - Q - R + V) - 0 - RV + (V^* - V^*R + V^*V) \\ &= (P - Q - R + V) - RV + (V^* - V^*R + Q) \\ &= (P - Q - R + V) - V + (V^* - V^* + Q) \\ &= P - R \end{aligned}$$

and

$$\begin{aligned}
WW^* &= (P-Q-R+V)(P-Q-R+V^*) \\
&= (P-Q-R+V^*) - QV^* - 0 + (V - VQ + VV^*) \\
&= (P-Q-R+V^*) - QV^* + (V - VQ + R) \\
&= (P-Q-R+V^*) - V^* + (V - V + R) \\
&= P - Q
\end{aligned}$$

as desired. □

Lemma 4.3. *Let $\mathfrak{A}$ be a C*-algebra and let Q, Q', and Q_0 be non-zero projections in $\mathfrak{A}$ such that $Q' < Q$ and $Q \sim Q_0$. Then there exists a non-zero projection $Q_0' \in \mathfrak{A}$ such that $Q_0' \sim Q'$, $Q_0' < Q_0$, and $Q - Q' \sim Q_0 - Q_0'$.*

Proof. Let $V \in \mathfrak{A}$ be such that $V^*V = Q$ and $VV^* = Q_0$. If $V' = VQ'$ then

$$(V')^*V' = Q'QQ' = Q' \qquad \text{and} \qquad V'(V')^* = VQ'V^* < VQV^* = Q_0$$

(where the strict inequality comes from the fact that $VQ'V^* = VQV^*$ implies $Q' = V^*VQ'V^*V = V^*VQV^*V = Q$ which is a contradiction). Let $Q_0' := V'(V')^*$. Thus $Q_0' \sim Q'$ (so Q_0' is non-zero) and $Q_0' < Q_0$. Moreover

$$(V - V')(V - V')^* = VV^* - V'V^* - V(V')^* + (V')(V')^* = Q_0 - VQ'V^* - VQ'V^* + Q_0' = Q_0 - Q_0'$$

and

$$(V - V')^*(V - V') = V^*V - (V')^*V - V^*(V') + (V')^*(V') = Q - Q'Q - QQ' + Q' = Q - Q'$$

so $Q - Q' \sim Q_0 - Q_0'$ as desired. □

The reason $K_0(\mathfrak{A})$ is special for unital, simple, purely infinite C*-algebras is the following.

Theorem 4.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Then $K_0(\mathfrak{A})$ (as defined in Construction 4.1) is a group.*

Proof. To show that $\mathfrak{A}$ is a group, it suffices to show that $\mathfrak{A}$ has an identity element and each element in $\mathfrak{A}$ has an additive inverse.

Fix an arbitrary non-zero projection $Q \in \mathfrak{A}$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, Lemma 2.3 implies there exists non-zero projections P' and Q' such that $I_{\mathfrak{A}} \sim P'$, $P' < I_{\mathfrak{A}}$, $Q \sim Q'$, and $Q' < Q$. Our goal is to show that

$$[I_{\mathfrak{A}} - P']_0 = [Q - Q']_0$$

and to use this to show that $[I_{\mathfrak{A}} - P']_0$ is the identity element of $K_0(\mathfrak{A})$ for any choice of projection P'.

By Proposition 2.6 implies there exists a non-zero projection Q_0 such that $Q \sim Q_0$ and $Q_0 \leq P'$. Hence, by Lemma 4.3, there exists non-zero projection $Q_0' \in \mathfrak{A}$ such that $Q_0' \sim Q'$, $Q_0' < Q_0$, and $Q - Q' \sim Q_0 - Q_0'$. Therefore, since $I_{\mathfrak{A}} - P'$ and $Q_0 - Q_0'$ are orthogonal projections (as $Q_0 \leq P'$), we obtain that

$$[I_{\mathfrak{A}} - P]_0 + [Q - Q']_0 = [I_{\mathfrak{A}} - P]_0 + [Q_0 - Q_0']_0 = [I_{\mathfrak{A}} - (P - Q_0 + Q_0')]_0.$$

However, since $Q_0 \sim Q_0'$, there exists a partial isometry $V \in \mathfrak{A}$ such that $V^*V = Q_0$ and $VV^* = Q_0'$. Since $Q_0' < Q_0 \leq P'$, we obtain that $VQ_0 = V = VP'$, and $Q_0'V = V = Q_0V = P'V$. Thus $Q_0V^* = V^* = P'V^*$ and $V^*Q_0' = V^* = V^*Q_0 = V^*P'$. Therefore, if $W := (P' - Q_0) + V$, we obtain that

$$W^*W = (P' - Q_0) + V^*(P' - Q_0) + (P' - Q_0)V + V^*V = (P' - Q_0) + 0 + 0 + Q_0 = P'$$

and

$$WW^* = (P' - Q_0) + V(P' - Q_0) + (P' - Q_0)V^* + VV^* = (P' - Q_0) + 0 + 0 + Q_0' = P' - Q_0 + Q_0'.$$

Therefore, Lemma 4.2 implies that

$$I_{\mathfrak{A}} - P' \sim I_{\mathfrak{A}} - (P' - Q_0 + Q_0')$$

so

$$[I_{\mathfrak{A}} - P']_0 = [I_{\mathfrak{A}} - (P' - Q_0 + Q_0')]_0 = [I_{\mathfrak{A}} - P]_0 + [Q - Q']_0.$$

However, the roles of $I_{\mathfrak{A}}$, P', Q, and Q' are easily interchanged in the above proof (we did not use any special properties of $I_{\mathfrak{A}}$) so we also obtain that

$$[Q - Q']_0 = [Q - Q']_0 + [I_{\mathfrak{A}} - P]_0.$$

Hence

$$[I_{\mathfrak{A}} - P']_0 = [I_{\mathfrak{A}} - P]_0 + [Q - Q']_0 = [Q - Q']_0$$

as addition is commutative.

Therefore, to see that $[I_{\mathfrak{A}} - P']_0$ is an identity element of $K_0(\mathfrak{A})$, we notice that

$$[Q]_0 + [I_{\mathfrak{A}} - P']_0 = [Q]_0 + [Q - Q']_0 = [Q + (Q - Q')]_0 = [Q]_0$$

by the definition of addition in $K_0(\mathfrak{A})$. Therefore, as Q was an arbitrary non-zero projection in $\mathfrak{A}$, we obtain that $[I_{\mathfrak{A}} - P']_0$ is an identity element in $K_0(\mathfrak{A})$.

To see that every element of $K_0(\mathfrak{A})$ has an additive inverse, fix a non-zero projection $Q \in \mathfrak{A}$. Since $\mathfrak{A}$ is purely infinite, Lemma 2.13 implies Q is properly infinite and thus Lemma 2.3 implies there exists non-zero orthogonal projections Q'' and Q' in $\mathfrak{A}$ such that $Q \sim Q' \sim Q''$ and $Q', Q'' < Q$. From the above proof $[Q - Q']_0$ is an identity element of $K_0(\mathfrak{A})$. Since

$$[Q]_0 + [Q - Q' - Q'']_0 = [Q'']_0 + [Q - Q' - Q'']_0 = [Q'' + (Q - Q' - Q'')]_0 = [Q - Q']_0$$

by the definition of addition in $K_0(\mathfrak{A})$, $[Q - Q' - Q'']_0$ is an additive inverse of $[Q]_0$ in $K_0(\mathfrak{A})$. As Q was an arbitrary non-zero projection, $K_0(\mathfrak{A})$ is a group. □

Remarks 4.5. For those familiar with general K-theory for C*-algebras, we will briefly outline why the above definition of $K_0(\mathfrak{A})$ is equivalent to the traditional definition. Recall that if $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra then $\mathcal{M}_n(\mathfrak{A})$ is a unital, simple, purely infinite C*-algebra by Theorem 3.11. Therefore, if $P, Q \in \mathcal{M}_n(\mathfrak{A})$, P and Q are equivalent to orthogonal projections inside the canonical copy of $\mathfrak{A} \subseteq \mathcal{M}_n(\mathfrak{A})$ (in the $(1,1)$-entry). Thus the abelian semigroup defined in the usual construction of $K_0(\mathfrak{A})$ is identically the $K_0(\mathfrak{A})$ constructed in Construction 4.1 and thus already a group (so nothing changes when the Grothendieck group of this semigroup is taken).

With the development of $K_0(\mathfrak{A})$ complete, we turn to the development of $K_1(\mathfrak{A})$.

Definition 4.6. Let $\mathfrak{A}$ be a unital C*-algebra. Let

$$\mathcal{U}(\mathfrak{A}) := \{U \in \mathfrak{A} \mid U \text{ is a unitary}\}$$

which will be called the unitary group of $\mathfrak{A}$.

We will say that $U, V \in \mathcal{U}(\mathfrak{A})$ are homotopically equivalent in $\mathcal{U}(\mathfrak{A})$ if there exists a continuous path $\gamma : [0,1] \to \mathcal{U}(\mathfrak{A})$ such that $\gamma(0) = U$ and $\gamma(1) = V$. We will use $U \sim_h V$ to denote U and V are homotopically equivalent.

Let $\mathcal{U}_0(\mathfrak{A})$ denote the path-connected component of $I_{\mathfrak{A}}$ in $\mathcal{U}(\mathfrak{A})$; that is

$$\mathcal{U}_0(\mathfrak{A}) := \{U \in \mathcal{U}(\mathfrak{A}) \mid U \sim_h I_{\mathfrak{A}}\}.$$

Remarks 4.7. Suppose $U, V, W \in \mathcal{U}(\mathfrak{A})$ are such that $U \sim_h V$ and $V \sim_h W$. Therefore there exists continuous functions $\gamma : [0, 1] \to \mathcal{U}(\mathfrak{A})$ and $\alpha : [0, 1] \to \mathcal{U}(\mathfrak{A})$ such that

$$\gamma(0) = U, \quad \gamma(1) = V = \alpha(0), \text{ and } \quad \alpha(1) = W.$$

If $\gamma_0 : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is defined by $\gamma_0(t) = \gamma(1 - t)$ for all $t \in [0, 1]$, γ_0 is a continuous function such that $\gamma_0(0) = V$ and $\gamma_0(1) = U$. Thus $V \sim_h U$. Moreover, if $\alpha_0 : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is defined by $\alpha_0(t) = \gamma(2t)$ for all $t \in [0, \frac{1}{2}]$ and $\alpha_0(t) = \alpha(2t - 1)$ for all $t \in [\frac{1}{2}, 1]$, then α_0 is a continuous function (as $\gamma(1) = \alpha(0)$) such that $\alpha_0(0) = U$ and $\alpha_0(1) = W$. Hence $U \sim_h W$. As $U \sim_h U$ is trivial (by taking the constant function with constant value U), $\sim_h$ is an equivalence relation on the set of unitaries.

Moreover, if $U_1 \sim_h V_1$ and $U_2 \sim_h V_2$ then $U_1U_2 \sim_h V_1V_2$ (as if $\gamma_i : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is a continuous function such that $\gamma(0) = U_i$ and $\gamma(1) = V_i$ then, if $\gamma_0 : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is defined by $\gamma_0(t) = \gamma_1(t)\gamma_2(t)$ for all $t \in [0, 1]$, γ_0 is a continuous function such that $\gamma_0(0) = U_1U_2$ and $\gamma_0(1) = V_1V_2$).

To begin our study of $K_1(\mathfrak{A})$, we will first investigate $\mathcal{U}_0(\mathfrak{A})$. Lemma 4.8 will be used in an essential part of the construction of $K_1(\mathfrak{A})$ and Lemma 4.9 is more of general interest (and provided some motivation in Chapter 5).

Lemma 4.8. *Let $\mathfrak{A}$ be a unital C*-algebra. Then the following are true:*

1. *If $A \in \mathfrak{A}$ is self-adjoint, $e^{iA} \in \mathcal{U}_0(\mathfrak{A})$.*
2. *If $U \in \mathcal{U}(\mathfrak{A})$ is such that $\sigma(U) \neq \mathbb{T}$, then $U = e^{iA}$ for some $A \in \mathfrak{A}_{sa}$ and thus $U \in \mathcal{U}_0(\mathfrak{A})$.*
3. *If $U, V \in \mathcal{U}(\mathfrak{A})$ are such that $\|U - V\| < 2$, then $V = Ue^{iA}$ for some $A \in \mathfrak{A}_{sa}$ and thus $U \sim_h V$.*

Proof. To see that (1) holds, let $A \in \mathfrak{A}$ be self-adjoint and define $\gamma : [0, 1] \to \mathcal{U}(\mathfrak{A})$ by $\gamma(t) = e^{itA}$. By the Continuous Functional Calculus, γ is a continuous function into $\mathcal{U}(\mathfrak{A})$ with $\gamma(0) = I_{\mathfrak{A}}$ and $\gamma(1) = e^{iA}$. Hence $e^{iA} \in \mathcal{U}_0(\mathfrak{A})$.

To see that (2) holds, notice that if $U \in \mathcal{U}(\mathfrak{A})$ is such that $\sigma(U) \neq \mathbb{T}$, then $U = e^{iA}$ for some self-adjoint $A \in \mathfrak{A}$ by the Continuous Functional Calculus (i.e. $A = -i \ln(U)$ for some choice of logarithmic branch). Thus (2) follows from (1).

To see that (3) holds, notice that if $U, V \in \mathcal{U}(\mathfrak{A})$ are such that $\|U - V\| < 2$ then $\|I_{\mathfrak{A}} - U^*V\| < 2$ so $-1 \notin \sigma(U^*V)$ by the Continuous Functional Calculus. Therefore, (2) implies $U^*V = e^{iA}$ for some $A \in \mathfrak{A}_{sa}$ and thus $U^*V \sim_h I_{\mathfrak{A}}$. Hence $V = U(U^*V) \sim_h U(I_{\mathfrak{A}}) = U$ as desired. □

Lemma 4.9. *Let $\mathfrak{A}$ be a unital C*-algebra. Then $\mathcal{U}_0(\mathfrak{A})$ is an open and closed normal subgroup of $\mathcal{U}(\mathfrak{A})$. Moreover*

$$\mathcal{U}_0(\mathfrak{A}) = \left\{ \prod_{j=1}^{n} e^{iA_j} \mid n \in \mathbb{N}, \{A_j\}_{j=1}^{n} \subseteq \mathfrak{A}_{sa} \right\}.$$

Proof. If $U, V \in \mathcal{U}_0(\mathfrak{A})$ then $U \sim_h I_{\mathfrak{A}}$ and $V \sim_h I_{\mathfrak{A}}$ so $UV \sim_h I_{\mathfrak{A}}^2 = I_{\mathfrak{A}}$ and thus $UV \in \mathcal{U}_0(\mathfrak{A})$. Moreover, if $U \in \mathcal{U}_0(\mathfrak{A})$ then $U \sim_h I_{\mathfrak{A}}$ so $U^{-1} \sim_h I_{\mathfrak{A}}$ (as if $\gamma : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is a continuous function such that $\gamma(0) = U$ and $\gamma(1) = I_{\mathfrak{A}}$ then, if $\gamma_0 : [0, 1] \to \mathcal{U}(\mathfrak{A})$ is defined by $\gamma_0(t) = \gamma(t)^*$ for all $t \in [0, 1]$, γ_0 is a continuous function such that $\gamma_0(0) = U^{-1}$ and $\gamma_0(1) = I_{\mathfrak{A}}$). Hence $\mathcal{U}_0(\mathfrak{A})$ is a subgroup of $\mathcal{U}(\mathfrak{A})$. Moreover, if $U \in \mathcal{U}_0(\mathfrak{A})$ and $V \in \mathcal{U}(\mathfrak{A})$ then

$$VUV^* \sim_h VI_{\mathfrak{A}}V^* = I_{\mathfrak{A}}$$

so $VUV^* \in \mathcal{U}_0(\mathfrak{A})$. Hence $\mathcal{U}_0(\mathfrak{A})$ is a normal subgroup of $\mathcal{U}(\mathfrak{A})$.

To see that $\mathcal{U}_0(\mathfrak{A})$ is open in $\mathcal{U}(\mathfrak{A})$, notice if $U \in \mathcal{U}_0(\mathfrak{A})$ and $V \in \mathcal{U}(\mathfrak{A})$ are such that $\|U - V\| < 2$ then $V \sim_h U \sim_h I$ by Lemma 4.8 so $V \in \mathcal{U}_0(\mathfrak{A})$. Thus $\mathcal{U}_0(\mathfrak{A})$ is open.

Since

$$\mathcal{U}(\mathfrak{A}) \setminus \mathcal{U}_0(\mathfrak{A}) = \bigcup_{U \in \mathcal{U}(\mathfrak{A}) \setminus \mathcal{U}_0(\mathfrak{A})} U\mathcal{U}_0(\mathfrak{A}),$$

$\mathcal{U}(\mathfrak{A}) \setminus \mathcal{U}_0(\mathfrak{A})$ is a union of open sets and thus open. Hence $\mathcal{U}_0(\mathfrak{A})$ is closed in $\mathcal{U}(\mathfrak{A})$.

Finally, to show that $\mathcal{U}_0(\mathfrak{A})$ is the desired set, we notice the inclusion $\supseteq$ is trivial as $\mathcal{U}_0(\mathfrak{A})$ is a subgroup and $e^{iA} \in \mathcal{U}_0(\mathfrak{A})$ for all $A \in \mathfrak{A}_{sa}$ by Lemma 4.8. To see the other inclusion, suppose to the contrary that there exists a unitary $U \in \mathcal{U}_0(\mathfrak{A})$ such that $U \notin \{\prod_{j=1}^n e^{iA_j} \mid n \in \mathbb{N}, \{A_j\}_{j=1}^n \subseteq \mathfrak{A}_{sa}\}$. Since $U \in \mathcal{U}_0(\mathfrak{A})$, there exists a continuous function $\gamma : [0,1] \to \mathcal{U}_0(\mathfrak{A})$ such that $\gamma(0) = I_{\mathfrak{A}}$ and $\gamma(1) = U$. Let

$$q = \inf \left\{ t \in [0,1] \mid \gamma(t) \notin \left\{ \prod_{j=1}^n e^{iA_j} \mid n \in \mathbb{N}, \{A_j\}_{j=1}^n \subseteq \mathfrak{A}_{sa} \right\} \right\}$$

which clearly exists as $\gamma(1) = U$. Since γ is continuous, there exists a $\delta > 0$ such that $\|\gamma(q) - \gamma(t)\| < 1$ for all $t \in [q-\delta, q+\delta]$. Thus $\|\gamma(q-\delta) - \gamma(t)\| < 2$ for all $t \in [q-\delta, q_\delta]$. Therefore Lemma 4.8 implies that for all $t \in [q-\delta, q+\delta]$ there exists an $A_t \in \mathfrak{A}_{sa}$ such that $\gamma(t) = e^{iA_t}\gamma(q-\delta)$. However, by the definition of q,

$$\gamma(q-\delta) = \prod_{j=1}^n e^{iA_j}$$

for some $n \in \mathbb{N}$ and $\{A_j\}_{j=1}^n \subseteq \mathfrak{A}_{sa}$ so

$$\gamma(t) = e^{iA_t} \prod_{j=1}^n e^{iA_j}$$

for all $t \in [q-\delta, q+\delta]$. As the above contradicts the definition of q, we have a contradiction so the result follows. $\square$

The construction of $K_1(\mathfrak{A})$ is more difficult than $K_0(\mathfrak{A})$ as we will need to consider an equivalent relation on the set of unitary operators in matrix algebras of $\mathfrak{A}$. In order to do this, we need the following essential lemma due to Whitehead.

Lemma 4.10 (Whitehead). *Let $\mathfrak{A}$ be a unital C*-algebra and let $U, V \in \mathcal{U}(\mathfrak{A})$. Then*

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \sim_h \begin{bmatrix} UV & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} \sim_h \begin{bmatrix} VU & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} \sim_h \begin{bmatrix} V & 0 \\ 0 & U \end{bmatrix}$$

in $\mathcal{M}_2(\mathfrak{A})$.

Proof. Let

$$W := \begin{bmatrix} 0 & I_{\mathfrak{A}} \\ I_{\mathfrak{A}} & 0 \end{bmatrix}.$$

Then it is clear that $W^2 = I_{\mathcal{M}_2(\mathfrak{A})}$ so $\sigma(W) \subseteq \{1, -1\}$. Hence $W \sim_h I_{\mathcal{M}_2(\mathfrak{A})}$. However, a trivial computation shows that

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} W \begin{bmatrix} V & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} W$$

and thus

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \sim_h \begin{bmatrix} U & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} I_{\mathcal{M}_2(\mathfrak{A})} \begin{bmatrix} V & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} I_{\mathcal{M}_2(\mathfrak{A})} = \begin{bmatrix} UV & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix}.$$

Repeating the above with U replaced with $I_{\mathfrak{A}}$ gives

$$\begin{bmatrix} I_{\mathfrak{A}} & 0 \\ 0 & V \end{bmatrix} \sim_h \begin{bmatrix} V & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix}$$

and thus

$$\begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} = \begin{bmatrix} I_{\mathfrak{A}} & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} \sim_h \begin{bmatrix} V & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix} = \begin{bmatrix} VU & 0 \\ 0 & I_{\mathfrak{A}} \end{bmatrix}.$$

Finally, the third $\sim_h$ in the statement of the theorem follows by symmetry. $\square$

We can now begin to define $K_1(\mathfrak{A})$. First we need to construct the following group.

Definition 4.11. Let $\mathfrak{A}$ be a unital C*-algebra. For two matrices $T \in \mathcal{M}_n(\mathfrak{A})$ and $S \in \mathcal{M}_m(\mathfrak{A})$ let $diag(T, S)$ denote the matrix in $\mathcal{M}_{n+m}(\mathfrak{A})$ where the upper-left $n \times n$ matrix is T, the lower-right $m \times m$ matrix is S, and all other entries are zero.

For each $n \in \mathbb{N}$ define the group homomorphism $\alpha_n : \mathcal{U}(\mathcal{M}_n(\mathfrak{A})) \to \mathcal{U}(\mathcal{M}_{n+1}(\mathfrak{A}))$ by $\alpha_n(T) = diag(T, I_{\mathfrak{A}})$ for all $T \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$. Let $\mathcal{U}_\infty(\mathfrak{A}) := \lim_{n\to\infty} \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$; that is $\mathcal{U}_\infty(\mathfrak{A})$ is the inductive limit of the unitary groups of $\mathcal{M}_n(\mathfrak{A})$ under the above inclusions. Recall that, abstractly, $\mathcal{U}_\infty(\mathfrak{A})$ can be viewed as the union of all $\mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$. Thus for any two $U, V \in \mathcal{U}_\infty$, there exists $n, m \in \mathbb{N}$ such that $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ and $V \in \mathcal{U}(\mathcal{M}_m(\mathfrak{A}))$.

We define a relation $\sim_1$ on $\mathcal{U}_\infty(\mathfrak{A})$ as follows: if $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ and $V \in \mathcal{U}(\mathcal{M}_m(\mathfrak{A}))$, $U \sim_1 V$ if and only if there exists a $k \geq \max\{m, n\}$ such that $diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})})$ and $diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})})$ are homotopically equivalent in $\mathcal{U}(\mathcal{M}_k(\mathfrak{A}))$.

The first step in developing $K_1(\mathfrak{A})$ is the following.

Theorem 4.12. *Let $\mathfrak{A}$ be a unital C*-algebra. The relation $\sim_1$ on $\mathcal{U}_\infty(\mathfrak{A})$ is an equivalence relation.*

Proof. It is clear that if $U \in \mathcal{U}_\infty(\mathfrak{A})$ then $U \sim_1 U$. Moreover, if $V \in \mathcal{U}_\infty(\mathfrak{A})$ and $U \sim_1 V$ then it is clear that $V \sim_1 U$ as homotopic equivalence is an equivalence relation on $\mathcal{U}(\mathcal{M}_k(\mathfrak{A}))$.

Finally, suppose $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$, $V \in \mathcal{U}(\mathcal{M}_m(\mathfrak{A}))$, and $W \in \mathcal{U}(\mathcal{M}_\ell(\mathfrak{A}))$ are such that $U \sim_1 V$ and $V \sim_1 W$. Then there exists $k_1 \geq \max\{m, n\}$ and $k_2 \geq \max\{m, \ell\}$ such that

$$diag(U, I_{\mathcal{M}_{k_1-n}(\mathfrak{A})}) \sim_h diag(V, I_{\mathcal{M}_{k_1-m}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_1}(\mathfrak{A}))$ and

$$diag(V, I_{\mathcal{M}_{k_2-m}(\mathfrak{A})}) \sim_h diag(W, I_{\mathcal{M}_{k_2-\ell}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_2}(\mathfrak{A}))$. Let $k := \max\{k_1, k_2\}$. It is then clear that

$$diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})}) \sim_h diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_k(\mathfrak{A}))$ and

$$diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})}) \sim_h diag(W, I_{\mathcal{M}_{k-\ell}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_k(\mathfrak{A}))$ as taking a direct sum with an identity will preserve homotopic equivalence (that is, take the direct sum of the continuous path with the constant path with constant value the identity to obtain the new continuous path). Therefore, as homotopic equivalence is an equivalence relation in $\mathcal{U}(\mathcal{M}_k(\mathfrak{A}))$, $U \sim_1 W$ as desired. □

Notation 4.13. Let $\mathfrak{A}$ be a unital C*-algebra and let $U \in \mathcal{U}_\infty(\mathfrak{A})$. Let $[U]_1$ denote the equivalence class of U in $\mathcal{U}_\infty(\mathfrak{A})$ with respect to the equivalence relation $\sim_1$ (see Definition 4.11 and Theorem 4.12).

Before we define $K_1(\mathfrak{A})$, we desire to describe the abelian operation on the $\sim_1$-equivalence classes of $\mathcal{U}_\infty(\mathfrak{A})$.

Proposition 4.14. *Let $\mathfrak{A}$ be a unital C*-algebra. Then*

1. *For all $n \in \mathbb{N}$ $[I_{\mathcal{M}_n(\mathfrak{A})}]_1 = [I_{\mathfrak{A}}]_1$.*
2. *For all $U, V \in \mathcal{U}_\infty(\mathfrak{A})$, the operation $[U]_1[V]_1 := [diag(U, V)]_1$ is well-defined.*
3. *For all $U, V \in \mathcal{U}_\infty(\mathfrak{A})$ $[U]_1[V]_1 = [V]_1[U]_1$ and $[U]_1[I_{\mathfrak{A}}]_1 = [U_1]$.*
4. *If $U, V \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ then $[U]_1[V]_1 = [UV]_1$ so $[U]_1[U^*]_1 = [I_{\mathfrak{A}}]_1$.*

Proof. Notice (1) is trivial by the definition of $\sim_1$.

To see (2), suppose $U_j \in \mathcal{U}(\mathcal{M}_{n_j}(\mathfrak{A}))$ and $V_j \in \mathcal{U}(\mathcal{M}_{m_j}(\mathfrak{A}))$ are such that $U \sim_1 U'$ and $V \sim_1 V'$. Therefore there exists a $k_1 \geq \max\{n_1, n_2\}$ and a $k_2 \geq \max\{m_1, m_2\}$ such that

$$diag(U_1, I_{\mathcal{M}_{k_1-n_1}(\mathfrak{A})}) \sim_h diag(U_2, I_{\mathcal{M}_{k_1-n_2}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_1}(\mathfrak{A}))$ and

$$diag(V_1, I_{\mathcal{M}_{k_2-m_1}(\mathfrak{A})}) \sim_h diag(V, I_{\mathcal{M}_{k_2-m_2}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_2}(\mathfrak{A}))$. As taking direct sums with the identity preserves homotopic equivalence, we can increase k_1 to assume that m_1 divides $k_1 - n_1$ and m_2 divides $k_1 - n_2$.

By taking direct sums of continuous paths, it is clear that

$$diag(U_1, I_{\mathcal{M}_{k_1-n_1}(\mathfrak{A})}, V_1, I_{\mathcal{M}_{k_2-m_1}(\mathfrak{A})}) \sim_h diag(U_2, I_{\mathcal{M}_{k_1-n_2}(\mathfrak{A})}, V_2, I_{\mathcal{M}_{k_2-m_2}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_1+k_2}(\mathfrak{A}))$. By the fact that m_1 divides $k_1 - n_1$ and m_2 divides $k_1 - n_2$, by applying Lemma 4.10 $\frac{k_1-n_1}{m_1}$ times, we see that

$$diag(U_1, I_{\mathcal{M}_{k_1-n_1}(\mathfrak{A})}, V_1, I_{\mathcal{M}_{k_2-m_1}(\mathfrak{A})}) \sim_h diag(U_1, V_1, I_{\mathcal{M}_{k_1-n_1}(\mathfrak{A})}, I_{\mathcal{M}_{k_2-m_1}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_1+k_2}(\mathfrak{A}))$ and, by applying Lemma 4.10 $\frac{k_1-n_2}{m_2}$ times,

$$diag(U_2, I_{\mathcal{M}_{k_1-n_2}(\mathfrak{A})}, V_2, I_{\mathcal{M}_{k_2-m_2}(\mathfrak{A})}) \sim diag(U_2, V_2, I_{\mathcal{M}_{k_1-n_2}(\mathfrak{A})}, I_{\mathcal{M}_{k_2-m_2}(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{k_1+k_2}(\mathfrak{A}))$. Hence $[diag(U_1, V_1)]_1 = [diag(U_2, V_2)]_1$ so this operation is well-defined.

To see (3), we note that $[U]_1[I_{\mathfrak{A}}]_1 = [diag(U, I_{\mathfrak{A}})]_1 = [U]_1$ is trivial by the definition of $\sim_1$. For the other equation, suppose $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ and $V \in \mathcal{U}(\mathcal{M}_m(\mathfrak{A}))$. Then, for any $k \geq \max\{m, n\}$, $[U]_1 = [diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})})]_1$ and $[V]_1 = [diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})})]_1$. However

$$diag(diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})}), diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})})) \sim_h diag(diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})}), diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})}))$$

in $\mathcal{M}_2(\mathcal{M}_k(\mathfrak{A}))$ by Lemma 4.10 so

$$[diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})})]_1[diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})}))]_1 = [diag(V, I_{\mathcal{M}_{k-m}(\mathfrak{A})})]_1[diag(U, I_{\mathcal{M}_{k-n}(\mathfrak{A})}))]_1$$

and thus $[U]_1[V]_1 = [V]_1[U]_1$ as desired.

To see (4), note if $U, V \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ then $diag(U, V)$ is homotopically equivalent to $diag(UV, I_{\mathcal{M}_n(\mathfrak{A})})$ in $\mathcal{M}_2(\mathcal{M}_n(\mathfrak{A}))$ by Lemma 4.10 and thus the result follows. ☐

Thus we can define $K_1(\mathfrak{A})$.

Definition 4.15. Let $\mathfrak{A}$ be a unital C*-algebra. We define $K_1(\mathfrak{A})$ to be the set

$$K_1(\mathfrak{A}) := \{[U]_1 \mid U \in \mathcal{U}_\infty(\mathfrak{A})\}$$

together with the well-defined (by Proposition 4.14) binary operation $[U]_1[V]_1 = [diag(U, V)]_1$. Thus $K_1(\mathfrak{A})$ is an abelian group by Proposition 4.14.

Note that the above holds for any unital C*-algebra. Moreover, it is clear that if $U, V \in \mathcal{U}(\mathfrak{A})$ are such that $U \sim_h V$ in $\mathfrak{A}$ then $[U]_1 = [V]_1$. Hence, as $\mathcal{U}_0(\mathfrak{A})$ is a subgroup of $\mathcal{U}(\mathfrak{A})$, there exists a well-defined group homomorphism from $\mathcal{U}(\mathfrak{A})/\mathcal{U}_0(\mathfrak{A})$ to $K_1(\mathfrak{A})$ defined by $U \mapsto [U]_1$. Our goal is to show that this group homomorphism is a group isomorphism for unital, simple, purely infinite C*-algebras.

To begin this proof, we have the following lemma which is stronger than what we currently need but will be of use in Chapter 5.

Lemma 4.16. *Let $\mathfrak{A}$ be a unital, purely infinite C*-algebra, let $U \in \mathcal{U}(\mathfrak{A})$, and let $\lambda_1, \ldots, \lambda_n \in \sigma(U)$ be distinct. Then for any $\epsilon > 0$ there exists a $V \in \mathcal{U}(\mathfrak{A})$ and infinite, orthogonal projections $P_1, \ldots, P_n \in \mathfrak{A}$ such that $\|U - V\| < \epsilon$, each P_j commutes with V, and $P_j V P_j = \lambda_j P_j$. Moreover $V = V' + \sum_{j=1}^n \lambda_j P_j$ where V' is a unitary operator in $\left(I_{\mathfrak{A}} - \sum_{j=1}^n P_j\right) \mathfrak{A} \left(I_{\mathfrak{A}} - \sum_{j=1}^n P_j\right)$.*

Proof. It is easy to choose non-zero, positive, continuous functions $f_1, \ldots, f_n$ on $\sigma(U)$ with the support of each f_j contained in the set $\{\lambda \mid |\lambda - \lambda_j| < \epsilon\}$. By choosing ϵ small enough, we can assume that the supports of $f_1, \ldots, f_n$ are disjoint.

Consider the hereditary C*-subalgebras $\overline{f_j(U)\mathfrak{A}f_j(U)}$ for all j. Since $\mathfrak{A}$ is purely infinite, there exists infinite projections $P_j \in \overline{f_j(U)\mathfrak{A}f_j(U)}$. Since the supports of $f_1, \ldots, f_n$ are disjoint, elements of different $\overline{f_j(U)\mathfrak{A}f_j(U)}$ are orthogonal so $P_1, \ldots, P_n \in \mathfrak{A}$ are infinite orthogonal projections. Moreover, it is clear that $P_j U P_i = 0$ for all $i \neq j$ as U commutes with each $f_k(U)$.

Let

$$V_0 := \sum_{j=1}^n \lambda_j P_j + \left(I_{\mathfrak{A}} - \sum_{j=1}^n P_j\right) U \left(I_{\mathfrak{A}} - \sum_{j=1}^n P_j\right) \in \mathfrak{A}.$$

Then

$$\|U - V_0\| \leq \max_j\{\|\lambda P_j - P_j U P_j\|\} \leq= \epsilon$$

since, by construction, $P_j U P_i = 0 = P_j V_0 P_i$ whenever $i \neq j$ and $P_j \leq E_U(\{\lambda \mid |\lambda - \lambda_j| < \epsilon\})$. Hence, by selecting ϵ small enough, we obtain that V_0 is invertible. Moreover, we can assume that $\|V_0 V_0^* - I_{\mathfrak{A}}\| < 2\epsilon$ so

$$\left\||V_0|^{-1} - I\right\| < 1 - \frac{1}{1 - 2\epsilon} = \frac{2\epsilon}{1 - 2\epsilon}.$$

Let V be the partial isometry in the polar decomposition of V_0. Since V_0 is invertible, $V \in \mathcal{U}(\mathfrak{A})$ and $V = V_0 |V_0|^{-1}$. Thus

$$\|U - V\| \leq \|U - V_0\| + \left\|V_0 - V_0|V_0|^{-1}\right\| \leq \epsilon + \frac{2\epsilon}{1 - 2\epsilon}$$

which can be made arbitrarily small. Finally, to see that $P_j V P_j = \lambda P_j$ for all j, we notice that $P_j V_0 = V_0 P_j = \lambda P_j$ for all j. Therefore P_j commutes with $C^*(V_0)$. Hence $P_j |V_0| P_j = (P_j V_0^* P_j V_0 P_j)^{\frac{1}{2}} = P_j$ so

$$P_j V P_j = P_j V_0 P_j |V_0|^{-1} P_j = \lambda P_j$$

for all j as desired. □

Corollary 4.17. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $U \in \mathcal{U}(\mathfrak{A})$. Then there exists a non-trivial projection $P \in \mathfrak{A}$ and a unitary $V \in P\mathfrak{A}P$ such that $U \sim_h V + (I_{\mathfrak{A}} - P)$.*

Proof. If $\sigma(U) = \{e^{i\alpha}\}$, let P be any non-trivial projection in $\mathfrak{A}$. Then $U = e^{i\alpha} I_{\mathfrak{A}} = e^{i\alpha} P + e^{i\alpha}(I_{\mathfrak{A}} - P)$. Let $V := e^{i\alpha} P \in \mathcal{U}(P\mathfrak{A}P)$ and define $\gamma : [0,1] \to \mathcal{U}(\mathfrak{A})$ by $\gamma(t) = e^{i\alpha} P + e^{i\alpha t}(I_{\mathfrak{A}} - P)$. Hence γ is a continuous path into $\mathcal{U}(\mathfrak{A})$ such that $\gamma(0) = V + (I_{\mathfrak{A}} - P)$ and $\gamma(1) = U$. Hence $U \sim_h V + (I_{\mathfrak{A}} - P)$.

Otherwise, let $\lambda_1, \lambda_2 \in \sigma(U)$ be distinct points. By Lemma 4.16 there exists infinite orthogonal projections $P_1, P_2 \in \mathfrak{A}$ and a unitary $V' \in (I_{\mathfrak{A}} - P_1 - P_2)\mathfrak{A}(I_{\mathfrak{A}} - P_1 - P_2)$ such that

$$\|U - (V' + \lambda_1 P_1 + \lambda_2 P_2)\| < 2.$$

Hence $U \sim_h V' + \lambda_1 P_1 + \lambda_2 P_2$ by Lemma 4.8. Thus, if we let $P := I_{\mathfrak{A}} - P_2$ and $V := V' + \lambda_1 P_1$, then $P \in \mathfrak{A}$ is a projection and $V \in \mathcal{U}(P\mathfrak{A}P)$ are such that $U \sim_h V + \lambda_2(I_{\mathfrak{A}} - P)$. By the same arguments as used in the above paragraph, $V + \lambda_2(I_{\mathfrak{A}} - P) \sim_h V + (I_{\mathfrak{A}} - P)$ so $U \sim_h V + (I_{\mathfrak{A}} - P)$. □

In the development of $K_1(\mathfrak{A})$ we used the matrix algebras $\mathcal{M}_n(\mathfrak{A})$ to construct $\mathcal{U}_\infty(\mathfrak{A})$ and the correct equivalence relation. There is an alternative method for investigating $K_1(\mathfrak{A})$ where the unitary group of the unitization $\mathfrak{K} \otimes_{\min} \mathfrak{A}$ is used. This is the motivation for the following lemma which will be essential in proving our desired result.

Lemma 4.18. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $U \in \mathcal{U}(\mathfrak{A})$, and let $E \in \mathfrak{K}$ be any rank one projection. Then $U \in \mathcal{U}_0(\mathfrak{A})$ if and only if $E \otimes U + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}$ in the unitization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$.*

Proof. It is clear (by taking direct sums) that if $U \in \mathcal{U}_0(\mathfrak{A})$ then $E \otimes U + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}$. Thus suppose $E \otimes U + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}$. By Corollary 4.17, $U \sim_h U' + (I_{\mathfrak{A}} - P)$ in $\mathfrak{A}$ for some non-trivial projection $P \in \mathfrak{A}$ and some unitary $U' \in P\mathfrak{A}P$. Hence

$$E \otimes (U' + (I_{\mathfrak{A}} - P)) + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}.$$

Since $I_{\mathfrak{A}} - P$ is non-zero and $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra, Lemma 2.3 implies that there exists a non-zero projection $Q < I_{\mathfrak{A}} - P$ such that $Q \sim I_{\mathfrak{A}} - P$. Since $I_{\mathfrak{A}} - P - Q$ is non-zero, Lemma 2.3 implies that there exists a collection of pairwise orthogonal projections $\{R_j\}_{j\geq 1}$ in $\mathfrak{A}$ such that $R_j \leq I_{\mathfrak{A}} - P - Q$ and $R_j \sim I_{\mathfrak{A}} - P - Q$ for all $j \in \mathbb{N}$. Let $R_0 := P + Q$ and let $V \in \mathfrak{A}$ be the partial isometry such that $V^*V = Q$ and $VV^* = I_{\mathfrak{A}} - P$. Therefore, since $Q < I_{\mathfrak{A}} - P$, $VP = PV = V^*P = PV^* = 0$. Therefore, if $W := P + V^*$ then

$$WW^* = (P + V^*)(P + V) = P + V^*V = P + Q = R_0$$

and

$$W^*W = (P + V)(P + V^*) = P + VV^* = I_{\mathfrak{A}}$$

so $R_0 \sim I_{\mathfrak{A}}$. Therefore $\{R_j\}_{j\geq 0}$ is a set of pairwise orthogonal projections all of which are equivalent to $I_{\mathfrak{A}}$.

For each $n \in \mathbb{N}$ let $V_n \in \mathfrak{A}$ be the isometry such that $V_nV_n^* = R_n$ with $V_0 = W$. For each $m \in \mathbb{N}$ let $F_m := R_0 + R_1 + \cdots + R_n \in \mathfrak{A}_m$ and define $\phi_m : \mathcal{M}_{m+1}(\mathfrak{A}) \to F_m\mathfrak{A}F_m$ by

$$\phi_m([A_{i,j}]) = \sum_{i,j=1}^{m+1} V_{i-1}A_{i,j}V_{j-1}^*.$$

To see that ϕ_m maps into $F_m\mathfrak{A}F_m$, we note that F_m is a projection so it suffices to show that

$$F_m\phi_m([A_{i,j}])F_m = \phi_m([A_{i,j}])$$

for all $[A_{i,j}] \in \mathcal{M}_{m+1}(\mathfrak{A})$. However, since $\{R_j\}_{j\geq 0}$ is a set of pairwise orthogonal projections and $V_jV_j^* = R_j$ for all $j \geq 0$, $V_j^*V_i = 0$ if $i \neq j$. Therefore $F_mV_j = V_j$ if $m \geq j$ and $V_i^*F_m = V_i^*$ if $m \geq i$. Therefore $F_m\phi_m([A_{i,j}])F_m = \phi_m([A_{i,j}])$ for all $[A_{i,j}] \in \mathcal{M}_{m+1}(\mathfrak{A})$ is clear. Moreover, as $V_j^*V_i = 0$ if $i \neq j$, it is trivial to verify that ϕ_m is a unital, injective *-homomorphism. To see that ϕ_m is surjective, we notice that if $A \in \mathfrak{A}$ then

$$F_mAF_m = \sum_{i,j=1}^{m+1} R_{i-1}AR_{j-1} = \sum_{i,j=1}^{m+1} V_{i-1}V_{i-1}^*AV_{j-1}V_{j-1}^* = \phi_m([V_{i-1}^*AV_{j-1}]).$$

Hence ϕ_m is a unital *-isomorphism and thus $\mathcal{M}_{m+1}(\mathfrak{A})$ and $F_m\mathfrak{A}F_m$ are isomorphic.

It is clear that $F_m\mathfrak{A}F_m$ embeds into $F_{m+1}\mathfrak{A}F_{m+1}$ for all $m \in \mathbb{N}$. Moreover, under the above isomorphism, this trivial imbedding corresponds to the canonical imbedding of $\mathcal{M}_{m+1}(\mathfrak{A})$ into the upper-left $(m+1) \times (m+1)$ entries of $\mathcal{M}_{m+2}(\mathfrak{A})$. Hence it is trivial to verify that the C*-algebra generated by $\bigcup_{m\geq 1} F_m\mathfrak{A}F_m$ and $I_{\mathfrak{A}}$ is canonically isomorphic to the untization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$. However, since

$$U' + (I_{\mathfrak{A}} - P) = (U' + Q) + (F_m - R_0) + (I_{\mathfrak{A}} - F_m)$$

for all $m \in \mathbb{N}$, by identifying R_0 with E, $U' + (I_{\mathfrak{A}} - P)$ corresponds to the unitary

$$E \otimes \phi_0^{-1}(U' + Q) + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}})$$

in the untization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$. However, as $V_0 = P + V^*$ where $VV^* = I_{\mathfrak{A}} - P$ and $V^*V = Q$ (so $VP = PV = V^*P = PV^* = 0$) and as $U' \in P\mathfrak{A}P$,

$$\begin{aligned}\phi_0(U' + (I_{\mathfrak{A}} - P)) &= (P + V^*)(U' + (I_{\mathfrak{A}} - P))(P + V) \\ &= U' + V^*(I_{\mathfrak{A}} - P)V \\ &= U' + V^*VV^*V \\ &= U' + Q.\end{aligned}$$

Hence $\phi_0^{-1}(U' + Q) = U' + (I_{\mathfrak{A}} - P)$.

Since $E \otimes (U' + (I_{\mathfrak{A}} - P)) + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}$ and $U' + (I_{\mathfrak{A}} - P)$ corresponds to the unitary $E \otimes \phi_0^{-1}(U' + Q) + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}})$ under the unital isomorphism of the C*-algebra generated by $\bigcup_{m\geq 1} F_m\mathfrak{A}F_m$ and $I_{\mathfrak{A}}$ is canonically isomorphic to the untization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$, $U' + (I_{\mathfrak{A}} - P) \sim_h I_{\mathfrak{A}}$ as a unital *-isomorphism applied to a path of unitaries is a path of unitaries. Hence $U \sim_h I_{\mathfrak{A}}$ as desired. □

With the above in-hand, to continue our quest in showing $K_1(\mathfrak{A}) = \mathcal{U}(\mathfrak{A})/\mathcal{U}_0(\mathfrak{A})$ we desire to show that any unitary $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ is $\sim_1$-equivalent to a unitary $V \in \mathcal{U}(\mathfrak{A}) \subseteq \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$. This, along with Lemma 4.18, will enable us to finish the proof. Our main tools are the following two lemmas.

Lemma 4.19. *Let $\mathfrak{A}$ be a unital C*-algebra and let $V \in \mathfrak{A}$ be a partial isometry such that $V^2 = 0$. Then for every unitary $U \in \mathcal{U}(V^*V\mathfrak{A}V^*V)$ the unitaries*

$$U_1 := U + (I_{\mathfrak{A}} - V^*V) \qquad \text{and} \qquad U_2 := VUV^* + (I_{\mathfrak{A}} - VV^*)$$

are homotopically equivalent in $\mathcal{U}(\mathfrak{A})$.

Proof. It is clear that U_1 and U_2 are unitaries in $\mathfrak{A}$. Moreover, it is clear that $W := V+V^*+(I_{\mathfrak{A}}-VV^*-V^*V)$ is a unitary since VV^* and V^*V are orthogonal projections (as $V^2 = 0$). Since $W^* = W$, $\sigma(W) \subseteq \mathbb{R} \cap \mathbb{T} = \{1, -1\}$ so Lemma 4.8 (part 2) implies that $W \in \mathcal{U}_0(\mathfrak{A})$.

Notice, since $V^*U = 0 = UV$, that

$$\begin{aligned} WU_1W^* &= (V + V^* + (I_{\mathfrak{A}} - VV^* - V^*V))(U + (I_{\mathfrak{A}} - V^*V))(V + V^* + (I_{\mathfrak{A}} - VV^* - V^*V)) \\ &= (VU + (V + V^*)(I_{\mathfrak{A}} - V^*V) + (I_{\mathfrak{A}} - VV^* - V^*V))(V + V^* + (I_{\mathfrak{A}} - VV^* - V^*V)) \\ &= (VU + V^* + (I_{\mathfrak{A}} - VV^* - V^*V))(V + V^* + (I_{\mathfrak{A}} - VV^* - V^*V)) \\ &= VU(V + V^*) + V^*V + (I_{\mathfrak{A}} - VV^* - V^*V)(V + V^*) + (I_{\mathfrak{A}} - VV^* - V^*V) \\ &= VUV^* + V^*V + 0 + (I_{\mathfrak{A}} - VV^* - V^*V) \\ &= VUV^* + I_{\mathfrak{A}} - VV^* = U_2. \end{aligned}$$

Hence

$$U_2 = WU_1W^* \sim_h I_{\mathfrak{A}}U_1I_{\mathfrak{A}} = U_1$$

as desired. □

Lemma 4.20. *Let $\mathfrak{A}$ be a unital C*-algebra and let $V \in \mathcal{M}_n(\mathfrak{A})$ be a partial isometry. Then for every unitary $U \in \mathcal{U}(V^*V\mathcal{M}_n(\mathfrak{A})V^*V)$, the unitaries*

$$U_1 := diag(U + (I_{\mathcal{M}_n(\mathfrak{A})} - V^*V), I_{\mathcal{M}_n(\mathfrak{A})}) \qquad \text{and} \qquad U_2 := diag(VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - VV^*), I_{\mathcal{M}_n(\mathfrak{A})})$$

are homotopically equivalent in $\mathcal{U}(\mathcal{M}_{2n}(\mathfrak{A}))$.

Proof. Consider the partial isometry $V_0 := diag(V, 0) \in \mathcal{M}_{2n}(\mathfrak{A})$. Then it is clear that $diag(U, 0) \in \mathcal{U}(V_0^*V_0\mathcal{M}_{2n}(\mathfrak{A})V_0V_0^*)$,

$$U_1 = diag(U, 0) + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_0^*V_0) = \begin{bmatrix} U + (I_{\mathcal{M}_n(\mathfrak{A})} - V^*V) & 0 \\ 0 & I_{\mathcal{M}_n(\mathfrak{A})} \end{bmatrix},$$

and

$$U_2 = V_0 diag(U, 0)V_0^* + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_0V_0^*) = \begin{bmatrix} VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - VV^*) & 0 \\ 0 & I_{\mathcal{M}_n(\mathfrak{A})} \end{bmatrix}.$$

However, if

$$V_1 := \begin{bmatrix} 0 & I_{\mathcal{M}_n(\mathfrak{A})} \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad V_2 := \begin{bmatrix} 0 & 0 \\ V & 0 \end{bmatrix},$$

then V_1 and V_2 are clearly partial isometries in $\mathcal{M}_{2n}(\mathfrak{A})$ such that $V_1^2 = 0 = V_2^2$, $V_0 = V_1V_2$, and $V_2^*V_2 = diag(V^*V, 0) = V_0^*V_0$. Therefore, by first applying Lemma 4.19 with $diag(U, 0) \in \mathcal{U}(V_0^*V_0\mathcal{M}_{2n}(\mathfrak{A})V_0V_0^*)$ and V_2, we obtain that

$$U_1 \sim_h V_2 diag(U, 0)V_2^* + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_2V_2^*).$$

However it is easy to verify that

$$W := V_2 diag(U, 0)V_2^* + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_2V_2^*) = \begin{bmatrix} 0 & 0 \\ 0 & VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - VV^*) \end{bmatrix}.$$

Hence, as $V_1^*V_1 = diag(0, I_{\mathcal{M}_n(\mathfrak{A})})$ and thus $diag(0, VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - VV^*)) \in \mathcal{U}(V_1^*V_1\mathcal{M}_{2n}(\mathfrak{A})V_1^*V_1)$, Lemma 4.19 implies

$$W \sim_h V_1WV_1^* + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_1V_1^*).$$

However

$$V_1WV_1^* + (I_{\mathcal{M}_{2n}(\mathfrak{A})} - V_1V_1^*) = U_2$$

so $U_1 \sim_h U_2$ as desired (alternatively $(V_1 + V_1^*)W(V_1 + V_1) = U_2$ and $V_1 + V_1^*$ is a self-adjoint unitary and thus homotopic to $I_{\mathcal{M}_2(\mathfrak{A})}$). ☐

With the above completed, we can now prove the necessary theorem about $K_1(\mathfrak{A})$ to move on with our studies.

Theorem 4.21. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. The group homomorphism $\phi : \mathcal{U}(\mathfrak{A})/\mathcal{U}_0(\mathfrak{A}) \to K_1(\mathfrak{A})$ defined by $U \mapsto [U]_1$ is a group isomorphism.*

Proof. To see that ϕ is injective, suppose $U, V \in \mathcal{U}(\mathfrak{A})$ are such that $[U]_1 = [V]_1$. By the definition of $\sim_1$, there exists a $k \in \mathbb{N}$ such that $diag(U, I_{\mathcal{M}_k(\mathfrak{A})}) \sim_h diag(U, I_{\mathcal{M}_k(\mathfrak{A})})$ in $\mathcal{U}(\mathcal{M}_{k+1}(\mathfrak{A}))$. Therefore, if $E \in \mathfrak{K}$ is any rank one projection then

$$E \otimes U + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h E \otimes V + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}})$$

in the unitization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$. Thus

$$E \otimes UV^* + (I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}} - E \otimes I_{\mathfrak{A}}) \sim_h I_{\mathfrak{K}\otimes_{\min}\mathfrak{A}}$$

in the unitization of $\mathfrak{K} \otimes_{\min} \mathfrak{A}$ so $UV^* \in \mathcal{U}_0(\mathfrak{A})$ by Lemma 4.18. Thus ϕ is injective.

To see that ϕ is surjective, suppose $U \in \mathcal{U}(\mathcal{M}_n(\mathfrak{A}))$ for some $n \in \mathbb{N}$. Thus it suffices to show that there exists a $W \in \mathcal{U}(\mathfrak{A})$ such that $[W]_1 = [U]_1$. Since $\mathcal{M}_n(\mathfrak{A})$ is a unital, simple, purely infinite C*-algebra by Theorem 3.11, there exists an isometry $V \in \mathcal{M}_n(\mathfrak{A})$ such that $VV^* = E_{1,1}$ where $E_{1,1}$ is the projection with $I_{\mathfrak{A}}$ in the $(1,1)$-entry and zeros elsewhere. Therefore

$$diag(U, I_{\mathcal{M}_n(\mathfrak{A})}) \sim_h diag(VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - E_{1,1}), I_{\mathcal{M}_n(\mathfrak{A})})$$

in $\mathcal{U}(\mathcal{M}_{2n}(\mathfrak{A}))$ by Lemma 4.20. Therefore, if $W := VUV^*$ (which can be viewed as a unitary operator in $\mathfrak{A}$), we obtain that

$$[U]_1 = [diag(U, I_{\mathcal{M}_n(\mathfrak{A})})]_1 = [diag(VUV^* + (I_{\mathcal{M}_n(\mathfrak{A})} - E_{1,1}), I_{\mathcal{M}_n(\mathfrak{A})})] = [W]_1$$

so ϕ is surjective. ☐

The main only use of the above in the rest of this paper will be done in conjunction with the following result which is essential in the proof of Theorem 5.11.

Lemma 4.22. *Let $\mathfrak{A}$ be a unital C^*-algebra such that there exists two isometries $T_1, T_2 \in \mathfrak{A}$ with $T_1T_1^* + T_2T_2^* = I_\mathfrak{A}$. Then*

$$[T_1UT_1^* + T_2UT_2^*]_1 = ([U]_1)^2$$

for any unitary $U \in \mathfrak{A}$.

Proof. The following proof is motivated by the ideas of Lemma 4.10. Since T_1 and T_2 are isometries in $\mathfrak{A}$ such that $T_1T_1^* + T_2T_2^* = I_\mathfrak{A}$, $T_1^*T_2 = 0 = T_2^*T_1$. Consider the operator $W := T_1T_2^* + T_2T_1^*$. Then $W = W^*$ and

$$W^2 = T_1T_2^*T_1T_2^* + T_1T_2^*T_2T_1^* + T_2T_1^*T_1T_2^* + T_2T_1^*T_2T_1^* = 0 + T_1T_1^* + T_2T_2^* + 0 = I_\mathfrak{A}.$$

Hence W is a self-adjoint unitary operator in $\mathfrak{A}$ so $W \sim_h I_\mathfrak{A}$ by Lemma 4.8. However

$$(T_1UT_1^* + T_2T_2^*)W(T_1UT_1^* + T_2T_2^*)W = (T_1UT_2^* + T_2T_1^*)(T_1UT_2^* + T_2T_1^*) = T_1UT_1^* + T_2UT_2^*.$$

Hence

$$T_1UT_1^* + T_2UT_2^* \sim_h (T_1UT_1^* + T_2T_2^*)I_\mathfrak{A}(T_1UT_1^* + T_2T_2^*)I_\mathfrak{A} = T_1U^2T_1^* + T_2T_2^*.$$

However, by applying Lemma 4.20 with $n = 1$ and the isometry $V = T_1$, we obtain that

$$diag(U^2, I_\mathfrak{A}) \sim_h diag(T_1U^2T_1^* + (I_\mathfrak{A} - T_1T_1^*), I_\mathfrak{A}) = diag(T_1U^2T_1^* + T_2T_2^*, I_\mathfrak{A})$$

in $\mathcal{U}(\mathcal{M}_2(\mathfrak{A}))$ and thus

$$[T_1UT_1^* + T_2UT_2^*]_1 = [T_1U^2T_1^* + T_2T_2^*]_1 = [diag(T_1U^2T_1^* + T_2T_2^*, I_\mathfrak{A})]_1 = [diag(U^2, I_\mathfrak{A})]_1 = [U^2]_1 = ([U]_1)^2$$

with the last equality coming from Proposition 4.14. $\square$

Remarks 4.23. To end this section, we note that $K_0(\mathcal{O}_n)$ and $K_1(\mathcal{O}_n)$ are known. In particular [Cu1] showed that $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n) = 0$ for all $n \geq 2$. It will be necessary in the final step of Theorem 11.11 to know that $K_0(\mathcal{O}_2)$ is trivial. However, we will obtain this fact as a corollary of Theorem 6.12.

5 Approximation Properties of Purely Infinite C*-Algebras

In this chapter we will begin to study purely infinite C*-algebras and their various properties. First we will look at a larger class of C*-algebras, known as the real rank zero C*-algebra, and see that purely infinite C*-algebras have real rank zero which will give us several interesting properties. Then we will show that purely infinite C*-algebras have the weak finite unitary property.

The results for the real rank zero portion of this chapter were developed from the excellent book [Da] (if you are reading these notes, you should definitely invest in this book). The portion of this chapter on the weak finite unitary property was developed from the original paper [Ph].

We begin with the definition of what it means for a C*-algebra to have real rank zero.

Definition 5.1. A unital C*-algebra $\mathfrak{A}$ is said to have real rank zero if the set of invertible self-adjoint elements, $\mathfrak{A}_{sa}^{-1}$, is dense in the set of all self-adjoint elements, $\mathfrak{A}_{sa}$.

A non-unital C*-algebra is said to have real rank zero if its unitization has real rank zero.

Example 5.2. Clearly $C(X)$ has real rank zero for a compact Hausdorff space X if and only if X is totally disconnected. Clearly every von Neumann algebra has real rank zero. Therefore all finite dimensional C*-algebras have real rank zero. Moreover, it is easy to see that the inductive limit of C*-algebras of real rank zero has real rank zero and thus AF C*-algebras have real rank zero. Moreover, the following proposition gives us more examples.

Proposition 5.3. *Let $\mathfrak{A}$ be a unital, purely infinite C*-algebra. Then $\mathfrak{A}$ has real rank zero.*

Proof. Let $A \in \mathfrak{A}$ be a self-adjoint operator with $\|A\| = 1$ and let $\epsilon > 0$. Consider the two functions on $\mathbb{R}$ defined by

$$f_\epsilon(x) = \max\{\epsilon - |x|, 0\} \qquad \text{and} \qquad g_\epsilon(x) = \begin{cases} 0 & \text{if } |x| < \epsilon \\ x - \epsilon & \text{if } x \geq \epsilon \\ x + \epsilon & \text{if } x \leq \epsilon \end{cases}.$$

Clearly $g_\epsilon(x) f_\epsilon(x) = 0$ for all $x \in \mathbb{R}$. Moreover, clearly $\|g_\epsilon(A) - A\| < \epsilon$.

Let $\mathfrak{B} := \overline{f_\epsilon(A)\mathfrak{A} f_\epsilon(A)}$ which is a hereditary C*-subalgebra of $\mathfrak{A}$. Therefore, since $\mathfrak{A}$ is purely infinite, there exists an infinite projection $P \in \mathfrak{B}$. Since $g_\epsilon(x) f_\epsilon(x) = 0$ for all $x \in \mathbb{R}$, $g_\epsilon(A)\mathfrak{B} = \{0\} = \mathfrak{B} g_\epsilon(A)$. Therefore $g_\epsilon(A)P = 0 = P g_\epsilon(A)$ so $g_\epsilon(A) = (I-P)g_\epsilon(A)(I-P)$. Hence, as $g_\epsilon(A) > 0$, $\mathfrak{B} \neq \mathfrak{A}$ so $P \neq I$. Hence $I - P \neq 0$. Therefore, since P is infinite, Proposition 2.6 implies that $I - P$ is equivalent to a subprojection of $\mathfrak{A}$. Therefore, there exists a partial isometry $V \in \mathfrak{A}$ such that $V^*V = I - P$ and $VV^* < P$. Let $Q = VV^*$.

Consider the element

$$B_\epsilon := g_\epsilon(A) + \epsilon(V + V^*) + \epsilon(P - Q).$$

By viewing $\mathfrak{A}$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$, we notice that the matrix decomposition of B_ϵ with respect to $(I-P)\mathcal{H} \oplus Q\mathcal{H} \oplus (P-Q)\mathcal{H}$ (where $V : (I-P)\mathcal{H} \to Q\mathcal{H}$ is viewed as the identity) is

$$B_\epsilon = \begin{bmatrix} g_\epsilon(A) & \epsilon & 0 \\ \epsilon & 0 & 0 \\ 0 & 0 & \epsilon \end{bmatrix}.$$

Therefore, since

$$\begin{bmatrix} g_\epsilon(A) & \epsilon \\ \epsilon & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & -\frac{1}{\epsilon^2} g_\epsilon(A) \end{bmatrix} = I_2 = \begin{bmatrix} 0 & \frac{1}{\epsilon} \\ \frac{1}{\epsilon} & -\frac{1}{\epsilon^2} g_\epsilon(A) \end{bmatrix} \begin{bmatrix} g_\epsilon(A) & \epsilon \\ \epsilon & 0 \end{bmatrix}$$

it is easy to see that B_ϵ is an invertible, self-adjoint operator. Moreover

$$\|B_\epsilon - A\| \leq \|g_\epsilon(A)\| + 3\epsilon \leq 4\epsilon$$

Hence, as $A \in \mathfrak{A}_{sa}$ was arbitrary, the proof is complete. ∎

Corollary 5.4. *$\mathcal{O}_n$ has real rank zero for all $n \geq 2$ and $n = \infty$.*

The most interesting properties of real rank zero C*-algebras are contained in the following.

Theorem 5.5. *Let $\mathfrak{A}$ be a unital C*-algebra. Then the following are equivalent:*

1. *(RR0) $\mathfrak{A}$ has real rank zero.*
2. *(FS) The element of $\mathfrak{A}$ with finite spectrum are dense in $\mathfrak{A}_{sa}$.*
3. *(HP) Every hereditary C*-subalgebra of $\mathfrak{A}$ has an approximate identity of projections (where the net of projections need not be increasing).*

Proof. We will show that (1) implies (2), (2) implies (3), and (3) implies (1). Note that (2) implies (1) is trivial.

Suppose $\mathfrak{A}$ has real rank zero. The idea is to take an arbitrary self-adjoint element, modify it a little using the real rank property so that its spectrum consists of disjoint intervals, and then approximate this new operator using spectral projections. Let $A \in \mathfrak{A}$ be a self-adjoint operator with norm 1 and let $\epsilon > 0$. Let $-1 = x_1, x_2, \ldots, x_n = 1$ be an increasing subset of $[-1, 1]$ that forms an $\frac{\epsilon}{2}$-net for $[-1, 1]$. Let $\epsilon_1 = \frac{\epsilon}{4}$. Since $\mathfrak{A}$ has real rank zero, there exists an element $A_1 \in \mathfrak{A}_{sa}$ such that $A_1 - x_1 I$ is invertible and $\|(A_1 - x_1 I) - (A - x_1 I)\| < \epsilon_1$. Hence $\|A - A_1\| < \epsilon_1$.

Next choose $0 < \epsilon_2 < \frac{\epsilon}{8}$ such that $[x_1 - \epsilon_2, x_1 + \epsilon_2]$ does not intersect $\sigma(A_1)$. Since $\mathfrak{A}$ has real rank zero, there exists an element $A_2 \in \mathfrak{A}_{sa}$ such that $A_2 - x_2 I$ is invertible and $\|(A_2 - x_2 I) - (A_1 - x_2 I)\| < \epsilon_2$. Hence $\|A_2 - A_1\| < \epsilon_2$. Moreover, by the choice of ϵ_2, $\left\|(A_1 - x_1 I)^{-1}\right\| < \epsilon_2^{-1}$ so

$$\left\|I - (A_1 - x_1 I)^{-1}(A_2 - x_1 I)\right\| \leq \left\|(A_1 - x_1 I)^{-1}\right\| \|A_2 - A_1\| < 1$$

Hence $(A_1 - x_1 I)^{-1}(A_2 - x_1 I)$ and thus $A_2 - x_1 I$ must be invertible. Therefore $x_1, x_2 \notin \sigma(A_2)$.

Next choose $0 < \epsilon_3 < \frac{\epsilon}{16}$ such that $[x_1 - \epsilon_3, x_1 + \epsilon_3]$ and $[x_2 - \epsilon_3, x_2 + \epsilon_3]$ do not intersect $\sigma(A_2)$. By repeating the above process ad nauseum, we eventually obtain a self-adjoint operator $A_n \in \mathfrak{A}$ such that $x_i \notin \sigma(A_n)$ for all i and

$$\|A - A_n\| \leq \sum_{i=1}^{n} \epsilon_i < \epsilon \left(\sum_{i=1}^{n} 2^{-i-1} \right) < \frac{\epsilon}{2}.$$

Since $x_i \notin \sigma(A_n)$ for all i, the operator

$$B := -E_{A_n}\left(-1 - \frac{\epsilon}{2}, -1\right] + \sum_{i=2}^{n} x_i E_{A_n}(x_{i-1}, x_i] + E_{A_n}\left(1, 1 + \frac{\epsilon}{2}\right)$$

(where $E_{A_n}(X)$ represents the spectral projection of A_n onto X) is an element of $\mathfrak{A}$. Clearly B is a self-adjoint operator with finite spectrum such that

$$\|A - B\| \leq \|A - A_n\| + \|A_n - B\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

as desired. Hence (1) implies (2).

Next suppose that (2) holds. Let $\mathfrak{B}$ be a hereditary C*-subalgebra of $\mathfrak{A}$. To show that $\mathfrak{B}$ has an approximate identity of projections, it suffices to show that for any finite set $B_1, \ldots, B_n \in \mathfrak{B}$ and any $0 < \epsilon < \frac{1}{2}$, there exists a projection $P \in \mathfrak{B}$ such that $\|B_i - B_i P\| < \epsilon$ for all i (and a two-sided approximation will be obtained by considering adjoints). To begin, fix $B_1, \ldots, B_n \in \mathfrak{B}$ and any $0 < \epsilon < \frac{1}{2}$. Then

$$\|B_i - B_i P\|^2 = \|(I - P)B_i^* B_i (I - P)\| \leq \|(I - P)B(I - P)\| \leq \|B - BP\|$$

where $B = \sum_{i=1}^{n} B_i^* B_i$. Therefore it suffices to consider the positive operator B. Moreover we may assume that $\|B\| = 1$.

We will first construct a projection in $\mathfrak{A}$ that has the desired property and then we will modify it slightly to obtain a projection in $\mathfrak{B}$. Fix $0 < \delta < \frac{1}{6}(\epsilon - \epsilon^2)$. Then fix $n \in \mathbb{N}$ such that $\delta^{\frac{2}{n}} > 1 - \delta$ (which is possible since $\delta < 1$). Since $\mathfrak{A}$ has property (2), there exists a positive operator $C \in \mathfrak{A}$ such that C has finite spectrum and $\left\|B^{\frac{1}{n}} - C\right\| < \frac{\delta}{n}$ and $\|C\| \leq 1$. Let $A = C^n$. Therefore

$$\|A - B\| \leq \sum_{i=1}^{n} \|C\|^i \left\|C - B^{\frac{1}{n}}\right\| \left\|B^{\frac{1}{n}}\right\|^{n-i} \leq n \left\|C - B^{\frac{1}{n}}\right\| < \delta.$$

Since C has finite spectrum, A also has finite spectrum and therefore the projection $Q := E_A[\delta, 1] \in \mathfrak{A}$. It is clear by the functional calculus that

$$\|A - AQ\| < \delta \qquad \text{and} \qquad \left\|A^{\frac{1}{n}} Q A^{\frac{1}{n}} - A\right\| \leq \max\{1 - \delta^{\frac{2}{n}}, \delta\} = \delta.$$

Let $X := B^{\frac{1}{n}} Q B^{\frac{1}{n}}$ (so X is positive with $\|X\| \leq 1$). Moreover, since $\mathfrak{B}$ is hereditary, we obtain that $X \in \mathfrak{B}$. We claim that X is almost a projection. Since

$$\|X - Q\| = \left\|B^{\frac{1}{n}} Q B^{\frac{1}{n}} - A^{\frac{1}{n}} Q A^{\frac{1}{n}} + A^{\frac{1}{n}} Q A^{\frac{1}{n}} - Q\right\| \leq 2\left\|B^{\frac{1}{n}} - A^{\frac{1}{n}}\right\| + \left\|A^{\frac{1}{n}} Q A^{\frac{1}{n}} - Q\right\| < 3\delta$$

we obtain that

$$\left\|X - X^2\right\| = \|(I - Q)(X - Q) - (X - Q)X\| \leq 6\delta < \epsilon - \epsilon^2.$$

Therefore, by the spectral theorem, $\sigma(X) \subseteq [0, \epsilon] \cup [1 - \epsilon, 1]$. Let $P := E_X[1 - \epsilon, 1]$ which is an element of $\mathfrak{B}$ by our choice of $\epsilon < \frac{1}{2}$. Clearly $\|X - P\| < \epsilon$ and

$$\|B - BP\| \leq \|B - A\| + \|A - AQ\| + \|AQ - BQ\| + \|BQ - BP\| < 3\delta + \|Q - P\| \leq 3\delta + (3\delta + \epsilon) < 2\epsilon - \epsilon^2 < 2\epsilon$$

as desired.

Finally suppose (3) holds and let $A \in \mathfrak{A}$ be an arbitrary self-adjoint operator of norm 1 and let $\epsilon > 0$. Let $A = A_+ - A_-$ be the decomposition of A into its positive and negative parts (where A_+ and A_- commute with $A_+ A_- = 0$). Let $\mathfrak{B} = \overline{A_+ \mathfrak{A} A_+}$ which is a hereditary C*-subalgebra of $\mathfrak{A}$ by Lemma 2.9. Since $\mathfrak{A}$ has property (3), there exists a projection $P \in \mathfrak{B}$ such that $\|A_+ - A_+ P\| < \epsilon$. Since $P \in \mathfrak{B}$ and $A_+ A_- = 0$, we obtain that $A_- P = 0 = P A_-$.

Let

$$B_\epsilon := PAP + (2\epsilon)P + (I - P)A(I - P) - (2\epsilon)(I - P) \in \mathfrak{A}.$$

Then B_ϵ is self-adjoint and

$$\|B_\epsilon - A\| \leq \|PA(I - P) + (I - P)AP\| + 2\epsilon \|P - (I - P)\| \leq \epsilon + 2\epsilon = 3\epsilon.$$

Moreover, since B_ϵ commutes with P, B_ϵ is invertible if and only if $PB_\epsilon P$ and $(I - P)B_\epsilon(I - P)$ are invertible. However

$$PB_\epsilon P = PAP + (2\epsilon)P \geq (2\epsilon)P$$

which is invertible as $2\epsilon > 0$ and

$$\begin{aligned}
(I - P)B_\epsilon(I - P) &= (I - P)A(I - P) - (2\epsilon)(I - P) \\
&= (I - P)A_+(I - P) - (I - P)A_-(I - P) - (2\epsilon)(I - P) \\
&\leq \epsilon(I - P) - (2\epsilon)(I - P) = -\epsilon(I - P)
\end{aligned}$$

(where $(I - P)A_+(I - P) \leq \epsilon(I - P)$ as $\|A_+ - A_+ P\| < \epsilon$) which is invertible as $-\epsilon < 0$. Hence $\mathfrak{A}$ has real rank zero. ☐

By the above theorem, we can see that the self-adjoint elements with finite spectrum are dense in the set of all self-adjoint elements in a real rank zero C*-algebra (this is known as property (FS)). However, we know that the spectrum of any unitary element of a C*-algebra has dimension one in $\mathbb{C}$ so it is natural to ask whether any unitary operator can be approximated by a unitary operator with finite spectrum in a real rank zero C*-algebra. We encapsulate this idea in the following definition.

Definition 5.6. A unital C*-algebra $\mathfrak{A}$ is said to have the finite unitary property (written property (FU)) if every unitary operator $U \in \mathfrak{A}$ is a limit of unitary operators with finite spectrum.

A unital C*-algebra $\mathfrak{A}$ is said to have the weak finite unitary property (written weak (FU)) if every unitary operator $U \in \mathfrak{A}$ in the connected component of the identity in the group of unitary operators of $\mathfrak{A}$ (denoted $\mathcal{U}_0(\mathfrak{A})$) is a limit of unitary operators with finite spectrum.

A non-unital C*-algebra is said to have (weak) property (FU) if its unitization has (weak) property (FU).

We will mainly be interesting in weak property (FU). Our goal is to show that every unital, simple, purely infinite C*-algebra has weak property (FU). The main idea of the proof is contained in the following (although the details will take us a fair amount of time to fill in).

Remarks 5.7. It is well known that if $\mathfrak{A}$ is a unital C*-algebra then $\mathcal{U}_0(\mathfrak{A})$ is the closure of $\{e^{iA_1}e^{iA_2}\cdots e^{iA_n} \mid A_j \in \mathfrak{A}_{sa}\}$ and is path-connected (see Lemma 4.9). However, if $U \in \mathcal{U}_0(\mathfrak{A})$ were a limit of elements of the form e^{iA_n} where $A_n \in \mathfrak{A}_{sa}$ and if $\mathfrak{A}$ had property (FS), then, by approximating each A_n by a self-adjoint element with finite spectrum, it is easy to see that U is the limit of unitaries with finite spectrum. Therefore, our goal is to show that in a unital, simple, purely infinite C*-algebra every unitary $U \in \mathcal{U}_0(\mathfrak{A})$ is a limit of elements of the form e^{iA_n} where $A_n \in \mathfrak{A}_{sa}$. One useful observation is that if $U \in \mathfrak{A}$ is a unitary operator such that $\sigma(U) \neq \mathbb{T}$, then $U = e^{iA}$ for some self-adjoint operator $A \in \mathfrak{A}$ by Lemma 4.8. The proof that unital, simple, purely infinite C*-algebras have weak property (FU) will involve a series of technical lemmas.

Lemma 5.8. *Let $\mathfrak{A}$ be a unital C*-algebra, let $\alpha : [0,1] \to \mathcal{U}_0(\mathfrak{A})$ be a piecewise C^1 path such that $\alpha(0) = I$, and let L be the length of α. Then $\sigma(\alpha(1)) \subseteq \{e^{i\theta} \mid -L \leq \theta \leq L\}$.*

Proof. To begin, we first claim that if $U, V \in \mathfrak{A}$ are unitaries and $\lambda \in \sigma(V)$, then there exists a $\mu \in \sigma(U)$ such that $|\mu - \lambda| \leq \|U - V\|$. To see this, suppose to the contrary that $\lambda \in \sigma(V)$ yet $|\mu - \lambda| > \|U - V\|$ for all $\mu \in \sigma(U)$. Then

$$\|I - (\lambda I - V)\| = \left\|(\lambda I - U)^{-1}(\lambda I - U) - (\lambda I - V)\right\| \leq \left\|(\lambda I - U)^{-1}\right\| \|U - V\| < 1$$

as $\left\|(\lambda I - U)^{-1}\right\| \leq dist(\lambda, \sigma(U))^{-1} < \|U - V\|^{-1}$. Hence $\lambda I - V$ is invertible which is a contradiction. Hence the claim must be true.

Let $0 = x_0 < x_1 < \cdots < x_n = 1$ be any partition of $[0,1]$. Suppose $\lambda \in \sigma(\alpha(1))$. By moving backwards along α, the above claim implies there are scalars $\mu_k \in \sigma(\alpha(x_k))$ such that $\mu_n = \lambda$ and $|\mu_k - \mu_{k-1}| \leq \|\alpha(x_k) - \alpha(x_{k+1})\|$. Hence

$$\sum_{k=1}^{n} |\mu_k - \mu_{k-1}| \leq \sum_{k=1}^{n} \|\alpha(x_k) - \alpha(x_{k+1})\|$$

By taking the infimum over all partitions of $[0,1]$, we see that the right hand side of the above equation must converge to L as α is C^1. Since $\max_k\{|\mu_k - \mu_{k-1}|\} \leq \max_k\{\|\alpha(x_k) - \alpha(x_{k+1})\|\}$, the limit of the left hand side of the above equation is at most the length of the path from 1 to λ along the unit circle. Therefore the result follows. □

The following is a result that is related to K-theory and is motivated by Lemma 4.10.

Lemma 5.9. *Let $\mathfrak{A}$ be a unital C*-algebra and let $U \in \mathfrak{A}$ be a unitary operator. Then for all $\epsilon > 0$ there exists an $A \in \mathcal{M}_2(\mathfrak{A})_{sa}$ such that $\left\|U \oplus U^* - e^{iA}\right\| < \epsilon$.*

Proof. The idea of the proof is to show that $U \oplus U^*$ can be approximated by unitary operators with a gap in their spectrum. Define the C^1 path $\alpha : [0, \frac{\pi}{2}] \to \mathcal{U}_0(\mathfrak{A})$ by

$$\alpha(x) = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}.$$

It is clear that α is a C^1 path, $\alpha(0) = I_2$ and $\alpha\left(\frac{\pi}{2}\right) = U \oplus U^*$. Moreover, if we take the derivative of α, we see, by the product rule, that $\alpha'(x)$ is the sum of two unitary operators so $\|\alpha'(x)\| \leq 2$ for all $x \in [0, \frac{\pi}{2}]$. Therefore, for any $x < \frac{\pi}{2}$, Lemma 5.8 implies that $-1 \notin \sigma(\alpha(x))$. Hence $U \oplus U^*$ can be approximated by unitary operators with a gap in their spectrum and thus the result follows by the functional calculus. □

Now we are at our main technical lemma before attempting to prove our desired result.

Lemma 5.10. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $E_1, E_2, E_3,$ and E_4 be non-zero orthogonal projections in $\mathfrak{A}$ such that $\sum_{j=1}^4 E_j = I$, and let $A \in \mathfrak{A}$ be a partial isometry with $A^*A = E_2$ and $AA^* = E_3$. Let $U \in \mathcal{U}(E_1\mathfrak{A}E_1)$ be such that $\sigma(U) = \mathbb{T}$, and let $V \in \mathcal{U}(E_2\mathfrak{A}E_2)$. Then for any $\epsilon > 0$ there exists a unitary $Z \in \mathcal{U}(\mathfrak{A})$ and a unitary $W \in \mathcal{U}(E_4\mathfrak{A}E_4)$ with finite spectrum such that*

$$\|Z^*(U + I - E_1)Z - (U + V + AV^*A^* + W)\| < \epsilon. \qquad (*)$$

It is helpful to consider the matrix decomposition of $\mathfrak{A}$ of the equation $(*)$ with respect to the $(i,j)^{th}$ coordinate being in $E_i\mathfrak{A}E_j$ and identifying $E_2\mathfrak{A}E_2$ and $E_3\mathfrak{A}E_3$ via the partial isometry A:

$$\left\| Z^* \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} Z - \begin{bmatrix} U & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V^* & 0 \\ 0 & 0 & 0 & W \end{bmatrix} \right\| < \epsilon$$

Proof. Consider the *-isomorphism $\varphi : \mathcal{M}_2(E_2\mathfrak{A}E_2) \to (E_2 + E_3)\mathfrak{A}(E_2 + E_3)$ defined by

$$\varphi\left(\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}\right) = A_{1,1} + A_{1,2}A^* + AA_{2,1} + AA_{2,2}A^*$$

(which is clearly a *-isomorphism). Therefore $V + AV^*A^* = \varphi(V \oplus V^*)$. Therefore, by Lemma 5.9, we can replace $V + AV^*A^*$ by e^{iH} where $H \in ((E_2+E_3)\mathfrak{A}(E_2+E_3))_{sa}$. Moreover, since $\mathfrak{A}$ is purely infinite, Lemma 2.13 implies that $(E_2 + E_3)\mathfrak{A}(E_2 + E_3)$ is purely infinite. Hence $(E_2 + E_3)\mathfrak{A}(E_2 + E_3)$ has real rank zero by Proposition 5.3. Hence we can assume that $V + AV^*A^* = \sum_{j=1}^n \lambda_j Q_j$ where $\lambda_j \in \mathbb{T}$ and Q_j are non-zero, orthogonal projections which sum to $E_2 + E_3$. Finally, by allowing another small perturbation by applying Lemma 4.16 to the unital, purely infinite C*-algebra $E_1\mathfrak{A}E_1$, we can assume that $U = U_0 + \sum_{j=1}^n \lambda_j P_j$ where $P_1, \ldots, P_n$ are infinite, orthogonal projections in $E_1\mathfrak{A}E_1$ and, if $P := E_1 - \sum_{j=1}^n P_j$, then $U_0 \in \mathcal{U}(P\mathfrak{A}P)$ (this is possible since $\sigma(U) = \mathbb{T}$).

Now we will find Z and W with the appropriate properties such that

$$Z^*(U + I - E_1)Z = U + \sum_{j=1}^n \lambda_j Q_j + W$$

(and thus the only approximations needed were the ones done above). Since $E_1\mathfrak{A}E_1$ is a unital, simple, purely infinite C*-algebra and every non-zero projection in $E_1\mathfrak{A}E_1$ is infinite by Lemma 2.13, there exists partial isometries $C_j \in P_1\mathfrak{A}P_1$ such that $C_j^*C_j = P_j$ and $C_jC_j^* < P_j$. Let $C := P + \sum_{j=1}^n C_j$. Then, since $P, P_1, \ldots, P_n$ are orthogonal projections with $P + \sum_{j=1}^n P_j = E_1$, we obtain that

$$C^*C = P + \sum_{j=1}^n P_j = E_1, \qquad CC^* = P + \sum_{j=1}^n C_jC_j^* = E_1 - \sum_{j=1}^n (P_j - C_jC_j^*),$$

and

$$CUC^* = \left(P + \sum_{j=1}^n C_j\right)\left(U_0 + \sum_{j=1}^n \lambda_j P_j\right)\left(P + \sum_{j=1}^n C_j^*\right) = U_0 + \sum_{j=1}^n \lambda_j C_jC_j^*.$$

Next, since each $P_j - C_jC_j^*$ is infinite, Proposition 2.6 implies we can choose partial isometries $D_j \in \mathfrak{A}$ such that $D_j^*D_j = Q_j$ and $D_jD_j^* \leq P_j - C_jC_j^*$. Therefore, since $Q_1, \ldots, Q_n$ are orthogonal, if we let $D := \sum_{j=1}^n D_j$, then D is a partial isometry with

$$D^*D = \sum_{j=1}^n Q_j = E_2 + E_3, \qquad DD^* = \sum_{j=1}^n D_jD_j^* \leq \sum_{j=1}^n (P_j - C_jC_j^*),$$

and

$$D\left(\sum_{j=1}^{n}\lambda_j Q_j\right)D^* = \sum_{j=1}^{n}\lambda_j D_j D_j^*.$$

Finally, by Theorem 2.14, there exists a partial isometry $B \in \mathfrak{A}$ such that

$$B^*B < E_4 \qquad \text{and} \qquad BB^* = \sum_{j=1}^{n}(P_j - C_jC_j^* - D_jD_j^*).$$

Let

$$W_0 := \sum_{j=1}^{n}\lambda_j B^*(P_j - C_jC_j^* - D_jD_j^*)B$$

which clearly is a unitary in $(B^*B)\mathfrak{A}(B^*B) \subseteq E_4\mathfrak{A}E_4$ with finite spectrum. Thus

$$BW_0B^* = \sum_{j=1}^{n}\lambda_j(P_j - C_jC_j^* - D_jD_j^*).$$

Moreover, if $Z_0 := B + C + D$, then

$$Z_0^*Z_0 = (B^* + C^* + D^*)(B + C + D) = B^*B + C^*C + D^*D = E_1 + E_2 + E_3 + B^*B$$

(where a little thought is needed to ensure cancellation),

$$Z_0Z_0^* = (B + C + D)(B^* + C^* + D^*) = BB^* + CC^* + DD^* = E_1$$

(where a little thought is needed to ensure cancellation), and

$$\begin{aligned} Z_0\left(U + \sum_{j=1}^{n}\lambda_j Q_j + W_0\right)Z_0^* &= CUC^* + D\left(\sum_{j=1}^{n}\lambda_j Q_j\right)D^* + BW_0B^* \\ &= \left(U_0 + \sum_{j=1}^{n}\lambda_j C_jC_j^*\right) + \left(\sum_{j=1}^{n}\lambda_j D_jD_j^*\right) + \left(\sum_{j=1}^{n}\lambda_j(P_j - C_jC_j^* - D_jD_j^*)\right) \\ &= U_0 + \sum_{j=1}^{n}\lambda_j P_j \\ &= U \end{aligned}$$

Therefore, Z_0 implies that

$$[E_1]_0 = [E_1 + E_2 + E_3 + B^*B]_0$$

in $K_0(\mathfrak{A})$. Thus

$$[I - E_1]_0 + [E_1]_0 = [I]_0 = [I - (E_1 + E_2 + E_3 + B^*B)]_0 + [E_1 + E_2 + E_3 + B^*B]_0 = [E_4 - B^*B]_0 + [E_1]_0.$$

Since $K_0(\mathfrak{A})$ is a group by Theorem 4.4, the above implies

$$[I - E_1]_0 = [E_4 - B^*B]_0$$

and thus $I - E_1$ is equivalent to $E_4 - B^*B$ in $\mathfrak{A}$. Hence there exists a partial isometry $Y \in \mathfrak{A}$ such that $YY^* = I - E_1$ and $Y^*Y = E_4 - B^*B$.

Let $Z := Z_0 + Y$ and $W := W_0 + E_4 - B^*B$. Then

$$Z^*Z = (Z_0^* + Y^*)(Z_0 + Y) = Z_0^*Z_0 + Y^*Y = (E_1 + E_2 + E_3 + B^*B) + (E_4 - B^*B) = I$$

and similarly

$$ZZ^* = Z_0 Z_0^* + YY^* = (E_1) + (I - E_1) = I$$

so Z is a unitary operator in $\mathfrak{A}$. Moreover $W \in E_4 \mathfrak{A} E_4$ being the sum of elements of $E_4 \mathfrak{A} E_4$. Moreover, since W_0 was a unitary in $(B^*B)\mathfrak{A}(B^*B)$ with finite spectrum, W is a unitary in $E_4 \mathfrak{A} E_4$ with finite spectrum. Finally

$$\begin{aligned}
Z^*(U + I - E_1)Z &= (Z_0^* + Y^*)(U + I - E_1)(Z_0 + Y) \\
&= Z_0^* U Z_0 + Y^*(I - E_1)Y \\
&= Z_0^* \left(Z_0 \left(U + \sum_{j=1}^{n} \lambda_j Q_j + W_0 \right) Z_0^* \right) Z_0 + (E_4 - B^*B) \\
&= (E_1 + E_2 + E_3 + B^*B) \left(U + \sum_{j=1}^{n} \lambda_j Q_j + W_0 \right) (E_1 + E_2 + E_3 + B^*B) + (E_4 - B^*B) \\
&= U + \sum_{j=1}^{n} \lambda_j Q_j + W_0 + (E_4 - B^*B) \\
&= U + \sum_{j=1}^{n} \lambda_j Q_j + W
\end{aligned}$$

as desired. □

Finally, we have the following.

Theorem 5.11. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let $U \in \mathcal{U}_0(\mathfrak{A})$. Then for all $\epsilon > 0$ there exists an $A \in \mathfrak{A}_{sa}$ such that $\left\| U - e^{iA} \right\| < \epsilon$. Hence $\mathfrak{A}$ has weak property (FU).*

Proof. Let $U \in \mathcal{U}_0(\mathfrak{A})$ and let $\epsilon > 0$. If $\sigma(U) \neq \mathbb{T}$, we are done by the Continuous Functional Calculus. Thus we may assume $\sigma(U) = \mathbb{T}$.

Since $1 \in \sigma(U)$, by Lemma 4.16 there exists an infinite projection P_0 and a unitary $U_{00} \in (I - P_0)\mathfrak{A}(I - P_0)$ such that $\|U - (U_{00} + P_0)\| < \frac{\epsilon}{4}$ and $U_{00} = (I - P_0)U(I - P_0)$. Since $\mathcal{U}_0(\mathfrak{A})$ is open, we can suppose without loss of generality that $U_{00} + P_0 \in \mathcal{U}_0(\mathfrak{A})$. Since P_0 is an infinite (and thus properly infinite) projection, there exists a partial isometry $X \in \mathfrak{A}$ such that $X^*X = P_0$ and $XX^* < P_0$. Since $U_{00} + P_0 \in \mathcal{U}_0(\mathfrak{A})$, there exists a continuous path $\alpha : [0, 1] \to \mathcal{U}_0(\mathfrak{A})$ such that $\alpha(0) = I$ and $\alpha(1) = U_{00} + P_0$. Let

$$\alpha_0 : [0, 1] \to (I - P_0 + XX^*)\mathfrak{A}(I - P_0 + XX^*)$$

be defined by

$$\alpha_0(x) = (I - P_0 + X)\alpha(x)(I - P_0 + X)^*.$$

It is trivial to note that

$$(I - P_0 + X)(I - P_0 + X)^* = I - P_0 + XX^*$$

so α_0 does indeed map into $(I - P_0 + XX^*)\mathfrak{A}(I - P_0 + XX^*)$. Moreover, we claim that $\alpha_0(x)$ is invertible for all x. To see this, we notice that

$$(I - P_0 + X)^*(I - P_0 + X) = I - P_0 + X^*X = I$$

so it is clear that since each $\alpha(x)$ is invertible, each $\alpha_0(x)$ is invertible. Therefore, if we define

$$\alpha_{00} : [0, 1] \to \mathcal{U}((I - P_0 + XX^*)\mathfrak{A}(I - P_0 + XX^*))$$

by $\alpha_{00}(x) = \alpha_0(x)|\alpha_0(x)|^{-1}$, then α_{00} is a continuous path. Moreover $\alpha_{00}(0) = (I - P_0 + XX^*)$ and since

$$\alpha_0(1) = (I - P_0 + X)(U_{00} + P_0)(I - P_0 + X)^* = (I - P_0)U_{00}(I - P_0) + XX^* = U_{00} + XX^*$$

which is a unitary element in $(I - P_0 + XX^*)\mathfrak{A}(I - P_0 + XX^*)$, we obtain that

$$U_{00} + XX^* \in \mathcal{U}_0((I - P_0 + XX^*)\mathfrak{A}(I - P_0 + XX^*)).$$

Therefore, if $P := P_0 - XX^*$ and $U_0 := U_{00} + XX^*$, then $U_0 \in \mathcal{U}_0((I-P)\mathfrak{A}(I-P))$ and $\|U - (U_0 + P)\| = \|U - (U_{00} + P_0)\| < \frac{\epsilon}{4}$. Again, if $\sigma(U_0) \neq \mathbb{T}$, we would be done by considering direct sums. Thus we may assume that $\sigma(U_0) = \mathbb{T}$.

Since $U_0 \in \mathcal{U}_0((I-P)\mathfrak{A}(I-P))$, we can find unitaries $U_0, U_1, \ldots, U_N \in \mathcal{U}_0((I-P)\mathfrak{A}(I-P))$ such that $\|U_j - U_{j+1}\| < \frac{\epsilon}{4}$ for all $j = 0, \ldots, N-1$, $U_0 = I - P$, and $U_N = I - P$. Since $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra by Lemma 2.13, Lemma 2.3 implies there exists partial isometries $\{C_j\}_{j=1}^N \cup \{D_j\}_{j=1}^N$ such that $C_j^*C_j = D_j^*D_j = I - P$ for all j and $P_j = C_jC_j^*$ and $Q_j = D_jD_j^*$ are all mutually orthogonal projections such that

$$\sum_{j=1}^N P_j + \sum_{j=1}^N Q_j < P.$$

Let $V := \sum_{j=1}^N C_jU_j^*C_j^*$, $A := \sum_{j=1}^N D_jC_j^*$, $E_1 := I - P$, $E_2 := \sum_{j=1}^N C_jC_j^*$, $E_3 := \sum_{j=1}^N D_jD_j^*$, and $E_4 := P - E_2 - E_3$. By construction, E_1, E_2, E_3, E_4 are non-zero orthogonal projections that sum to the identity. Moreover, it is clear that

$$A^*A = \sum_{j=1}^N C_jC_j^* = E_2 \qquad \text{and} \qquad AA^* = \sum_{j=1}^N D_jD_j^* = E_3.$$

By construction $U_0 \in U(E_1\mathfrak{A}E_1)$ is such that $\sigma(U_0) = \mathbb{T}$. Lastly, it is clear that $V \in E_2\mathfrak{A}E_2$ and since

$$V^*V = \sum_{j=1}^N C_jU_jC_j^*C_jU_j^*C_j^* = \sum_{j=1}^N C_jU_j(I-P)U_j^*C_j^* = \sum_{j=1}^N C_jU_jU_j^*C_j^* = \sum_{j=1}^N C_j(I-P)C_j^* = E_2$$

and similarly $VV^* = E_2$. Hence $V \in \mathcal{U}(E_2\mathfrak{A}E_2)$. Hence, by Lemma 5.10, there exists a unitary $Z \in \mathcal{U}(\mathfrak{A})$ and a unitary $W \in E_4\mathfrak{A}E_4$ with finite spectrum such that

$$\|Z^*(U_0 + P)Z - (U_0 + V + AV^*A + W)\| < \frac{\epsilon}{4}.$$

Let $D_0 := I - P$ and let $B := \sum_{j=1}^N D_{j-1}C_j^*$. Then

$$B^*B = \sum_{j=1}^N C_jC_j^* = \sum_{j=1}^N P_j \qquad \text{and } BB^* = \sum_{j=1}^n D_{j-1}D_{j-1}^* = (I-P) + \sum_{j=1}^{N-1} Q_j.$$

Moreover, we notice that

$$AV^*A^* = \sum_{i,j,k=1}^N D_iC_i^*C_jU_jC_j^*C_kD_k^* = \sum_{j=1}^N D_jU_jD_j^*$$

and

$$BV^*B^* = \sum_{i,j,k=1}^N D_{i-1}C_i^*C_jU_jC_j^*C_kD_{k-1}^* = \sum_{j=1}^N D_{j-1}U_jD_{j-1}^*.$$

Therefore

$$\begin{aligned} &\|(U_0 + (AV^*A^* - Q_N) + V) - (BV^*B^* + V)\| \\ = &\left\| D_0U_0D_0^* + \sum_{j=1}^N D_jU_jD_j^* - \sum_{j=1}^N D_{j-1}U_jD_{j-1}^* - Q_N \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{N} D_{j-1}(U_{j-1} - U_j)D_{j-1}^* + D_N U_N D_N - Q_N \right\| \\
&= \left\| \sum_{j=1}^{N} D_{j-1}(U_{j-1} - U_j)D_{j-1}^* \right\| \qquad \text{since } U_N = I - P \text{ so } D_N U_N D_n^* = D_N D_N^* = Q_N \\
&= \max\{\|U_{j-1} - U_j\| \mid 1 \leq j \leq N\} \quad \text{since } D_{j-1}(U_{j-1} - U_j)D_{j-1}^* = Q_{j-1}D_{j-1}(U_{j-1} - U_j)D_{j-1}^* Q_{j-1} \\
&< \frac{\epsilon}{4}.
\end{aligned}
$$

Let

$$
R := \left(I - P + \sum_{j=1}^{N-1} Q_j \right) + \sum_{j=1}^{N} P_j = I - E_4 - Q_N
$$

Since $R = BB^* + B^*B$, by applying the same proof used at the beginning of the proof of Lemma 5.10, there exists a self-adjoint element $H_0 \in R\mathfrak{A}R$ such that $\left\|e^{iH_0} - (BV^*B^* + V)\right\| < \frac{\epsilon}{4}$ where $e^{iH_0} \in R\mathfrak{A}R$.

Since W has finite spectrum, there exists an $H_1 \in E_4\mathfrak{A}E_4$ such that $W = e^{iH_1}$ (the exponential in $E_4\mathfrak{A}E_4$). Let $H := H_0 + H_1$. By the construction of R, $H_0H_1 = 0$ and $HQ_N = 0 = Q_NH$. Therefore $e^{iH} = e^{iH_0} + Q_N + W$ (where we view $e^{iH_0} \in R\mathfrak{A}R$). Therefore

$$
\begin{aligned}
\left\|e^{iH} - Z^*UZ\right\| \leq \left\|e^{iH_0} - (BV^*B + V)\right\| &+ \|(BV^*B + V) - (U_0 + AV^*A^* - Q_N + V)\| \\
&+ \|(U_0 + V + AV^*A + W) - Z^*(U_0 + P)Z\| + \|(U_0 + P) - U\| \\
&\leq \epsilon
\end{aligned}
$$

Hence $\left\|e^{iZHZ^*} = U\right\| \leq \epsilon$ as desired.

Hence, by Remarks 5.7, $\mathfrak{A}$ has weak property (FU). □

It should be noted that H. Lin has proven that all unital C*-algebra of real rank zero have the weak (FU) property (see [Li]). However, to prove this would take us too far from our goal.

6 *-Homomorphisms From $\mathcal{O}_2$

In this chapter we will study the unital *-homomorphisms from $\mathcal{O}_2$ into a unital, simple, purely infinite C*-algebra. The main goal of this chapter is to prove that all such *-homomorphisms are approximately unitarily equivalent.

Most of the results for this chapter were developed from the book [Ro2] and the additional papers referenced there. Lemma 6.4 is from the excellent book [Da]. The details of Lemma 6.7 are from [Ro1].

We begin with a definition that will be interesting for C*-algebra with weak property (FU).

Definition 6.1. Let $\mathfrak{A}$ be a unital C*-algebra. We say that $\mathfrak{A}$ has finite exponential length L if each unitary $U \in \mathcal{U}_0(\mathfrak{A})$ can be written as $U = e^{iH_1} \cdots e^{iH_n}$ where $H_j \in \mathfrak{A}$ are self-adjoint elements such that $\sum_{j=1}^n \|H_j\| \leq L$.

Lemma 6.2. *Let $\mathfrak{A}$ be a unital C*-algebra with weak property (FU). Then $\mathfrak{A}$ has finite exponential length 4.*

Proof. Let $U \in \mathcal{U}_0(\mathfrak{A})$ be arbitrary. Since $\mathfrak{A}$ has weak property (FU), there exists a unitary $V \in \mathfrak{A}$ such that V has finite spectrum and $\|U - V\| < \frac{2}{\pi}(4-\pi)$. Therefore UV^* is a unitary in $\mathfrak{A}$ such that $\|UV^* - I\| < \frac{2}{\pi}(4-\pi) < 1$ so that $-1 \notin \sigma(UV^*)$ by the Continuous Functional Calculus. Hence there exists a self-adjoint element $H_1 \in \mathfrak{A}$ such that $UV^* = e^{iH_1}$. In fact, we can choose $H_1 = \ln(UV^*)$ for a suitable branch of ln and thus we can force $\sigma(H_1) = \ln(\sigma(UV^*))$ for this choice of ln. However, since $\|UV^* - I\| < \frac{2}{\pi}(4-\pi)$, we see that

$$\|H_1\| \leq \arccos\left(\frac{2\pi - 4}{\pi}\right) \leq 4 - \pi$$

by the Continuous Functional Calculus. Hence $U = e^{iH_1}V$ with $\|H_1\| \leq 4-\pi$. Since V has finite spectrum, we can write $V = e^{iH_2}$ where H_2 is self-adjoint and $\|H_2\| \leq \pi$. Therefore we have that $U = e^{iH_1}e^{iH_2}$. Since $(4-\pi)+\pi = 4$, the result is complete. $\square$

Lemma 6.3. *Let $\mathfrak{A}$ be a unital C*-algebra with weak property (FU) and let $k \in \mathbb{N}$. Every unitary $U \in \mathcal{U}_0(\mathfrak{A})$ can be written as $U = V_1 \cdots V_{2k}$ where V_j are unitaries in $\mathfrak{A}$ such that $\|V_j - I\| \leq \frac{\pi}{k}$.*

Proof. From the proof of Lemma 6.2 each unitary $U \in \mathcal{U}_0(\mathfrak{A})$ can be written as $U = e^{iH_1}e^{iH_2}$ where $\|H_j\| \leq \pi$. Hence $k = 1$ $\left\|e^{iH_j} - I\right\| \leq 2 \leq \pi$ which completes the proof when $k = 1$.

By the Continuous Functional Calculus it is easy to see that if $W \in \mathfrak{A}$ is a unitary with finite spectrum then

$$\left\|W^{\frac{1}{k}} - I\right\| \leq \sqrt{2 - 2\cos\left(\frac{\pi}{k}\right)} \leq \sqrt{2}\sqrt{1 - \cos\left(\frac{\pi}{k}\right)} \leq \sqrt{2}\sqrt{\frac{\left(\frac{\pi}{k}\right)^2}{2}} = \frac{\pi}{k}.$$

Therefore, for each $k \in \mathbb{N}$, $U = e^{i\frac{H_1}{k}}e^{i\frac{H_1}{k}} \cdots e^{i\frac{H_1}{k}}e^{i\frac{H_2}{k}}e^{i\frac{H_2}{k}} \cdots e^{i\frac{H_2}{k}}$ where $\left\|e^{i\frac{H_j}{k}} - I\right\| \leq \frac{\pi}{k}$. $\square$

Next we will need some specific properties of a 'shift' action that is similar to the one used to prove that $\mathcal{O}_2$ was nuclear in Construction 1.18. To prove this, we will need to view the 2^∞ UHF C*-algebra in an alternative manner.

Lemma 6.4. *Let $\alpha : \ell_2(\mathbb{N}) \to \mathcal{B}(\ell_2(\mathbb{N}))$ be any continuous linear map such that*

$$\alpha(\xi)\alpha(\eta) + \alpha(\eta)\alpha(\xi) = 0 \qquad \text{and} \qquad \alpha(\xi)^*\alpha(\eta) + \alpha(\eta)\alpha(\xi)^* = \langle \eta, \xi \rangle I$$

for all $\xi, \eta \in \ell_2(\mathbb{N})$ (these two equations are known as the Canonical Anticommutation Relations). If $\mathfrak{A}$ is the C*-algebra generated by $\{\alpha(\xi) \mid \xi \in \ell_2(\mathbb{N})\} \subseteq \mathcal{B}(\ell_2(\mathbb{N}))$ then $\mathfrak{A}$ is *-isomorphic to the 2^∞ UHF C*-algebra.

Proof. Let $(e_j)_{j\geq 1}$ be the canonical orthonormal basis for $\ell_2(\mathbb{N})$. For each unit vector $\xi \in \ell_2(\mathbb{N})$ let $E(\xi) := \alpha(\xi)^*\alpha(\xi)$. Using the CARs, we see that $\alpha(\xi)^2 = 0$ and $\alpha(\xi)^*\alpha(\xi) + \alpha(\xi)\alpha(\xi)^* = I$. If we multiply the second equation by $\alpha(\xi)^*\alpha(\xi)$, we see that $(\alpha(\xi)^*\alpha(\xi))^2 = \alpha(\xi)^*\alpha(\xi)$ and thus $E(\xi)$ is a projection. Moreover $\alpha(\xi)\alpha(\xi)^*$ is a projection which will be orthogonal to $E(\xi)$ and $\alpha(\xi)\alpha(\xi)^* + E(\xi) = I$ by the CARs. Hence $\alpha(\xi)\alpha(\xi)^* = E(\xi)^\perp (:= I - E(\xi))$.

Therefore $\alpha(e_1)$ is a partial isometry with domain $E(e_1)$ and range $E(e_1)^\perp$. Therefore

$$C^*(\alpha(e_1)) = span\{\alpha(e_1), \alpha(e_1)^*, E(e_1), E(e_1)^\perp\} \simeq \mathcal{M}_2(\mathbb{C}).$$

Let $E^{(1)}_{2,1} := \alpha(e_1)$, $E^{(1)}_{1,2} := \alpha(e_1)^*$, $E^{(1)}_{1,1} := E(e_1)$, and $E^{(1)}_{2,2} := E(e_1)^\perp$ (so that $\{E^{(1)}_{i,j}\}$ are matrix units for $C^*(\alpha(e_1))$).

Next we notice that if ξ and η are orthogonal unit vectors then

$$\alpha(\eta)E(\xi) - E(\xi)\alpha(\eta) = \alpha(\eta)\alpha(\xi)^*\alpha(\xi) + \alpha(\xi)^*\alpha(\eta)\alpha(\xi) = \langle \eta, \xi\rangle \alpha(\xi) = 0$$

(by the CARs) so that $\alpha(\eta)$ commutes with $E(\xi)$. Hence it is easy to see that $E(\eta)$ commutes with $E(\xi)$.

Let $V_1 := I - 2E(e_1) = E(e_1)^\perp - E(e_1)$. Then (by the CARs)

$$V_1\alpha(e_2)\alpha(e_1) = -V_1\alpha(e_1)\alpha(e_2) = -\alpha(e_1)\alpha(e_1)^*\alpha(e_1)\alpha(e_2) = \alpha(e_1)V_1\alpha(e_2)$$

and

$$V_1\alpha(e_2)\alpha(e_1)^* = -V_1\alpha(e_1)^*\alpha(e_2) = -\alpha(e_1)^*\alpha(e_1)\alpha(e_1)^*\alpha(e_2) = \alpha(e_1)^*V_1\alpha(e_2)$$

so that $V_1\alpha(e_2)$ commutes with $C^*(\alpha(e_1))$.

Since $V_1 = V_1^*$, $V_1^2 = I$, and V_1 commutes with $\alpha(e_2)$, we see that

$$C^*(V_1\alpha(e_2)) = span\{V_1\alpha(e_2), V_1\alpha(e_2)^*, E(e_2), E(e_2)^\perp\} \simeq \mathcal{M}_2(\mathbb{C})$$

and the matrix units of $C^*(V_1\alpha(e_2))$ commute with the matrix units of $C^*(\alpha(e_1))$. Let $E^{(2)}_{2,1} := V_1\alpha(e_2)$, $E^{(2)}_{1,2} := V_1\alpha(e_2)^*$, $E^{(2)}_{1,1} := E(e_2)$, and $E^{(2)}_{2,2} := E(e_2)^\perp$. Therefore we can see that $C^*(\alpha(e_1), \alpha(e_2)) = C^*(\alpha(e_1), V_1\alpha(e_2)) \simeq \mathcal{M}_4(\mathbb{C})$ with a standard basis $\{E^{(1)}_{i,j}E^{(2)}_{k,l}\}$ for $1 \leq i, j, k, l \leq 2$.

Therefore, if we define $V_n := \prod_{j=1}^n (I - 2E(e_j))$ for all $n \geq 2$ and we define the matrix units $E^{(n)}_{1,1} := E(e_n)$, $E^{(n)}_{2,1} := V_{n+1}\alpha(e_n)$, $E^{(n)}_{1,2} := V_{n+1}\alpha(e_n)^*$, and $E^{(n)}_{2,2} := E(e_n)^\perp$ for all $n \geq 1$, we can repeat the above proof to see that $\mathfrak{A}_n := C^*(\{\alpha(e_j) \mid 1 \leq j \leq n\})$ are an increasing sequence of matrix algebras with $\mathfrak{A}_n$ isomorphic to $\mathcal{M}_{2^n}(\mathbb{C})$ with the matrix units $E_{\phi,\psi} := \prod_{k=1}^n E^{(k)}_{\phi(k),\psi(k)}$ for all functions ϕ and ψ from $\{1, \ldots, n\}$ to $\{1, 2\}$. Since α is continuous, $\bigcup_{n\geq 1} \mathfrak{A}_n$ is dense in $\mathfrak{A}$ and thus $\mathfrak{A}$ is isomorphic to the 2^∞ UHF C*-algebra. □

Our next result is to show that such relations exists.

Lemma 6.5. *There exists a continuous linear map $\alpha : \ell_2(\mathbb{N}) \to \mathcal{B}(\ell_2(\mathbb{N}))$ such that*

$$\alpha(\xi)\alpha(\eta) + \alpha(\eta)\alpha(\xi) = 0 \qquad \text{and} \qquad \alpha(\xi)^*\alpha(\eta) + \alpha(\eta)\alpha(\xi)^* = \langle \eta, \xi\rangle I$$

for all $\xi, \eta \in \ell_2(\mathbb{N})$.

Proof. Let $\mathcal{H}$ be a separable Hilbert space and consider the Fock space $\mathcal{F}(\mathcal{H})$ of $\mathcal{H}$; that is

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n\geq 0} \mathcal{H}^{\otimes n}.$$

For each $n \in \mathbb{N}$ and $\{\xi_j\}_{j=1}^n \subseteq \mathcal{H}$ define

$$\xi_1 \wedge \cdots \wedge \xi_n := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} sgn(\sigma)\xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \otimes \cdots \otimes \xi_{\sigma(n)}$$

where S_n is the permutation group on n elements and $sgn(\sigma)$ is the signature (that is (-1) to the power of the number of inversions) of a permutation σ. Therefore, it is clear that if $\sigma \in S_n$ then

$$\xi_{\sigma(1)} \wedge \cdots \wedge \xi_{\sigma(n)} = sgn(\sigma)\xi_1 \wedge \cdots \wedge \xi_n.$$

Let

$$\mathcal{H}_{0,n} := \overline{span\{\xi_1 \wedge \cdots \wedge \xi_n \mid \{\xi_j\}_{j=1}^n \subseteq \mathcal{H}\}} \subseteq \mathcal{H}^{\otimes n}.$$

Therefore, we can view

$$\mathcal{F}_a(\mathcal{H}) := \bigoplus_{n\geq 0} \mathcal{H}_{0,n}$$

as a Hilbert subspace of $\mathcal{F}(\mathcal{H})$. The Hilbert space $\mathcal{F}_a(\mathcal{H})$ is called the anti-symmetric Fock space of $\mathcal{H}$. Notice that since $\mathcal{H}$ is separable, $\mathcal{F}_a(\mathcal{H})$ is separable and thus isomorphic to $\ell_2(\mathbb{N})$.

Define $\alpha : \mathcal{H} \to \mathcal{B}(\mathcal{F}_a(\mathcal{H}))$ by

$$\alpha(\xi)\,(\xi_1 \wedge \cdots \wedge \xi_n) = \xi \wedge \xi_1 \wedge \cdots \wedge \xi_n$$

for all $\xi_1 \wedge \cdots \wedge \xi_n \in \mathcal{H}_{0,n}$. A moments consideration shows that α is a well-defined, continuous, linear map (as wedging with something is clearly well-defined, linear, and continuous). Moreover, we notice that

$$\alpha(\xi)\alpha(\eta)\,(\xi_1 \wedge \cdots \wedge \xi_n) = \xi \wedge \eta \wedge \xi_1 \wedge \cdots \wedge \xi_n = -\eta \wedge \xi \wedge \xi_1 \wedge \cdots \wedge \xi_n = -\alpha(\eta)\alpha(\xi)\,(\xi_1 \wedge \cdots \wedge \xi_n)$$

so $\alpha(\xi)\alpha(\eta) + \alpha(\xi)(\eta) = 0$.

Next we claim that

$$\alpha(\xi)^*\,(\xi_1 \wedge \cdots \wedge \xi_n) = \sum_{k=1}^n (-1)^{k+1}\langle \xi_k, \xi\rangle(\xi_1 \wedge \cdots \wedge \widehat{\xi_k} \wedge \cdots \wedge \xi_n)$$

where $\widehat{\xi_k}$ represents that ξ_k is missing. To see this, we notice that

$$\begin{array}{rl} & \left\langle \sum_{k=1}^n (-1)^{k+1}\langle \xi_k, \eta_n\rangle(\xi_1 \wedge \cdots \wedge \widehat{\xi_k} \wedge \cdots \wedge \xi_n), \eta_1 \wedge \cdots \wedge \eta_{n-1}\right\rangle \\ = & \sum_{k=1}^n (-1)^{k+1}\langle \xi_k, \eta_n\rangle \left\langle \xi_1 \wedge \cdots \wedge \widehat{\xi_k} \wedge \cdots \wedge \xi_n, \eta_1 \wedge \cdots \wedge \eta_{n-1}\right\rangle \\ = & \frac{1}{(n-1)!}\sum_{k=1}^n (-1)^{k+1}\langle \xi_k, \eta_n\rangle \sum_{\sigma,\tau\in S_n, \sigma(k)=k, \tau(k)=n} sgn(\sigma)((-1)^{n-k}sgn(\tau))\prod_{j=1, j\neq k}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \\ = & \frac{1}{(n-1)!}\sum_{k=1}^n (-1)^{n-1}\langle \xi_k, \eta_n\rangle \sum_{\sigma,\tau\in S_n, \sigma(k)=k, \tau(k)=n} sgn(\sigma)sgn(\tau)\prod_{j=1, j\neq k}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \end{array}$$

and

$$\begin{array}{rl} & \langle \alpha(\eta_n)^*(\xi_1 \wedge \cdots \wedge \xi_n), \eta_1 \wedge \cdots \wedge \eta_{n-1}\rangle \\ = & \langle \xi_1 \wedge \cdots \wedge \xi_n, \eta_n \wedge \eta_1 \wedge \cdots \wedge \eta_{n-1}\rangle \\ = & (-1)^{n-1}\,\langle \xi_1 \wedge \cdots \wedge \xi_n, \eta_1 \wedge \cdots \wedge \eta_n\rangle \\ = & \frac{1}{n!}(-1)^{n-1}\sum_{\sigma,\tau\in S_n} sgn(\sigma)sgn(\tau)\prod_{j=1}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \\ = & \frac{1}{n!}\sum_{\ell,k=1}^n (-1)^{n-1}\sum_{\sigma,\tau\in S_n, \sigma(\ell)=k, \tau(\ell)=n} sgn(\sigma)sgn(\tau)\prod_{j=1}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \\ = & \frac{1}{n!}\sum_{\ell,k=1}^n (-1)^{n-1}\langle \xi_k, \eta_n\rangle\sum_{\sigma,\tau\in S_n, \sigma(\ell)=k, \tau(\ell)=n} sgn(\sigma)sgn(\tau)\prod_{j=1, j\neq\ell}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \\ = & \frac{1}{n!}\sum_{\ell,k=1}^n (-1)^{n-1}\langle \xi_k, \eta_n\rangle\sum_{\sigma,\tau\in S_n, \sigma(k)=k, \tau(k)=n} sgn(\sigma)sgn(\tau)\prod_{j=1, j\neq\ell}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \\ = & \frac{1}{(n-1)!}\sum_{k=1}^n (-1)^{n-1}\langle \xi_k, \eta_n\rangle\sum_{\sigma,\tau\in S_n, \sigma(k)=k, \tau(k)=n} sgn(\sigma)sgn(\tau)\prod_{j=1, j\neq\ell}^n \langle \xi_{\sigma(j)}, \eta_{\tau(j)}\rangle \end{array}$$

so $\alpha(\eta_n)^*$ has the desired form. Finally, we notice that

$$(\alpha(\xi)^*\alpha(\eta) + \alpha(\eta)\alpha(\xi)^*)(1) = \alpha(\xi)^*(\eta) + \alpha(\eta)(0) = \langle \eta, \xi\rangle 1$$

and

$$\begin{array}{rcl} \alpha(\xi)^*\alpha(\eta)(\zeta_1 \wedge \cdots \wedge \zeta_n) & = & \alpha(\xi)^*(\eta \wedge \zeta_1 \wedge \cdots \wedge \zeta_n) \\ & = & \langle \eta, \xi\rangle + \sum_{k=1}^n (-1)^{k+2}\langle \eta_k, \xi\rangle(\eta \wedge \zeta_1 \wedge \cdots \wedge \widehat{\zeta_k} \wedge \cdots \wedge \zeta_n) \end{array}$$

whereas

$$\begin{array}{rcl} \alpha(\eta)\alpha(\xi)^*(\zeta_1 \wedge \cdots \wedge \zeta_n) & = & \alpha(\eta)\left(\sum_{k=1}^n (-1)^k\langle \eta_k, \xi\rangle(\zeta_1 \wedge \cdots \wedge \widehat{\zeta_k} \wedge \cdots \wedge \zeta_n)\right) \\ & = & \sum_{k=1}^n (-1)^{k+1}\langle \eta_k, \xi\rangle(\eta \wedge \zeta_1 \wedge \cdots \wedge \widehat{\zeta_k} \wedge \cdots \wedge \zeta_n). \end{array}$$

Thus we obtain that

$$\alpha(\xi)^*\alpha(\eta) + \alpha(\eta)\alpha(\xi)^* = \langle \eta, \xi\rangle I$$

for all $\xi, \eta \in \mathcal{H}$. $\square$

Now that the existence of maps satisfying the CARs are known to exists, we can use Lemma 6.4 to describe the 2^∞ UHF algebra.

Proposition 6.6 (The Rokhlin Property of the Bernoulli Shift)**.** *Let σ be the one-sided Bernoulli shift on the 2^∞ UHF algebra $\mathfrak{A} = \otimes_{n=1}^\infty \mathcal{M}_2(\mathbb{C})$; that is*

$$\sigma(A_1 \otimes A_2 \otimes \cdots) = I \otimes A_1 \otimes A_2 \otimes \cdots.$$

For each $k \in \mathbb{N}$ let $\mathfrak{A}_k := \otimes_{n=1}^k \mathcal{M}_2(\mathbb{C}) \subseteq \mathfrak{A}$ (which is a unital C-subalgebra). Then for each $\epsilon > 0$ and for each $r \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ and projections $P_0, P_1, \ldots, P_{2^r-1}, P_{2^r} = P_0$ in $\mathfrak{A}_k$ such that $\sum_{j=1}^{2^r} P_j = I$ and $\|\sigma(P_j) - P_{j+1}\| < \epsilon$ for all $j = 0, 1, \ldots, 2^r - 1$.*

Proof. For those familiar with the Rokhlin Lemma for a free action on a probability space, the conclusion of this result has a very similar flavour.

Let $\alpha : \ell_2(\mathbb{N}) \to \mathcal{B}(\ell_2(\mathbb{N}))$ be a continuous linear map satisfying the CARs (whose existence is guaranteed by Lemma 6.5). Therefore $\mathfrak{A}$ can be viewed as the C*-algebra generated by the image of α. Let S be the unilateral forward shift on $\ell_2(\mathbb{N})$. Notice if $\alpha' : \ell_2(\mathbb{N}) \to \mathcal{B}(\ell_2(\mathbb{N}))$ is defined by $\alpha'(\xi) = \alpha(S\xi)$ then α' also a continuous linear map which satisfies the CARs. Hence there is a *-homomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$ defined by $\rho(\alpha(\xi)) = \alpha(S\xi)$ for all $\xi \in \ell_2(\mathbb{N})$.

We claim that σ and ρ agree on the C*-algebra generated by all elements of the form $\alpha(\xi)\alpha(\eta)$ and $\alpha(\xi)\alpha(\eta)^*$ where $\eta, \xi \in \ell_2(\mathbb{N})$. To see this, we notice that if $k < m$ then (with the notation as in the proof of Lemma 6.4)

$$\sigma(V_{k+1}\alpha(e_k)) = V_{k+2}\alpha(e_{k+1}) \qquad \text{and} \qquad \sigma(V_{m+1}\alpha(e_m)) = V_{m+2}\alpha(e_{m+1})$$

by the description that σ is a forward shift. Moreover, it is clear that $\sigma(E(e_n)) = E(e_{n+1})$ for all n and thus (as σ is unital)

$$\sigma(V_n) = \sigma\left(\prod_{j=1}^{n}(I - 2E(e_j))\right) = \prod_{j=2}^{n+1}(I - 2E(e_j))$$

for all $n \geq 1$. Hence, as $V_n\alpha(e_j) = -\alpha(e_j)V_n$ for all $j \leq n$,

$$\begin{array}{rcl} \sigma(V_{k+1}\alpha(e_k)V_{m+1}\alpha(e_m)) & = & -\sigma(V_{k+1}V_{m+1}\alpha(e_k)\alpha(e_m)) \\ & = & -\left(\prod_{j=2}^{k+2}(I - 2E(e_j))\right)\left(\prod_{j=2}^{m+2}(I - 2E(e_j))\right)\sigma(\alpha(e_k)\alpha(e_m)) \end{array}$$

where as

$$\sigma(V_{k+1}\alpha(e_k)V_{m+1}\alpha(e_m)) = V_{k+2}\alpha(e_{k+1})V_{m+2}\alpha(e_{m+1}) = -V_{k+2}V_{m+2}\alpha(e_{k+1})\alpha(e_{m+1})$$

However, it is easy to see that

$$\left(\prod_{j=2}^{k+2}(I - 2E(e_j))\right)\left(\prod_{j=2}^{m+2}(I - 2E(e_j))\right) = V_{k+2}V_{m+2}$$

is invertible and thus

$$\sigma(\alpha(e_k)\alpha(e_m)) = \alpha(e_{k+1})\alpha(e_{m+1}) = \rho(\alpha(e_k)\alpha(e_m)).$$

Similarly

$$\sigma(\alpha(e_k)\alpha(e_m)^*) = \alpha(e_{k+1})\alpha(e_{m+1})^* = \rho(\alpha(e_k)\alpha(e_m)^*)$$

by simply adding a * on every $\alpha(e_m)$ in the above computation. Moreover, similar computations hold with $k > m$ (and trivially when $k = m$). Then, by considering adjoints, linearity, continuity of maps, and the CARs, the claim is complete.

Let $\omega_j := e^{\frac{2\pi i}{2^j}}$ for all $j \in \{1, 2, \ldots, r\}$. Then it is possible to choose an orthonormal set $\xi_0, \xi_1, \ldots, \xi_r \in \ell_2(\mathbb{N})$ such that

$$\|S\xi_0 - \xi_0\| < \frac{\epsilon}{4(\|\alpha\|^2 + 1)} \qquad \text{and} \qquad \|S\xi_j - \omega_j \xi_j\| < \frac{\epsilon}{4(\|\alpha\|^2 + 1)}$$

for all $j \in \{1, \ldots, r\}$ (i.e. $\xi_0 = \frac{1}{n}\sum_{j=1}^{n} e_j$, $\xi_1 = \frac{1}{n}\sum_{j=n+1}^{2n} \omega_1^{-j} e_j$, etc. for some large choice of n). Let $T_j := \alpha(\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*)$ for $j \in \{1, 2, \ldots, r\}$. Then clearly $\rho(T_j) = \sigma(T_j)$ by the above results and thus (as α is continuous)

$$\begin{aligned}
\|\sigma(T_j) - \omega_j T_j\| &= \|\alpha(S\xi_j)(\alpha(S\xi_0) + \alpha(S\xi_0)^*) - \omega_j \alpha(\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*)\| \\
&\leq \|\alpha(S\xi_j)(\alpha(S\xi_0) + \alpha(S\xi_0)^*) - \alpha(S\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*)\| \\
&\quad + \|\alpha(S\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*) - \omega_j \alpha(\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*)\| \\
&\leq 2\|\alpha\|^2 \frac{\epsilon}{4(\|\alpha\|^2 + 1)} + 2\|\alpha\|^2 \frac{\epsilon}{4(\|\alpha\|^2 + 1)} = \epsilon.
\end{aligned}$$

Next we notice that

$$T_j T_j^* = \alpha(\xi_j)(\alpha(\xi_0) + \alpha(\xi_0)^*)^2 \alpha(\xi_j)^* = \alpha(\xi_j)\alpha(\xi_j)^*$$

by the CARs, and similarly

$$T_j^* T_j = \alpha(\xi_j)^* \alpha(\xi_j) = E(\xi_j)$$

as $E(\xi_j)$ and $\alpha(\xi_0)$ commute as ξ_j and ξ_0 are orthogonal. Hence $C^*(T_j) \simeq \mathcal{M}_2(\mathbb{C})$ where the isomorphism takes T_j to $E_{2,1}$.

Moreover we notice that

$$\begin{aligned}
(I - 2E(\xi_1))T_2 T_1 &= (I - 2E(\xi_1))\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*)\alpha(\xi_1)(\alpha(\xi_0) + \alpha(\xi_0)^*) \\
&= (-1)^3 \alpha(\xi_1)(I - 2E(\xi_1))\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*)^2 \\
&= \alpha(\xi_1)(I - 2E(\xi_1))(\alpha(\xi_0) + \alpha(\xi_0)^*)\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*) \\
&= T_1(I - 2E(\xi_1))T_2
\end{aligned}$$

and

$$\begin{aligned}
(I - 2E(\xi_1))T_2 T_1^* &= (I - 2E(\xi_1))\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*)^2 \alpha(\xi_1)^* \\
&= -(\alpha(\xi_0) + \alpha(\xi_0)^*)(I - 2E(\xi_1))\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*)\alpha(\xi_1)^* \\
&= (-1)^4 (\alpha(\xi_0) + \alpha(\xi_0)^*)\alpha(\xi_1)^*(I - 2E(\xi_1))\alpha(\xi_2)(\alpha(\xi_0) + \alpha(\xi_0)^*) \\
&= T_1^*(I - 2E(\xi_1))T_2.
\end{aligned}$$

Therefore, by applying the same idea as in the proof of Lemma 6.4, we see that $C^*(T_1, \ldots, T_r) \simeq \otimes_{j=1}^r \mathcal{M}_2(\mathbb{C})$. Moreover, using the fact that $\|\sigma(T_j) - \omega_j T_j\|$ is small (and thus $\|\sigma(I - 2E(\xi_j)) - (I - 2E(\xi_j))\|$ is small), we see that the restriction of σ to $C^*(T_1, \ldots, T_r) \simeq \otimes_{j=1}^r \mathcal{M}_2(\mathbb{C})$ is close to the inner automorphism $Ad(U)$ where $U = U_1 \otimes U_2 \otimes \cdots \otimes U_r$ where $U_k = diag(1, \omega_k)$. Therefore the spectrum of U is precisely the 2^r-th roots of unity. Hence U is unitarily equivalent to a cyclic shift on $\mathcal{M}_{2^r}(\mathbb{C})$.

Let $P_0, P_1, \ldots, P_{2^r-1}, P_{2^r} = P_0$ be the projections in $\mathcal{M}_{2^r}(\mathbb{C})$ corresponding to the above cyclic shift. Therefore $\sum_{j=1}^{2^r} P_j = I$ and $\|\sigma(P_j) - P_{j+1}\|$ is small for all $j = 0, 1, \ldots, 2^r - 1$. Moreover, $P_j \in C^*(T_1, \ldots, T_r)$ for all j. By the choice of ξ_j, we obtain that $C^*(T_1, \ldots, T_r) \subseteq \mathfrak{A}_k$ for some k (i.e. we used $\{e_1, \ldots, e_k\}$ to create all of the ξ_j's for some k). Hence the proof is complete. $\square$

The key lemma for our main result is the following mess.

Lemma 6.7. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C^*-algebra and let T_1 and T_2 be isometries in $\mathfrak{A}$ such that $T_1 T_1^* + T_2 T_2^* = I_{\mathfrak{A}}$. Let $\gamma : \mathfrak{A} \to \mathfrak{A}$ be the *-homomorphism defined by $\gamma(A) := T_1 A T_1^* + T_2 A T_2^*$. Then for each $U \in \mathcal{U}(\mathfrak{A})$ and for each $\epsilon > 0$ there exists a unitary $V \in \mathfrak{A}$ such that $\|V\gamma(V)^* - U\| < \epsilon$.*

Proof. For this proof, to easy notation and clarity, we will prove several lemma along the way. Fix a $U \in \mathcal{U}(\mathfrak{A})$ and let $\epsilon > 0$. It is clear that γ is indeed a unital *-homomorphism as T_1 and T_2 are isometries with orthogonal ranges. Moreover, γ must be isometric since if $A \in \mathfrak{A}$ was in the kernel of γ, then $0 = T_1^*\gamma(A)T_1 = A$.

For the rest of this proof, we will use the notation developed in Chapter 1 for $\mathcal{O}_2$. Define $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ by $\lambda(A) := S_1AS_1^* + S_2AS_2^*$ where S_1 and S_2 are the isometries that generate $\mathcal{O}_2$. Clearly λ is a unital *-homomorphism. By the proof of Lemma 1.7, we can see that $\lambda|_{\mathfrak{F}^2} = \sigma$ where σ is the Bernoulli shift in Proposition 6.6. By the Universality of the Cuntz Algebras (Theorem 1.14), there exists *-homomorphism $\varphi, \psi : \mathcal{O}_2 \to \mathfrak{A}$ such that $\varphi(S_i) = T_i$ and $\psi(S_i) = UT_i$. Note $\varphi \circ \lambda = \gamma$ as $\varphi(S_j) = T_j$.

Lemma 6.8. *For all $A \in \mathfrak{A}$ and for all $k \in \mathbb{N}$, $\gamma^k(A) = \sum_{|\mu|=k} \varphi(S_\mu)A\varphi(S_\mu^*)$ and, if*

$$U_k := \sum_{|\mu|=k} \psi(S_\mu)\varphi(S_\mu)^*,$$

then U_k is a unitary with $U_1 = U$ and $\psi(S_\nu) = U_k\varphi(S_\nu)$ for all words ν such that $|\nu| = k$.

Proof. The proof that $\gamma^k(A) = \sum_{|\mu|=k} \varphi(S_\mu)A\varphi(S_\mu^*)$ for all $A \in \mathfrak{A}$ is trivial from the definition. Moreover, since ψ and φ are unital *-homomorphisms, we obtain that

$$U_kU_k^* = \sum_{k=|\mu|=|\nu|} \psi(S_\mu)\varphi(S_\mu^*S_\nu)\psi(S_\nu^*) = \sum_{k=|\mu|} \psi(S_\mu S_\mu^*) = I_{\mathfrak{A}}$$

and similarly $U_k^*U_k = I_{\mathfrak{A}}$. Finally, it is clear that $U_1 = UT_1T_1^* + UT_2T_2^* = U$ and

$$U_k\varphi(S_\nu) = \sum_{|\mu|=k} \psi(S_\mu)\varphi(S_\mu^*S_\nu) = \psi(S_\nu)$$

as desired. □

Lemma 6.9. *For each $k \in \mathbb{N}$, $Im(\gamma^k) = \varphi(\mathfrak{F}_k^2)' \cap \mathfrak{A}$.*

Proof. It is easy to see that $\gamma^k(A)\varphi(S_\mu) = \varphi(S_\mu)A$ and $\varphi(S_\mu)^*\gamma^k(A) = A\varphi(S_\mu)^*$ for all $A \in \mathfrak{A}$ and words μ with $|\mu| = k$ (as $\gamma^k(A) = \sum_{|\mu|=k} \varphi(S_\mu)A\varphi(S_\mu^*)$ from above). Hence

$$\gamma^k(A)\varphi(S_\mu S_\nu^*) = \varphi(S_\mu)A\varphi(S_\nu)^* = \varphi(S_\mu S_\nu^*)\gamma^k(A)$$

for all $A \in \mathfrak{A}$ and $|\mu| = |\nu| = k$. Hence $Im(\gamma^k) \subseteq \varphi(\mathfrak{F}_k^2)' \cap \mathfrak{A}$.

To prove the other inclusion, let $B \in \varphi(\mathfrak{F}_k^2)' \cap \mathfrak{A}$. Then, for any two words μ and ν with $|\mu| = |\nu| = k$, we have that $\gamma^k(B)\varphi(S_\mu S_\nu^*) = \varphi(S_\mu S_\nu^*)\gamma^k(B)$. Hence, as φ is unital, $\varphi(S_\mu)^*B\varphi(S_\mu) = \varphi(S_\nu)^*B\varphi(S_\nu)$. Let $A := \varphi(S_\nu)^*B\varphi(S_\nu)$ for some fixed word ν of length k. Then

$$\gamma^k(A) = \sum_{|\mu|=k} \varphi(S_\mu)\varphi(S_\nu)^*B\varphi(S_\nu)\varphi(S_\mu^*) = \sum_{|\mu|=k} B\varphi(S_\mu)\varphi(S_\nu)^*\varphi(S_\nu)\varphi(S_\mu^*) = \sum_{|\mu|=k} B\varphi(S_\mu)\varphi(S_\mu^*) = B$$

as desired. □

Lemma 6.10. *For each $k \in \mathbb{N}$, $U = U_k\gamma^k(U)\gamma(U_k)^*$.*

Proof. We will proceed by induction on k. When $k = 1$ we have that $U_1\gamma(U)\gamma(U_1)^* = U\gamma(U)\gamma(U)^* = U$ (as γ is a unital *-homomorphism). Now suppose that $U = U_k\gamma^k(U)\gamma(U_k)^*$ so $\gamma^k(U) = U_k^*U\gamma(U_k)$. Notice that

$$\gamma(U_k) = \sum_{j=1}^{2} \varphi(S_j)U_k\varphi(S_j)^* = U^*\sum_{j=1}^{2} \psi(S_j)U_k\varphi(S_j)^* = U^*\sum_{j=1}^{2}\sum_{|\mu|=k} \psi(S_j)\psi(S_\mu)\varphi(S_\mu)^*\varphi(S_j)^* = U^*U_{k+1}.$$

Therefore $U = U_{k+1}\gamma(U_k)^*$ (which gives us some hope we are on the right track to proving the theorem). Hence

$$\gamma^{k+1}(U) = \gamma(U_k^* U\gamma(U_k)) = \gamma(U_k^* U U^* U_{k+1}) = \gamma(U_k^* U_{k+1})$$

and thus

$$U_{k+1}\gamma^{k+1}(U)\gamma(U_{k+1})^* = U_{k+1}\gamma(U_k^*) = U$$

as desired. □

Now, until otherwise stated, suppose that $U \in \mathcal{U}_0(\mathfrak{A})$. Then we have the following.

Lemma 6.11. *Let $k, m \in \mathbb{N}$ be arbitrary and let $\ell := k + 2m - 1$. Then there exists unitaries $\{W_j\}_{j=0}^{2m-1} \subseteq \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$ such that*

$$W_0\gamma(W_1)\gamma^2(W_2)\cdots\gamma^{2m-1}(W_{2m-1}) = I_{\mathfrak{A}}$$

and $\left\|\gamma^\ell(U) - W_j\right\| \leq \frac{\pi}{m}$ for all $j \in \{0, 1, \ldots, 2m-1\}$.

Proof. Let

$$X_j := \gamma^\ell(U)\gamma^{\ell+1}(U)\cdots\gamma^{\ell+j}(U)$$

for $j \in \{0, 1, \ldots, 2m-1\}$ and let

$$V := U\gamma(U)\cdots\gamma^{2m-1}(U).$$

Clearly $X_{2m-1} = \gamma^\ell(V)$. Moreover, since γ is a unital *-homomorphism, we obtain that $V \in \mathcal{U}_0(\mathfrak{A})$ as $U \in \mathcal{U}_0(\mathfrak{A})$. Since $\mathfrak{A}$ has weak property (FU) by Theorem 5.11, Lemma 6.3 implies that we can write $V = V_1V_2\cdots V_{2m}$ for some unitaries $V_j \in \mathfrak{A}$ such that $\|V_j - I_{\mathfrak{A}}\| \leq \frac{\pi}{m}$ for all j. By applying γ^ℓ to each V_j, we obtain unitaries $Y_0, Y_1, \ldots, Y_{2m-1} \in \mathfrak{A} \cap \varphi(\mathfrak{F}_\ell^2)'$ such that $\|Y_j - I_{\mathfrak{A}}\| \leq \frac{\pi}{m}$ for all j and $X_{2m-1} = Y_{2m-1}Y_{2m-2}\cdots Y_1Y_0$ (we have reversed the indexing).

Since each X_j is in the image of γ^ℓ, we obtain that $X_j^*Y_j^*X_j \in \mathfrak{A} \cap \varphi(\mathfrak{F}_\ell^2)'$ by Lemma 6.9. Moreover, we notice that

$$\mathfrak{A} \cap \varphi(\mathfrak{F}_\ell^2)' = Im(\gamma^\ell) \subseteq Im(\gamma^{j+k}) = \gamma^j(\mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)')$$

for all $j \in \{0, 1, \ldots, 2m-1\}$. Hence, as γ^ℓ is injective, there are unitaries $Z_0, Z_1, \ldots, Z_{2m-1} \in \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$ such that $\gamma^j(Z_j) = X_j^*Y_j^*X_j$ for all $\in \{0, 1, \ldots, 2m-1\}$.

Let $W_j := \gamma^\ell(U)Z_j \in \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$. Since $X_{j-1}^*X_j = \gamma^{\ell+j}(U)$ and each X_j is a unitary, we obtain that

$$\begin{aligned}
&W_0\gamma(W_1)\gamma^2(W_2)\cdots\gamma^{2m-1}(W_{2m-1}) \\
&= \gamma^\ell(U)Z_0\gamma^{\ell+1}(U)\gamma(Z_1)\cdots\gamma^{\ell+2m-1}(U)\gamma^{2m-1}(Z_{2m-1}) \\
&= X_0Z_0(X_0^*X_1)\gamma(Z_1)(X_1^*X_2)\cdots(X_{2m-2}X_{2m-1}^*)\gamma^{2m-1}(Z_{2m-1})(X_{2m-1}^*X_{2m-1}) \\
&= X_0(X_0^*Y_0^*X_0)(X_0^*X_1)(X_1^*Y_1^*X_1)(X_1^*X_2)\cdots(X_{2m-2}X_{2m-1}^*)(X_{2m-1}^*Y_{2m-1}^*X_{2m-1})(X_{2m-1}^*X_{2m-1}) \\
&= Y_0^*Y_1^*Y_2^*\cdots Y_{2m-1}^*X_{2m-1} = I_{\mathfrak{A}}
\end{aligned}$$

as desired. Moreover, since γ is isometric

$$\left\|\gamma^\ell(U) - W_j\right\| = \|I_{\mathfrak{A}} - Z_j\| = \left\|\gamma^j(I_{\mathfrak{A}} - Z_j)\right\| = \left\|I_{\mathfrak{A}} - X_j^*Y_j^*X_j\right\| = \left\|I_{\mathfrak{A}} - Y_j^*\right\| \leq \frac{\pi}{m}$$

which completes the proof. □

By the proof of Lemma 1.7 we can see that $\lambda|_{\mathfrak{F}^2} = \sigma$ where σ is the Bernoulli shift in Proposition 6.6. Therefore Proposition 6.6 implies that there exists a $k \in \mathbb{N}$, an $r \in \mathbb{N}$ with $2^{3-r} < \frac{\epsilon}{2}$, and projections $P_0, P_1, \ldots, P_{2^r} = P_0$ in $\mathfrak{F}_k^2 \subseteq \mathcal{O}_2$ such that $\sum_{j=0}^{2r-1} P_j = I_{\mathcal{O}_2}$ and $\|\lambda(P_j) - P_{j-1}\| < \frac{\epsilon}{2^{r+1}}$ (where we simply reversed the order of the projections in Proposition 6.6).

Let $Q_j := \varphi(P_j) \in \mathfrak{A}$ for all $j \in \{1, \ldots, 2^r\}$. Then $Q_j \in \varphi(\mathfrak{F}_k^2)$ are orthogonal projections such that $\sum_{j=0}^{2r-1} Q_j = I_{\mathfrak{A}}$ and

$$\|\gamma(Q_j) - Q_{j-1}\| = \|\varphi(\lambda(P_j)) - \varphi(P_{j+1})\| \leq \frac{\epsilon}{2^{r+1}}$$

(since $\varphi \circ \lambda = \gamma$ as $\varphi(S_j) = T_j$) for all $j \in \{1, \ldots, 2^r\}$.

Let $\ell := k + 2^r - 1$. Hence, by Lemma 6.11, there exists unitaries $\{W_j\}_{j=0}^{2^r-1} \subseteq \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$ such that

$$W_0\gamma(W_1)\gamma^2(W_2)\cdots\gamma^{2^r-1}(W_{2^r-1}) = I_{\mathfrak{A}}$$

and $\left\|\gamma^\ell(U) - W_j\right\| \leq \frac{\pi}{2^{r-1}} < \frac{\epsilon}{2}$.

Let $V_{2^r} := I_{\mathfrak{A}}$ and define

$$V_j := W_j\gamma(W_{j+1})\cdots\gamma^{2^r-j-1}(W_{2^r-1})$$

for $j \in \{1, \ldots, 2^r - 1\}$. Therefore $V_j \in \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$ by Lemma 6.9 and clearly $V_{2^r} = I_{\mathfrak{A}} = V_0$. Moreover, it is clear that $W_j = V_j\gamma(V_{j+1})^*$ for all $j \in \{0, 1, \ldots, 2^r - 1\}$.

Let

$$V := \sum_{j=1}^{2^r} V_jQ_j \in \mathfrak{A}.$$

Since $\sum_{j=1}^{2^r} Q_j = I_{\mathfrak{A}}$ and each $Q_j \in \varphi(\mathfrak{F}_k^2)$, we obtain that each V_j commutes with each Q_i (as $V_j \in \mathfrak{A} \cap \varphi(\mathfrak{F}_k^2)'$) and thus

$$V = \sum_{j=1}^{2^r} V_jQ_j = \sum_{j=1}^{2^r} Q_jV_jQ_j$$

is the direct sum of unitaries. Hence $V \in \mathcal{U}(\mathfrak{A})$.

We claim that $\left\|\gamma^\ell(U) - V\gamma(V)^*\right\| < \epsilon$. To see this, let

$$\Delta := V\sum_{j=1}^{2^r}(\gamma(Q_j) - Q_{j-1})\gamma(V_j)^*.$$

Since $\|\gamma(Q_j) - Q_{j-1}\| \leq \frac{\epsilon}{2^{r+1}}$ for all $j \in \{1, \ldots, 2^r\}$, we see that $\|\Delta\| < \frac{\epsilon}{2}$. Moreover

$$\begin{aligned}
V\gamma(V)^* &= V\gamma\left(\sum_{j=1}^{2^r} Q_jV_j^*\right) \\
&= V\sum_{j=1}^{2^r} Q_{j-1}\gamma(V_j)^* + \Delta \\
&= \sum_{j=0}^{2^r-1} V_jQ_j\gamma(V_{j+1})^* + \Delta \\
&= \sum_{j=0}^{2^r-1} W_jQ_j + \Delta.
\end{aligned}$$

Since

$$\gamma^\ell(U) \in Im(\gamma^\ell) \subseteq Im(\gamma^k) = \varphi(\mathfrak{F}_k^2)' \cap \mathfrak{A}$$

and each $Q_j \in\in \varphi(\mathfrak{F}_k^2)$,

$$\begin{aligned}
\left\|\gamma^\ell(U) - V\gamma(V)^*\right\| &\leq \left\|\sum_{j=0}^{2^r-1}(\gamma^\ell(U) - W_j)Q_j\right\| + \|\Delta\| \\
&= \left\|\sum_{j=0}^{2^r-1} Q_j(\gamma^\ell(U) - W_j)Q_j\right\| + \|\Delta\| \\
&= \max_j\{\left\|(\gamma^\ell(U) - W_j)\right\|\} + \|\Delta\| < \epsilon
\end{aligned}$$

as claimed.

We have shown that $\left\|\gamma^\ell(U) - V\gamma(V)^*\right\| < \epsilon$. Note $U = U_\ell \gamma^\ell(U)\gamma(U_\ell)^*$ by Lemma 6.10. Therefore

$$\|U - (U_\ell V)\gamma(U_\ell V)^*\| \leq \left\|\gamma^\ell(U) - V\gamma(V)^*\right\| < \epsilon$$

which completes the proof in the case that $U \in \mathcal{U}_0(\mathfrak{A})$.

Finally let $U \in \mathcal{U}(\mathfrak{A})$ be arbitrary. By Lemma 4.22 we know that

$$[\gamma(U)]_1 = [T_1 U T_1^* + T_2 U T_2^*]_1 = ([U]_1)^2$$

in $K_1(\mathfrak{A})$. Therefore

$$[U^2\gamma(U)^*]_1 = ([U]_1)^2([\gamma(U)]_1)^{-1} = ([U]_1)^2(([U]_1)^2)^{-1} = 0$$

in $K_1(\mathfrak{A})$. Hence Theorem 4.21 implies $U^2\gamma(U)^* \in \mathcal{U}_0(\mathfrak{A})$. Thus, by the first part of the proof, there exists a unitary $V_0 \in \mathfrak{A}$ such that $\left\|V_0\gamma(V_0)^* - U^2\gamma(U)^*\right\| < \epsilon$. Hence $\|V\gamma(V)^* - U\| < \epsilon$ where $V := U^*V_0$. ☐

Theorem 6.12. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Then any two unital *-homomorphism $\varphi, \psi : \mathcal{O}_2 \to \mathfrak{A}$ are approximately unitarily equivalent (that is, there exists a sequence $(V_n)_{n\geq 1}$ of unitaries in $\mathfrak{A}$ such that $V_n\varphi(T)V_n^* \to \psi(T)$ for all $T \in \mathcal{O}_2$).*

Proof. Let $\varphi, \psi : \mathcal{O}_2 \to \mathfrak{A}$ be unital *-homomorphism and let $U := \psi(S_1)\varphi(S_1)^* + \psi(S_2)\varphi(S_2)^* \in \mathfrak{A}$. Notice that the proof of Lemma 6.8 implies that U is a unitary operator in $\mathfrak{A}$. Let $T_j := \varphi(S_j) \in \mathfrak{A}$ for $j \in \{1, 2\}$. Therefore, since φ is unital, each T_j is an isometry and $T_1T_1^* + T_2T_2^* = I_\mathfrak{A}$. Define $\gamma : \mathfrak{A} \to \mathfrak{A}$ by $\gamma(A) := T_1AT_1^* + T_2AT_2^*$. Thus, by Lemma 6.7, there exists a sequence of unitaries $(V_n)_{n\geq1}$ in $\mathfrak{A}$ such that $V_n\gamma(V_n)^* \to U$. However

$$\gamma(V_n)^*T_j = (T_1V_n^*T_1^* + T_2V_n^*T_2^*)T_j = T_jV_n^*$$

so

$$V_n\varphi(S_j)V_n^* = V_nT_jV_n^* = V_n\gamma(V_n)^*T_j \to UT_j = (\psi(S_1)\varphi(S_1)^* + \psi(S_2)\varphi(S_2)^*)\varphi(S_j) = \psi(S_j)$$

for $j \in \{1, 2\}$. Hence, as $\mathcal{O}_2 = C^*(S_1, S_2)$, we see that φ and ψ are approximately unitarily equivalent as desired. ☐

To complete this chapter, we desire to use Theorem 6.12 to compute $K_0(\mathcal{O}_2)$. Recall that $\mathcal{O}_2$ is a unital, simple (Theorem 1.15), purely infinite (Corollary 2.12) C*-algebra so Theorem 6.12 implies that any two unital *-homomorphisms from $\mathcal{O}_2$ to itself are approximately unitarily equivalent. The following results enables us to conclude that two unitarily equivalent projections are equivalent and two projections that are 'close' are equivalent.

Lemma 6.13. *Let $\mathfrak{A}$ be a unital C*-algebra and let $P, Q \in \mathfrak{A}$ be projections such that $P = VQV^*$ for some isometry $V \in \mathfrak{A}$. Then $P \sim Q$.*

Proof. Let $W := QV^* \in \mathfrak{A}$. Then $W^*W = VQV^* = P$ and $WW^* = QV^*VQ = QI_\mathfrak{A}Q = Q$ so $P \sim Q$. ☐

Lemma 6.14. *Let $\mathfrak{A}$ be a unital C*-algebra and let $P, Q \in \mathfrak{A}$ be projections such that $\|P - Q\| < \frac{1}{2}$. Then $P \sim Q$.*

Proof. Let $Z := PQ + (I_\mathfrak{A} - P)(I_\mathfrak{A} - Q) \in \mathfrak{A}$. Then it is clear that

$$\begin{aligned}
\|Z - I_\mathfrak{A}\| &= \|(PQ + (I_\mathfrak{A} - P)(I_\mathfrak{A} - Q)) - (Q + (I_\mathfrak{A} - Q))\| \\
&\leq \|(P - I_\mathfrak{A})Q\| + \|((I_\mathfrak{A} - P) - I_\mathfrak{A})(I_\mathfrak{A} - Q)\| \\
&= \|(P - Q)Q\| + \|((I_\mathfrak{A} - P) - (I_\mathfrak{A} - Q))(I_\mathfrak{A} - Q)\| \\
&\leq \|P - Q\| + \|Q - P\| < 1
\end{aligned}$$

so Z is invertible in $\mathfrak{A}$. Therefore, if U is the partial isometry in the polar decomposition of Z, $Z = U|Z|$ and U is a unitary element of $\mathfrak{A}$.

We claim that $UQU^* = P$. To see this, we notice that $U = Z|Z|^{-1}$, $ZQ = PQ = PZ$, and

$$Z^*Z = QPQ + (I_{\mathfrak{A}} - Q)(I_{\mathfrak{A}} - P)(I_{\mathfrak{A}} - Q).$$

Thus $QZ^*Z = QPQ = Z^*ZQ$ so Q commutes with Z^*Z. Hence Q commutes with $C^*(Z^*Z)$ and thus Q commutes with $|Z|^{-1}$. Thus

$$\begin{aligned} UQU^* &= Z|Z|^{-1}Q|Z|^{-1}Z^* \\ &= ZQ|Z|^{-2}Z^* \\ &= PZ|Z|^{-2}Z^* \\ &= P|Z^*|^{-2}ZZ^* = P \end{aligned}$$

as claimed.

Since $P = UQU^*$ and $U \in \mathfrak{A}$ is a unitary, Lemma 6.13 implies $P \sim Q$. □

Theorem 6.15. *The group $K_0(\mathcal{O}_2)$ is trivial.*

Proof. Let $P \in \mathcal{O}_2$ be an arbitrary non-zero projection. Define $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ by $\lambda(T) := S_1TS_1^* + S_2TS_2^*$ where S_1 and S_2 are the canonical isometries generating $\mathcal{O}_2$. It is trivial to verify that λ is an unital *-homomorphism (that is also injective). Therefore, by Theorem 6.12, λ is approximately unitarily equivalent to the identity map on $\mathcal{O}_2$. Hence there exists a unitary $U \in \mathcal{O}_2$ such that $\|P - U\lambda(P)U^*\| < \frac{1}{2}$. Hence Lemma 6.13, Lemma 6.14, and the fact that $S_1PS_1^*$ and S_2PS_2 are non-zero orthogonal projections imply that

$$[P]_0 = [U\lambda(P)U^*]_0 = [\lambda(P)]_0 = [S_1PS_1^* + S_2PS_2^*]_0 = [S_1PS_1^*]_0 + [S_2PS_2^*]_0 = [P]_0 + [P]_0.$$

Hence $[P]_0$ must be the trivial element of $K_0(\mathfrak{A})$. Therefore, as $P \in \mathcal{O}_2$ was an arbitrary non-zero projection, $K_0(\mathcal{O}_2)$ is trivial. □

7 On $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$

In this chapter we will study the C*-algebra $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ (as $\mathcal{O}_2$ is nuclear by Theorem 1.20, we need not specify the tensor product). The main goal of this chapter is to prove that $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$.

Most of the results for this chapter were developed from the book [Ro2] and the additional papers referenced there.

In order to prove that $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$, we first recall that $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ is a unital, simple, purely infinite C*-algebra by Theorem 3.11 and the fact that $\mathcal{O}_2$ is a unital, simple (Theorem 1.15), purely infinite (Corollary 2.12) C*-algebra. Therefore Theorem 6.12 implies that the two unital *-homomorphism $T \mapsto T \otimes I_{\mathcal{O}_2}$ and $T \mapsto I_{\mathcal{O}_2} \otimes T$ from $\mathcal{O}_2$ to $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ are unitarily equivalent. We will use this fact to construct an isomorphism. Thus we begin by demonstrating one way of showing that two separable C*-algebras are isomorphic.

Lemma 7.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable C*-algebras such that $\mathfrak{B}$ is unital and let $\pi : \mathfrak{A} \to \mathfrak{B}$ be an injective *-homomorphism. Suppose that there exists a sequence of unitaries $(U_n)_{n\geq 1}$ in $\mathfrak{B}$ such that*

$$\lim_{n\to\infty} \|U_n\pi(A) - \pi(A)U_n\| = 0 \qquad \textit{and} \qquad \lim_{n\to\infty} dist(U_n^* B U_n, \pi(\mathfrak{A})) = 0$$

*for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Then there exists a *-isomorphism $\sigma : \mathfrak{A} \to \mathfrak{B}$ that is approximately unitarily equivalent to π.*

Proof. Since $\mathfrak{A}$ and $\mathfrak{B}$ are separable we can find countable dense subsets $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ for $\mathfrak{A}$ and $\mathfrak{B}$ respectively. By the assumptions given in the statement of the lemma, we can inductively find unitaries $V_n \in \mathfrak{B}$ and elements $\{A_{j,n}\}_{j=1}^n \in \mathfrak{A}$ such that

1. $\left\|V_n^*(V_{n-1}^* \cdots V_1^* B_j V_1 \cdots V_{n-1})V_n - \pi(A_{j,n})\right\| \leq \frac{1}{n}$ for $j \in \{1, \dots, n\}$
2. $\|V_n\pi(A_j) - \pi(A_j)V_n\| \leq \frac{1}{2^n}$ for $j \in \{1, \dots, n\}$
3. $\|V_n\pi(A_{j,m}) - \pi(A_{j,m})V_n\| \leq \frac{1}{2^n}$ for $m \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, m\}$.

By 2) we obtain that $(V_1V_2\cdots V_n\pi(A_j)V_n^*\cdots V_2^*V_1^*)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{B}$ for all $j \in \mathbb{N}$. Therefore, since $\{A_j\}_{j\geq 1}$ is dense in $\mathfrak{A}$, it is easy to see that $(V_1V_2\cdots V_n\pi(A)V_n^*\cdots V_2^*V_1^*)_{n\geq 1}$ is a Cauchy sequence in $\mathfrak{B}$ for all $A \in \mathfrak{A}$. Hence we define $\sigma : \mathfrak{A} \to \mathfrak{B}$ by $\sigma(A) := \lim_{n\to\infty} V_1V_2\cdots V_n\pi(A)V_n^*\cdots V_2^*V_1^*$.

It is clear by the definition of σ and the fact that π is a *-homomorphism that σ is linear and self-adjoint. Moreover, since each V_n is a unitary element of $\mathfrak{B}$, it is easy to see that σ is a *-homomorphism. Moreover, since π is injective, $\|V_1V_2\cdots V_n\pi(A)V_n^*\cdots V_2^*V_1^*\| = \|A\|$ for all $A \in \mathfrak{A}$ so σ is injective.

To see that σ is surjective, we notice from 3) that for any $j \leq n$

$$\|\sigma(A_{j,n}) - V_1V_2\cdots V_n\pi(A_{j,n})V_n^*\cdots V_2^*V_1^*\| \leq \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n}.$$

Therefore, we obtain from 1) that

$$\|B_j - \sigma(A_{j,n})\| \leq \frac{1}{2^n} + \|B_j - V_1V_2\cdots V_n\pi(A_{j,n})V_n^*\cdots V_2^*V_1^*\| \leq \frac{1}{2^n} + \frac{1}{n}$$

for all $j \in \mathbb{N}$. Hence, since the range of a *-homomorphism is closed, $B_j \in \sigma(\mathfrak{A})$ for all $j \in \mathbb{N}$. Therefore, since $\{B_n\}_{n\geq 1}$ is dense in $\mathfrak{B}$, $\mathfrak{B} = \sigma(\mathfrak{A})$ so σ is surjective and thus a *-isomorphism. ☐

The above lemma gives us a way to show that $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ are isomorphic as clearly there exists an injective *-homomorphism from $\mathcal{O}_2$ into $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$. Thus we need only a way to construct the unitaries $U_n \in \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ as described in the lemma. To show this, we will look at a particular property of a sequence of *-homomorphism.

Definition 7.2. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. A sequence $(B_n)_{n\geq 1} \subseteq \mathfrak{B}$ is said to be asymptotically central if $\lim_{n\to\infty} \|BB_n - B_nB\| = 0$ for all $B \in \mathfrak{B}$.

A sequence $(\pi_n)_{n\geq 1}$ of *-homomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is said to be asymptotically central if $(\pi_n(A))_{n\geq 1}$ is an asymptotically central sequence in $\mathfrak{B}$ for all $A \in \mathfrak{A}$.

Example 7.3. It is well-known that if $\mathfrak{J}$ is an ideal in $\mathfrak{A}$ then there exists a C*-bounded approximate identity for $\mathfrak{J}$ that is asymptotically central in $\mathfrak{A}$.

The main result we need is the following.

Lemma 7.4. *There exists an asymptotically central sequence of unital *-homomorphisms from $\mathcal{O}_2$ to $\mathcal{O}_2$.*

Proof. Let $\lambda : \mathcal{O}_2 \to \mathcal{O}_2$ be the injective, unital *-homomorphisms defined by

$$\lambda(A) := S_1AS_1^* + S_2AS_2^*$$

for all $A \in \mathcal{O}_2$. We claim that a bounded sequence $(A_n)_{n\geq 1} \in \mathcal{O}_2$ is asymptotically central if

$$\lim_{n\to\infty} \|\lambda(A_n) - A_n\| = 0.$$

To see this, suppose $\lim_{n\to\infty} \|\lambda(A_n) - A_n\| = 0$. Then

$$\lim_{n\to\infty} \|S_jA_n - A_nS_j\| = \lim_{n\to\infty} \|(\lambda(A_n) - A_n)S_j\| = 0$$

and

$$\lim_{n\to\infty} \left\|S_j^*A_n - A_nS_j^*\right\| = \lim_{n\to\infty} \left\|S_j^*(A_n - \lambda(A_n))\right\| = 0$$

for all $j \in \{1, 2\}$. Therefore it is easy to see that

$$\lim_{n\to\infty} \|p(S_1, S_2, S_1^*, S_2^*)A_n - A_np(S_1, S_2, S_1^*, S_2^*)\| = 0$$

for any polynomial p in four non-commuting variables. Therefore, since *-alg(S_1, S_2) is dense in $\mathcal{O}_2$ and $(A_n)_{n\geq 1}$ is a bounded sequence, it is easy to see that $(A_n)_{n\geq 1}$ is an asymptotically central sequence.

Let $U := \sum_{i,j=1}^2 S_iS_jS_i^*S_j^*$. Then U is self-adjoint and

$$U^2 = \sum_{i,j,k,\ell=1}^2 S_iS_jS_i^*S_j^*S_kS_\ell S_k^*S_\ell^* = \sum_{i,j=1}^2 S_iS_jS_j^*S_i^* = I_{\mathcal{O}_2}$$

so U is a unitary in $\mathcal{O}_2$. Moreover, we notice that

$$US_k = \sum_{i,j=1}^2 S_iS_jS_i^*S_j^*S_k = \sum_{i=1}^2 S_iS_kS_i^* = \lambda(S_k)$$

for all $k \in \{1, 2\}$.

Since $\mathcal{O}_2$ is unital, simple, and purely infinite, Lemma 6.7 implies that there exists a sequence of unitaries $(V_n)_{n\geq 1} \in \mathcal{O}_2$ such that $\lim_{n\to\infty} V_n\lambda(V_n)^* = U$. Since U is self-adjoint, this implies that $\lim_{n\to\infty} \lambda(V_n)V_n^* = U$. Therefore, by replacing V_n with V_n^*, we obtain a sequence of unitaries $(V_n)_{n\geq 1} \in \mathcal{O}_2$ such that $\lim_{n\to\infty} \lambda(V_n)^*V_n = U$.

Since each V_n is a unitary, by the Universal Property of the Cuntz algebras there exists unital *-homomorphism $\lambda_{V_n} : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\lambda_{V_n}(S_j) = V_nS_j$ for $j \in \{1, 2\}$. We claim that $(\lambda_{V_n})_{n\geq 1}$ is an asymptotically central sequence of *-homomorphisms. To see this, we will show that

$$\lim_{n\to\infty} \|\lambda(\lambda_{V_n}(A)) - \lambda_{V_n}(A)\| = 0$$

for each $A \in \mathcal{O}_2$ which will complete the proof from the above observation. To see this, we notice for $j \in \{1, 2\}$ that

$$\begin{aligned} \|\lambda(\lambda_{V_n}(S_j)) - \lambda_{V_n}(S_j)\| &= \|\lambda(V_n S_j) - V_n S_j\| \\ &= \|\lambda(V_n)\lambda(S_j) - V_n S_j\| \\ &= \|\lambda(S_j) - \lambda(V_n)^* V_n S_j\| \\ &= \|U S_j - \lambda(V_n)^* V_n S_j\| \\ &\leq \|U - \lambda(V_n)^* V_n\| \end{aligned}$$

which converges to zero as $n \to \infty$. Since λ and each λ_{V_n} is a *-homomorphism, the above also implies that

$$\lim_{n\to\infty} \left\|\lambda(\lambda_{V_n}(S_j^*)) - \lambda_{V_n}(S_j^*)\right\| = 0$$

for $j \in \{1, 2\}$. Therefore, by taking algebraic combinations, density, and the fact that *-homomorphisms are contractive, we obtain that

$$\lim_{n\to\infty} \|\lambda(\lambda_{V_n}(A)) - \lambda_{V_n}(A)\| = 0$$

for each $A \in {}^*\text{-alg}(S_1, S_2)$ and thus

$$\lim_{n\to\infty} \|\lambda(\lambda_{V_n}(A)) - \lambda_{V_n}(A)\| = 0$$

for each $A \in \mathcal{O}_2$. Hence $(\lambda_{V_n})_{n\geq 1}$ is an asymptotically central sequence of *-homomorphisms. □

We are now ready to prove our main result.

Theorem 7.5. $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$.

Proof. Let $\pi : \mathcal{O}_2 \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ be defined by $\pi(T) = T \otimes I_{\mathcal{O}_2}$. Clearly π is a unital, injective *-homomorphism between unital, separable C*-algebras. Therefore to show that $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ are isomorphic it suffices to verify the conditions of Lemma 7.1.

We claim that it suffices to show that there exists a sequence of unitaries $(V_n)_{n\geq 1} \in \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ such that

$$\|V_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})V_n\| < \frac{1}{n} \qquad \text{and} \qquad dist(V_n^*(I_{\mathcal{O}_2} \otimes S_j)V_n, \mathcal{O}_2 \otimes I_{\mathcal{O}_2}) < \frac{1}{n} \qquad (*)$$

for $j \in \{1, 2\}$. To see this, we notice

$$\left\|V_n(S_j^* \otimes I_{\mathcal{O}_2}) - (S_j^* \otimes I_{\mathcal{O}_2})V_n\right\| = \left\|(S_j^* \otimes I_{\mathcal{O}_2})V_n^* - V_n^*(S_j \otimes I_{\mathcal{O}_2})\right\| = \|V_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})V_n\|$$

so $(*)$ implies

$$\lim_{n\to\infty} \|V_n(p(S_1, S_2, S_1^*, S_2^*) \otimes I_{\mathcal{O}_2}) - (p(S_1, S_2, S_1^*, S_2^*) \otimes I_{\mathcal{O}_2})V_n\| = 0$$

for all polynomials p in four non-commuting variables. Therefore $\lim_{n\to\infty} \|V_n\pi(A) - \pi(A)V_n\| = 0$ for all $A \in \mathcal{O}_2$ by density. Moreover, since $\pi(A)$ asymptotically commutes with V_n, we obtain that

$$\limsup_{n\to\infty} dist(V_n^*(A \otimes S_j)V_n, \mathcal{O}_2 \otimes I_{\mathcal{O}_2}) \leq \|A\| \lim_{n\to\infty} dist(V_n^*(I_{\mathcal{O}_2} \otimes S_j)V_n, \mathcal{O}_2 \otimes I_{\mathcal{O}_2}) = 0$$

for all $A \in \mathcal{O}_2$ and $j \in \{1, 2\}$. Therefore, again by taking adjoints, algebraic combinations, and using density, we obtain that

$$\lim_{n\to\infty} dist(V_n^* T V_n, \mathcal{O}_2 \otimes I_{\mathcal{O}_2}) = 0$$

for all $T \in \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$. Hence the conditions of Lemma 7.1 will be satisfied for π if we can construct unitaries $(V_n)_{n\geq 1}$ in $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ such that $(*)$ holds and thus we will obtain $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$ from Lemma 7.1.

To verify $(*)$, let $\epsilon > 0$. Since $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ is unital, simple, and purely infinite, Theorem 6.12 implies that the unital *-homomorphisms π and $\sigma : \mathcal{O}_2 \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ defined by $\sigma(T) := I_{\mathcal{O}_2} \otimes T$ are approximately unitarily equivalent. Therefore there exists a unitary $W \in \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ such that

$$\|W(S_j \otimes I_{\mathcal{O}_2})W^* - (I_{\mathcal{O}_2} \otimes S_j)\| < \epsilon$$

for all $j \in \{1, 2\}$.

Let $\rho_n : \mathcal{O}_2 \to \mathcal{O}_2$ be the asymptotically central sequence of unital *-homomorphisms from Lemma 7.4 and let $\psi_n : \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ be the *-homomorphisms $\psi_n := \rho_n \otimes Id$. Therefore each ψ_n is a unital *-homomorphism so $W_n := \psi_n(W) \in \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ is a unitary.

Next we claim that

$$\lim_{n\to\infty} \|W_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})W_n\| = 0.$$

To see this, we notice that for all $\delta > 0$ we can select $A_k, B_k \in \mathcal{O}_2$ such that $\|W - \sum_{k=1}^m A_k \otimes B_k\| < \delta$. Hence $\|W_n - \sum_{k=1}^m \rho_n(A_k) \otimes B_k\| < \delta$ so

$$\|W_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})W_n\| \leq 2\delta + \sum_{k=1}^m \|\rho_n(A_k)S_j - S_j\rho_n(A_k)\| \, \|B_k\| \, .$$

Hence

$$\limsup_{n\to\infty} \|W_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})W_n\| \leq 2\delta$$

for all $\delta > 0$ and thus

$$\lim_{n\to\infty} \|W_n(S_j \otimes I_{\mathcal{O}_2}) - (S_j \otimes I_{\mathcal{O}_2})W_n\| = 0.$$

However, we notice that

$$\begin{aligned} dist(W_n^*(I_{\mathcal{O}_2} \otimes S_j)W_n, \mathcal{O}_2 \otimes I_{\mathcal{O}_2}) &\leq \|W_n^*(I_{\mathcal{O}_2} \otimes S_j)W_n - \rho_n(S_j) \otimes I_{\mathcal{O}_2}\| \\ &= \|\psi_n(W^*(I_{\mathcal{O}_2} \otimes S_j)W - S_j \otimes I_{\mathcal{O}_2})\| < \epsilon. \end{aligned}$$

Therefore, by selecting select ϵ small enough and n large enough, W_n can be used to construct unitaries that satisfy $(*)$ thus completing the proof. $\square$

8 States on Purely Infinite C*-Algebras

In this chapter we will enhance the theory of the set of all states on a unital, simple, purely infinite C*-algebra. Along the way, we will need to develop several general results about pure states and irreducible representations of C*-algebras. Our goal is to show that every state on a unital, simple, purely infinite C*-algebra $\mathfrak{A}$ is close to a compression map on $\mathfrak{A}$. To prove this, we desire to make use of the excision results of Chapter 3 and thus we will first need to prove that every state on a unital, simple, purely infinite C*-algebra is a weak*-limit of pure states.

Most of the results for this chapter were developed from the book [Di].

To begin, we desire to know when the weak*-closure of a set of states on a C*-algebra $\mathfrak{A}$ contains all pure states on $\mathfrak{A}$. To prove the desired result, we need the following commonly used result from functional analysis (which we provide for completeness).

Theorem 8.1 (Milman's). *Let K be a compact set in a locally convex topological vector space X such that $\overline{co}(K)$ is compact. Then every extreme point of $\overline{co}(K)$ lies in K.*

Proof. Suppose that there is an extreme point p of $\overline{co}(K)$ that is not in K. Since $\{p\}$ is a compact set and K is closed, there exists an open neighbourhood U of 0_X such that $(\overline{p+U})\cap K = \emptyset$. As X is locally convex, there exists a convex neighbourhood $V' \subseteq U$ of 0_X. Hence there exists a balanced, convex neighbourhood $V \subseteq V'$ of 0_X that also satisfies $(p+\overline{V})\cap K = \emptyset$. Since K is compact, we may choose $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n}(x_i + V)$. Let $A_i := \overline{co}(K \cap (x_i + V))$. Thus each A_i is convex and is also compact since it is a closed subset of the compact set $\overline{co}(K)$. Moreover $K \subseteq \bigcup_{i=1}^{n} A_i$. Therefore

$$\overline{co}(K) \subseteq \overline{co}\left(\bigcup_{i=1}^{n} A_i\right) = co\left(\bigcup_{i=1}^{n} A_i\right)$$

since $co\left(\bigcup_{i=1}^{n} A_i\right)$ is compact (as the convex hull of two compact sets K_1, K_2 is compact as it is the image of $[0,1]\times K_1 \times K_2$ under a continuous map) and hence closed. However, since $\overline{co}(K)$ is convex and $A_i \subseteq \overline{co}(K)$, we have that

$$\overline{co}(K) = co\left(\bigcup_{i=1}^{n} A_i\right).$$

Hence, since $p \in \overline{co}(K)$, by rearranging the order of the A_i there exists an $N \in \{1,\ldots,n\}$, $y_i \in A_i$, $t_i \geq 0$, and $t_1 > 0$ such that $\sum_{i=1}^{N} t_i = 1$ and $p = \sum_{i=1}^{N} t_i y_i$. However, notice that

$$p = t_1 y_1 + (1-t_1)\frac{t_2 y_2 + \cdots + t_N y_N}{t_2 + \cdots + t_N}$$

so p is a convex combination of two elements of $\overline{co}(K)$. Since p is an extreme point of $\overline{co}(K)$, we must have that $y_1 = p$. Thus, for some A_i, we have that

$$p \in A_i \subseteq \overline{co}(x_i + V) \subseteq x_i + \overline{V} \subseteq K + \overline{V}$$

since V is convex. However, this contradicts the fact that $(p+\overline{V})\cap K = \emptyset$ since if $p = k + v$ where $k \in K$, $v \in \overline{V}$, then $p - v = k$ and $-v \in \overline{V}$ since $\overline{V}$ is balanced. Hence we have our contradiction. ☐

Corollary 8.2. *Let $\mathfrak{A}$ be a unital C*-algebra, let $\mathfrak{A}_{sa}$ the self-adjoint elements of $\mathfrak{A}$, let $\mathcal{S}(\mathfrak{A})$ the set of state on $\mathfrak{A}$, let $\mathcal{PS}(\mathfrak{A})$ the set of pure states on $\mathfrak{A}$, and let $Q \subseteq \mathcal{S}(\mathfrak{A})$ be such that if $A \in \mathfrak{A}_{sa}$ satisfies $\varphi(A) \geq 0$ for all $\varphi \in Q$ then $A \geq 0$. Then $\overline{conv(Q)}^{w^*} = \mathcal{S}(\mathfrak{A})$ and $\mathcal{PS}(\mathfrak{A}) \subseteq \overline{Q}^{w^*}$.*

Proof. To see that $\overline{conv(Q)}^{w^*} = \mathcal{S}(\mathfrak{A})$, suppose to the contrary that $\overline{conv(Q)}^{w^*} \neq \mathcal{S}(\mathfrak{A})$. Therefore there exists a state φ on $\mathfrak{A}$ such that $\varphi \notin \overline{conv(Q)}^{w^*}$. By the separation version of the Hahn-Banach Theorem, there exists $\alpha < \beta$ in $\mathbb{R}$ and a weak*-continuous linear functional on $\mathfrak{A}^*$ (which is just an element $A \in \mathfrak{A}$) such that

$$Re(\varphi(A)) \leq \alpha < \beta \leq Re(\psi(A))$$

for all $\psi \in \overline{conv(Q)}^{w^*}$. However, by considering $B := Re(A)$ and the fact that φ and each ψ under consideration is positive and thus self-adjoint, we obtain that

$$\varphi(B) \leq \alpha < \beta \leq \psi(B)$$

for all $\psi \in Q$. Therefore $\psi(B - \beta I_{\mathfrak{A}}) \geq 0$ for all $\psi \in Q$. Therefore, as B is self-adjoint, we obtain that $B - \beta I_{\mathfrak{A}} \geq 0$ by our assumptions on Q. Hence, as φ is positive, $\varphi(B) \geq \beta$ which is a contradiction. Hence $\overline{conv(Q)}^{w^*} = \mathcal{S}(\mathfrak{A})$ as desired.

To see that $\mathcal{PS}(\mathfrak{A}) \subseteq \overline{Q}^{w^*}$, we notice that since $\overline{Q}^{w^*}$ is a weak*-closed and thus weak*-compact subset of $\mathcal{S}(\mathfrak{A})$, Milman's Theorem implies that every extreme point of $\mathcal{S}(\mathfrak{A}) = \overline{conv\left(\overline{Q}^{w^*}\right)}^{w^*}$ is contained in $\overline{Q}^{w^*}$. As $\mathcal{PS}(\mathfrak{A})$ are the extreme points of $\mathcal{S}(\mathfrak{A})$, the result follows. $\square$

Next we desire to be able to use the kernels of representations to determine that certain states are weak*-limits of convex combinations of states of vector states.

Lemma 8.3. *Let $\mathfrak{A}$ be a C*-algebra and let $\{\pi_\alpha : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_\alpha)\}_{\alpha \in I}$ be a family of representations of $\mathfrak{A}$. Then*

1. *Each state on $\mathfrak{A}$ that vanishes on $\bigcap_{\alpha\in I} ker(\pi_\alpha)$ is a weak*-limit of states of the form $\omega_{\xi_1}\circ\pi_{\alpha_1}+\cdots+\omega_{\xi_n}\circ\pi_{\alpha_n}$ where $\alpha_j \in I$, $\xi_j \in \mathcal{H}_{\alpha_j}$ are vectors such that $\sum_{i=1}^n \|\xi_i\|^2 = 1$, and $\omega_{\xi_j}\circ\pi_{\alpha_j}(A) = \langle \pi_{\alpha_j}(A)\xi_j, \xi_j\rangle_{\mathcal{H}_{\alpha_j}}$ for all $A \in \mathfrak{A}$ and for all j.*

2. *Each pure state on $\mathfrak{A}$ that vanishes on $\bigcap_{\alpha\in I} ker(\pi_\alpha)$ is a weak*-limit of states of the form $\omega_\xi \circ \pi_\alpha$ where $\alpha \in I$, $\xi \in \mathcal{H}_\alpha$ is a unit vector, and $\omega_\xi \circ \pi_\alpha(A) = \langle \pi_\alpha(A)\xi, \xi\rangle_{\mathcal{H}_\alpha}$ for all $A \in \mathfrak{A}$.*

Proof. Let φ be a (pure) state that vanishes on $\bigcap_{\alpha\in I} ker(\pi_\alpha)$. By common representation theory results, we can assume that each π_α is non-degenerated. Moreover, by moding $\mathfrak{A}$ out by $\bigcap_{\alpha\in I} ker(\pi_\alpha)$, we may assume that $\bigcap_{\alpha\in I} ker(\pi_\alpha) = \{0\}$ as we can view φ as a (pure) state on this quotient C*-algebra and φ will be a weak*-limit of states on $\mathfrak{A}$ that vanish on $\bigcap_{\alpha\in I} ker(\pi_\alpha)$ if and only if it is a weak*-limit of the same states as viewed on the quotient algebra. Therefore we can assume that $\rho = \oplus_{\alpha\in I}\pi_\alpha$ is a non-degenerate faithful representation of $\mathfrak{A}$ and thus we can view $\mathfrak{A}$ as a non-degenerate C*-subalgebra of $\mathcal{B}(\oplus_{\alpha\in I}\mathcal{H}_\alpha)$. Thus we can assume without loss of generality that $\mathfrak{A}$ is unital by adding in the unit of $\mathcal{B}(\oplus_{\alpha\in I}\mathcal{H}_\alpha)$ if necessary (as we will still get states on $\mathfrak{A}$ as $\mathfrak{A}$ is non-degenerate). Let Q be the set of all state on $\mathfrak{A}$ of the form $\omega_\xi \circ \pi_\alpha$ for some $\xi \in \mathcal{H}_\alpha$ of norm one. However, if $A \in \mathfrak{A}_{sa}$ and $\varphi(A) \geq 0$ for all $\varphi \in Q$, then $\langle \pi_\alpha(A)\xi, \xi\rangle_{\mathcal{H}_\alpha} \geq 0$ for all $\xi \in \mathcal{H}_\alpha$ and thus $\pi_\alpha(A) \geq 0$ for all $\alpha \in I$. As ρ is a faithful representation of $\mathfrak{A}$, we obtain that $A \geq 0$ and thus Q satisfies the conditions of Lemma 8.2. Hence the result follows. $\square$

Next we will need a common result known as Glimm's Lemma. Note that this lemma is usually stated for separable C*-algebras on a separable Hilbert space (where it is possible to use sequences of vector states). However, we will need the full version of this lemma.

Lemma 8.4 (Glimm's Lemma)**.** *Let $\mathcal{H}$ be a Hilbert space, let $\mathfrak{K}$ be the compact operators on $\mathcal{H}$, and let $\mathfrak{A}$ be a C*-subalgebra of $\mathcal{B}(\mathcal{H})$ with $I_{\mathcal{H}} \in \mathfrak{A}$. If φ is a state on $\mathfrak{A}$ that vanishes on $\mathfrak{A} \cap \mathfrak{K}$ then φ is a weak*-limit of vector states on $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ is irreducible in $\mathcal{H}$ then φ is a weak*-limit of pure state of $\mathfrak{A}$.*

Proof. The result that "if $\mathfrak{A}$ is irreducible in $\mathcal{H}$ then φ is a weak*-limit of pure state of $\mathfrak{A}$" follows from the first statement since if $\mathfrak{A}$ is irreducible on $\mathcal{H}$, every vector space defined on $\mathfrak{A}$ by $\mathcal{H}$ is a pure state on $\mathfrak{A}$. Thus we focus on proving the first statement. If $\mathfrak{K} \nsubseteq \mathfrak{A}$ we can define a state φ' on $\mathfrak{A} + \mathfrak{K}$ (which is a C*-algebra) by $\varphi'(A + K) = \varphi(A)$ for all $A \in \mathfrak{A}$ and $K \in \mathfrak{K}$ (which is a well-defined state on $\mathfrak{A} + \mathfrak{K}$ as φ vanishes on $\mathfrak{A} \cap \mathfrak{K}$). Thus, without loss of generality, $\mathfrak{K} \subseteq \mathfrak{A}$.

Let $q : \mathfrak{A} \to \mathfrak{A}/\mathfrak{K}$ be the canonical quotient map. Since φ vanishes on $\mathfrak{K}$, φ defines a state φ' on $\mathfrak{A}/\mathfrak{K}$ by $\varphi'(A + \mathfrak{K}) = \varphi(A)$ for all $A \in \mathfrak{A}$. Thus φ' is a weak*-limit (in $\mathfrak{A}/\mathfrak{K}$) of states of the form $\lambda_1\varphi'_1 + \cdots + \lambda_n\varphi'_n$ where φ'_j are pure states on $\mathfrak{A}/\mathfrak{K}$ and $\lambda_j \in (0, 1)$ are such that $\sum_{j=1}^n \lambda_j = 1$. Each φ'_j defines a state $\varphi_j = \varphi'_j \circ q$ on $\mathfrak{A}$ and it is easy to see that φ is a weak*-limit of the states of the form $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$ where $\varphi_j = \varphi'_j \circ q$ where φ'_j are pure states on $\mathfrak{A}/\mathfrak{K}$ and $\lambda_j \in (0, 1)$ are such that $\sum_{j=1}^n \lambda_j = 1$. However, we claim that each φ_j as listed above must be a pure state on $\mathfrak{A}$. To see this, suppose $\varphi_j = \lambda\psi_1 + (1 - \lambda)\psi_2$ where $\lambda \in (0, 1)$ and $\psi_1, \psi_2 \in \mathcal{S}(\mathfrak{A})$. If $K \in \mathfrak{K}$ is positive then $\psi_1(K) \geq 0$, $\psi_2(K) \geq 0$, and $0 = \lambda\psi_1(K) + (1 - \lambda)\psi_2(K)$. Hence $\psi_1(K) = \psi_2(K) = 0$ for all $K \in \mathfrak{K}$ positive and thus for all $K \in \mathfrak{K}$. Hence ψ_1 and ψ_2 define states ψ'_1 and ψ'_2 on $\mathfrak{A}/\mathfrak{K}$ by $\psi'_i(A+\mathfrak{K}) = \psi_i(A)$ for all $A \in \mathfrak{A}$. It is easy to see that $\varphi'_j = \lambda\psi'_1 + (1-\lambda)\psi'_2$. However, as φ'_j was assumed to be a pure state on $\mathfrak{A}/\mathfrak{K}$, $\psi'_i = \varphi'_j$ for $i \in \{1, 2\}$ and thus $\psi_1 = \psi_2 = \varphi_j$. Hence φ_j is a pure state on $\mathfrak{A}$. Therefore φ is a weak*-limit of states of the form $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$ where φ_j are pure states on $\mathfrak{A}$ such that $\varphi_j(\mathfrak{K}) = \{0\}$ and $\lambda_j \in (0, 1)$ are such that $\sum_{j=1}^n \lambda_j = 1$. Therefore, to prove the lemma, it suffices to consider states of the form $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$ where φ_j are pure states on $\mathfrak{A}$ such that $\varphi_j(\mathfrak{K}) = \{0\}$ and $\lambda_j \in (0, 1)$ are such that $\sum_{j=1}^n \lambda_j = 1$.

To show that $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n$ is a weak*-limit of vector states, we will show for any finite set $\{A_1, \ldots, A_m\}$ of self-adjoint elements of $\mathfrak{A}$ such that $A_1 = I$ there exists unit vectors $\xi_1, \ldots, \xi_n \in \mathcal{H}$ such that $\langle A_i\xi_j, \xi_k\rangle = 0$ for $j < k$ and such that $|\varphi_j(A_i) - \langle A_i\xi_j, \xi_j\rangle| \leq 1$ for all $i \in \{1, \ldots, m\}$ and for all $j \in \{1, \ldots, n\}$. To begin, as $A_1 = I$, let ξ_1 be any unit vector.

Suppose that ξ_j have been constructed with the desired properties for $j < \ell$. Let

$$\mathcal{K}_0 := \mathit{span}\{A_i\xi_j \mid 1 \leq i \leq m, j < \ell\}$$

so that $\mathcal{K}_0$ is a finite dimensional subspace of $\mathcal{H}$. Let $\mathcal{K} := \mathcal{H} \ominus \mathcal{K}_0$. Since $\mathcal{K}_0$ is a finite dimensional subspace of $\mathcal{H}$, the projection onto $\mathcal{K}_0$, denoted $P_{\mathcal{K}_0}$, is an element of $\mathfrak{K} \subseteq \mathfrak{A}$. Clearly $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$ is a C*-subalgebra of $\mathfrak{A}$ (as $I \in \mathfrak{A}$) and $\varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}}$ is a state on $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$ since it is clearly a positive linear functional and

$$\varphi_\ell(P_\mathcal{K}) = \varphi_\ell(P_\mathcal{K}) + 0 = \varphi_\ell(P_\mathcal{K}) + \varphi_\ell(P_{\mathcal{K}_0}) = \varphi(I) = 1$$

as φ_ℓ vanishes on $\mathfrak{K}$.

We claim that $\varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}}$ is a pure state on $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$. To see this, suppose $\varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}} = \lambda\psi_1 + (1 - \lambda)\psi_2$ where $\lambda \in (0, 1)$ and ψ_1 and ψ_2 are states on $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$. Define $\psi'_1, \psi'_2 : \mathfrak{A} \to \mathbb{C}$ by $\psi'_q(A) = \psi_q(P_\mathcal{K}AP_\mathcal{K})$. Clearly ψ'_1 and ψ'_2 are state on $\mathfrak{A}$ such that $\varphi_\ell = \lambda\psi'_1 + (1 - \lambda)\psi'_2$ (as φ_ℓ vanishes on $\mathfrak{K}$ and thus lives on $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$). Therefore, as φ_ℓ was a pure state on $\mathfrak{A}$, $\psi'_1 = \psi'_2 = \varphi_\ell$ and thus $\psi_1 = \psi_2 = \varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}}$. Hence $\varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}}$ is a pure state on $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$.

Since $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$ contains all of the compact operators on $\mathcal{K}$ as $\mathfrak{A}$ contains all of the compact operators on $\mathcal{H}$, $P_\mathcal{K}\mathfrak{A}P_\mathcal{K}$ is irreducible in $\mathcal{K}$. Thus, by Lemma 8.3, $\varphi_\ell|_{P_\mathcal{K}\mathfrak{A}P_\mathcal{K}}$ is a weak*-limit of states of the form ω_{ζ_α} where ζ_α are unit vectors in $\mathfrak{K}$. Hence, for $1 \leq i \leq m$, $\varphi_\ell(A_i) = \varphi_\ell(P_\mathcal{K}A_iP_\mathcal{K})$ is a weak*-limit of $\langle P_\mathcal{K}A_iP_\mathcal{K}\zeta_\alpha, \zeta_\alpha\rangle = \langle A_i\zeta_\alpha, \zeta_\alpha\rangle$. Hence we can find an $\xi_\ell \in \mathcal{K}$ such that $|\varphi_\ell(A_i) - \langle A_i\xi_\ell, \xi_\ell\rangle| \leq 1$ for all $1 \leq i \leq m$. Moreover, as $\xi_\ell \in \mathcal{K} = \mathcal{K}_0^\perp$, $\langle A_i\xi_j, \xi_\ell\rangle = 0$ for all $j < \ell$. Hence the construction of the ξ_j's proceeds by recursion.

Fix a set $\{A_1, \ldots, A_m\}$ of self-adjoint elements of $\mathfrak{A}$ such that $A_1 = I$ and let $\xi_1, \ldots, \xi_n \in \mathcal{H}$ be the unit vectors constructed above such that $\langle A_i\xi_j, \xi_k\rangle = 0$ for $j < k$ and such that $|\varphi_j(A_i) - \langle A_i\xi_j, \xi_j\rangle| \leq 1$ for all $i \in \{1, \ldots, m\}$ and for all $j \in \{1, \ldots, n\}$. Let $\xi := \sum_{j=1}^n \sqrt{\lambda_j}\xi_j$. Since $A_1 = I$, $\langle \xi_j, \xi_k\rangle = 0$ for all $j \neq k$ and thus ξ is a unit vector. Moreover, as each A_i is self-adjoint, $\langle A_i\xi_j, \xi_k\rangle = 0$ for all $j \neq k$. Hence

$$\begin{aligned}
\left|\sum_{j=1}^n \lambda_j\varphi_j(A_i) - \langle A_i\xi, \xi\rangle\right| &= \left|\sum_{j=1}^n \lambda_j\varphi_j(A_i) - \sum_{j,k=1}^n \left\langle A_i\sqrt{\lambda_j}\xi_j, \sqrt{\lambda_k}\xi_k\right\rangle\right| \\
&= \left|\sum_{j=1}^n \lambda_j\varphi_j(A_i) - \sum_{j=1}^n \lambda_j\langle A_i\xi_j, \xi_j\rangle\right| \\
&\leq \sum_{j=1}^n \lambda_j = 1
\end{aligned}$$

for all $i > 1$.

Therefore for any finite subset of self-adjoint elements of $\mathfrak{A}$ we have found a vector state on $\mathfrak{A}$ that differs from $\sum_{j=1}^n \lambda_j \varphi_j$ by at most one on all of the self-adjoint elements in our finite subset. Therefore, by scaling our self-adjoint elements and noting that the span of the self-adjoint elements of $\mathfrak{A}$ is $\mathfrak{A}$, $\sum_{j=1}^n \lambda_j \varphi_j$ is a weak*-limit of vector states as desired. □

Now we may begin to discuss states on unital, simple, purely infinite C*-algebras. First we desire to show that every state on a unital, simple, purely infinite C*-algebra is a weak*-limit of pure states. We make the following notation that will be used in the few remaining proofs.

Notation 8.5. Let $\mathfrak{A}$ be a C*-algebra. We denote the set of all irreducible representations of $\mathfrak{A}$ by $Irr(\mathfrak{A})$.

If $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathfrak{A}$, we denote by C_π the set of all $A \in \mathfrak{A}$ such that $\pi(A)$ is a compact operator. Thus C_π is an ideal of $\mathfrak{A}$.

Lemma 8.6. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Then $\bigcap_{\pi \in Irr(\mathfrak{A})} C_\pi = \{0\}$.*

Proof. Suppose to the contrary that $\bigcap_{\pi \in Irr(\mathfrak{A})} C_\pi$ is non-zero. Therefore, since $\mathfrak{A}$ is simple, $C_\pi = \mathfrak{A}$ for all $\pi \in Irr(\mathfrak{A})$. By Lemma 2.3 there exists a non-zero isometry $V \in \mathfrak{A}$ such that $P = VV^* < I_\mathfrak{A}$. Therefore $\pi(I_\mathfrak{A}) = \pi(V^*V) = \pi(V)^*\pi(V)$ and $\pi(P) = \pi(V)\pi(V)^*$ for all $\pi \in Irr(\mathfrak{A})$. Hence $\pi(I_\mathfrak{A})$ and $\pi(V)$ are equivalent projections. However, since $C_\pi = \mathfrak{A}$, $\pi(I_\mathfrak{A})$ and $\pi(P)$ must be compact operators. Therefore, as $\pi(P) \leq \pi(I_\mathfrak{A})$ and $\pi(I_\mathfrak{A})$ and $\pi(P)$ are equivalent compact projections, $\pi(I_\mathfrak{A}) = \pi(P)$ for all irreducible representations π. However, since the irreducible representations of $\mathfrak{A}$ separate points in $\mathfrak{A}$, $P = I_\mathfrak{A}$ which is a contradiction. Hence $\bigcap_{\pi \in Irr(\mathfrak{A})} C_\pi = \{0\}$. □

Theorem 8.7. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Then $\overline{\mathcal{PS}(\mathfrak{A})}^{w^*} = \mathcal{S}(\mathfrak{A})$.*

Proof. Clearly $\overline{\mathcal{PS}(\mathfrak{A})}^{w^*} \subseteq \mathcal{S}(\mathfrak{A})$. Let $\varphi \in \mathcal{S}(\mathfrak{A})$ be arbitrary. Since $\bigcap_{\pi \in Irr(\mathfrak{A})} C_\pi = \{0\}$ and $\mathfrak{A}$ is simple, there exists an irreducible representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $C_\pi = \{0\}$. Therefore $\varphi(C_\pi) = \{0\}$. Hence φ defines a state φ' on $\pi(\mathfrak{A})$ that vanishes on $\pi(\mathfrak{A}) \cap \mathfrak{K} = \{0\}$. Therefore, by Glimm's Lemma, φ' is a weak*-limit of pure states on $\pi(\mathfrak{A})$ (which must define pure state on $\mathfrak{A}$ as π is irreducible) and thus φ is a weak*-limit of pure states on $\mathfrak{A}$. □

With the above result in-hand, we are able to prove our final result for this chapter.

Theorem 8.8. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let φ be a state on $\mathfrak{A}$. Then for every $\epsilon > 0$ and every finite subset $\mathcal{F} \subseteq \mathfrak{A}$ there exists a non-zero projection $P \in \mathfrak{A}$ such that $\|PAP - \varphi(A)P\| < \epsilon$ for all $A \in \mathcal{F}$.*

Proof. Let φ be a state on $\mathfrak{A}$. By Theorem 8.7, φ is a weak*-limit of pure states. Therefore, by Proposition 3.8, φ can be excised. Hence there exists a net of positive elements $(A_\alpha)_\Lambda$ with $\|A_\alpha\| = 1$ such that $\lim_\Lambda \left\|A_\alpha A A_\alpha - \varphi(A)A_\alpha^2\right\| = 0$ for all $A \in \mathfrak{A}$.

Let $\epsilon > 0$ and let $\mathcal{F}$ be a finite subset of $\mathfrak{A}$. Choose α such that $\left\|A_\alpha A A_\alpha - \varphi(A)A_\alpha^2\right\| < \frac{\epsilon}{2}$ for all $A \in \mathcal{F}$. Since $A_\alpha \geq 0$ and since $\mathfrak{A}$ has real rank zero by Proposition 5.3, Theorem 5.5 plus some thought implies that there exists a positive element $X \in \mathfrak{A}$ with finite spectrum such that $\|X\| = 1$ and $\left\|XAX - \varphi(A)X^2\right\| < \epsilon$ for all $A \in \mathcal{F}$. Since X has finite spectrum and $\|X\| = 1$, $P := \chi_{\{1\}}(X)$ is a non-zero projection in $\mathfrak{A}$. Moreover

$$\|PAP - \varphi(A)P\| = \left\|PXAXP - P(\varphi(A)X^2)P\right\| \leq \left\|XAX - \varphi(A)X^2\right\| < \epsilon$$

for all $A \in \mathcal{F}$. Hence the result follows. □

9 Non-Standard Results on Completely Positive Maps

In this chapter we will develop the necessary theory of completely positive maps needed in subsequent chapters. We assume the reader is already familiar with the standard theory of completely positive and completely bounded maps and we focus on some non-standard results.

Most of the results for this chapter were developed from the paper [EH] and from the books [Pa] and [BO].

We begin with a brief study of completely bounded maps on finite dimensional operator systems. We begin with a technical lemma from Banach spaces.

Lemma 9.1. *Let $\mathfrak{X}$ be a Banach space with dimension n. Then there exists a basis $\{e_1, \ldots, e_n\}$ for $\mathfrak{X}$ such that $\|e_i\| = 1$ for all $i \in \{1, \ldots, n\}$ and, if $\{f_1, \ldots, f_n\}$ is the corresponding dual basis, $\|f_i\| = 1$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let $\mathcal{B} = \{y_1, \ldots, y_n\}$ be any basis for $\mathfrak{X}$. For any n vectors $z_1, \ldots, z_n$ in $\mathfrak{X}$ with $\|z_i\| = 1$ for all $i \in \{1, \ldots, n\}$, define $V(z_1, \ldots, z_n) := det([a_{i,j}]_{i,j})$ where $(a_{i,1}, a_{i,2}, \ldots, a_{i,n})$ is the coordinates of z_i with respect to $\mathcal{B}$. Therefore it is clear that $|V|$ is a continuous function on a compact subset and thus obtains its maximum at a set $\{e_1, \ldots, e_n\}$ with $\|e_j\| = 1$ for all $j \in \{1, \ldots, n\}$. Notice that $V(y_1, \ldots, y_n) = 1$ so $|V(e_1, \ldots, e_n)| > 0$ and thus $\{e_1, \ldots, e_n\}$ must be a basis for $\mathfrak{X}$.

Let $\{f_1, \ldots, f_n\}$ be the dual basis of $\{e_1, \ldots, e_n\}$. It is trivial to verify that

$$f_j(x) = \frac{V(e_1, \ldots, e_{j-1}, x, e_{j+1}, \ldots, e_n)}{V(e_1, \ldots, e_n)}$$

for all $x \in \mathfrak{X}$ with $\|x\| = 1$ and for all $j \in \{1, \ldots, n\}$. Therefore, due to the maximality of $|V|$ at $\{e_1, \ldots, e_n\}$, $|f_j(x)| \leq 1$ for all $x \in \mathfrak{X}$ with $\|x\| = 1$. Hence the result follows. $\square$

Lemma 9.2. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be operator spaces and suppose $\mathcal{S}_1$ is finite dimensional. Then any linear map $\varphi : \mathcal{S}_1 \to \mathcal{S}_2$ is completely bounded with $\|\varphi\|_{cb} \leq dim(\mathcal{S}_1) \|\varphi\|$.*

Proof. Note that φ must be bounded being a linear map with a finite dimensional domain and thus a finite dimensional range. To see that $\|\varphi\|_{cb} \leq n \|\varphi\|$, let $\{x_1, \ldots, x_n\}$ be a basis for $\mathcal{S}_1$ with dual basis $\{f_1, \ldots, f_n\}$ such that $\|x_j\| = 1 = \|f_j\|$ for all $j \in \{1, \ldots, n\}$ (whose existence is guaranteed by Lemma 9.1). Then for all $x \in \mathcal{S}_1$ $x = \sum_{j=1}^n f_j(x) x_j$ and thus $\varphi(x) = \sum_{j=1}^n f_j(x)\varphi(x_j)$. However, it is easy to see that the linear maps $x \mapsto f_j(x)\varphi(x_j)$ have completely bounded norm at most $\|f_j\|_{cb} \|\varphi(x_j)\| = \|f_j\| \|\varphi(x_j)\| \leq \|\varphi\|$. Hence

$$\|\varphi\|_{cb} \leq \sum_{j=1}^n \|x \mapsto f_j(x)\varphi(x_j)\|_{cb} \leq n \|\varphi\|$$

as desired. $\square$

Lemma 9.3. *Let $\mathfrak{A}$ be a unital C*-algebra, let $A_1, \ldots, A_m \in \mathfrak{A}$ be linearly independent, and suppose that $\mathcal{S} := span(A_1, \ldots, A_n)$ is an operator system in $\mathfrak{A}$. Let*

$$M := \sup\left\{ \max_{1 \leq j \leq m} |\alpha_j| \mid \left\| \sum_{k=1}^m \alpha_k A_k \right\| \leq 1 \right\}.$$

Then for any $B_1, \ldots, B_m \in \mathfrak{A}$ the linear map $\Phi : \mathcal{S} \to span(B_1, \ldots, B_m)$ defined by $\Phi(A_j) := B_j$ for all $1 \leq j \leq m$ is completely bounded with

$$\|\Phi\|_{cb} \leq 1 + mM \sum_{j=1}^m \|A_j - B_j\|.$$

If $mM\sum_{j=1}^m \|A_j - B_j\| < 1$ *then* Φ^{-1} *exists and*

$$\left\|\Phi^{-1}\right\|_{cb} \leq \left(1 - mM\sum_{j=1}^m \|A_j - B_j\|\right)^{-1}.$$

Proof. Note that $M < \infty$ as $\mathcal{S}$ is an n-dimensional vector space and all norms (specifically $\|\cdot\|_{\mathfrak{A}}$ and $\|\cdot\|_\infty$) are equivalent. Consider $\mathbb{C}^m$ with the ℓ_∞-norm. Define $Q : \mathcal{S} \to \mathbb{C}^n$ by $Q(A_j) := e_j$ and define $R : \mathbb{C}^n \to \mathfrak{A}$ by $R(e_j) := B_j - A_j$. Due to the norm defined on $\mathbb{C}^n$, $\|Q\| = M$ and $\|R\| \leq \sum_{j=1}^m \|A_j - B_j\|$. Therefore, by Lemma 9.2,

$$\|R \circ Q\|_{cb} \leq m\|R \circ Q\| \leq m\|R\|\|Q\| = mM\sum_{j=1}^m \|A_j - B_j\|.$$

Therefore, since $\Phi = Id + R \circ Q$, we obtain that

$$\|\Phi\|_{cb} \leq 1 + mM\sum_{j=1}^m \|A_j - B_j\|$$

as desired. However, using the above norm estimate when $mM\sum_{j=1}^m \|A_j - B_j\| < 1$, we obtain that

$$\|\Phi_n(A)\| \geq \|A\| - \|(R \circ Q)_n(A)\| \geq \|A\|\left(1 - mM\sum_{j=1}^m \|A_j - B_j\|\right)$$

for all $A \in \mathcal{M}_n(\mathfrak{A})$. Therefore Φ_n is bounded below for all n and thus is invertible. Moreover, the above norm estimate implies

$$\left\|(\Phi_n)^{-1}\right\| \leq \left(1 - mM\sum_{j=1}^m \|A_j - B_j\|\right)^{-1}$$

for all $n \in \mathbb{N}$ and thus the result follows (as $(\Phi_n)^{-1} = (\Phi^{-1})_n$ for all $n \in \mathbb{N}$). $\square$

The above lemma will be essential for us to construct completely bounded maps with well-behaved norms. Next we desire to show that a unital, self-adjoint, completely bounded map is 'close' to a completely positive map.

Theorem 9.4. *Let* $\mathfrak{A}$ *be a unital* C^**-algebra, let* $\mathcal{S} \subseteq \mathfrak{A}$ *be an operator system, let* $\mathcal{H}$ *be a Hilbert space, and let* $\Phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ *be a unital, self-adjoint, completely bounded map. Then there exists a unital, completely positive map* $\Psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ *such that* $\|\Psi|_{\mathcal{S}} - \Phi\|_{cb} \leq 2(\|\Phi\|_{cb} - 1)$.

Proof. Since Φ is unital, $\|\Phi\|_{cb} \geq 1$. By Wittstock's Extension Theorem there exists a completely bounded map $\Psi_0 : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ such that $\Psi_0|_{\mathcal{S}} = \Phi$ and $\|\Psi_0\|_{cb} = \|\Phi\|_{cb}$. By Wittstock's Theorem there exists a Hilbert space $\mathcal{K}$, a unital *-homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$, and isometries $V_i : \mathcal{H} \to \mathcal{K}$ (for $i \in \{1, 2\}$) $\Psi_0(A) = \|\Phi\|_{cb} V_1^*\pi(A)V_2$ for all $A \in \mathfrak{A}$.

Since Φ is self-adjoint, we obtain that

$$\Phi(A) = \Psi_0(A) = \|\Phi\|_{cb} V_1^*\pi(A)V_2 = \|\Phi\|_{cb} V_2^*\pi(A)V_1$$

for all $A \in \mathcal{S}$. Define $\Psi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ by

$$\Psi(A) := \frac{1}{2}(V_1^*\pi(A)V_1 + V_2^*\pi(A)V_2)$$

for all $A \in \mathfrak{A}$. Clearly Ψ is a completely positive map with $\Psi(I) = \frac{1}{2}(V_1^*V_1 + V_2^*V_2) = I$. Moreover we notice for all $A \in \mathcal{S}$ that

$$\begin{aligned}\frac{1}{2}\left\|\Phi\right\|_{cb}(V_1 - V_2)^*\pi(A)(V_1 - V_2) &= \frac{1}{2}\left\|\Phi\right\|_{cb}(V_1^*\pi(A)V_1 + V_2^*\pi(A)V_2 - V_1^*\pi(A)V_2 - V_2^*\pi(A)V_1)\\ &= \left\|\Phi\right\|_{cb}\Psi(A) - \Phi(A).\end{aligned}$$

Therefore

$$\left\|\Psi|_{\mathcal{S}} - \Phi\right\|_{cb} \leq \left\|\Psi - \left\|\Phi\right\|_{cb}\Psi\right\|_{cb} + \left\|\left\|\Phi\right\|_{cb}\Psi|_{\mathcal{S}} - \Phi\right\|_{cb} \leq (\left\|\Phi\right\|_{cb} - 1) + \frac{1}{2}\left\|\Phi\right\|_{cb}\left\|V_1 - V_2\right\|^2.$$

However, since $\Phi(I_{\mathfrak{A}}) = I_{\mathcal{H}}$, $I_{\mathcal{H}} = \left\|\Phi\right\|_{cb} V_1^*V_2$ so

$$\frac{1}{2}\left\|\Phi\right\|_{cb}\left\|V_1 - V_2\right\|^2 = \frac{1}{2}\left\|\left\|\Phi\right\|_{cb} I_{\mathcal{H}} - 2I_{\mathcal{H}} + \left\|\Phi\right\|_{cb} I_{\mathcal{H}}\right\| = \left\|\Phi\right\|_{cb} - 1$$

and thus

$$\left\|\Psi|_{\mathcal{S}} - \Phi\right\|_{cb} \leq 2(\left\|\Phi\right\|_{cb} - 1)$$

as desired. □

Later we will need some of the theory of lifting completely positive maps. These following results are based mainly on the work contained in [EH]. We begin by studying when contractive, completely positive maps into a quotient can be lifted to completely positive maps.

Lemma 9.5. *Let $\mathfrak{J}$ be an ideal in a unital C*-algebra $\mathfrak{B}$ and let $\mathcal{S}$ be a separable operator system. The set of contractive, completely positive maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$ with a contractive, completely positive lifting to $\mathfrak{B}$ is closed in the point-norm topology on all bounded linear maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$. Thus the set of unital, completely positive maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$ with a unital, completely positive lifting to $\mathfrak{B}$ is closed in the point-norm topology on all bounded linear maps from $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$.*

Proof. Let $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the canonical quotient map. Let $\varphi : \mathcal{S} \to \mathfrak{B}/\mathfrak{J}$ be a bounded linear map such that there exists contractive (unital), completely positive maps $\psi'_n : \mathcal{S} \to \mathfrak{B}$ such that $(q \circ \psi'_n)_{n\geq 1}$ converges to φ in the point-norm topology. Clearly this implies φ is completely positive and contractive (unital). Let $\{A_k\}_{k\geq 1}$ be a dense subset of $\mathcal{S}$. Therefore, by passing to a subsequence, we may assume that $\|q(\psi'_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n$.

We claim that it suffices to construct a sequence $\psi_n : \mathcal{S} \to \mathfrak{B}$ of contractive (unital), completely positive maps such that $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n$ and $\|\psi_{n+1}(A_k) - \psi_n(A_k)\| < \frac{1}{2^{n-3}}$ for all $k \leq n-1$. If such a sequence exists, then it is clear that $(\psi_n(A_k))_{n\geq 1}$ is a Cauchy sequence for all $k \in \mathbb{N}$ and thus, as $\{A_k\}_{k\geq 1}$ is a dense subset of $\mathcal{S}$, $\psi(A) := \lim_{k\to\infty}\psi_n(A)$ exists for all $A \in \mathcal{S}$. Clearly ψ will be a contractive (unital), completely positive map (being the point-norm limit of contractive (unital), completely positive maps) and, since $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \geq 1$, $q(\psi(A_k)) = \varphi(A_k)$ for all $k \in \mathbb{N}$. Therefore, by density, $q \circ \psi = \varphi$ as desired.

To construct such a sequence, we proceed by induction. Let $\psi_1 := \psi'_1$. Suppose we have constructed $\psi_n : \mathcal{S} \to \mathfrak{B}$ such that $\|q(\psi_n(A_k)) - \varphi(A_k)\| \leq \frac{1}{2^n}$ for all $k \leq n$ and $\|\psi_n(A_k) - \psi_{n-1}(A_k)\| < \frac{1}{2^{n-3}}$ for all $k \leq n-1$. Let $(E_\lambda)_\Lambda$ be a quasicentral C*-bounded approximate identity for $\mathfrak{J}$ in $\mathfrak{B}$. Then

$$\lim_\Lambda \left\|(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}\psi_n(A)(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} + E_\lambda^{\frac{1}{2}}\psi_n(A)E_\lambda^{\frac{1}{2}} - \psi_n(A)\right\| = 0$$

for all $A \in \mathcal{S}$, and, if $B_k := \psi'_{n+1}(A_k) - \psi_n(A_k)$, then

$$\lim_\Lambda \left\|(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}B_k(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}\right\| = \|q(B_k)\| < \frac{2}{2^n}$$

if $k \leq n$. Hence there exists an $E := E_\lambda \in \mathfrak{J}$ so that

$$\left\|(I_{\mathfrak{B}} - E)^{\frac{1}{2}}\psi_n(A_k)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} + E^{\frac{1}{2}}\psi_n(A_k)E^{\frac{1}{2}} - \psi_n(A_k)\right\| < \frac{1}{2^{n+1}}$$

for all $k \leq n+1$ and $\left\|(I_{\mathfrak{B}} - E)^{\frac{1}{2}} B_k (I_{\mathfrak{B}} - E)^{\frac{1}{2}}\right\| < \frac{1}{2^{n-1}}$ for all $k \leq n$. Define $\psi_{n+1} : \mathcal{S} \to \mathfrak{B}$ by

$$\psi_{n+1}(A) := (I_{\mathfrak{B}} - E)^{\frac{1}{2}} \psi'_{n+1}(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} + E^{\frac{1}{2}} \psi_n(A) E^{\frac{1}{2}}$$

for all $A \in \mathcal{S}$. Clearly ψ_{n+1} is a completely positive map. In the contractive case, to see that ψ_{n+1} is contractive we note that ψ'_{n+1} and ψ_n are contractive maps and $(I_{\mathfrak{B}} - E) + E = I_{\mathfrak{B}}$ so $\|\psi_{n+1}(I_{\mathcal{S}})\| \leq 1$. In the unital case, to see that ψ_{n+1} is unital we note that ψ'_{n+1} and ψ_n are unital maps and $(I_{\mathfrak{B}} - E) + E = I_{\mathfrak{B}}$ so $\psi_{n+1}(I_{\mathcal{S}}) = I_{\mathfrak{B}}$. To see that ψ_{n+1} has the desired properties, we notice that $q \circ \psi_{n+1} = q \circ \psi'_{n+1}$ so $\|q(\psi_n(A_k)) - \varphi(A_k)\| < \frac{1}{2^n}$ for all $k \leq n+1$. Moreover

$$\begin{aligned} \|\psi_{n+1}(A_k) - \psi_n(A_k)\| &\leq \tfrac{1}{2^{n+1}} + \left\|(I_{\mathfrak{B}} - e)^{\frac{1}{2}} \psi'_{n+1}(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} - (I_{\mathfrak{B}} - E)^{\frac{1}{2}} \psi_n(A_k)(I_{\mathfrak{B}} - E)^{\frac{1}{2}}\right\| \\ &= \tfrac{1}{2^{n+1}} + \|B_k\| \leq \tfrac{1}{2^{n-2}}. \end{aligned}$$

for all $k \leq n$ as desired. □

The above is useful as to show that a unital, completely positive maps into a quotient is liftable, it suffices to show that the map is a limit of liftable maps in the point-norm topology. One example of this is the following (although the proof is annoying).

Lemma 9.6. *Let $\mathfrak{J}$ be an ideal in a unital C*-algebra $\mathfrak{B}$ such that for every C*-algebra $\mathfrak{C}$ the kernel of $q \otimes Id_{\mathfrak{C}} : \mathfrak{B} \otimes_{\min} \mathfrak{C} \to (\mathfrak{B}/\mathfrak{J}) \otimes_{\min} \mathfrak{C}$ is equal to $\mathfrak{J} \otimes_{\min} \mathfrak{C}$ (where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ is the canonical quotient map). If $\mathcal{S}$ is a finite dimensional operator system and $\varphi : \mathcal{S} \to \mathfrak{B}/\mathfrak{J}$ is a unital, completely positive map then φ has a unital, completely positive lifting $\psi : \mathcal{S} \to \mathfrak{B}$.*

Proof. Since $\mathcal{S}$ is finite dimensional, there exists an algebraic lifting $\psi : \mathcal{S} \to \mathfrak{B}$ of φ. By replacing $\psi(A)$ with $\frac{1}{2}(\psi(A) + \psi(A)^*)$ for all $A \in \mathcal{S}$ we may assume that ψ is a self-adjoint lifting of φ. Of course, the idea of the proof is to correct ψ.

Let $(E_\lambda)_\Lambda$ be a quasicentral C*-bounded approximate identity of $\mathfrak{J}$ in $\mathfrak{B}$. If $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ is the canonical quotient map then $q(I_{\mathfrak{B}} - E_\lambda) = I_{\mathfrak{B}/\mathfrak{J}}$ for all $\lambda \in \Lambda$. Hence $q\left((I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}\right)$ is the positive square root of $I_{\mathfrak{B}/\mathfrak{J}}$ in $\mathfrak{B}/\mathfrak{J}$ so $q\left((I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}\right) = I_{\mathfrak{B}/\mathfrak{J}}$ (alternatively, this can be obtained by taking limits of polynomials).

For each $\lambda \in \Lambda$ define $\psi_\lambda : \mathcal{S} \to \mathfrak{B}$ by

$$\psi_\lambda(A) := (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \psi(A)(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}$$

for all $A \in \mathcal{S}$. Since $q\left((I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}\right) = I_{\mathfrak{B}/\mathfrak{J}}$ and ψ is a lifting of φ, each ψ_λ is a self-adjoint lifting of φ.

We claim that if we can show $\lim_\Lambda \|\psi_\lambda\|_{cb} = 1$ then the proof will be complete. To see this, let $\epsilon > 0$. We will use our completely bounded norm estimates to correct ψ. By our assumptions on the limit, we can assume that $\|\psi\|_{cb} \leq 1 + \epsilon$ by replacing ψ with one of the self-adjoint liftings ψ_λ of φ. Let $\phi : \mathcal{S} \to \mathbb{C}$ be an arbitrary state. For each $\lambda \in \Lambda$ define $\psi'_\lambda : \mathcal{S} \to \mathfrak{B}$ by

$$\psi'_\lambda(A) := (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \psi(A)(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} + \phi(A) E_\lambda$$

for all $A \in \mathcal{S}$. Clearly each ψ'_λ is a self-adjoint lifting of φ. Since the map

$$B_1 \oplus \alpha \mapsto (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} B_1 (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} + \alpha E_\lambda$$

from $\mathfrak{B} \oplus \mathbb{C}$ to $\mathfrak{B}$ is a unital linear map that is the sum of two completely positive maps, it is a unital, completely positive map. Since each ψ'_λ is the composition of the above map with the map $\psi \oplus \phi : \mathcal{S} \to \mathfrak{B} \oplus \mathfrak{B}$, ψ'_λ is a self-adjoint, completely bounded map with $\|\psi'\|_{cb} \leq \|\psi\|_{cb} \leq 1 + \epsilon$.

Notice that

$$\begin{aligned} I_{\mathfrak{B}} - \psi'_\lambda(I_{\mathcal{S}}) &= I_{\mathfrak{B}} - (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \psi(I_{\mathcal{S}})(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} - E_\lambda \\ &= (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} (I_{\mathfrak{B}} - \psi(I_{\mathcal{S}}))(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}}. \end{aligned}$$

However, since ψ is a lifting of φ and φ is unital, $I_{\mathfrak{B}} - \psi(I_{\mathcal{S}}) \in \mathfrak{J}$. Hence, as $(E_\lambda)_\Lambda$ is quasicentral, the above implies

$$\lim_{\Lambda} I_{\mathfrak{B}} - \psi'_\lambda(I_{\mathcal{S}}) = 0.$$

Choose $\lambda_0 \in \Lambda$ such that $\left\|I_{\mathfrak{B}} - \psi'_{\lambda_0}(I_{\mathcal{S}})\right\| < \epsilon$. Define $\psi'' : \mathcal{S} \to \mathfrak{B}$ by

$$\psi''(A) := \psi'_{\lambda_0}(A) + (I_{\mathfrak{B}} - \psi'_{\lambda_0}(I_{\mathcal{S}}))\phi(A)$$

for all $A \in \mathcal{S}$. Since ψ'_{λ_0} is a lifting for φ and $\left\|I_{\mathfrak{B}} - \psi'_{\lambda_0}(I_{\mathcal{S}})\right\| < \epsilon$ by the above computations, it is clear that $\|\varphi - q \circ \psi''\| < \epsilon$. Note ψ'' is a unital, self-adjoint completely bounded map such that

$$\|\psi''\|_{cb} \leq 1 + \epsilon + \left\|I_{\mathfrak{B}} - \psi'_{\lambda_0}(I_{\mathcal{S}})\right\| \leq 1 + 2\epsilon.$$

By Theorem 9.4 there exists a unital, completely positive map $\theta : \mathcal{S} \to \mathfrak{B}$ such that $\|\theta - \psi''\|_{cb} \leq 4\epsilon$. Thus $q \circ \theta : \mathcal{S} \to \mathfrak{B}/\mathfrak{J}$ is a unital, completely positive map with a unital, completely positive lifting $\theta : \mathcal{S} \to \mathfrak{B}$ such that $\|q \circ \theta - \varphi\| \leq 5\epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, Lemma 9.5 implies φ has a unital, completely positive lifting to $\mathfrak{B}$.

Therefore, to complete the proof, it suffices to show that $\lim_\Lambda \|\psi_\lambda\|_{cb} = 1$. Suppose otherwise that $\lim_\Lambda \|\psi_\lambda\|_{cb} \neq 1$. Then there exists an $\epsilon > 0$ such that $\limsup_\Lambda \|\psi_\lambda\|_{cb} > 1 + 4\epsilon$. By replacing ψ_λ with a subnet, we may assume that $\|\psi_\lambda\|_{cb} \geq 1 + 2\epsilon$ for all $\lambda \in \Lambda$. Hence for each $\lambda \in \Lambda$ there exists an $n_\lambda \in \mathbb{N}$ and an $A_\lambda \in \mathcal{M}_{n_\lambda}(\mathcal{S})$ such that $\|A_\lambda\| = 1$ and $\|(\psi_\lambda)_{n_\lambda}(A_\lambda)\| \geq \|\psi_\lambda\|_{cb} - \epsilon \geq 1 + \epsilon$.

Now we will use this sequence of matrix algebras of $\mathcal{S}$ to construct an operator in a C*-algebra such that we can use the exact sequence condition to obtain a contradiction. Consider $\mathcal{S}_0 := \prod_\Lambda(\mathcal{M}_{n_\lambda}(\mathcal{S}))$. Therefore $\mathcal{S}_0$ is an operator system on $\prod_\Lambda(\mathcal{H}\otimes\mathbb{C}^{n_\lambda}) \simeq \mathcal{H}\otimes(\prod_\Lambda \mathbb{C}^{n_\lambda})$ (where $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$). Under this unitary equivalence of Hilbert spaces, we claim that $\mathcal{S}_0$ is $\mathcal{S} \odot \mathfrak{C}$ where $\mathfrak{C} := \prod_\Lambda \mathcal{M}_{n_\lambda}(\mathbb{C})$. To see this, we notice that $\mathcal{S} \odot \mathfrak{C}$ is clearly a subspace of $\mathcal{S}_0$ under this unitary equivalence. To see the other direction, let $\{e_1, \ldots, e_m\}$ be a basis of unit vectors for $\mathcal{S}$ and let $\{f_1, \ldots, f_m\}$ be the dual basis. Therefore the maps $(f_j)_{n_\lambda} : \mathcal{M}_{n_\lambda}(\mathcal{S}) \to \mathcal{M}_{n_\lambda}(\mathbb{C})$ are completely bounded and, if $T = \sum_{k=1}^m e_k \otimes A_k \in \mathcal{M}_{n_\lambda}(\mathcal{S})$, then $A_k = (f_k)_{n_\lambda}(T)$. Therefore, if $(y_\lambda)_\Lambda \in \mathcal{S}_0$ then

$$(y_\lambda)_\Lambda = \left(\sum_{k=1}^m e_k \otimes (f_k)_{n_\lambda}(y_\lambda)\right)_\Lambda = \sum_{k=1}^m e_k \otimes ((f_k)_{n_\lambda}(y_\lambda))_\Lambda$$

which is an element of $\mathcal{S} \odot \mathfrak{C}$ as each $(f_k)_{n_\lambda}$ is completely bounded. Hence we will view $\mathcal{S}_0$ as $\mathcal{S} \odot \mathfrak{C}$.

Let $A := \sum_{k=1}^m (e_k \otimes ((f_k)_{n_\lambda}(A_\lambda))_\Lambda) \in \mathcal{S}_0$ (so A corresponds to the operator $(A_\lambda)_\Lambda$ which has norm at most 1). Then for any $\nu \in \Lambda$

$$\begin{aligned} (\psi_\nu \otimes Id_{\mathfrak{C}})(A) &= \textstyle\sum_{k=1}^m (\psi_\nu(e_k) \otimes ((f_k)_{n_\lambda}(A_\lambda))_\Lambda) \\ &= (\textstyle\sum_{k=1}^m \psi_\nu(e_k) \otimes (f_k)_{n_\lambda}(A_\lambda))_\Lambda \\ &= ((\psi_\nu)_{n_\lambda}(A_\lambda))_\Lambda \end{aligned}$$

so

$$\|(\psi_\nu \otimes Id_{\mathfrak{C}})(A)\| \geq \|(\psi_\nu)_{n_\nu}(A_\nu)\| \geq 1 + \epsilon.$$

However

$$\begin{aligned} (\psi_\nu \otimes Id_{\mathfrak{C}})(A) &= ((\psi_\nu)_{n_\lambda}(A_\lambda))_\Lambda \\ &= \left(\left((I_{\mathfrak{B}} - E_\nu)^{\frac{1}{2}} \otimes I_{n_\lambda}\right)\psi_{n_\lambda}(A_\lambda)\left((I_{\mathfrak{B}} - E_\nu)^{\frac{1}{2}} \otimes I_{n_\lambda}\right)\right)_\Lambda \\ &= \left((I_{\mathfrak{B}} - E_\nu)^{\frac{1}{2}} \otimes I_{\mathfrak{C}}\right)(\psi \otimes Id_{\mathfrak{C}})(A)\left((I_{\mathfrak{B}} - E_\nu)^{\frac{1}{2}} \otimes I_{\mathfrak{C}}\right). \end{aligned}$$

However, it is clear that $(E_\lambda \otimes I_{\mathfrak{C}})_\Lambda$ is a quasicentral C*-bounded approximate identity of $\mathfrak{J} \otimes_{\min} \mathfrak{C}$ inside $\mathfrak{B} \otimes_{\min} \mathfrak{C}$ (that is, first check it on the span of the elementary tensors). Therefore

$$\begin{aligned} \limsup_\nu \|(\psi_\nu \otimes Id_{\mathfrak{C}})(A)\| &= \|(\psi \otimes Id_{\mathfrak{C}})(A) + \mathfrak{J} \otimes_{\min} \mathfrak{C}\| \\ &= \|(q \otimes I_{\mathfrak{C}})((\psi \otimes Id_{\mathfrak{C}})(A))\| \\ &= \|(\varphi \otimes Id_{\mathfrak{C}})(A)\| \\ &\leq 1 \end{aligned}$$

(as $\|A\| \leq 1$ and $\varphi \otimes Id_{\mathfrak{C}}$ is a unital, completely positive map) which is a contradiction. Hence the result is complete. $\square$

Although the converse of the above result is true, we shall not present the proof as we do not require it. Next we will look at a weak form of injectivity that will enable us to lift completely positive map into quotients by ideals with this property.

Definition 9.7. Let $\mathfrak{A}$ be a C*-algebra. We say that $\mathfrak{A}$ is approximately injective if for every finite dimensional operator systems $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{B}(\mathcal{H})$, any completely positive map $\varphi_1 : \mathcal{S}_1 \to \mathfrak{A}$, and any $\epsilon > 0$ there exists a completely positive map $\varphi_2 : \mathcal{S}_2 \to \mathfrak{A}$ such that $\|\varphi_2|_{\mathcal{S}_1} - \varphi_1\| < \epsilon$.

Clearly every injective C*-algebra and every nuclear C*-algebra is approximately injective. Our goal is to upgrade Lemma 9.6 from finite dimensional operator systems to separable operator systems provided that our ideals $\mathfrak{J}$ are approximately injective. We proceed with the following two results.

Lemma 9.8. *Let $\mathfrak{J}$ be an approximately injective ideal in a unital C*-algebra $\mathfrak{B}$ such that for every unital C*-algebra $\mathfrak{C}$ the kernel of $q \otimes Id_{\mathfrak{C}} : \mathfrak{B} \otimes_{\min} \mathfrak{C} \to (\mathfrak{B}/\mathfrak{J}) \otimes_{\min} \mathfrak{C}$ is equal to $\mathfrak{J} \otimes_{\min} \mathfrak{C}$ (where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ is the canonical quotient map).*

Let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{B}(\mathcal{H})$ be finite dimensional operator systems (with the same unit as $\mathcal{B}(\mathcal{H})$) and $\varphi_2 : \mathcal{S}_2 \to \mathfrak{B}/\mathfrak{J}$ be a unital, completely positive map. If the restriction $\varphi_1 = \varphi_2|_{\mathcal{S}_1}$ has a unital, completely positive lifting $\psi_1 : \mathcal{S}_1 \to \mathfrak{B}$ then for any $\epsilon > 0$ there exists a unital, completely positive lifting $\psi_2 : \mathcal{S}_2 \to \mathfrak{B}$ of φ_2 such that $\|\psi_2|_{\mathcal{S}_1} - \psi_1\| < \epsilon$.

Proof. The idea of the proof is to use Lemma 9.6 to get a lifting of φ_2 and then use the approximate injectivity of $\mathfrak{J}$ to correct this lifting. Let $(E_\lambda)_\Lambda$ be a quasicentral C*-bounded approximate identity for $\mathfrak{J}$ inside $\mathfrak{B}$. By Lemma 9.6 there exists a unital, completely positive lifting $\psi : \mathcal{S}_2 \to \mathfrak{B}$ such that $q \circ \psi = \psi_2$. If $\psi_1 : \mathcal{S}_1 \to \mathfrak{B}$ is a unital, completely positive lifting of $\varphi_1 = \varphi_2|_{\mathcal{S}_1}$, then $\psi_1(A) - \psi(A) \in \mathfrak{J}$ for all $A \in \mathcal{S}_1$. Therefore, since $(E_\lambda)_\Lambda$ is a quasicentral C*-bounded approximate identity for $\mathfrak{J}$ inside $\mathfrak{B}$,

$$\lim_\Lambda \left\| (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} (\psi_1(A) - \psi(A))(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \right\| = 0$$

for all $A \in \mathcal{S}_1$ and

$$\lim_\Lambda \left\| \psi_1(A) - (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \psi_1(A)(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} - E_\lambda^{\frac{1}{2}} \psi_1(A) E_\lambda^{\frac{1}{2}} \right\| = 0$$

for all $A \in \mathcal{S}_1$. Hence

$$\lim_\Lambda \left\| \psi_1(A) - (I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} \psi(A)(I_{\mathfrak{B}} - E_\lambda)^{\frac{1}{2}} - E_\lambda^{\frac{1}{2}} \psi_1(A) E_\lambda^{\frac{1}{2}} \right\| = 0$$

for all $A \in \mathcal{S}_1$.

Fix $0 < \delta < 1$. Therefore, since $\mathcal{S}_1$ is finite dimensional, there exists a $\lambda \in \Lambda$ such that if $E := E_\lambda$ then

$$\left\| \psi_1(A) - (I_{\mathfrak{B}} - E)^{\frac{1}{2}} \psi(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} - E^{\frac{1}{2}} \psi_1(A) E^{\frac{1}{2}} \right\| \leq \delta \|A\|$$

for all $A \in \mathcal{S}_1$. However, since $E \in \mathfrak{J}$ and $\mathfrak{J}$ is an ideal of $\mathfrak{B}$, the map $A \mapsto E^{\frac{1}{2}} \psi_1(A) E^{\frac{1}{2}}$ is a contractive, completely positive map of $\mathcal{S}_1$ into $\mathfrak{J}$. Therefore, since $\mathfrak{J}$ is approximately injective, there exists a completely positive map $\theta : \mathcal{S}_2 \to \mathfrak{J}$ such that

$$\left\| \theta(A) - E^{\frac{1}{2}} \psi_1(A) E^{\frac{1}{2}} \right\| < \delta \|A\|$$

for all $A \in \mathcal{S}_1$. Since ψ_1 is unital, $\|\theta(I_{\mathcal{H}}) - E\| \leq \delta$.

Define $\psi' : \mathcal{S}_2 \to \mathfrak{B}$ by

$$\psi'(A) := (I_{\mathfrak{B}} - E)^{\frac{1}{2}} \psi(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} + \theta(A)$$

for all $A \in \mathcal{S}_2$. Since $\theta(A) \in \mathfrak{J}$ for all $A \in \mathcal{S}_2$, it is clear that $q \circ \psi' = q \circ \psi = \varphi_2$. Moreover

$$\|\psi_1(A) - \psi'(A)\| \leq \left\|\theta(A) - E^{\frac{1}{2}}\psi_1(A)E^{\frac{1}{2}}\right\| + \left\|\psi_1(A) - (I_{\mathfrak{B}} - E)^{\frac{1}{2}}\psi(A)(I_{\mathfrak{B}} - E)^{\frac{1}{2}} - E^{\frac{1}{2}}\psi_1(A)E^{\frac{1}{2}}\right\| \leq 2\delta \|A\|$$

for all $A \in \mathcal{S}_1$ and

$$\|I_{\mathfrak{B}} - \psi'(I_{\mathcal{H}})\| = \|I_{\mathfrak{B}} - ((I_{\mathfrak{B}} - E) + \theta(I_{\mathcal{H}}))\| \leq \delta < 1.$$

Therefore, if $B := \psi'(I_{\mathcal{H}})$, then B is invertible with $\left\|B^{-\frac{1}{2}}\right\| \leq \frac{1}{\sqrt{1-\delta}}$. Therefore, the map $\psi_2 : \mathcal{S}_2 \to \mathfrak{B}$ defined by

$$\psi_2(A) := B^{-\frac{1}{2}}\psi'(A)B^{-\frac{1}{2}}$$

for all $A \in \mathcal{S}_2$ is clearly a unital, completely positive map. To see that ψ_2 is a lifting of φ_2, we notice that

$$q(B) = q(\psi'(I_{\mathcal{H}})) = \varphi_2(I_{\mathcal{H}}) = I_{\mathcal{H}}$$

(as ψ' was a lifting of φ_2) so $q(B^{-\frac{1}{2}}) = I_{\mathcal{H}}$. Hence it is clear that ψ_2 is a lifting of φ_2. Moreover

$$\begin{aligned} \|\psi_2(A) - \psi'(A)\| &\leq \left(\left\|I_{\mathfrak{B}} - B^{-\frac{1}{2}}\right\| \|\varphi'\| + \left\|B^{-\frac{1}{2}}\right\| \|\varphi'\| \left\|I_{\mathfrak{B}} - B^{-\frac{1}{2}}\right\|\right) \|A\| \\ &\leq \left((1+\delta)\left(1 - \tfrac{1}{\sqrt{1+\delta}}\right) + (1+\delta)\left(1 - \tfrac{1}{\sqrt{1+\delta}}\right)\tfrac{1}{\sqrt{1-\delta}}\right) \|A\| \end{aligned}$$

for all $A \in \mathcal{S}_1$ so, as $\|\psi_1(A) - \psi'(A)\| \leq 2\delta \|A\|$ for all $A \in \mathcal{S}_1$,

$$\|\psi_2(A) - \psi_1(A)\| < \epsilon$$

for all $A \in \mathcal{S}_1$ by making δ suitably small. ☐

With the above result in hand, the following is simply to apply Lemma 9.8 recursively.

Lemma 9.9. *Let $\mathfrak{J}$ be an approximately injective ideal in a unital C^*-algebra $\mathfrak{B}$ such that for every unital C^*-algebra $\mathfrak{C}$ the kernel of $q \otimes Id_{\mathfrak{C}} : \mathfrak{B} \otimes_{\min} \mathfrak{C} \to (\mathfrak{B}/\mathfrak{J}) \otimes_{\min} \mathfrak{C}$ is equal to $\mathfrak{J} \otimes_{\min} \mathfrak{C}$ (where $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ is the canonical quotient map). Then any unital, completely positive map φ from a separable operator system $\mathcal{S}$ into $\mathfrak{B}/\mathfrak{J}$ has a unital, completely positive lifting $\psi : \mathcal{S} \to \mathfrak{B}$.*

Proof. Let $\mathcal{S}$ be a separable operator system and let $\varphi : \mathcal{S} \to \mathfrak{B}/\mathfrak{J}$ be a unital, completely positive map. Choose a sequence $(A_n)_{n\geq 1}$ of self-adjoint elements of $\mathcal{S}$ with $A_1 := I$ and dense span in $\mathcal{S}$. For each $k \in \mathbb{N}$ let $\mathcal{S}_k := span\{A_1, A_2, \ldots, A_k\}$. Therefore, each $\mathcal{S}_k$ is a finite dimensional operator system with $\mathcal{S}_k \subseteq \mathcal{S}_{k+1}$ for all $k \in \mathbb{N}$.

By Lemma 9.8 there exists a sequence $\psi_n : \mathcal{S}_n \to \mathfrak{B}$ of unital, completely positive maps such that $q \circ \psi_n = \varphi|_{\mathcal{S}_n}$ and $\|\psi_{n+1}(A) - \psi_n(A)\| \leq \frac{1}{2^n}\|A\|$ for all $A \in \mathcal{S}_n$. Therefore, since $\bigcup_{k\geq 1}\mathcal{S}_k$ is dense in $\mathcal{S}$, the inequalities $\|\psi_{n+1}(A) - \psi_n(A)\| \leq \frac{1}{2^n}\|A\|$ for all $A \in \mathcal{S}_n$ imply that $\psi(A) := \lim_{k\to\infty}\psi_k(A)$ exists for all $A \in \bigcup_{k\geq 1}\mathcal{S}_k$ and extends to a linear map on $\mathcal{S}$ by continuity. Since each ψ_k is a unital, completely positive map, ψ is a unital, completely positive map. Moreover, since $q \circ \psi_n = \varphi|_{\mathcal{S}_n}$, it is clear that ψ is a lifting of φ by density. ☐

Finally we verify that the conditions of the above lemma can be reduced to assuming $\mathfrak{C} = \mathcal{B}(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$ provided φ is a *-homomorphism.

Theorem 9.10. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital C^*-algebras with $\mathfrak{A}$ separable, let $\mathfrak{J}$ be an approximately injective ideal of $\mathfrak{B}$, and let $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ be an injective, unital *-homomorphism. Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space and suppose that the induced map of algebraic tensor products*

$$\mathfrak{A} \odot \mathcal{B}(\mathcal{H}) \to \frac{\mathfrak{B} \odot \mathcal{B}(\mathcal{H})}{\mathfrak{J} \odot \mathcal{B}(\mathcal{H})}$$

*extends continuously to a (necessarily injective) unital *-homomorphism*

$$\mathfrak{A} \otimes_{\min} \mathcal{B}(\mathcal{H}) \to \frac{\mathfrak{B} \otimes_{\min} \mathcal{B}(\mathcal{H})}{\mathfrak{J} \otimes_{\min} \mathcal{B}(\mathcal{H})}.$$

Then there exists a unital, completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$ which lifts φ.

Proof. Let $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the canonical quotient map. We claim that we may assume $\mathfrak{A} = \mathfrak{B}/\mathfrak{J}$ and $\varphi = id_{\mathfrak{B}/\mathfrak{J}}$. Indeed let $\mathfrak{B}_0 := q^{-1}(\varphi(\mathfrak{A})) \subseteq \mathfrak{B}$ which is a C*-algebra (as φ is a *-homomorphism). Since the minimal tensor product preserves inclusions,

$$\mathfrak{J} \otimes_{\min} \mathcal{B}(\mathcal{H}) \subseteq \mathfrak{B}_0 \otimes_{\min} \mathcal{B}(\mathcal{H}) \subseteq \mathfrak{B} \otimes_{\min} \mathcal{B}(\mathcal{H})$$

so the hypotheses of the lemma still hold with $\mathfrak{B}_0$ replacing $\mathfrak{B}$.

By our above assumptions, it suffices to prove that for every unital C*-algebra $\mathfrak{C}$ the kernel of $q \otimes Id_{\mathfrak{C}} : \mathfrak{B} \otimes_{\min} \mathfrak{C} \to (\mathfrak{B}/\mathfrak{J}) \otimes_{\min} \mathfrak{C}$ is equal to $\mathfrak{J} \otimes_{\min} \mathfrak{C}$. If $\mathfrak{C}$ is separable, then we may view $\mathfrak{C}$ as a unital C*-subalgebra of $\mathcal{B}(\mathcal{H})$. Therefore, since the kernel of

$$\mathfrak{B} \otimes_{\min} \mathfrak{C} \subseteq \mathfrak{B} \otimes_{\min} \mathcal{B}(\mathcal{H}) \to (\mathfrak{B}/\mathfrak{J}) \otimes_{\min} \mathcal{B}(\mathcal{H})$$

is precisely $(\mathfrak{B} \otimes_{\min} \mathfrak{C}) \cap (\mathfrak{J} \otimes_{\min} \mathcal{B}(\mathcal{H})) = \mathfrak{J} \otimes_{\min} \mathfrak{C}$ by considering the C*-bounded approximate identity $(E_\lambda \otimes I)_\Lambda$ for $\mathfrak{J} \otimes_{\min} \mathcal{B}(\mathcal{H})$ (where $(E_\lambda)_\Lambda$ is any C*-bounded approximate identity for $\mathfrak{J}$) the separable case is complete.

For a general C*-algebra $\mathfrak{C}$, we need only show that $ker(q \circ Id_{\mathfrak{C}}) \subseteq \mathfrak{J} \otimes_{\min} \mathfrak{C}$. If $T \in ker(q \otimes Id_{\mathfrak{C}})$ then there exists a separable C*-subalgebra $\mathfrak{C}_0 \subseteq \mathfrak{C}$ such that $T \in \mathfrak{B} \otimes_{\min} \mathfrak{C}_0 \subseteq \mathfrak{B} \otimes_{\min} \mathfrak{C}$. Hence $T \in ker(q \otimes Id_{\mathfrak{C}_0})$ so $T \in \mathfrak{J} \otimes_{\min} \mathfrak{C}_0 \subseteq \mathfrak{J} \otimes_{\min} \mathfrak{C}$ by the separable case. Hence the proof is complete. $\square$

To complete this chapter, we desire to show that every unital completely positive map from a unital, separable, nuclear C*-algebra into a quotient C*-algebra has an unital completely positive lifting. The easiest way to prove this is to use Lemma 9.5, nuclearity, and show that algebraic liftings of unital, completely positive maps from matrix algebras can be taken to be positive. We begin with the following result.

Lemma 9.11. *Let $\mathfrak{B}$ be a unital C*-algebra, let $\phi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}$ be a linear map, and let $\{E_{i,j}\}_{i,j=1}^n$ denote the standard matrix units for $\mathcal{M}_n(\mathbb{C})$. Then the following are equivalent:*

1. *ϕ is completely positive.*
2. *ϕ is n-positive.*
3. *$[\phi(E_{i,j})]$ is positive in $\mathcal{M}_n(\mathfrak{B})$.*

Proof. It is clear that 1) implies 2). To see that 2) implies 3), we notice that $[E_{i,j}] \in \mathcal{M}_n(\mathcal{M}_n(\mathbb{C}))$ is self-adjoint (as $[E_{i,j}]^* = [E_{j,i}^*] = [E_{i,j}]$) and $[E_{i,j}]^2 = [\sum_{k=1}^n E_{i,k}E_{k,j}] = n[E_{i,j}]$. Hence $z^2 - nz = 0$ on $\sigma([E_{i,j}])$ and thus $\sigma([E_{i,j}]) \subseteq \{0, n\}$. Hence $[E_{i,j}]$ is positive. Therefore, since ϕ is n-positive, $\phi([E_{i,j}])$ is positive in $\mathcal{M}_n(\mathfrak{B})$.

Suppose 3) holds. Let $k \in \mathbb{N}$ be arbitrary. Without loss of generality we may assume $\mathfrak{B} \subseteq \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. To show that ϕ is k-positive, let $A_1, \ldots, A_k \in \mathcal{M}_n(\mathbb{C})$ be arbitrary. Since $A_s \in \mathcal{M}_n(\mathbb{C})$, there exists $a_{i,j,s} \in \mathbb{C}$ ($s \in \{1, \ldots k\}$, $i, j \in \{1, \ldots n\}$) such that $A_s = \sum_{i,j=1}^n a_{i,j,s}E_{i,j}$. Thus, a simple computation shows that

$$A_i^* A_j = \left(\sum_{l,m=1}^n \overline{a_{l,m,i}} E_{m,l} \right) \left(\sum_{s,t=1}^n a_{s,t,j} E_{s,t} \right) = \sum_{l,m,t=1}^n \overline{a_{l,m,i}} a_{l,t,j} E_{m,t}.$$

Fix $h = (h_1, h_2, \dots, h_k) \in \mathcal{H}^{\oplus k}$ and let $x_{l,m} := \sum_{j=1}^{k} a_{l,m,j} h_j \in \mathcal{H}$ for $l, m = \{1, \dots n\}$. Then

$$\begin{aligned}
\sum_{i,j=1}^{k} \langle \phi(A_i^* A_j) h_j, h_i \rangle &= \sum_{i,j=1}^{k} \sum_{l,m,t=1}^{n} \langle \phi(\overline{a_{l,m,i}} a_{l,t,j} E_{m,t}) h_j, h_i \rangle \\
&= \sum_{i,j=1}^{k} \sum_{l,m,t=1}^{n} \langle \phi(E_{m,t}) a_{l,t,j} h_j, a_{l,m,i} h_i \rangle \\
&= \sum_{i=1}^{k} \sum_{l,m,t=1}^{n} \langle \phi(E_{m,t}) x_{l,t}, a_{l,m,i} h_i \rangle \\
&= \sum_{l,m,t=1}^{n} \langle \phi(E_{m,t}) x_{l,t}, x_{l,m} \rangle \\
&= \sum_{l=1}^{n} \sum_{m,t=1}^{n} \langle \phi(E_{m,t}) x_{l,t}, x_{l,m} \rangle.
\end{aligned}$$

However, $[\phi(E_{i,j})]$ is positive in $\mathcal{M}_n(\mathfrak{B})$ and hence $\sum_{m,t=1}^{n} \langle \phi(E_{m,t}) x_{l,t}, x_{l,m} \rangle = \langle \phi_n([E_{m,t}]) x, x \rangle \geq 0$ where $x = (x_{l,1}, \dots, x_{l,n}) \in \mathcal{H}^n$. Hence, since the sum of positive numbers is positive, $\sum_{i,j=1}^{k} \langle \phi(A_i^* A_j) h_j, h_i \rangle \geq 0$. Hence ϕ is k-positive and, as k was arbitrary, ϕ is completely positive as desired. □

Theorem 9.12. *Let $\mathfrak{A}$ be a unital, separable, nuclear C*-algebra, let $\mathfrak{B}$ be a unital C*-algebra, let $\mathfrak{J}$ be an ideal of $\mathfrak{B}$, and let $q : \mathfrak{B} \to \mathfrak{B}/\mathfrak{J}$ be the canonical quotient map. Then for every unital, completely positive map $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ there exists a unital, completely positive map $\Phi : \mathfrak{A} \to \mathfrak{B}$ such that $q \circ \Phi = \varphi$.*

Proof. Let $\varphi : \mathfrak{A} \to \mathfrak{B}/\mathfrak{J}$ be a unital, completely positive map. Since $\mathfrak{A}$ is separable, Lemma 9.5 implies that the set of unital, completely positive maps from $\mathfrak{A}$ into $\mathfrak{B}/\mathfrak{J}$ that have liftings is closed in the point-norm topology. Thus it suffices to show that φ is a point-norm limit of unital, completely positive maps into $\mathfrak{B}/\mathfrak{J}$ with unital, completely positive liftings. Since $\mathfrak{A}$ is nuclear, φ is a point-norm limit of unital completely positive maps of the form $\psi \circ \phi$ where $\phi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ and $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}/\mathfrak{J}$ are unital, completely positive maps. If we can show that ψ has a lifting to a unital, completely positive map, then $\psi \circ \phi$ has a lifting to a unital, completely positive map and thus we are done by Lemma 9.5.

To see that ψ has a completely positive lifting, let $\{E_{i,j}\}_{i,j=1}^{n}$ denote the standard matrix units for $\mathcal{M}_n(\mathbb{C})$. Note that $[\psi(E_{i,j})] \in \mathcal{M}_n(\mathfrak{B}/\mathfrak{J}) \simeq \mathcal{M}_n(\mathfrak{B})/\mathcal{M}_n(\mathfrak{J})$ is positive. Therefore, standard functional calculus results imply that there exists a positive matrix $[B_{i,j}] \in \mathcal{M}_n(\mathfrak{B})$ such that $q_n([B_{i,j}]) = [\varphi(E_{i,j})]$. Define $\Psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{B}$ by $\Psi([a_{i,j}]) := \sum_{i,j=1}^{n} a_{i,j} B_{i,j}$ for all $[a_{i,j}] \in \mathcal{M}_n(\mathbb{C})$. Clearly Ψ is a linear map. Notice that $\Psi_n([E_{i,j}]) = [B_{i,j}] \geq 0$ so Ψ is a completely positive map by Lemma 9.11. Moreover

$$q(\Psi([a_{i,j}])) = q\left(\sum_{i,j=1}^{n} a_{i,j} B_{i,j} \right) = \sum_{i,j=1}^{n} a_{i,j} \varphi(E_{i,j}) = \psi([a_{i,j}])$$

so Ψ is a lifting of ψ.

However, Ψ need not be unital. To fix this, we notice that $q(\Psi(I_n)) = \psi(I_n) = I_{\mathfrak{B}/\mathfrak{J}}$. Since $\Psi(I_n)$ is self-adjoint, $\Psi(I_n) = I_{\mathfrak{B}} + A$ where $A \in \mathfrak{J}_{sa}$. Using the Continuous Functional Calculus, write $A = A_+ - A_-$ where $A_+, A_- \in \mathfrak{J}_+$ are such that $A_+ A_- = 0$. Let $f : \mathcal{M}_n(\mathbb{C}) \to \mathbb{C}$ be any state on $\mathcal{M}_n(\mathbb{C})$ and define $\Psi' : \mathcal{M}_n(\mathbb{C})) \to \mathfrak{B}$ by

$$\Psi'(T) := (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} (\Psi(T) + f(T) A_-)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}$$

for all $T \in \mathcal{M}_n(\mathbb{C}))$. Clearly

$$\Psi'(I_n) = (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} (\Psi(I_n) + A_-)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} = (I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} (I_{\mathfrak{B}} + A_+)(I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}} = I_{\mathfrak{B}}$$

so Ψ' is a unital, completely positive map. Since $q\left((I_{\mathfrak{B}} + A_+)^{-\frac{1}{2}}\right) = I_{\mathfrak{B}/\mathfrak{J}}$ and

$$q(\Psi(T) + f(T)A_-) = q(\Psi(T)) = \psi(T)$$

for all $T \in \mathcal{M}_n(\mathbb{C})$, Ψ' is the desired unital, completely positive lifting of ψ. $\square$

10 Completely Positive Maps on Purely Infinite C*-Algebras

In this chapter we will develop some theory on completely positive maps between unital, simple, purely infinite C*-algebra. In particular, we desire to prove the opposite of Theorem 6.12: any two injective *-homomorphisms from a unital, separable, exact C*-algebra into $\mathcal{O}_2$ are approximately unitarily equivalent.

Most of the results for this chapter were developed from the paper [KP]

To begin, we need a slightly technical result relating to the polar decomposition of non-invertible operators in a C*-algebra.

Lemma 10.1. *Let $\mathfrak{A}$ be a C*-algebra, let $P \in \mathfrak{A}$ be a projection, and let $A \in \mathfrak{A}$ be such that $AP = A$ and $\|A^*A - P\| < 1$. Then $T := (PA^*AP)^{-\frac{1}{2}}$ exists in $P\mathfrak{A}P$ and $V := AT$ is a partial isometry in $\mathfrak{A}$ such that $V^*V = P$. Moreover*

$$\|V - A\| \leq 1 - (1 - \|A^*A - P\|)^{\frac{1}{2}} \leq \|A^*A - P\|.$$

Proof. Since $AP = A$, $PA^*AP = A^*A$. Therefore, since $\|A^*A - P\| < 1$, PA^*AP is an invertible positive operator in $P\mathfrak{A}P$ and thus T exists. Let $V := AT$. Then, as $T \in P\mathfrak{A}P$,

$$V^*V = TA^*AT = TPA^*APT = (PA^*AP)^{-\frac{1}{2}}PA^*AP(PA^*AP)^{-\frac{1}{2}} = P$$

as claimed. Hence V is a partial isometry in $\mathfrak{A}$.

To obtain the norm estimates, we notice that $1-x \leq (1-x)^{\frac{1}{2}}$ for all $x \in [0, 1)$. Hence, as $\|A^*A - P\| < 1$, the inequality $1 - (1 - \|A^*A - P\|)^{\frac{1}{2}} \leq \|A^*A - P\|$ is trivial. To obtain the other inequality, we notice that

$$\begin{aligned}
\|AT - A\|^2 &= \|AP(T - P)\|^2 \\
&= \|(T - P)(PA^*AP)(T - P)\| \\
&= \left\|(P - (PA^*AP)^{\frac{1}{2}})^2\right\| \\
&= \left\|P - (PA^*AP)^{\frac{1}{2}}\right\|^2
\end{aligned}$$

so $\|AT - A\| = \left\|P - (PA^*AP)^{\frac{1}{2}}\right\|$. However, $\|A^*A - P\| = \|PA^*AP - P\|$ so

$$P - \|A^*A - P\| \leq PA^*AP \leq P + \|A^*A - P\|$$

and thus

$$\begin{aligned}
\|AT - A\| &\leq \sup\{|1 - \sqrt{x}| \mid x \in [1 - \|A^*A - P\|, 1 + \|A^*A - P\|]\} \\
&= \max\{1 - (1 - \|A^*A - P\|)^{\frac{1}{2}}, (1 + \|A^*A - P\|)^{\frac{1}{2}} - 1\}.
\end{aligned}$$

A moment of consideration about the calculus of $x \mapsto \sqrt{1+x}$ show that the difference $1 - \sqrt{1-\alpha}$ is larger than the difference of $\sqrt{1+\alpha} - 1$ for all $\alpha \in [0, 1)$ and thus $\|AT - A\| \leq 1 - (1 - \|A^*A - P\|)^{\frac{1}{2}}$ as desired. □

The following lemma is fairly technical. The idea behind the proof is to consider $\mathcal{M}_n(\mathfrak{A})$ (which is a unital, simple, purely infinite C*-algebra if $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra by Theorem 3.11), excise a certain state (via Theorem 8.8), use equivalent projections to create the correct partial isometries, and then cut back down to $\mathfrak{A}$.

Lemma 10.2. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra, let $\varphi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ be a unital, completely positive map, and let $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{A}$ be a *-homomorphism. Then for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{A}$ there exists a partial isometry $V \in \mathfrak{A}$ such that $V^*V = \psi(I_n)$ and $\|V^*AV - \psi(\varphi(A))\| < \epsilon$ for all $A \in \mathcal{F}$.*

Proof. Let $\epsilon > 0$ be arbitrary and let $\mathcal{F}$ be an arbitrary finite subset of $\mathfrak{A}$. Without loss of generality, we may suppose $I_{\mathfrak{A}} \in \mathcal{F}$ and every element of $\mathcal{F}$ has norm at most one. Let $\delta = \min\left\{\frac{1}{n^3}, \frac{\epsilon}{4n^3}\right\} > 0$.

Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis on $\mathbb{C}^n$ and let $\{E_{i,j}\}_{i,j=1}^n$ be the canonical matrix units. Since $\varphi(I_{\mathfrak{A}}) = I_n$, it is a standard result that the formula

$$f_\varphi\left(\sum_{i,j=1}^n E_{i,j} \otimes A_{i,j}\right) = \frac{1}{n}\sum_{i,j=1}^n \langle \varphi(A_{i,j})e_j, e_i\rangle$$

gives rise to a well-defined state f_φ on $\mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$ such that $\varphi(A) = n\sum_{i,j=1}^n f_\varphi(E_{i,j} \otimes A)E_{i,j}$ for all $A \in \mathfrak{A}$.

Since $\mathcal{M}_n(\mathfrak{A})$ is a unital, simple, purely infinite (by Theorem 3.11) C*-algebra, Theorem 8.8 implies that there exists a projection $P_0 \in \mathcal{M}_n(\mathfrak{A})$ such that

$$\|P_0(E_{i,j} \otimes A)P_0 - f_\varphi(E_{i,j} \otimes A)P_0\| < \delta$$

for all $A \in \mathcal{F}$ and for all $1 \leq i, j \leq n$. However, $E_{1,1} \otimes \psi(E_{1,1})$ is a non-zero projection (as $\psi(E_{1,1}) \neq 0$ or else $\psi(E_{j,j})$ would be equivalent in $\mathfrak{A}$ to a zero projection for all j and thus $\psi(I_n) = 0 \neq I_{\mathfrak{A}}$) and $\mathcal{M}_n(\mathfrak{A})$ is a unital, simple, purely infinite C*-algebra, there exists a non-zero projection $P \leq P_0$ in $\mathcal{M}_n(\mathfrak{A})$ such that P is equivalent to $E_{1,1} \otimes \psi(E_{1,1})$. Let $V_1 \in \mathcal{M}_n(\mathfrak{A})$ be the partial isometry such that $V_1V_1^* = P$ and $V_1^*V_1 = E_{1,1} \otimes \psi(E_{1,1})$ and, for each $j \in \{2, \ldots, n\}$, let $V_j \in \mathcal{M}_n(\mathfrak{A})$ be defined by

$$V_j = V_1(E_{1,1} \otimes \psi(E_{1,j})).$$

Therefore, since

$$V_i^*V_j = (E_{1,1} \otimes \psi(E_{i,1}))V_1^*V_1(E_{1,1} \otimes \psi(E_{1,j})) = (E_{1,1} \otimes \psi(E_{i,j}))$$

for all $1 \leq i, j \leq n$ as ψ is a *-homomorphism, each V_j is a partial isometry in $\mathcal{M}_n(\mathfrak{A})$. Moreover, we notice that $PV_j = V_j$ for all j as $PV_1 = V_1$. Therefore, for all $1 \leq i, j, k, l \leq n$ and for all $A \in \mathcal{F}$,

$$\begin{aligned}
\|V_i^*(E_{k,l} \otimes A)V_j - f_\varphi(E_{k,l} \otimes A)(E_{1,1} \otimes \psi(E_{i,j}))\| &= \|V_i^*(E_{k,l} \otimes A)V_j - f_\varphi(E_{k,l} \otimes A)V_i^*V_j\| \\
&= \|V_i^*P(E_{k,l} \otimes A)PV_j - f_\varphi(E_{k,l} \otimes A)V_i^*PV_j\| \\
&\leq \|P(E_{k,l} \otimes A)P - f_\varphi(E_{k,l} \otimes A)P\| \\
&\leq \|P_0(E_{k,l} \otimes A)P_0 - f_\varphi(E_{k,l} \otimes A)P_0\| < \delta.
\end{aligned}$$

We desire to remove the $f_\varphi(E_{k,l} \otimes A)$'s from the above expression. Let $C := \sum_{k=1}^n (E_{1,k} \otimes I)V_k \in \mathcal{M}_n(\mathfrak{A})$. Then for all $A \in \mathcal{F}$

$$C^*(E_{1,1} \otimes A)C = \sum_{i,j=1}^n V_i^*(E_{i,1} \otimes I)(E_{1,1} \otimes A)(E_{1,j} \otimes I)V_j = \sum_{i,j=1}^n V_i^*(E_{i,j} \otimes A)V_j.$$

Thus

$$\begin{aligned}
\|nC^*(E_{1,1} \otimes A)C - E_{1,1} \otimes \psi(\varphi(A))\| &= \left\|nC^*(E_{1,1} \otimes A)C - E_{1,1} \otimes \psi\left(n\sum_{i,j=1}^n f_\varphi(E_{i,j} \otimes A)E_{i,j}\right)\right\| \\
&= \left\|nC^*(E_{1,1} \otimes A)C - n\sum_{i,j=1}^n f_\varphi(E_{i,j} \otimes A)(E_{1,1} \otimes \psi(E_{i,j}))\right\| \\
&\leq n\sum_{i,j=1}^n \|V_i^*(E_{i,j} \otimes A)V_j - f_\varphi(E_{i,j} \otimes A)(E_{1,1} \otimes \psi(E_{i,j}))\| < n^3\delta
\end{aligned}$$

for all $A \in \mathcal{F}$. Therefore, as $I_{\mathfrak{A}} \in \mathcal{F}$, we obtain that

$$\|nC^*(E_{1,1} \otimes I_{\mathfrak{A}})C - E_{1,1} \otimes \psi(I_n)\| < n^3\delta$$

as $\varphi(I_{\mathfrak{A}}) = I_n$.

Next we will use C to construct our partial isometry with the aid of Lemma 10.1. Let

$$D := \sqrt{n}(E_{1,1} \otimes I_{\mathfrak{A}})C(E_{1,1} \otimes \psi(I_n)) \in (E_{1,1} \otimes I_{\mathfrak{A}})(\mathcal{M}_n(\mathfrak{A}))(E_{1,1} \otimes I_{\mathfrak{A}}).$$

Therefore

$$D^*D = n(E_{1,1} \otimes \psi(I_n))C^*(E_{1,1} \otimes I_{\mathfrak{A}})C(E_{1,1} \otimes \psi(I_n))$$

and

$$D(E_{1,1} \otimes \psi(I_n)) = D.$$

Moreover, as $(E_{1,1} \otimes \psi(I_n))$ is a projection in $\mathcal{M}_n(\mathfrak{A})$, we obtain that

$$\|D^*D - (E_{1,1} \otimes \psi(I_n))\| \leq \|nC^*(E_{1,1} \otimes I)C - E_{1,1} \otimes \psi(I_n)\| < n^3\delta \leq 1.$$

Hence Lemma 10.1 implies (where the functional calculus is taken in $(E_{1,1} \otimes \psi(I_n))(\mathcal{M}_n(\mathfrak{A}))(E_{1,1} \otimes \psi(I_n)))$ that $D(D^*D)^{-\frac{1}{2}}$ is a partial isometry in $(E_{1,1} \otimes I_{\mathfrak{A}})(\mathcal{M}_n(\mathfrak{A}))(E_{1,1} \otimes I_{\mathfrak{A}})$ such that

$$\left(D(D^*D)^{-\frac{1}{2}}\right)^* \left(D(D^*D)^{-\frac{1}{2}}\right) = E_{1,1} \otimes \psi(I_n)$$

and

$$\left\|D(D^*D)^{-\frac{1}{2}} - D\right\| \leq \|D^*D - (E_{1,1} \otimes \psi(I_n))\| < n^3\delta.$$

Moreover, we notice for all $A \in \mathcal{F}$ that

$$\begin{aligned}\|D^*(E_{1,1} \otimes A)D - E_{1,1} \otimes \psi(\varphi(A))\| &= \|n(E_{1,1} \otimes \psi(I_n))^*C^*(E_{1,1} \otimes A)C(E_{1,1} \otimes \psi(I_n)) - E_{1,1} \otimes \psi(\varphi(A))\| \\ &\leq \|nC^*(E_{1,1} \otimes A)C - E_{1,1} \otimes \psi(\varphi(A))\| < n^3\delta.\end{aligned}$$

Since $D(D^*D)^{-\frac{1}{2}}$ is in $(E_{1,1} \otimes I_{\mathfrak{A}})(\mathcal{M}_n(\mathfrak{A}))(E_{1,1} \otimes I_{\mathfrak{A}}) \simeq \mathfrak{A}$, there exists an element $V \in \mathfrak{A}$ such that $D(D^*D)^{-\frac{1}{2}} = E_{1,1} \otimes V$ and since $D(D^*D)^{-\frac{1}{2}}$ is a partial isometry, we obtain that V is also a partial isometry. Moreover $\|E_{1,1} \otimes V - D\| = \left\|D(D^*D)^{-\frac{1}{2}} - D\right\| \leq n^3\delta < 1$ so $\|D\| \leq 2$.

We claim that V is the desired partial isometry. To see this, we notice that

$$E_{1,1} \otimes V^*V = \left(D(D^*D)^{-\frac{1}{2}}\right)^* \left(D(D^*D)^{-\frac{1}{2}}\right) = E_{1,1} \otimes \psi(I_n)$$

and thus $V^*V = \psi(I_n)$. Moreover, for all $A \in \mathcal{F}$ we assumed that $\|A\| \leq 1$ so

$$\begin{aligned}\|V^*AV - \psi(\varphi(A))\| &= \|E_{1,1} \otimes V^*AV - E_{1,1} \otimes \psi(\varphi(A))\| \\ &= \|(E_{1,1} \otimes V)^*(E_{1,1} \otimes A)(E_{1,1} \otimes V) - E_{1,1} \otimes \psi(\varphi(A))\| \\ &\leq n^3\delta + 2n^3\delta + \|D^*(E_{1,1} \otimes A)D - E_{1,1} \otimes \psi(\varphi(A))\| \\ &< n^3\delta + 2n^3\delta + n^3\delta = 4n^3\delta < \epsilon\end{aligned}$$

as desired. □

Next we need the following technical lemma which has a simple, yet difficult to conceive proof.

Lemma 10.3. *Let $\mathfrak{A}$ be a unital C^*-algebra and let $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{A}$ be a unital, completely positive map. Let $\{E_{i,j}\}_{i,j=1}^n$ be the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$. There exists a partial isometry $V \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$ such that*

$$V^*V = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}} \qquad \text{and} \qquad V^*(T \otimes I_n \otimes I_{\mathfrak{A}})V = E_{1,1} \otimes E_{1,1} \otimes \psi(T)$$

for all $T \in \mathcal{M}_n(\mathbb{C})$.

Proof. Let $X := \sum_{i,j=1}^n E_{i,j} \otimes E_{i,j} \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C})$. Then X is a self-adjoint element such that

$$X^2 = \sum_{i,j,k,\ell=1}^n (E_{i,j} \otimes E_{i,j})(E_{k,\ell} \otimes E_{k,\ell}) = \sum_{i,j,\ell=1}^n E_{i,\ell} \otimes E_{i,\ell} = nX$$

so that $\sigma(X) \subseteq \{0, \sqrt{n}\}$ by the Continuous Functional Calculus. Hence X is a positive element of $\mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C})$.

Thus, as ψ is a completely positive map,

$$Y := \psi_n(X) = \sum_{i,j=1}^n E_{i,j} \otimes \psi(E_{i,j}) \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$$

is also a positive element. Write $Y^{\frac{1}{2}} = \sum_{i,j=1}^n E_{i,j} \otimes A_{i,j}$ where $A_{i,j} \in \mathfrak{A}$. Since $Y^{\frac{1}{2}}$ is positive and thus self-adjoint, $A_{i,j}^* = A_{j,i}$ for all i, j and

$$\sum_{i,j=1}^n E_{i,j} \otimes \psi(E_{i,j}) = Y = \left(\sum_{i,\ell=1}^n E_{i,\ell} \otimes A_{i,\ell}\right)\left(\sum_{k,j=1}^n E_{k,j} \otimes A_{k,j}\right) = \sum_{i,j=1}^n E_{i,j} \otimes \left(\sum_{k=1}^n A_{i,k}A_{k,j}\right)$$

so $\sum_{k=1}^n A_{i,k}A_{k,j} = \psi(E_{i,j})$ for all i, j.

Let $V := \sum_{i,j=1}^n E_{i,1} \otimes E_{j,1} \otimes A_{j,i} \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$. Then

$$\begin{aligned} V^*(E_{i,j} \otimes I_n \otimes I_{\mathfrak{A}})V &= \left(\sum_{p,k=1}^n E_{1,p} \otimes E_{1,k} \otimes A_{k,p}^*\right)(E_{i,j} \otimes I_n \otimes I)\left(\sum_{q,\ell=1}^n E_{q,1} \otimes E_{\ell,1} \otimes A_{\ell,q}\right) \\ &= \sum_{i,j,k,\ell=1}^n E_{1,1} \otimes E_{1,k}E_{\ell,1} \otimes A_{k,i}^*A_{\ell,j} \\ &= \sum_{i,j,k=1}^n E_{1,1} \otimes E_{1,1} \otimes A_{i,k}A_{k,j} \\ &= E_{1,1} \otimes E_{1,1} \otimes \psi(E_{i,j}) \end{aligned}$$

for all i, j so $V^*(T \otimes I_n \otimes I_{\mathfrak{A}})V = E_{1,1} \otimes E_{1,1} \otimes \psi(T)$ for all $T \in \mathcal{M}_n(\mathbb{C})$ by linearity. Therefore

$$V^*V = V^*(I_n \otimes I_n \otimes I_{\mathfrak{A}})V = E_{1,1} \otimes E_{1,1} \otimes \psi(I) = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$$

as claimed. Moreover, as $V^*V = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$ is a projection, V^*V must be a partial isometry as desired. ☐

Next we can show that nuclear maps between unital, simple, purely infinite C*-algebras have a nice form.

Proposition 10.4. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra and let $\Phi : \mathfrak{A} \to \mathfrak{A}$ be a unital, nuclear, completely positive map. For every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{A}$ there exists a non-unitary isometry $V \in \mathfrak{A}$ such that $\|V^*AV - \Phi(A)\| < \epsilon$ for all $A \in \mathcal{F}$.*

Proof. Since Φ is nuclear, Φ is the pointwise norm limit of maps $\psi \circ \varphi : \mathfrak{A} \to \mathfrak{A}$ where $\varphi : \mathfrak{A} \to \mathcal{M}_n(\mathbb{C})$ and $\psi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{A}$ are unital, completely positive maps. Thus it suffices to consider $\Phi = \psi \circ \varphi$.

Let $\epsilon > 0$ be arbitrary and let $\mathcal{F}$ be an arbitrary finite subset of $\mathfrak{A}$. We desire to 'correct' ψ to get a *-homomorphism so we can apply Lemma 10.2. Let $\{E_{i,j}\}_{i,j=1}^n$ be the canonical matrix units of $\mathcal{M}_n(\mathbb{C})$. Let $W_0 \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$ be the partial isometry from Lemma 10.3 such that

$$W_0^*W_0 = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}} \qquad \text{and} \qquad W_0^*(T \otimes I_n \otimes I_{\mathfrak{A}})W_0 = E_{1,1} \otimes E_{1,1} \otimes \psi(T)$$

for all $T \in \mathcal{M}_n(\mathbb{C})$. Moreover, since $\mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$ is a unital, simple (Proposition 3.4), purely infinite (Theorem 3.11) C*-algebra, Proposition 2.6 implies that there exists an non-zero isometry $W_1 \in \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A}$ such that $0 < W_1 W_1^* < E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$.

Notice that $\mathfrak{A}$ is isomorphic to

$$\mathfrak{B} := (E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}})(\mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathcal{M}_n(\mathbb{C}) \otimes_{\min} \mathfrak{A})(E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}).$$

Therefore, since $0 < W_1 W_1^* < E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$, the map π_0 defined by $\pi_0(T) := W_1(T \otimes I_n \otimes I_{\mathfrak{A}})W_1^*$ maps $\mathcal{M}_n(\mathbb{C})$ into $\mathfrak{B} \simeq \mathfrak{A}$. Moreover, since W_1 is an isometry, $\pi_0 : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{A}$ is a *-homomorphism.

Let $W := W_1 W_0$. Then, as $W_0^* W_0 = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$ and $W_1 W_1^* < E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$, $W \in \mathfrak{B}$ so we can view W as an element of $\mathfrak{A}$. Moreover

$$W^* W = W_0^* W_1^* W_1 W_0 = W_0^* W_0 = E_{1,1} \otimes E_{1,1} \otimes I_{\mathfrak{A}}$$

so W (when viewed in $\mathfrak{A}$) is an isometry. Moreover, we notice for all $T \in \mathcal{M}_n(\mathbb{C})$ that

$$\begin{aligned} W^* \pi_0(T) W &= W_0^* W_1^* W_1 (T \otimes I_n \otimes I_{\mathfrak{A}}) W_1^* W_1 W_0 \\ &= W_0^* (T \otimes I_n \otimes I_{\mathfrak{A}}) W_0 = E_{1,1} \otimes E_{1,1} \otimes \psi(T) \end{aligned}$$

in $\mathfrak{B}$ so $W^* \pi_0(T) W = \psi(T)$ when viewed as elements of $\mathfrak{A}$.

Let $P := W_1 W_1^*$ which is a projection that we can view as an element of $\mathfrak{A}$ such that $0 < P < I_{\mathfrak{A}}$. Since $\mathfrak{A}$ is purely infinite and simple, Lemma 2.3 implies that $I_{\mathfrak{A}} - P$ is a properly infinite projection. Hence there exists n partial isometries $\{V_j\}_{j=1}^n$ such that $V_j^* V_j = I_{\mathfrak{A}} - P$ and $\sum_{j=1}^n V_j V_j^* \leq I_{\mathfrak{A}} - P$. Therefore $E_{i,j} := V_i V_j*$ defines a system of matrix units inside $(I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$ and thus there exists an injective *-homomorphism $\pi_1 : \mathcal{M}_n(\mathbb{C}) \to (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P)$. Therefore, if we defined $\pi : \mathcal{M}_n(\mathbb{C}) \to \mathfrak{A}$ by $\pi(T) := \pi_0(T) + \pi_1(T)$ for all $T \in \mathcal{M}_n(\mathbb{C})$ then, since π_0 and π_1 are *-homomorphisms with orthogonal ranges (as $(I_{\mathfrak{A}} - P)W_1 = 0 = W_1^*(I_{\mathfrak{A}} - P)$), π is a *-homomorphism. Moreover, since

$$W^* \pi_1(T) W \in W_0^* W_1^* (I_{\mathfrak{A}} - P)\mathfrak{A}(I_{\mathfrak{A}} - P) W_1 W_0 = \{0\},$$

we obtain that $W^* \pi(T) W = \psi(T)$ when viewed as an element of $\mathfrak{A}$.

By Lemma 10.2 there exists a partial isometry $V_0 \in \mathfrak{A}$ such that $V_0^* V_0 = \pi(I_n)$ and $\|V_0^* A V_0 - \pi(\varphi(A))\| < \epsilon$ for all $A \in \mathcal{F}$. Let $V := V_0 W \in \mathfrak{A}$. Then

$$V^* V = W^* V_0^* V_0 W = W^* \pi(I_n) W = \psi(I_n) = I_{\mathfrak{A}}$$

as ψ is unital. Hence V is an isometry. Moreover, we notice that

$$\|V^* A V - \psi(\varphi(A))\| = \|W^* V_0^* A V_0 W - W^* \pi(\varphi(A)) W\| \leq \|V_0^* A V_0 - \pi(\varphi(A))\| < \epsilon$$

for all $A \in \mathcal{F}$ as desired. Finally, to see that V is not a unitary, we notice that

$$V_0^* V_0 = \pi(I_n) = \pi_0(I_n) + \pi_1(I_n) > \pi_0(I_n) = W_1 W_1^*$$

and

$$V V^* = V_0 W_1 W_0 W_0^* W_1^* V_0^* \leq V_0 W_1 W_1^* V_0^*.$$

If $VV^* = I_{\mathfrak{A}}$, then

$$V_0^* V_0 = V_0^* V V^* V_0 \leq V_0^* V_0 W_1 W_1^* V_0^* V_0 = \pi(I_n) W_1 W_1^* \pi(I_n) = W_1 W_1^*$$

which is clearly a contradiction. □

With the above completed, we can begin to prove our next major technical result. This result enables us to connect unital, completely positive maps from a unital, separable, exact C*-algebra into unital, separable, nuclear C*-algebras on finite dimensional operator spaces. The idea is to construct the unital, completely positive map Θ by going through two finite dimensional operator systems of matrix algebras.

Lemma 10.5. *Let $\mathfrak{A}$ be a unital, separable, exact C*-algebra, let $\mathcal{S} \subseteq \mathfrak{A}$ be a finite dimensional operator system, and let $\epsilon > 0$. For every $0 < \delta < \frac{\epsilon}{2}$ there exists an integer n such that whenever $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are separable, unital C*-algebras with $\mathfrak{B}_2$ nuclear and whenever $\Phi : \mathcal{S} \to \mathfrak{B}_1$ and $\Psi : \mathcal{S} \to \mathfrak{B}_2$ are unital, completely positive maps such that Φ is injective and $\Phi^{-1} : \Phi(\mathcal{S}) \to \mathcal{S}$ satisfies $\left\|\Phi^{-1}\right\|_n \leq 1+\delta$, then there is a unital, completely positive map $\Theta : \mathfrak{B}_1 \to \mathfrak{B}_2$ such that $\|\Theta \circ \Phi - \Psi\| < \epsilon$.*

Proof. Fix $0 < \delta < \frac{\epsilon}{2}$ and let

$$\rho := \frac{\epsilon - 2\delta}{3(1+\delta)} > 0.$$

Since $\mathfrak{A}$ is exact, we can view $\mathfrak{A}$ as a unital C*-subalgebra of some $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is separable and the inclusion is nuclear. Let $\{A_1, \ldots, A_m\}$ be a basis for $\mathcal{S}$ with $A_1 = I_{\mathfrak{A}}$ and choose $\mu > 0$ small enough so that if $B_1, \ldots, B_m \in \mathcal{B}(\mathcal{H})$ and $\|A_j - B_j\| < \mu$ for all $j \in \{1, \ldots, m\}$ then the map $T : \mathcal{S} \to span\{B_1, \ldots, B_m\}$ defined by $T(A_j) := B_j$ for all $j \in \{1, \ldots, m\}$ satisfies $\left\|T^{-1}\right\|_{cb} < 1 + \rho$ by Lemma 9.3.

Since the inclusion of $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$ is nuclear, there exists an $n \in \mathbb{N}$ and unital, completely positive maps $S_1 : \mathcal{S} \to \mathcal{M}_n(\mathbb{C})$ and $\tilde{S}_2 : \mathcal{M}_n(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ such that the elements $B_j := \tilde{S}_2(S_1(A_j))$ satisfy $\|A_j - B_j\| < \mu$ for all $j \in \{1, \ldots, m\}$. Let T be the map listed in the above paragraph for this choice of $\{B_1, \ldots, B_m\}$ and let $\mathcal{S}_0 := S_1(\mathcal{S})$ (which is an operator space in $\mathcal{M}_n(\mathbb{C})$). Define $S_2 : \mathcal{S}_0 \to \mathcal{S}$ by $S_2 := T^{-1} \circ \tilde{S}_2$. Hence S_2 is unital, $S_2 \circ S_1 = Id_{\mathcal{S}}$, and $\|S_2\|_{cb} < 1 + \rho$. Moreover, notice for all $j \in \{1, \ldots, m\}$ that if $C_j := S_1(A_j)$ then $\mathcal{S}_0 = span\{C_1, \ldots, C_m\}$ and

$$S_2(C_j^*) = S_2(S_1(A_j^*)) = A_j^* = (S_2(S_1(A_j)))^* = S_2(C_j)^*$$

so $S_2(X^*) = S_2(X)^*$ for all $X \in \mathcal{S}_0$. Thus S_2 is a self-adjoint map on $\mathcal{S}_0$. Since $B_1 = \tilde{S}_2(S_1(I_{\mathfrak{A}})) = I_{\mathfrak{A}}$, S_2 is also unital.

Let $\Phi : \mathcal{S} \to \mathfrak{B}_1$ and $\Psi : \mathcal{S} \to \mathfrak{B}_2$ be unital, completely positive maps such that Φ is injective and $\Phi^{-1} : \Phi(\mathcal{S}) \to \mathcal{S}$ satisfies $\left\|(\Phi^{-1})_n\right\| \leq 1 + \delta$.

Since $\mathfrak{B}_2$ is nuclear, there exists an $r \in \mathbb{N}$ and unital, completely positive maps $W_1 : \mathcal{S} \to \mathcal{M}_r(\mathbb{C})$ and $W_2 : \mathcal{M}_r(\mathbb{C}) \to \mathfrak{B}_2$ such that $\|W_2 \circ W_1 - \Psi\| < \rho$. Since $W_1 \circ S_2 : \mathcal{S}_0 \to \mathcal{M}_r(\mathbb{C})$ is unital, self-adjoint map with $\|W_1 \circ S_2\|_{cb} < 1 + \rho$, Theorem 9.4 implies that there exists a unital, completely positive map $Q : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_r(\mathbb{C})$ such that $\|Q|_{\mathcal{S}_0} - W_1 \circ S_2\| < 2\rho$.

Consider $S_1 \circ \Phi^{-1} : \Phi(\mathcal{S}) \to \mathcal{S}_0 \subseteq \mathcal{M}_n(\mathbb{C})$. Since Φ is an injective, unital, completely positive map, Φ^{-1} is a unital, self-adjoint map. Therefore $S_1 \circ \Phi^{-1}$ is a unital, self-adjoint map such that

$$\left\|S_1 \circ \Phi^{-1}\right\|_n \leq \|S_1\|_{cb} \left\|\Phi^{-1}\right\|_n \leq 1 + \delta.$$

Hence, as the completely bounded norm of a linear map into $\mathcal{M}_n(\mathbb{C})$ is determined by the n-norm, this implies that $\left\|S_1 \circ \Phi^{-1}\right\|_{cb} \leq 1 + \delta$ and thus Theorem 9.4 implies that there exists a unital, completely positive map $R : \mathfrak{B}_1 \to \mathcal{M}_n(\mathbb{C})$ such that $\left\|R|_{\Phi(\mathcal{S})} - S_1 \circ \Phi^{-1}\right\| \leq 2\delta$.

Let $\Theta := W_2 \circ Q \circ R : \mathfrak{B}_1 \to \mathfrak{B}_2$. Then Θ is a unital, completely positive map such that

$$\begin{aligned}
&\left\|\Psi \circ \Phi^{-1} - \Theta|_{\Phi(\mathcal{S})}\right\| \\
\leq\ &\left\|\Psi \circ \Phi^{-1} - W_2 \circ W_1 \circ \Phi^{-1}\right\| + \left\|W_2 \circ W_1 \circ S_2 \circ S_1 \circ \Phi^{-1} - W_2 \circ Q \circ R|_{\Phi(\mathcal{S})}\right\| \\
\leq\ &\|\Psi - W_2 \circ W_1\| \left\|\Phi^{-1}\right\| + \|W_2\| \left\|W_1 \circ S_2 \circ S_1 \circ \Phi^{-1} - Q \circ R|_{\Phi(\mathcal{S})}\right\| \\
\leq\ &\rho(1+\delta) + \left\|W_1 \circ S_2 \circ S_1 \circ \Phi^{-1} - Q \circ S_1 \circ \Phi^{-1}\right\| + \left\|Q \circ S_1 \circ \Phi^{-1} - Q \circ R|_{\Phi(\mathcal{S})}\right\| \\
\leq\ &\rho(1+\delta) + \|W_1 \circ S_2 - Q|_{\mathcal{S}_0}\| \left\|S_1 \circ \Phi^{-1}\right\| + \|Q\| \left\|S_1 \circ \Phi^{-1} - R|_{\Phi(\mathcal{S})}\right\| \\
\leq\ &\rho(1+\delta) + 2\rho(1+\delta) + 2\delta < \epsilon
\end{aligned}$$

as desired. ∎

The following is a simple application of the above result.

Corollary 10.6. *Let $\mathfrak{A}$ be a unital, separable, exact C*-algebra. Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be unital, separable C*-algebras with $\mathfrak{B}_2$ nuclear and let $\varphi_j : \mathfrak{A} \to \mathfrak{B}_j$ be unital *-homomorphisms such that φ_1 is injective. Then there exists a sequence of unital, completely positive maps $\psi_n : \mathfrak{B}_1 \to \mathfrak{B}_2$ such that $\psi_n(\varphi_1(A)) \to \varphi_2(A)$ for all $A \in \mathfrak{A}$.*

Proof. As $\mathfrak{A}$ is separable, there exists an increasing sequence of finite dimensional operator systems in $\mathfrak{A}$ with dense union in $\mathfrak{A}$. As φ_1 is a unital, injective *-homomorphism, the inverse of φ_1 has completely bounded norm 1. Hence the result follows by applying Lemma 10.5. □

As the ψ_n's are almost 'conjugation by isometries' when $\mathfrak{B}_1 = \mathfrak{B}_2$ is a unital, separable simple, purely infinite, nuclear C*-algebra by Proposition 10.4, if we can change these isometries into unitaries, the proof of Theorem 10.10 will be complete. This leads us to our last two technical lemmas. The later requires the first which will be used to approximate the 'off-diagonal' components of a unitary.

Lemma 10.7. *Let $\mathfrak{A}$ be a unital C*-algebra, let $U \in \mathfrak{A}$ be a unitary, and let $V \in \mathfrak{A}$ be an isometry with range projection $P := VV^*$. Then*

$$\|U - (PUP + (I_{\mathfrak{A}} - P)U(I_{\mathfrak{A}} - P))\| \leq \inf\{(2\|V^*UV - U_0\|)^{\frac{1}{2}} \mid U_0 \in \mathcal{U}(\mathfrak{A})\}.$$

Proof. Notice

$$\begin{aligned}\|U - (PUP + (I_{\mathfrak{A}} - P)U(I_{\mathfrak{A}} - P))\| &= \|PU(I_{\mathfrak{A}} - P) + (I_{\mathfrak{A}} - P)UP\| \\ &= \|PU^*(I_{\mathfrak{A}} - P)UP + (I_{\mathfrak{A}} - P)U^*PU(I_{\mathfrak{A}} - P)\|^{\frac{1}{2}} \\ &= \max\{\|PU^*(I_{\mathfrak{A}} - P)UP\|^{\frac{1}{2}}, \|(I_{\mathfrak{A}} - P)U^*PU(I_{\mathfrak{A}} - P)\|^{\frac{1}{2}}\}.\end{aligned}$$

However, if $U_0 \in \mathfrak{A}$ is a unitary, then

$$\begin{aligned}\|PU^*(I_{\mathfrak{A}} - P)UP\| &= \|PU^*UP - PU^*PUP\| \\ &= \|P - (PUP)^*PUP\| \\ &= \|VU_0^*V^*VU_0V^* - (PUP)^*PUP\| \\ &\leq 2\|VU_0V^* - PUP\| \\ &= 2\|VU_0V^* - VV^*UVV^*\| \\ &\leq 2\|U_0 - V^*UV\|\end{aligned}$$

and since

$$\|(I_{\mathfrak{A}} - P)U^*PU(I_{\mathfrak{A}} - P)\|^{\frac{1}{2}} = \|PU(I_{\mathfrak{A}} - P)\| = \|PU(I_{\mathfrak{A}} - P)U^*P\|^{\frac{1}{2}}$$

we may repeat the above computations with U and U_0 replaced with U^* and U_0^* to obtain that

$$\|(I_{\mathfrak{A}} - P)U^*PU(I_{\mathfrak{A}} - P)\| \leq 2\|U_0 - V^*UV\|$$

which completes the proof. □

Lemma 10.8. *Let $\mathfrak{A}$ be a unital C*-algebra, let S and T be two isometries in $\mathfrak{A}$, and let $\mathfrak{D}$ be a C*-subalgebra of $\mathfrak{A}$ such that $I_{\mathfrak{A}} \in \mathfrak{D}$, $\mathfrak{D} \simeq \mathcal{O}_2$, and every element of $\mathfrak{D}$ commutes with S and T. Then there exists a unitary $W \in \mathfrak{A}$ such that whenever $U, V \in \mathcal{U}(\mathfrak{A})$ commute with every element of $\mathfrak{D}$, then*

$$\|W^*VW - U\| \leq 15\,(\max\{\|S^*US - V\|, \|T^*UT - V\|\})^{\frac{1}{2}}.$$

Proof. Let $\mathfrak{B}$ be the relative commutant of $\mathfrak{D}$ in $\mathfrak{A}$. Since $\mathcal{O}_2$ is nuclear by Theorem 1.19, by the properties of the maximal tensor norm there exists a unital *-homomorphism $\pi : \mathcal{O}_2 \otimes_{\min} \mathfrak{B} \to \mathfrak{A}$ such that $\pi(I_{\mathcal{O}_2} \otimes B) = B$ for all $B \in \mathfrak{B}$ and $\pi|_{\mathcal{O}_2 \otimes I_{\mathfrak{A}}}$ is an isomorphism of $\mathcal{O}_2$ and $\mathfrak{D}$. Hence we can view $S, T \in \mathfrak{B}$ and it suffices to prove that there exists a $W \in \mathcal{O}_2 \otimes_{\min} \mathfrak{B}$ such that

$$\|W^*(I_{\mathcal{O}_2} \otimes V)W - (I_{\mathcal{O}_2} \otimes U)\| \leq 15\,(\max\{\|S^*US - V\|, \|T^*UT - V\|\})^{\frac{1}{2}}$$

for all $U, V \in \mathfrak{B}$.

The remainder of the proof is fairly technical. The idea is to create a bunch of orthogonal projections and partial isometries to explicitly write down a unitary W that intertwines the sum of compressions of U to a sum of compressions of V.

To begin, let

$$e_1 := SS^* \quad \text{and} \quad f_1 := TT^*.$$

Let

$$e_2 := Sf_1S^* \leq e_1, \quad f_2 := Te_1T^* \leq f_1, \quad \text{and} \quad f_3 := Te_2T^* \leq f_2.$$

Consider the two sets of mutually orthogonal projections summing to $I_{\mathfrak{B}}$

$$P_1 := I_{\mathfrak{B}} - e_1, \quad P_2 := e_1 - e_2, \quad \text{and} \quad P_3 := e_2$$

and

$$Q_1 := I_{\mathfrak{B}} - f_1, \quad Q_2 := f_1 - f_2, \quad Q_3 := f_2 - f_3, \quad \text{and} \quad Q_4 := f_3.$$

Consider the operators

$$C_1 := P_2SQ_1, \quad C_2 := P_1T^*Q_2, \quad C_3 := P_2T^*Q_3, \quad \text{and} \quad C_4 := P_3T^*Q_4.$$

Then C_1, C_2, C_3, and C_4 are partial isometries with

$$\begin{aligned}
C_1^*C_1 &= Q_1S^*P_2SQ_1 = Q_1S^*(SS^* - Sf_1S^*)SQ_1 = Q_1(I_{\mathfrak{B}} - f_1)Q_1 = Q_1 \\
C_1C_1^* &= P_2SQ_1S^*P_2 = P_2S(I_{\mathfrak{B}} - f_1)S^*P_2 = P_2(e_1 - e_2)P_2 = P_2 \\
C_2^*C_2 &= Q_2TP_1T^*Q_2 = Q_2T(I_{\mathfrak{B}} - e_1)T^*Q_2 = Q_2(f_2 - f_3)Q_2 = Q_2 \\
C_3^*C_3 &= Q_3TP_2T^*Q_3 = Q_3T(e_1 - e_2)T^*Q_3 = Q_3(f_3 - f_4)Q_3 = Q_3 \\
C_4^*C_4 &= Q_4TP_3T^*Q_4 = Q_4Te_2T^*Q_4 = Q_4f_3Q_4 = Q_4 \\
C_2C_2^* &= P_1T^*Q_2TP_1 = P_1T^*(f_1 - f_2)TP_1 = P_1T^*(TT^* - Te_1T^*)TP_1 = P_1(I_{\mathfrak{B}} - e_1)P_1 = P_1 \\
C_3C_3^* &= P_2T^*Q_3TP_2 = P_2T^*(f_2 - f_3)TP_2 = P_2T^*(Te_1T^* - Te_2T^*)TP_2 = P_2(e_1 - e_2)P_2 = P_2 \\
C_4C_4^* &= P_3T^*Q_4TP_3 = P_3T^*(f_3)TP_3 = P_3T^*(Te_2T^*)TP_3 = P_3(e_2)P_3 = P_3.
\end{aligned}$$

Hence $C_jC_k^* = 0$ for all $j \neq k$, $C_1^*C_k = 0$ for all $k \in \{2, 4\}$, and $C_3^*C_k = 0$ for all $k \in \{2, 4\}$.

Let S_1 and S_2 be the standard generators of $\mathcal{O}_2$. Let

$$W := S_1 \otimes C_1 + I_{\mathcal{O}_2} \otimes C_2 + S_2 \otimes C_3 + I_{\mathcal{O}_2} \otimes C_4 \in \mathcal{O}_2 \otimes_{\min} \mathfrak{B}.$$

Then, since $C_1^*C_k = 0$ for all $k \in \{2, 4\}$ and $C_3^*C_k = 0$ for all $k \in \{2, 4\}$,

$$W^*W = (S_1^*S_1 \otimes Q_1 + I_{\mathcal{O}_2} \otimes Q_2 + S_2^*S_2 \otimes Q_3 + I_{\mathcal{O}_2} \otimes Q_4) + (S_1^*S_2 \otimes C_1^*C_3 + S_2^*S_1 \otimes C_3^*C_1) = I_{\mathcal{O}_2} \otimes I_{\mathfrak{B}}$$

and, since $C_jC_k^* = 0$ for all $j \neq k$,

$$WW^* = S_1S_1^* \otimes P_2 + I_{\mathcal{O}_2} \otimes P_1 + S_2S_2^* \otimes P_2 + I_{\mathcal{O}_2} \otimes P_3 = I_{\mathcal{O}_2} \otimes I_{\mathfrak{B}}.$$

Hence W is a unitary operator in $\mathcal{O}_2 \otimes_{\min} \mathfrak{B}$. Moreover

$$\begin{aligned}
W(I_{\mathcal{O}_2} \otimes Q_1)W^* &= S_1S_1^* \otimes C_1Q_1C_1^* = S_1S_1^* \otimes (C_1C_1^*)^2 = S_1S_1^* \otimes P_2, \\
W(I_{\mathcal{O}_2} \otimes Q_2)W^* &= I_{\mathcal{O}_2} \otimes C_2Q_2C_2^* = I_{\mathcal{O}_2} \otimes (C_2C_2^*)^2 = I_{\mathcal{O}_2} \otimes P_1, \\
W(I_{\mathcal{O}_2} \otimes Q_3)W^* &= S_2S_2^* \otimes C_3Q_3C_3^* = S_2S_2^* \otimes (C_3C_3^*)^2 = S_2S_2^* \otimes P_2, \text{ and} \\
W(I_{\mathcal{O}_2} \otimes Q_4)W^* &= I_{\mathcal{O}_2} \otimes C_4Q_4C_4^* = I_{\mathcal{O}_2} \otimes (C_4C_4^*)^2 = I_{\mathcal{O}_2} \otimes P_3.
\end{aligned}$$

Now fix unitaries $U, V \in \mathfrak{B}$ and let $\delta := \max\{\|S^*US - V\|, \|T^*VT - U\|\} \leq 2$ (note that δ may be zero if S and T are unitaries). Notice that

$$P_2S = P_2(e_1 - e_2)S = P_2(SS^* - Sf_1S^*)S = P_2(S - Sf_1) = P_2S(I_{\mathfrak{B}} - f_1) = P_2SQ_1 = C_1Q_1$$

and $P_2 \leq e_1 = SS^*$. Hence

$$\|C_1(Q_1VQ_1)C_1^* - P_2UP_2\| = \|P_2SVS^*P_2 - P_2SS^*USS^*P_2\| \leq \|V - S^*US\| \leq \delta.$$

Notice that

$$C_2Q_2 = P_1T^*Q_2 = P_1T^*(f_1 - f_2) = P_1T^*(TT^* - Te_1T^*) = P_1(T^* - e_1T^*) = P_1(I_{\mathfrak{B}} - e_1)T^* = P_1T^*$$

so

$$\|C_2(Q_2VQ_2)C_2^* - P_1UP_1\| = \|P_1T^*VTP_1 - P_1UP_1\| \leq \|T^*VT - U\| \leq \delta.$$

Similarly

$$C_3Q_3 = P_2T^*Q_3 = P_2T^*(f_2 - f_3) = P_2T^*(Te_1T^* - Te_2T^*) = P_2(e_1T^* - e_1T^*) = P_2(e_1 - e_2)T^* = P_2T^*$$

so

$$\|C_3(Q_3VQ_3)C_3^* - P_2UP_2\| = \|P_2T^*VTP_2 - P_2UP_2\| \leq \|T^*VT - U\| \leq \delta.$$

Finally

$$C_4Q_4 = P_3T^*Q_4 = P_3T^*f_3 = P_3T^*Te_2T^* = P_3e_2T^* = P_3T^*$$

so

$$\|C_4(Q_4VQ_4)C_4^* - P_3UP_3\| = \|P_3T^*VTP_3 - P_3UP_3\| \leq \|T^*VT - U\| \leq \delta.$$

Notice (by the same arguments used to compute $W(I_{\mathcal{O}_2} \otimes Q_j)W^*$ for all j) that

$$\begin{aligned} &W(I_{\mathcal{O}_2} \otimes (Q_1VQ_1 + Q_2VQ_2 + Q_3VQ_3 + Q_4VQ_4))W^* \\ &= S_1S_1^* \otimes C_1(Q_1VQ_1)C_1^* + I_{\mathcal{O}_2} \otimes C_2(Q_2VQ_2)C_2^* + S_2S_2^* \otimes C_3(Q_3VQ_3)C_3^* + I_{\mathcal{O}_2} \otimes C_4(Q_4VQ_4)C_4^* \end{aligned}$$

and thus if

$$V_0 := Q_1VQ_1 + Q_2VQ_2 + Q_3VQ_3 + Q_4VQ_4 \qquad \text{and} \qquad U_0 := P_1UP_1 + P_2UP_2 + P_3UP_3$$

then

$$\begin{aligned} &\|W(I_{\mathcal{O}_2} \otimes V)W^* - I_{\mathcal{O}_2} \otimes U\| \\ &\leq \|U - U_0\| + \|V - V_0\| + \|W(I_{\mathcal{O}_2} \otimes V_0)W^* - I_{\mathcal{O}_2} \otimes U_0\| \\ &\leq \|U - U_0\| + \|V - V_0\| + \|I_{\mathcal{O}_2} \otimes (C_2(Q_2VQ_2)C_2^* - P_1UP_1)\| + \|I_{\mathcal{O}_2} \otimes (C_4(Q_4VQ_4)C_4^* - P_3UP_3)\| \\ &\quad + \|S_1S_1^* \otimes C_1(Q_1VQ_1)C_1^* + S_2S_2^* \otimes C_3(Q_3VQ_3)C_3^* - (S_1^*S_1 + S_2S_2^*) \otimes P_2UP_2\| \\ &\leq \|U - U_0\| + \|V - V_0\| + 4\delta \end{aligned}$$

Thus it suffices to approximate the norms of $\|U - U_0\|$ and $\|V - V_0\|$.

Recall that $e_1 = SS^*$ so

$$\|U - (e_1Ue_1 + P_1UP_1)\| \leq \sqrt{2\,\|S^*US - V\|} \leq \sqrt{2\delta}$$

by Lemma 10.7. Since ST is also an isometry, $STT^*S^* = e_2$, and

$$\|(ST)^*U(ST) - U\| \leq \|T^*S^*UST - T^*VT\| + \|T^*VT - U\| \leq 2\delta,$$

Lemma 10.7 applied with the isometry ST gives

$$\|U - (e_2Ue_2 + (I_{\mathfrak{B}} - e_2)U(I_{\mathfrak{B}} - e_2))\| \leq \sqrt{2\,\|(ST)^*U(ST) - U\|} \leq \sqrt{4\delta}.$$

By compression the above expression by $e_1 \geq e_2$, we obtain that

$$\|e_1Ue_1 - (P_2UP_2 + P_1UP_1)\| \leq \sqrt{4\delta}$$

so

$$\|U - U_0\| \le \|U - (e_1 U e_1 + P_1 U P_1)\| + \|e_1 U e_1 - (P_2 U P_2 + P_1 U P_1)\| \le (\sqrt{2} + 2)\sqrt{\delta}.$$

Similarly recall that $f_1 = TT^*$ so

$$\|V - (f_1 V f_1 + Q_1 V Q_1)\| \le \sqrt{2\,\|T^* V T - U\|} \le \sqrt{2\delta}$$

by Lemma 10.7. Since TS is also an isometry, $TSS^*T^* = f_2$, and

$$\|(TS)^* V (TS) - V\| \le \|S^* T^* V T S - S^* U S\| + \|S^* U S - V\| \le 2\delta,$$

Lemma 10.7 applied with the isometry TS gives

$$\|V - (f_2 V f_2 + (I_{\mathfrak{B}} - f_2) V (I_{\mathfrak{B}} - f_2))\| \le \sqrt{2\,\|(TS)^* V (TS) - V\|} \le \sqrt{4\delta}.$$

By compression the above expression by $f_1 \ge f_2$, we obtain that

$$\|f_1 V f_1 - (f_2 V f_2 + Q_2 V Q_2)\| \le \sqrt{4\delta}.$$

Since TST is also an isometry, $TSTT^*S^*T^* = f_3$, and

$$\|(TST)^* V (TST) - U\| \le \|T^* S^* T^* V T S T - T^* S^* U S T\| + \|T^* S^* U S T - T^* V T\| + \|T^* V T - U\| \le 3\delta,$$

Lemma 10.7 applied with the isometry TST gives

$$\|V - (f_3 V f_3 + (I_{\mathfrak{B}} - f_3) V (I_{\mathfrak{B}} - f_3))\| \le \sqrt{6\delta}.$$

By compression the above expression by $f_2 \ge f_3$, we obtain that

$$\|f_2 V f_2 - (Q_4 V Q_4 + Q_3 V Q_3)\| \le \sqrt{6\delta}.$$

Hence

$$\begin{aligned}\|V - V_0\| &= \|V - (f_1 V f_1 + Q_1 V Q_1)\| + \|f_1 V f_1 - (f_2 V f_2 + Q_2 V Q_2)\| + \|f_2 V f_2 - (Q_4 V Q_4 + Q_3 V Q_3)\| \\ &\le (\sqrt{2} + 2 + \sqrt{6})\sqrt{\delta}\end{aligned}$$

Thus, as $\delta \le 2$ so $\delta \le \sqrt{2\delta}$,

$$\|W(I \otimes V)W^* - U\| \le 4\sqrt{2}\sqrt{\delta} + (\sqrt{2} + 2 + \sqrt{6})\sqrt{\delta} + (\sqrt{2} + 2)\sqrt{\delta} \le 15\sqrt{\delta}$$

as desired. □

Combining Proposition 10.4, Corollary 10.6, and Lemma 10.8, we obtain the following lemma that will easily enable us to prove our main result.

Lemma 10.9. *Let $\mathfrak{A}$ be a unital, separable, exact C*-algebra and let $\mathfrak{B}$ be a unital, separable, nuclear, simple, purely infinite C*-algebra. Let $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ be two injective, unital *-homomorphisms. Then the unital *-homomorphism $\Phi, \Psi : \mathfrak{A} \to \mathcal{O}_2 \otimes_{\min} \mathfrak{B}$ defined by $\Phi(A) := I_{\mathcal{O}_2} \otimes \varphi(A)$ and $\Psi(A) := I_{\mathcal{O}_2} \otimes \psi(A)$ for all $A \in \mathfrak{A}$ are approximately unitarily equivalent.*

Proof. Let $n \in \mathbb{N}$ be arbitrary, let $U_1, \ldots, U_n \in \mathfrak{A}$ be unitaries, and let $\epsilon > 0$. Since $\mathfrak{A}$ is a unital, separable C*-algebra and thus the span of the set of unitaries is dense in $\mathfrak{A}$, it suffices to show that there exists a unitary $W \in \mathcal{O}_2 \otimes_{\min} \mathfrak{B}$ such that

$$\|W(I_{\mathcal{O}_2} \otimes \varphi(U_j))W^* - I_{\mathcal{O}_2} \otimes \psi(U_j)\| < \epsilon$$

for all $j \in \{1, \ldots, n\}$. By Corollary 10.6 there exists completely positive maps $S, T : \mathfrak{B} \to \mathfrak{B}$ such that

$$\|S(\varphi(U_j)) - \psi(U_j)\| < \frac{1}{2}\left(\frac{\epsilon}{15}\right)^2 \qquad \text{and} \qquad \|T(\psi(U_j)) - \psi(U_j)\| < \frac{1}{2}\left(\frac{\epsilon}{15}\right)^2$$

for all $j \in \{1, \ldots, n\}$. Since $\mathfrak{B}$ is a unital, simple, purely infinite, nuclear C*-algebra, Proposition 10.4 implies that there exists isometries $S_0, T_0 \in \mathfrak{B}$ such that

$$\|S_0^*(\varphi(U_j))S_0 - S(\varphi(U_j))\| < \frac{1}{2}\left(\frac{\epsilon}{15}\right)^2 \qquad \text{and} \qquad \|T_0^*(\psi(U_j))T_0 - T(\psi(U_j))\| < \frac{1}{2}\left(\frac{\epsilon}{15}\right)^2$$

for all $j \in \{1, \ldots, n\}$. Hence

$$\|S_0^*(\varphi(U_j))S_0 - \psi(U_j)\| < \left(\frac{\epsilon}{15}\right)^2 \qquad \text{and} \qquad \|T_0^*(\psi(U_j))T_0 - \varphi(U_j)\| < \left(\frac{\epsilon}{15}\right)^2$$

for all $j \in \{1, \ldots, n\}$. Thus Lemma 10.8 gives the desired unitary W. ☐

Theorem 10.10. *Let $\mathfrak{A}$ be a unital, separable, exact C*-algebra. Any two injective, unital *-homomorphisms from $\mathfrak{A}$ to $\mathcal{O}_2$ are approximately unitarily equivalent.*

Proof. Let $\pi, \sigma : \mathfrak{A} \to \mathcal{O}_2$ be two injective, unital *-homomorphisms. Let $\psi : \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism from Theorem 7.5. Let $\phi : \mathcal{O}_2 \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ be the injective, unital *-homomorphism defined by $\phi(A) = I_{\mathcal{O}_2} \otimes A$ for all $A \in \mathcal{O}_2$. Since $\psi \circ \phi : \mathcal{O}_2 \to \mathcal{O}_2$ is a unital, injective *-homomorphism and $\mathcal{O}_2$ is a unital, simple, purely infinite C*-algebra, Theorem 6.12 implies that $\psi \circ \phi$ is approximately unitarily equivalent to $Id_{\mathcal{O}_2}$.

Let $\Phi, \Psi : \mathfrak{A} \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$ be defined by $\Phi(A) := I_{\mathcal{O}_2} \otimes \pi(A) = \phi(\pi(A))$ and $\Psi(A) := I_{\mathcal{O}_2} \otimes \sigma(A) = \phi(\sigma(A))$ for all $A \in \mathfrak{A}$. Since π and σ are unital, injective *-homomorphisms, $\mathfrak{A}$ is a unital, separable, exact C*-algebra, and $\mathcal{O}_2$ is a unital, separable, nuclear, simple, purely infinite C*-algebra, Lemma 10.9 implies that Φ and Ψ are approximately unitarily equivalent. Hence $\phi \circ \pi$ and $\phi \circ \sigma$ are approximately unitarily equivalent.

Hence $\pi = Id_{\mathcal{O}_2} \circ \pi$ is approximately unitarily equivalent to $(\psi \circ \phi) \circ \pi = \psi \circ (\phi \circ \pi)$ which is approximately unitarily equivalent to $\psi \circ (\phi \circ \sigma) = (\psi \circ \phi) \circ \sigma$ which is approximately unitarily equivalent to σ as desired. ☐

11 Embedding into $\mathcal{O}_2$

In this chapter we will finally prove our main result(Theorem 11.11) that every unital, separable, exact C*-algebra has a unital embedding into $\mathcal{O}_2$. The idea of the proof is to first prove that if $\mathfrak{A}$ is a unital, separable, exact C*-algebra that embeds into the ultraproduct of the Cuntz algebra then $\mathfrak{A}$ embeds into $\mathcal{O}_2$. This easily enables us to show that separable, exact, quasidiagonal C*-algebras embed into the Cuntz algebra. The remainder of the proof is to upgrade this result to separable, exact C*-algebras by showing that every separable, exact C*-algebra embeds into the reduced cross product of a separable, quasidiagonal, exact C*-algebra by $\mathbb{Z}$ and by showing such algebras embed into $\mathcal{O}_2$.

Most of the results for this chapter were developed from the paper [KP].

We begin with some simple notation for the chapter.

Notation 11.1. Let $\mathfrak{A}$ be a C*-algebra. We define

$$\ell_\infty(\mathfrak{A}) := \{(A_n)_{n\geq 1} \mid A_n \in \mathfrak{A}, \sup_{n\geq 1} \|A_n\| < \infty\}$$

and

$$c_0(\mathfrak{A}) := \{(A_n)_{n\geq 1} \mid A_n \in \mathfrak{A}, \lim_{n\to\infty} \|A_n\| = 0\}.$$

Let $\mathfrak{A}_\infty := \ell_\infty(\mathfrak{A})/c_0(\mathfrak{A})$ and let $q_\infty : \ell_\infty(\mathfrak{A}) \to \mathfrak{A}_\infty$ be the canonical quotient map.

Our first major step in the proof is the following lemma.

Lemma 11.2. *Let $\mathfrak{A}$ be a unital, separable, exact C*-algebra such that there exists an injective, unital *-homomorphism $\varphi : \mathfrak{A} \to (\mathcal{O}_2)_\infty$ with a lifting to a unital, completely positive map from $\mathfrak{A}$ to $\ell_\infty(\mathcal{O}_2)$. Then there is an injective, unital *-homomorphism from $\mathfrak{A}$ to $\mathcal{O}_2$.*

Proof. Since $\mathfrak{A}$ is unital and separable, there exists a sequence $(U_n)_{n\geq 1} \in \mathcal{U}(\mathfrak{A})$ with dense span in $\mathfrak{A}$. For each $n \geq 1$ let

$$\mathcal{S}_n := span\{I_{\mathfrak{A}}, U_1, U_1^*, U_2, U_2^*, \ldots, U_n, U_n^*\}$$

which is a finite dimensional operator system in $\mathfrak{A}$. Clearly $\mathbb{C}I_{\mathfrak{A}} \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots$ and the union of the $\mathcal{S}_n$'s is dense in $\mathfrak{A}$.

We claim that there exists a unital, injective *-homomorphism $\psi : \mathfrak{A} \to (\mathcal{O}_2)_\infty$ with a unital completely positive lifting $V : \mathfrak{A} \to \ell_\infty(\mathcal{O}_2)$ defined by

$$V(A) = (V_1(A), V_2(A), \ldots)$$

where $V_j : \mathfrak{A} \to \mathcal{O}_2$ are unital, completely positive maps such that for each fixed $n \in \mathbb{N}$ there exists an $N_n \in \mathbb{N}$ such that if $m \geq N_n$ then the restriction $V_m|_{\mathcal{S}_n}$ is injective and $\lim_{m\to\infty} \left\|(V_m|_{\mathcal{S}_n})^{-1}\right\|_k = 1$ for all $k \in \mathbb{N}$. Once ψ is constructed, we will be able to apply Lemma 10.5 to intertwine the V_m's on the unitaries U_j where $1 \leq j \leq m$ by unital, completely positive maps between $\mathcal{O}_2$. We will then be able to apply Proposition 10.4 and Lemma 10.8 to construct unitaries that intertwine the $I_{\mathcal{O}_2} \otimes V_m(U_j)$'s. Then it will be a simple matter to construct an injective *-homomorphism into $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$.

By the assumptions on φ there exists a unital, completely positive lifting $Q : \mathfrak{A} \to \ell_\infty(\mathcal{O}_2)$ of φ. Therefore there exists unital, completely positive maps $Q_j : \mathfrak{A} \to \mathcal{O}_2$ such that

$$Q(A) = (Q_1(A), Q_2(A), \ldots)$$

for all $A \in \mathfrak{A}$.

First we remark that even though φ is injective, it need not be the case that

$$\lim_{m\to\infty} \|(Q_m)_k(A)\| = \|A\|$$

for all $A \in \mathcal{M}_k(\mathfrak{A})$ as the limit on the left need not exists. However, by grouping the Q_m's into blocks, we will be close.

For each $N \in \mathbb{N}$ define the map $\varphi^{(N)} : \mathfrak{A} \to (\mathcal{O}_2)_\infty$ by

$$\varphi^{(N)}(A) := q_\infty(Q_{N+1}(A), Q_{N+2}(A), \ldots)$$

for all $A \in \mathfrak{A}$. Since Q is a lifting of φ and φ is a unital, injective *-homomorphism, it is trivial to verify that $\varphi^{(N)}$ is a unital, injective *-homomorphism. Moreover, it is trivial to see for every $N, k \in \mathbb{N}$ and $A \in \mathcal{M}_k(\mathfrak{A})$ that

$$\lim_{m\to\infty} \|((Q_{N+1})_k(A), (Q_{N+2})_k(A), \ldots, (Q_{N+m})_k(A))\| \geq \left\|\varphi_k^{(N)}(A)\right\| = \|A\|.$$

Since each $\mathcal{S}_n$ is finite dimensional, we can construct recursively a sequence

$$0 = N_1 < N_2 < \cdots < N_m < N_{m+1} < \cdots$$

of natural numbers such that

$$\left\|((Q_{N_m+1})_k(A), (Q_{N_m+2})_k(A), \ldots, (Q_{N_{m+1}})_k(A))\right\| \geq (1 - 2^{-m}) \|A\|$$

for all $k \leq m$ and $A \in \mathcal{M}_k(\mathcal{S}_m)$ (that is, choose a suitable approximation on a basis for each space and extend it to the entire space by using the fact that norms are equivalent).

By Theorem 1.23, for each $m \in \mathbb{N}$ there exists a unital, injective *-homomorphism $\sigma_m : \mathcal{O}_2^{\oplus N_{m+1}-N_m} \to \mathcal{M}_{N_{m+1}-N_m}(\mathcal{O}_2) \simeq \mathcal{O}_2$ by embedding along the diagonal. Define $V_m : \mathfrak{A} \to \mathcal{O}_2$ by

$$V_m(A) := \sigma_m(Q_{N_m+1}(A), Q_{N_m+2}(A), \ldots, Q_{N_{m+1}}(A)))$$

for all $A \in \mathfrak{A}$. Clearly V_m is a unital, completely positive map as σ_m is a *-homomorphism and each Q_j is a unital, completely positive map. Finally we define $V : \mathfrak{A} \to \ell_\infty(\mathcal{O}_2)$ by

$$V(A) := (V_1(A), V_2(A), \ldots)$$

for each $A \in \mathfrak{A}$. Clearly V is a unital, completely positive map. Moreover, for each $k, n \in \mathbb{N}$ we notice that for sufficiently large m that $V_m|_{\mathcal{S}_n}$ is injective and

$$\lim_{m\to\infty} \left\|(V_m|_{\mathcal{S}_n})^{-1}\right\|_k = 1$$

by the choice of the N_m's. By letting $k = 1$, we see that the contractive linear map $\psi := q_\infty \circ V : \mathfrak{A} \to (\mathcal{O}_2)_\infty$ has the property that $\|\psi(A)\| \geq \|A\|$ for all $A \in \bigcup_{n\geq 1} \mathcal{S}_n$. Hence, as $\bigcup_{n\geq 1} \mathcal{S}_n$ is dense in $\mathfrak{A}$, ψ is an isometric linear map and thus is injective. Moreover, since

$$\lim_{j\to\infty}(Q_j(AB) - Q_j(A)Q_j(B)) = 0$$

for all $A, B \in \mathfrak{A}$ as Q is a lifting of the *-homomorphism φ,

$$\lim_{m\to\infty}(V_m(AB) - V_m(A)V_m(B)) = 0$$

for all $A, B \in \mathfrak{A}$. Hence $\psi = q_\infty \circ V$ is a unital *-homomorphism and its lifting V satisfies the desired conditions.

Select a decreasing sequence of strictly positive scalars $(\delta_m)_{m\geq 1}$ such that $\delta_0 < 1$, $2\delta_m + 15\sqrt{5\delta_m} < 2^{-m}$. Since $\mathfrak{A}$ is unital, separable, and exact and $\mathcal{O}_2$ is unital, separable, and nuclear, by Lemma 10.5 there exists an increasing sequence of positive natural numbers $(k(m))_{m\geq 1}$ such that whenever $\Phi, \Psi : \mathcal{S}_m \to \mathcal{O}_2$ are unital, completely positive maps such that Φ is injective and

$$\left\|\Phi^{-1}\right\|_{k(m)} \leq 1 + \delta_m$$

then there exists a unital, completely positive map $\Theta : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\|\Theta \circ \Phi - \Psi\| < 2\delta_m$. The conditions on V clearly imply that we may pass to a subsequence in the variable m in such a way that $V_m|_{\mathcal{S}_n}$ is injective for all $n \leq m$, and we have the norm estimates

$$\begin{array}{rcl} \left\|(V_m|_{\mathcal{S}_n})^{-1}\right\|_{k(m)} & \leq & 1+\delta_m, \\ \|V_m(U_n)^*V_m(U_n) - I_{\mathcal{O}_2}\| & < & \delta_m, \text{ and} \\ \|V_m(U_n)V_m(U_n)^* - I_{\mathcal{O}_2}\| & < & \delta_m \end{array}$$

for all m and for all $n \leq m$.

By the above estimates, Lemma 10.5 implies that there exists unital, completely positive maps $\Psi_m, \Phi_m : \mathcal{O}_2 \to \mathcal{O}_2$ such that

$$\|\Phi_m \circ V_m|_{\mathcal{S}_m} - V_{m+1}|_{\mathcal{S}_m}\| \leq 2\delta_m \qquad \text{and} \qquad \|\Phi_m \circ V_{m+1}|_{\mathcal{S}_m} - V_m|_{\mathcal{S}_m}\| \leq 2\delta_{m+1} \leq 2\delta_m$$

as $V_m|_{\mathcal{S}_m}$ and $V_{m+1}|_{\mathcal{S}_m}$ are injective. Moreover, since $\delta_m < 1$ for all m, the above norm estimates imply that for all $1 \leq j \leq m$ the operator $X_m^{(j)} := V_m(U_j)|V_m(U_j)|^{-1}$ exists and is a unitary operator in $\mathcal{O}_2$. Moreover, Lemma 9.2 implies that

$$\left\|X_m^{(j)} - V_m(U_j)\right\| \leq \|V_m(U_n)^*V_m(U_n) - I_{\mathcal{O}_2}\| < \delta_m$$

for all $1 \leq j \leq m$. Hence

$$\left\|\Phi_m(X_m^{(j)}) - X_{m+1}^{(j)}\right\| \leq 2\delta_m + \|\Phi_m(V_m(U_j)) - V_{m+1}(U_j)\| < 4\delta_m$$

and

$$\left\|\Psi_m(X_{m+1}^{(j)}) - X_m^{(j)}\right\| \leq 2\delta_m + \|\Psi_m(V_{m+1}(U_j)) - V_m(U_j)\| < 4\delta_m$$

for all $1 \leq j \leq m$ by the above norm estimates. However, since $\mathcal{O}_2$ is a unital, simple, nuclear, purely infinite C*-algebra and $\Phi_m, \Psi_m : \mathcal{O}_2 \to \mathcal{O}_2$ are unital, completely positive maps, Proposition 10.4 implies that there exists isometries $S_m, T_m \in \mathcal{O}_2$ such that

$$\left\|T_m^*X_m^{(j)}T_m - \Phi_m(X_m^{(j)})\right\| < \delta_m \qquad \text{and} \qquad \left\|S_m^*X_{m+1}^{(j)}S_m - \Psi_m(X_{m+1}^{(j)})\right\| < \delta_m$$

for all $1 \leq j \leq m$. Hence we obtain that

$$\left\|T_m^*X_m^{(j)}T_m - X_{m+1}^{(j)}\right\| < 5\delta_m \qquad \text{and} \qquad \left\|S_m^*X_{m+1}^{(j)}S_m - X_m^{(j)}\right\| < 5\delta_m$$

for all $1 \leq j \leq m$. Since each $X_m^{(j)}$ is a unitary for all $1 \leq j \leq m$, by applying Lemma 10.8 with the C*-algebra $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$, $\mathfrak{D} := \mathcal{O}_2 \otimes (\mathbb{C}I_{\mathcal{O}_2}) \subseteq \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$, and the isometries $I_{\mathcal{O}_2} \otimes T_m$ and $I_{\mathcal{O}_2} \otimes S_m$, there exists unitaries $Z_m \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\begin{array}{rcl} \left\|Z_m(I_{\mathcal{O}_2} \otimes X_m^{(j)})Z_m^* - I_{\mathcal{O}_2} \otimes X_{m+1}^{(j)}\right\| & \leq & 15\left(\max\left\{\left\|T_m^*X_m^{(j)}T_m - X_{m+1}^{(j)}\right\|, \left\|S_m^*X_{m+1}^{(j)}S_m - X_m^{(j)}\right\|\right\}\right)^{\frac{1}{2}} \\ & \leq & 15\sqrt{5\delta_m}. \end{array}$$

Therefore, since $\left\|X_m^{(j)} - V_m(U_j)\right\| < \delta_m$ for all $1 \leq j \leq m$,

$$\|Z_m(I_{\mathcal{O}_2} \otimes V_m(U_j))Z_m^* - I_{\mathcal{O}_2} \otimes V_{m+1}(U_j)\| \leq 2\delta_m + 15\sqrt{5\delta_m} < 2^{-m}$$

for all $1 \leq j \leq m$.

For each $n \in \mathbb{N}$ define $Y_n := Z_1^*Z_2^* \cdots Z_n^*$ which is a unitary element of $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$. Moreover, by the above computation, $(Y_n(I_{\mathcal{O}_2} \otimes V_n(U_j))Y_n^*)_{n\geq 1}$ is a Cauchy sequence for all $j \in \mathbb{N}$. By linearity and as each map is contractive, $(Y_n(I_{\mathcal{O}_2} \otimes V_n(A))Y_n^*)_{n\geq 1}$ is a Cauchy sequence for all $A \in \bigcup_{n\geq 1} \mathcal{S}_n$. Therefore the map $\psi_0 : \bigcup_{n\geq 1} \mathcal{S}_n \to \mathcal{O}_2 \otimes \mathcal{O}_2$ defined by

$$\psi_0(A) = \lim_{n\to\infty} Y_n(I_{\mathcal{O}_2} \otimes V_n(A))Y_n^*$$

extends to a unital (as each V_n is unital), completely positive map $\psi : \mathfrak{A} \to \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2$. Since

$$\lim_{m\to\infty} (V_m(AB) - V_m(A)V_m(B)) = 0$$

for all $A, B \in \mathfrak{A}$, it is trivial to see that ψ is a *-homomorphism on $\bigcup_{n\geq 1} \mathcal{S}_n$ and thus is a *-homomorphism by continuity. Finally, for all $A \in \bigcup_{n\geq 1} \mathcal{S}_n$ we notice

$$\|\psi(A)\| = \lim_{n\to\infty} \|V_n(A)\| = \|A\|$$

(where the last equality follows since $\|V_n(A)\| \leq \|A\|$ for all $A \in \bigcup_{n\geq 1} \mathcal{S}_n$ and $\lim_{m\to\infty} \left\|(V_m|_{\mathcal{S}_n})^{-1}\right\| = 1$ for all $n \in \mathbb{N}$). Hence ψ is isometric on a dense subset of $\mathfrak{A}$ so ψ is isometric and thus injective.

Since $\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$ by Theorem 7.5, the proof is complete. $\square$

As mentioned in the introduction, the above lemma is directly suited to prove that every unital, separable, quasidiagonal, exact C*-algebra has a unital embedding into $\mathcal{O}_2$. The following is a definition of a quasidiagonal C*-algebra (although it is not the definition from the paper and probably not the original definition).

Definition 11.3. A unital, separable C*-algebra $\mathfrak{A}$ is said to be quasidiagonal if there exists a sequence of unital completely positive maps $\varphi_n : \mathfrak{A} \to \mathcal{M}_{k_n}(\mathbb{C})$ such that $\lim_{n\to\infty} \|\varphi_n(A)\| = \|A\|$ for all $A \in \mathfrak{A}$ and $\lim_{n\to\infty} \|\varphi_n(A)\varphi_n(B) - \varphi_n(AB)\| = 0$ for all $A, B \in \mathfrak{A}$.

Remarks 11.4. It is easy to see that the above definition is equivalent to the statement that there exists a unital *-isomorphism $\pi : \mathfrak{A} \to \left(\prod_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right) / \left(\bigoplus_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right)$ that has a unital, completely positive lifting $\Phi : \mathfrak{A} \to \prod_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})$. As a corollary of this, we have the following.

Corollary 11.5. *Let $\mathfrak{A}$ be a unital, separable, quasidiagonal, exact C*-algebra. Then there exists a unital, injective *-homomorphism from $\mathfrak{A}$ to $\mathcal{O}_2$.*

Proof. Since $\mathfrak{A}$ is unital, quasidiagonal C*-algebra there exists a unital *-isomorphism

$$\pi : \mathfrak{A} \to \left(\prod_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right) / \left(\bigoplus_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right)$$

that has a unital, completely positive lifting $\Phi : \mathfrak{A} \to \prod_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})$. Since there exists a unital copy of $\mathcal{M}_n(\mathbb{C})$ inside $\mathcal{O}_2$ for all $n \in \mathbb{N}$ (by Theorem 1.23 as $\mathcal{M}_n(\mathcal{O}_2) \simeq \mathcal{O}_2$), $\left(\prod_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right) / \left(\bigoplus_{n\geq 1} \mathcal{M}_{k_n}(\mathbb{C})\right)$ has a unital isometric embedding inside $(\mathcal{O}_2)_\infty$. Hence the conditions of Lemma 10.2 are satisfied so there exists a unital, injective *-homomorphism from $\mathfrak{A}$ to $\mathcal{O}_2$. $\square$

With the above case completed, we begin our preparations to prove the general case. To prove the general case, we will show that certain reduced cross products of separable, quasidiagonal, exact C*-algebras by $\mathbb{Z}$ embed into $\mathcal{O}_2$. In order to prove this, we will demonstrate the existence of certain injective *-homomorphisms and unital, completely positive liftings of maps from such reduced cross products.

Definition 11.6. Let $\mathfrak{A}$ be a C*-algebra, let G be a discrete group, and let $\alpha : G \to Aut(\mathfrak{A})$ be a *-homomorphism. A covariant representation (U, φ) of the system $(\mathfrak{A}, G, \alpha)$ on a unital C*-algebra $\mathfrak{B}$ is a group homomorphism $U : G \to \mathcal{U}(\mathfrak{B})$ together with a *-homomorphism $\varphi : \mathfrak{A} \to \mathfrak{B}$ such that

$$U(g)\varphi(A)U(g)^* = \varphi(\alpha_g(A))$$

for all $A \in \mathfrak{A}$ and $g \in G$.

We will prove the following result for any amenable, discrete group G although we will only apply this result when $G = \mathbb{Z}$.

Lemma 11.7. *Let G be a discrete amenable group and let $\alpha : G \to Aut(\mathfrak{A})$ be an action of G on a unital C^*-algebra $\mathfrak{A}$. Let (U, φ) be a covariant representation of the system $(\mathfrak{A}, G, \alpha)$ on a unital C^*-algebra $\mathfrak{B}$ with φ injective. Then there exists an injective unital *-homomorphism $\psi : \mathfrak{A} \rtimes_{\alpha,r} G \to C^*_\lambda(G) \otimes_{\min} \mathfrak{B}$ such that $\psi(A) = I_{C^*_\lambda(G)} \otimes \varphi(A)$ and $\psi(g) = g \otimes U(g)$ for all $A \in \mathfrak{A}$ and $g \in G$.*

Proof. Since G is a discrete amenable group, $\mathfrak{A} \rtimes_{\alpha,r} G = \mathfrak{A} \rtimes_\alpha G$. Therefore, to show that there exists a *-homomorphism $\psi : \mathfrak{A} \rtimes_{\alpha,r} G \to C^*_\lambda(G) \otimes_{\min} \mathfrak{B}$ such that $\psi(A) = I_{C^*_\lambda(G)} \otimes \varphi(A)$ and $\psi(g) = g \otimes U(g)$ for all $A \in \mathfrak{A}$ and $g \in G$, it suffices to show that the pair $A \mapsto I_{C^*_\lambda(G)} \otimes \varphi(A)$ and $g \mapsto g \otimes U(g)$ is a covariant representation of $(\mathfrak{A}, G, \alpha)$. However, it is clear that

$$(g \otimes U(g))(I_{C^*_\lambda(G)} \otimes \varphi(A))(g \otimes U(g))^* = I_{C^*_\lambda(G)} \otimes U(g)\varphi(A)U(g)^* = \varphi(\alpha_g(A))$$

for all $A \in \mathfrak{A}$ and $g \in G$. Hence the *-homomorphism ψ exists.

To show that ψ is injective, it suffices to show that ψ is unitarily equivalent to the canonical representation of $\mathfrak{A} \rtimes_{\alpha,r} G$. Let $\pi_0 : \mathfrak{B} \to \mathcal{B}(\mathcal{H}_0)$ be an injective unital representation of $\mathfrak{B}$ and let $\lambda : C^*_\lambda(G) \to \mathcal{B}(\ell_2(G))$ be the left regular representation. Let $\sigma := (\lambda \otimes \pi_0) \circ \psi : \mathfrak{A} \rtimes_{\alpha,r} G \to \mathcal{B}(\ell_2(G) \otimes \mathcal{H}_0)$ which is a *-homomorphism and let $\pi := \pi_0 \circ \varphi : \mathfrak{A} \to \mathcal{B}(\mathcal{H}_0)$ (which is a injective *-homomorphism as π_0 and φ are injective). If σ is unitarily equivalent to the canonical representation of $\mathfrak{A} \rtimes_{\alpha,r} G$ on $\ell_2(G) \otimes \mathcal{H}_0$ given by π, then σ will be injective and thus ψ will be injective.

Notice for all $A \in \mathfrak{A}$, $g \in G$, and $\xi \in \mathcal{H}_0$ that $A \in \mathfrak{A} \rtimes_{\alpha,r} G$ acts as

$$A(\delta_g \otimes \xi) = \delta_g \otimes \pi(\alpha_{g^{-1}}(A))\xi = \delta_g \otimes \pi_0(\varphi(\alpha_{g^{-1}}(A)))\xi$$

whereas

$$\sigma(A)(\delta_g \otimes \xi) = \delta_g \otimes \pi_0(\varphi(A))\xi.$$

Similarly, if $g \in G$, $g \in \mathfrak{A} \rtimes_{\alpha,r} G$ acts on $\ell_2(G) \otimes \mathcal{H}_0$ by $\lambda(g) \otimes I_{\mathcal{H}_0}$ whereas

$$\sigma(g) = (\lambda \otimes \pi_0)(\psi(g)) = \lambda(g) \otimes \pi_0(U(g)).$$

Define $V \in \mathcal{B}(\ell_2(G) \otimes \mathcal{H}_0)$ by $V(\delta_g \otimes \xi) = \delta_g \otimes \pi_0(U(g))\xi$ for all $g \in G$ and $\xi \in \mathcal{H}_0$ and by extending by linearity and density. Since π_0 is unital, it is clear that V is a unitary operator. Moreover

$$V^*\sigma(A)V(\delta_g \otimes \xi) = \delta_g \otimes \pi_0(U(g)^{-1})\pi_0(\varphi(A))\pi_0(U(g))\xi = \delta_g \otimes \pi_0(\varphi(\alpha_{g^{-1}}(A)))\xi = A(\delta_g \otimes \xi)$$

whereas

$$V^*\sigma(g)V(\delta_h \otimes \xi) = V^*\sigma(g)(\delta_h \otimes \pi_0(U(h))\xi) = V^*(\delta_{gh} \otimes \pi_0(U(gh))\xi) = \delta_{gh} \otimes \xi = (\lambda(g) \otimes I)(\delta_h \otimes \xi)$$

for all $A \in \mathfrak{A}$, and all $g, h \in G$. Hence σ is unitarily equivalent to the canonical representation of $\mathfrak{A} \rtimes_{\alpha,r} G$ on $\ell_2(G) \otimes \mathcal{H}_0$ so ψ is injective. ☐

Lemma 11.8. *Let $\mathfrak{B}$ be a unital C^*-algebra, let $\mathfrak{A}$ be a C^*-subalgebra of $\mathfrak{B}$ such that $I_{\mathfrak{B}} \in \mathfrak{A}$, and let $\sigma \in Aut(\mathfrak{A})$. Suppose that σ is approximately inner in $\mathfrak{B}$; that is, there exists a sequence $(V_n)_{n\geq 1} \in \mathcal{U}(\mathfrak{B})$ such that $\lim_{n\to\infty} V_n A V_n^* = \sigma(A)$ for all $A \in \mathfrak{A}$. Let z be the standard generator of $C(\mathbb{T})$ and let U be the canonical unitary in $\mathfrak{A} \rtimes_\sigma \mathbb{Z}$ which implements σ on $\mathfrak{A}$. Then the maps*

$$A \mapsto I_{\mathbb{C}(\mathbb{T})} \otimes q_\infty(A, A, A, \ldots) \qquad \textit{and} \qquad U \mapsto z \otimes q_\infty(V_1, V_2, V_3, \ldots)$$

(where $q_\infty : \ell_\infty(\mathfrak{B}) \to \mathfrak{B}_\infty$ is the canonical quotient map) define an injective, unital *-homomorphism $\varphi : \mathfrak{A} \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes_{\min} \mathfrak{B}_\infty$. Moreover, for any unital C^*-algebra $\mathfrak{C}$, this *-homomorphism extends continuously to an injective, unital *-homomorphism from $(\mathfrak{A} \rtimes_\sigma \mathbb{Z}) \otimes_{\min} \mathfrak{C}$ to $C(\mathbb{T}) \otimes_{\min} ((\ell_\infty(\mathfrak{B}) \otimes_{\min} \mathfrak{C})/(c_0(\mathfrak{B}) \otimes_{\min} \mathfrak{C}))$.

Proof. To show that these two maps define an injective, unital *-homomorphism $\varphi : \mathfrak{A} \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes_{\min} \mathfrak{B}_\infty$, it suffices by Lemma 11.7 to show that these two maps are a covariant representation of the system $(\mathfrak{A}, \mathbb{Z}, \sigma)$ on $\mathfrak{B}_\infty$ as the map for $\mathfrak{A}$ is clearly injective. However, by the assumptions of this lemma, it is clear that

$$(V_1, V_2, V_3, \ldots) \cdot (A, A, A, \ldots,) \cdot (V_1, V_2, V_3, \ldots)^* - (\sigma(A), \sigma(A), \sigma(A), \ldots) \in c_0(\mathfrak{B})$$

for all $A \in \mathfrak{A}$ and thus the two maps define a covariant representation of the system $(\mathfrak{A}, \mathbb{Z}, \sigma)$ on $\mathfrak{B}_\infty$.

For the second part of the lemma, we desire to apply the first part of the lemma to $(\mathfrak{A} \rtimes_\sigma \mathbb{Z}) \otimes_{\min} \mathfrak{C}$. By considering the definition of the reduced cross product C*-algebra, it is clear that

$$(\mathfrak{A} \rtimes_\sigma \mathbb{Z}) \otimes_{\min} \mathfrak{C} \simeq (\mathfrak{A} \otimes_{\min} \mathfrak{C}) \rtimes_{\sigma \otimes Id} \mathbb{Z}$$

and it is clear that

$$\lim_{n\to\infty} (V_n \otimes I_{\mathfrak{C}}) T (V_n \otimes I_{\mathfrak{C}})^* = (\sigma \otimes Id)(T)$$

for all $T \in \mathfrak{A} \otimes_{\min} \mathfrak{C}$ (as it clearly holds on the elementary tensors and thus extends to the span and then closure of the span of the elementary tensors). Hence $\mathfrak{A} \otimes_{\min} \mathfrak{C}$ is a C*-subalgebra of $\mathfrak{B} \otimes_{\min} \mathfrak{C}$ that contains the identity and $\sigma \otimes Id \in Aut(\mathfrak{A} \otimes_{\min} \mathfrak{C})$ is approximately inner in $\mathfrak{B} \otimes_{\min} \mathfrak{C}$ through the unitaries $(V_n \otimes I_{\mathfrak{C}})_{n\geq 1}$. Therefore the first part of the lemma implies that there exists an injective *-homomorphism

$$\psi : (\mathfrak{A} \rtimes_\sigma \mathbb{Z}) \otimes_{\min} \mathfrak{C} \to C(\mathbb{T}) \otimes_{\min} ((\ell_\infty(\mathfrak{B} \otimes_{\min} \mathfrak{C}))/(c_0(\mathfrak{B} \otimes_{\min} \mathfrak{C})))$$

where, if $q'_\infty : \ell_\infty(\mathfrak{B} \otimes_{\min} \mathfrak{C} \to (\ell_\infty(\mathfrak{B} \otimes_{\min} \mathfrak{C}))/(c_0(\mathfrak{B} \otimes_{\min} \mathfrak{C}))$ is the canonical quotient map

$$\psi(A) = I_{\mathbb{C}(\mathbb{T})} \otimes q'_\infty(A \otimes I_{\mathfrak{C}}, A \otimes I_{\mathfrak{C}}, A \otimes I_{\mathfrak{C}}, \ldots) \qquad \text{and} \qquad \psi(U) \otimes q'_\infty(V_1 \otimes I_{\mathfrak{C}}, V_2 \otimes I_{\mathfrak{C}}, V_3 \otimes I_{\mathfrak{C}}, \ldots).$$

However, a moments consideration of the minimal tensor product of C*-algebras implies that the canonical *-homomorphism

$$\Psi : (\ell_\infty(\mathfrak{B}) \otimes_{\min} \mathfrak{C})/(c_0(\mathfrak{B}) \otimes_{\min} \mathfrak{C}) \to (\ell_\infty(\mathfrak{B} \otimes_{\min} \mathfrak{C}))/(c_0(\mathfrak{B} \otimes_{\min} \mathfrak{C}))$$

is injective. Therefore, since the range of ψ (on the elementary tensors) is contained in the image of Ψ, the result follows. ☐

Lemma 11.9. *Let $\mathfrak{B}$ be a unital, separable, nuclear C*-algebra, let $\mathfrak{A}$ be a C*-subalgebra of $\mathfrak{B}$ such that $I_{\mathfrak{B}} \in \mathfrak{A}$, and let $\sigma \in Aut(\mathfrak{A})$ be approximately inner in $\mathfrak{B}$. Then the injective, unital *-homomorphism $\varphi : \mathfrak{A} \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes_{\min} \mathfrak{B}_\infty$ from Lemma 11.8 has a lifting to a unital, completely positive map $\psi : \mathfrak{A} \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathfrak{B})$.*

Proof. The proof of this result follows from Lemma 11.8 and Theorem 9.10 where, in Theorem 9.10, $\mathfrak{A}$ is $\mathfrak{A} \rtimes_\sigma \mathbb{Z}$, $\mathfrak{B}$ is $C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathfrak{B})$, $\mathfrak{J}$ is $C(\mathbb{T}) \otimes_{\min} c_0(\mathfrak{B})$, and φ is φ. To see this, notice that $C(\mathbb{T})$ is nuclear and $c_0(\mathfrak{B})$ is nuclear as $\mathfrak{B}$ is nuclear. Therefore $C(\mathbb{T}) \otimes_{\min} c_0(\mathfrak{B})$ is nuclear and thus an approximately injective ideal in $C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathfrak{B})$. By Lemma 11.8 φ extends to an injective *-homomorphism

$$\tilde{\varphi} : (\mathfrak{A} \rtimes_\sigma \mathbb{Z}) \otimes_{\min} \mathcal{B}(\mathcal{H}) \to C(\mathbb{T}) \otimes_{\min} \left(\frac{\ell_\infty(\mathfrak{B}) \otimes_{\min} \mathcal{B}(\mathcal{H})}{c_0(\mathfrak{B}) \otimes_{\min} \mathcal{B}(\mathcal{H})} \right).$$

Since $C(\mathbb{T})$ is nuclear,

$$C(\mathbb{T}) \otimes_{\min} \left(\frac{\ell_\infty(\mathfrak{B}) \otimes_{\min} \mathcal{B}(\mathcal{H})}{c_0(\mathfrak{B}) \otimes_{\min} \mathcal{B}(\mathcal{H})} \right) \simeq \left(\frac{(C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathfrak{B})) \otimes_{\min} \mathcal{B}(\mathcal{H})}{(C(\mathbb{T}) \otimes_{\min} c_0(\mathfrak{B})) \otimes_{\min} \mathcal{B}(\mathcal{H})} \right)$$

so Theorem 9.10 implies that φ has a unital, completely positive lifting $\psi : \mathfrak{A} \rtimes_\sigma \mathbb{Z} \to C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathfrak{B})$. ☐

Finally we have the following lemma that will enable us to create a copy of a C*-algebra $\mathfrak{A}$ inside a certain cross product.

Lemma 11.10. *Define $\tau_1 \in Aut(C_0(\mathbb{R}))$ by $\tau_1(f)(x) = f(x+1)$ for all $x \in \mathbb{R}$ and for all $f \in C_0(\mathbb{R})$. Then $C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}$ contains a non-zero projection.*

Proof. We will explicitly write down a non-zero projection. To begin, we define two elements $f, g \in C_0(\mathbb{R})$ by

$$f(x) = \begin{cases} 1+x & \text{if } x \in [-1,0] \\ 1-x & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(x) = \begin{cases} \sqrt{f(x) - f(x)^2} & \text{if } x \in [-1,0] \\ 0 & \text{otherwise} \end{cases}.$$

Clearly f and g are well-defined, positive elements of $C_0(\mathbb{R})$. Let U be the unitary in (the multiplier algebra of) $C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}$ such that $UhU^* = \tau_1(h)$ for all $h \in C_0(\mathbb{R})$ (to avoid using multiplier algebras, we can extend τ_1 to an automorphism of the unitization $\widetilde{C_0(\mathbb{R})}$ of $C_0(\mathbb{R})$ and view $C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z} \subseteq \widetilde{C_0(\mathbb{R})} \rtimes_{\tau_1} \mathbb{Z}$).

Let $P := gU + f + U^*g$ which is an element of $C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}$. We claim that P is a non-zero projection. Indeed if $\mathcal{E} : C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z} \to C_0(\mathbb{R})$ is the canonical conditional expectation then $\mathcal{E}(P) = f \neq 0$ so $P \neq 0$. To see that P is a projection, we note that P is clearly self-adjoint so it suffices to show that $P^2 = P$.

To show that $P^2 = P$, we will show several small facts that will enable us to show that $P^2 = P$. First notice that since g lives on $[-1,0]$ and $\tau_1(g)$ lives on $[0,1]$, $g\tau(g) = 0$. Moreover it is clear that

$$\tau_1(f)(x) = \begin{cases} 2+x & \text{if } x \in [-2,-1] \\ -x & \text{if } x \in [-1,0] \\ 0 & \text{otherwise} \end{cases}$$

and thus $(\tau_1(f) + f)|_{[-1,0]} = 1$. Next we notice that if $x \in [0,1]$ then

$$f(x-1) - f(x-1)^2 = x - x^2 = (1-x) - (1-x)^2 = f(x) - f(x)^2$$

so

$$\tau_1^{-1}(g^2)(x) = \begin{cases} f(x-1) - f(x-1)^2 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases} = \begin{cases} f(x) - f(x)^2 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}.$$

Finally, since

$$g^2(x) = \begin{cases} f(x) - f(x)^2 & \text{if } x \in [-1,0] \\ 0 & \text{otherwise} \end{cases}$$

it is clear that

$$g^2 + f^2 + \tau_1^{-1}(g^2) = f.$$

Hence

$$\begin{aligned} P^2 &= (gU + f + U^*g)(gU + f + U^*g) \\ &= gUgU + gUf + g^2 + fgU + f^2 + fU^*g + U^*g^2U + U^*gf + U^*gU^*g \\ &= g\tau_1(g)U^2 + gUf + g^2 + fgU + f^2 + fU^*g + \tau_1^{-1}(g^2) + U^*gf + U^*U^*\tau_1(g)g \\ &= gUf + g^2 + fgU + f^2 + fU^*g + \tau_1^{-1}(g^2) + U^*gf \\ &= f + gUf + fgU + fU^*g + U^*gf \\ &= f + \tau_1(f)gU + fgU + U^*\tau_1(f)g + U^*gf \\ &= f + (\tau_1(f) + f)gU + U^*g(\tau_1(f) + f) \\ &= f + gU + U^*g \end{aligned}$$

as g lives on $[-1,0]$ and $(\tau_1(f) + f)|_{[-1,0]} = 1$. ☐

Now, onto the star attraction.

Theorem 11.11. *Let $\mathfrak{A}$ be a unital, separable, exact C^*-algebra. Then there exists an injective, unital *-homomorphism from $\mathfrak{A}$ to $\mathcal{O}_2$.*

Proof. The idea of the proof is to embed $\mathfrak{A}$ into the cross product of a $\mathcal{O}_2$-embeddable C*-algebra against $\mathbb{Z}$ (in a not necessarily unital way) and to construct a unital embedding of this cross product into $\mathcal{O}_2$.

First we remark that the cone $C_0([0,1)) \otimes_{\min} \mathfrak{A}$ is separable, quasidiagonal, and exact (see Corollary 7.3.7 [BO] for the quasidiagonal claim). Let $\mathfrak{B}_0$ be the unitization of $C_0(\mathbb{R}) \otimes_{\min} \mathfrak{A}$. Since $\mathfrak{B}_0$ is the unitization of a subalgebra of $C_0([0,1)) \otimes_{\min} \mathfrak{A}$ (namely $C_0((0,1)) \otimes_{\min} \mathfrak{A}$), $\mathfrak{B}_0$ is a unital, quasidiagonal, separable, exact C*-algebra. Hence Corollary 11.5 implies that there exists a unital, injective *-homomorphism $\varphi_0 : \mathfrak{B}_0 \to \mathcal{O}_2$.

Define $\tau_1 \in Aut(C_0(\mathbb{R}))$ by $\tau_1(f)(x) = f(x+1)$ for all $x \in \mathbb{R}$ and for all $f \in C_0(\mathbb{R})$. Define $\tau \in Aut(\mathfrak{B}_0)$ by $\tau(I_{\mathfrak{B}_0}) = I_{\mathfrak{B}_0}$ and $\tau(f \otimes A) = \tau_1(f) \otimes A$ for all $f \in C_0(\mathbb{R})$ and $A \in \mathfrak{A}$. Finally let $\mathfrak{B} := \mathfrak{B}_0 \rtimes_\tau \mathbb{Z}$ which is a unital, separable, exact C*-algebra. We desire to construct an injective, unital *-homomorphism of $\mathfrak{B}$ into $(\mathcal{O}_2)_\infty$ that has a unital, completely positive lifting to $\ell_\infty(\mathcal{O}_2)$ so that we can apply Lemma 11.2 to get an injective, unital *-homomorphism of $\mathfrak{B}$ into $\mathcal{O}_2$.

To begin let $\psi_0 := \varphi_0 \circ \tau_1 : \mathcal{B}_0 \to \mathcal{O}_2$ which is an injective, unital *-homomorphism and let $\mu : \mathcal{O}_2 \otimes_{\min} \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism (that exists by Theorem 7.5). Therefore the unital *-homomorphisms $\varphi, \psi : \mathcal{B}_0 \to \mathcal{O}_2$ defined by $\varphi(B) = \mu(\varphi_0(B) \otimes I_{\mathcal{O}_2})$ and $\psi(B) = \mu(\psi_0(B) \otimes I_{\mathcal{O}_2})$ are injective. Therefore, by Theorem 10.10, ψ and φ are approximately unitarily equivalent. Hence, if we view $\mathfrak{B}_0$ as a unital C*-subalgebra of $\mathcal{O}_2$ via φ, then the automorphism $\tau_1 \in Aut(\mathfrak{B}_0)$ is approximately inner inside $\mathcal{O}_2$ and thus Lemma 11.8 implies that there exists an injective, unital *-homomorphism from $\mathfrak{B}$ into $C(\mathbb{T}) \otimes_{\min} (\mathcal{O}_2)_\infty$. Moreover Lemma 11.9 implies that there exists a unital, completely positive lifting of this *-homomorphism into $C(\mathbb{T}) \otimes_{\min} (\mathcal{O}_2)_\infty$.

It is clear to see that $C(\mathbb{T})$ has a canonical inclusion inside $\mathcal{O}_2$ (that is, just exhibit a unitary element with spectrum $\mathbb{T}$ which can easily be obtained by a positive element with spectrum $[0,1]$). Therefore, using the above maps, there exists a composition of injective *-homomorphisms

$$\mathfrak{B} \to C(\mathbb{T}) \otimes_{\min} (\mathcal{O}_2)_\infty \to \mathcal{O}_2 \otimes_{\min} (\mathcal{O}_2)_\infty \to (\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2)_\infty \simeq_\mu (\mathcal{O}_2)_\infty$$

with a unital, completely positive lifting

$$\mathfrak{B} \to C(\mathbb{T}) \otimes_{\min} \ell_\infty(\mathcal{O}_2) \to \mathcal{O}_2 \otimes_{\min} \ell_\infty(\mathcal{O}_2) \to \ell_\infty(\mathcal{O}_2 \otimes_{\min} \mathcal{O}_2) \simeq_\mu \ell_\infty(\mathcal{O}_2).$$

Hence there exists an injective, unital *-homomorphism $\gamma : \mathfrak{B} \to \mathcal{O}_2$ by Lemma 11.2.

Notice that $\mathfrak{B}$ contains the C*-algebra

$$(C_0(\mathbb{R}) \otimes_{\min} \mathfrak{A}) \rtimes_\tau \mathbb{Z} \simeq (C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}) \otimes_{\min} \mathfrak{A}.$$

However $C_0(\mathbb{R}) \rtimes_{\tau_1} \mathbb{Z}$ contains a non-zero projection by Lemma 11.10. Hence $(C_0(\mathbb{R}) \otimes_{\min} \mathfrak{A}) \rtimes_\tau \mathbb{Z}$ contains an isomorphic copy $\mathfrak{A}_0$ of $\mathfrak{A}$. Let $P \in \mathfrak{B}$ be the identity of $\mathfrak{A}_0$. Therefore the map $\gamma|_{\mathfrak{A}_0} : \mathfrak{A}_0 \to \gamma(P)\mathcal{O}_2\gamma(P)$ is an injective, unital *-homomorphism from $\mathfrak{A}$ into $\gamma(P)\mathcal{O}_2\gamma(P)$.

Since $\gamma(P) \neq 0$, we can consider the K_0-element of $\gamma(P)$. However, by Theorem 6.15, $[\gamma(P)]_0 = 0 = [I_{\mathcal{O}_2}]_0$ and thus $\gamma(P)$ is equivalent to the identity of $\mathcal{O}_2$ by K-Theory. Therefore there exists an isometry $V \in \mathcal{O}_2$ such that $VV^* = \gamma(P)$. Therefore, if S_1 and S_2 are the generators of $\mathcal{O}_2$, it is easy to see that VS_1V^* and VS_2V^* are isometries in $\gamma(P)\mathcal{O}_2\gamma(P)$ that generate $\gamma(P)\mathcal{O}_2\gamma(P)$ so $\gamma(P)\mathcal{O}_2\gamma(P) \simeq \mathcal{O}_2$ completing the proof. $\square$

12 $\mathcal{O}_2 \otimes_{\min} \mathfrak{A} \simeq \mathcal{O}_2$

In this chapter we will classify all C*-algebras $\mathfrak{A}$ such that $\mathcal{O}_2 \otimes_{\min} \mathfrak{A} \simeq \mathcal{O}_2$. We will easily be able to give necessary conditions on $\mathfrak{A}$ in order for $\mathcal{O}_2 \otimes_{\min} \mathfrak{A} \simeq \mathcal{O}_2$. To establish the converse, the idea is to show that if $\mathfrak{A}$ has the above properties (which are preserved under taking tensor products and direct limits) and $\mathfrak{A}$ has an asymptotically central inclusion of $\mathcal{O}_2$, then $\mathfrak{A} \simeq \mathcal{O}_2$. This later result will be proven by using approximate unitary equivalence of injective *-homomorphism along with examining the relative commutant of $\mathfrak{A}$ in its natural embedding into an ultraproduct of $\mathfrak{A}$.

Most of the results for this chapter were developed from the paper [KP]. Definition 12.6 and Proposition 12.7 are from the book [Ro2].

We begin by establishing the necessary conditions.

Remarks 12.1. If $\mathfrak{A}$ is a C*-algebra such that $\mathfrak{A}\otimes_{\min}\mathcal{O}_2 \simeq \mathcal{O}_2$, then $\mathfrak{A}$ must be unital. To see this, represent $\mathfrak{A}$ and $\mathcal{O}_2$ faithful and non-degenerately on separable Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Since $\mathfrak{A} \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$, $\mathfrak{A} \otimes_{\min} \mathcal{O}_2$ is unital. Let $I_0 \in \mathfrak{A} \otimes_{\min} \mathcal{O}_2$ be the unit and let $(E_\lambda)_\Lambda$ be a C*-bounded approximate identity of $\mathfrak{A}$. Then $\lim_\Lambda (E_\lambda \otimes I_{\mathcal{O}_2})(\xi \otimes \eta) = \xi \otimes \eta$ for all $\xi \otimes \eta \in \mathcal{H} \otimes \mathcal{K}$. Hence, by linearity and density, $\lim_\Lambda (E_\lambda \otimes I_{\mathcal{O}_2})(\zeta) = \zeta$ for all $\zeta \in \mathcal{H} \otimes \mathcal{K}$. Thus

$$I_0\zeta = \lim_\Lambda I_0(E_\lambda \otimes I_{\mathcal{O}_2})\zeta = \lim_\Lambda (E_\lambda \otimes I_{\mathcal{O}_2})\zeta = \zeta$$

for all $\zeta \in \mathcal{H} \otimes \mathcal{K}$. Hence $I_0 = I_\mathcal{H} \otimes I_\mathcal{K}$. Therefore for all $\epsilon > 0$ there exists $A_i \in \mathfrak{A}$ and $B_i \in \mathcal{O}_2$ such that $\|I_0 - \sum_{i=1}^n A_i \otimes B_i\| < \epsilon$. Therefore, as $I_0 = I_\mathcal{H} \otimes I_\mathcal{K}$, by taking the inner product against all vectors of the form $\xi \otimes \eta$ where η is a fixed unit vector in $\mathcal{K}$, we obtain that $\|I_\mathcal{H} - \sum_{i=1}^n \lambda_i A_i\| < \epsilon$ where λ_i are scalars. Hence $I_\mathcal{H} \in \mathfrak{A}$ so $\mathfrak{A}$ is unital as desired.

Moreover, if $\mathfrak{A}$ is a C*-algebra such that $\mathfrak{A} \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$ then it is clear that $\mathfrak{A}$ is separable and simple (as $\mathcal{O}_2$ is simple by Theorem 1.15 so $\mathfrak{A} \otimes_{\min} \mathcal{O}_2$ is simple if and only if $\mathfrak{A}$ is simple by Proposition 3.4). We also claim that $\mathfrak{A}$ must be nuclear. To see this, we notice that if $\mathfrak{B}$ is any C*-algebra and the canonical *-homomorphism $\pi : \mathfrak{B} \otimes_{\max} \mathfrak{A} \to \mathfrak{B} \otimes_{\min} \mathfrak{A}$ has non-trivial kernel, then the canonical *-homomorphism $\pi \otimes Id_{\mathcal{O}_2} : (\mathfrak{B}\otimes_{\max}\mathfrak{A})\otimes_{\max}\mathcal{O}_2 \to (\mathfrak{B}\otimes_{\min}\mathfrak{A})\otimes_{\max}\mathcal{O}_2 = (\mathfrak{B}\otimes_{\min}\mathfrak{A})\otimes_{\min}\mathcal{O}_2$ has non-trivial kernel. Therefore, since the maximal and minimal tensor products of C*-algebras are associative, the canonical inclusion

$$\mathfrak{B} \otimes_{\max} (\mathfrak{A} \otimes_{\min} \mathcal{O}_2) = \mathfrak{B} \otimes_{\max} (\mathfrak{A} \otimes_{\max} \mathcal{O}_2) \hookrightarrow \mathfrak{B} \otimes_{\min} (\mathfrak{A} \otimes_{\min} \mathcal{O}_2)$$

has non-trivial kernel which contradicts the fact that $\mathfrak{A} \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$ is nuclear.

To see that the above conditions are sufficient for $\mathfrak{A} \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$, we begin by examining the relative commutant of a C*-algebra $\mathfrak{A}$ inside its ultraproduct. We will show that, under the necessary conditions on $\mathfrak{A}$, there exists non-unitary isometries in relative commutant of $\mathfrak{A}$ inside its ultraproduct and this will enable us to show that the relative commutant is a unital, simple, purely infinite C*-algebra.

Notation 12.2. Let $\mathfrak{A}$ be a C*-algebra and let ω be an ultrafilter. We define

$$c_\omega(\mathfrak{A}) := \{(A_n)_{n\geq 1} \mid A_n \in \mathfrak{A}, \lim_{n\to\omega} \|A_n\| = 0\}.$$

Let $\mathfrak{A}_\omega := \ell_\infty(\mathfrak{A})/c_\omega(\mathfrak{A})$ and let $q_\omega : \ell_\infty(\mathfrak{A}) \to \mathfrak{A}_\omega$ be the canonical quotient map. Thus $\|q_\omega((A_n)_{n\geq 1})\| = \lim_{n\to\omega} \|A_n\|$.

Remarks 12.3. Consider the *-homomorphism $\pi : \mathfrak{A} \to \ell_\infty(\mathfrak{A})$ defined by $\pi(A) = (A)_{n\geq 1}$. It is easy to see that $q_\omega \circ \pi : \mathfrak{A} \to \mathfrak{A}_\omega$ is a unital, injective *-homomorphism for all ultrafilters ω. Thus there exists a canonical inclusion of $\mathfrak{A}$ inside $\mathfrak{A}_\omega$. Hence we will view $\mathfrak{A} \subseteq \mathfrak{A}_\omega$ via this canonical inclusion.

Moreover, we will use $\mathfrak{A}' \cap \mathfrak{A}_\omega$ to denote the set of all $T \in \mathfrak{A}_\omega$ such that T commutes with $q_\omega(\pi(A))$ for all $A \in \mathfrak{A}$.

Our first goal is to prove Proposition 12.5 which states $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is unital, simple, and purely infinite provided $\mathfrak{A}$ is a unital, separable, simple, nuclear, purely infinite C*-algebra. The main technical requirement is the following lemma.

Lemma 12.4. *Let $\mathfrak{A}$ be a unital, separable, simple, nuclear, purely infinite C*-algebra and let ω be an ultrafilter. Suppose $A, B \in \mathfrak{A}' \cap \mathfrak{A}_\omega$ are self-adjoint operators such that $\sigma(B) \subseteq \sigma(A)$. Then there exists a non-unitary isometry $S \in \mathfrak{A}' \cap \mathfrak{A}_\omega$ such that SS^* commutes with A and $S^*AS = B$.*

Proof. The idea of the proof is to modify A and B into unitaries with certain spectra. Then we will use certain *-homomorphisms into $\mathfrak{A}_\omega$ from nuclear C*-algebras to obtain liftings to sequence of unital, completely positive maps into $\mathfrak{A}$. We will then use the fact that $\mathfrak{A}$ is unital, simple, and purely infinite along with Lemma 10.5 to intertwine these completely positive maps and then an application of Proposition 10.4 will enable us to construct our non-unitary partial isometry.

First, by scaling A and B by the same non-zero positive scalar, we may assume that that $\|A\|, \|B\| \leq \frac{\pi}{2}$. Let $W := e^{iA}$ which is a unitary element of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ such that $X := \sigma(W)$ is contained in the intersection of the right half plane with the unit circle. Let $z \in C(X)$ be the standard generating unitary. Since $C(X)$ is nuclear and the *-homomorphisms on $\mathfrak{A}$ and $C(X)$ defined by

$$T \mapsto q_\omega(T, T, T, \ldots) \qquad \text{and} \qquad f(z) \mapsto f(W)$$

have commuting ranges (as $W \in \mathfrak{A}' \cap \mathfrak{A}_\omega$), these *-homomorphisms define a unital *-homomorphism $\varphi : C(X) \otimes_{\min} \mathfrak{A} \to \mathfrak{A}_\omega$. Similarly, $C(X)$ is nuclear and the *-homomorphisms on $\mathfrak{A}$ and $C(X)$ defined by

$$T \mapsto q_\omega(T, T, T, \ldots) \qquad \text{and} \qquad f(z) \mapsto f(e^{iB})$$

have commuting ranges (as $B \in \mathfrak{A}' \cap \mathfrak{A}_\omega$), these *-homomorphisms define a unital *-homomorphism $\psi : C(X) \otimes_{\min} \mathfrak{A} \to \mathfrak{A}_\omega$.

We claim that φ is an injective *-homomorphism. To see this, suppose to the contrary that φ is not injective. Note that all of the ideals of $C(X) \otimes_{\min} \mathfrak{A}$ are of the form $C_0(U) \otimes_{\min} \mathfrak{A}$ where U is an open subset of X. Thus, if φ is not injective, $ker(\varphi) = C_0(U) \otimes_{\min} \mathfrak{A}$ for some non-empty open subset U of X. Since U is non-empty, there exists a non-zero element $f \in C_0(U)$. However, this implies that $0 = \varphi(f \otimes I_{\mathfrak{A}}) = f(W)$ which implies $f = 0$ on $\sigma(W) = X \supseteq U$ which is a contradiction. Hence φ is injective. We note that ψ need not be injective when $\sigma(B) \subsetneq \sigma(A)$.

Since $C(X) \otimes_{\min} \mathfrak{A}$ is nuclear (being the tensor product of nuclear C*-algebras), Theorem 9.12 implies that there exists unital, completely positive map $\Phi, \Psi : C(X) \otimes_{\min} \mathfrak{A} \to \ell_\infty(\mathfrak{A})$ that lift φ and ψ. Therefore there exists unital, completely positive maps $\Phi_m, \Psi_m : C(X) \otimes_{\min} \mathfrak{A} \to \mathfrak{A}$ such that

$$\Phi(T) = (\Phi_1(T), \Phi_2(T), \ldots) \qquad \text{and} \qquad \Psi(T) = (\Psi_1(T), \Psi_2(T), \ldots)$$

for all $T \in C(X) \otimes_{\min} \mathfrak{A}$.

In order to apply Lemma 10.5, we need to choose an increasing sequence of finite dimensional operator systems and choose suitable Φ_m and Ψ_m to satisfy the conditions of the lemma. Then we can apply Proposition 10.4 to obtain our sequence of non-unitary isometries.

Since $\mathfrak{A}$ is a unital, separable C*-algebra, there exists a sequence of unitaries $(U_n)_{n\geq 1}$ with dense span in $\mathfrak{A}$. Consider the increasing sequence of operator systems of $C(X) \otimes_{\min} \mathfrak{A}$ defined by

$$\mathcal{S}_n := span\{I_{C(X)} \otimes I_{\mathfrak{A}}, z \otimes I_{\mathfrak{A}}, z^* \otimes I_{\mathfrak{A}}, I_{C(X)} \otimes U_1, I_{C(X)} \otimes U_1^*, \ldots I_{C(X)} \otimes U_n, I_{C(X)} \otimes U_n^*\}.$$

For each $n, k \in \mathbb{N}$ and $T \in \mathcal{M}_k(\mathcal{S}_n)$ it is clear that

$$\lim_{m\to\omega} \|(\Phi_m)_k(T)\| = \|(\varphi)_k(T)\| = \|T\|$$

since φ was an injective *-homomorphism. Therefore, since each $\mathcal{S}_n$ was finite dimensional, if follows that there exists a neighbourhood V of ω in $\beta\mathbb{N}$ such that for all $m \in V \cap \mathbb{N}$ the map $\Phi_m|_{\mathcal{S}_n}$ is invertible. Moreover we obtain that $\lim_{m\to\omega} \left\|(\Phi_m|_{\mathcal{S}_n})^{-1}\right\|_k = 1$ for all $n, k \in \mathbb{N}$.

With the above construction in hand, we are in a perfect position to apply Lemma 10.5. Since $\mathfrak{A}$ is a unital, separable, nuclear C*-algebra, Lemma 10.5 implies there exists an increasing sequence of positive integers $(k(m))_{m\geq 1}$ such that whenever $\theta_1, \theta_2 : \mathcal{S}_m \to \mathfrak{A}$ are unital, completely positive maps with θ_1

injective and $\left\|\theta_1^{-1}\right\|_{k(m)} \leq 1 + \frac{1}{m}$ then there exists a unital completely positive map $\Theta : \mathfrak{A} \to \mathfrak{A}$ such that $\|\Theta \circ \theta_1 - \theta_2\| < \frac{2}{m}$. Choose a decreasing sequence of neighbourhoods $V_1 \supseteq V_2 \supseteq \cdots$ of ω in $\beta\mathbb{N}$ such that $\left\|(\Phi_\ell|_{\mathcal{S}_m})^{-1}\right\|_{k(m)} \leq 1 + \frac{1}{m}$ for all $\ell \in V_m \cap \mathbb{N}$. By replacing V_m with $V_m \setminus \{1, 2, \ldots, m\}$, we may assume that $\mathbb{N} \cap (\bigcap_{m=1}^\infty V_m) = \emptyset$.

Note by the structure of the topology on $\beta\mathbb{N}$, $V_m \cap \mathbb{N} \neq \emptyset$ for all $m \in \mathbb{N}$. For each $m \in \mathbb{N}$ and $\ell \in (V_m \setminus V_{m+1}) \cap \mathbb{N}$, Lemma 10.5 implies there exists a unital, completely positive map $\Theta_\ell : \mathfrak{A} \to \mathfrak{A}$ such that $\|\Theta_\ell \circ \Phi_\ell|_{\mathcal{S}_m} - \Psi_\ell|_{\mathcal{S}_m}\| < \frac{2}{m}$. Therefore, by applying Proposition 10.4 to Θ_ℓ (as $\mathfrak{A}$ is unital, nuclear, simple, and purely infinite) and by applying the fact that $\mathcal{S}_m$ is finite dimensional, there exists non-unitary isometries $S_\ell \in \mathfrak{A}$ such that $\|S_\ell^* \Phi_\ell(T) S_\ell - \Psi_\ell(T)\| \leq \frac{3\|T\|}{m}$ for all $T \in \mathcal{S}_m$. Since the $\mathcal{S}_n$'s are increasing and the V_m's are decreasing, we obtain that $\|S_\ell^* \Phi_\ell(T) S_\ell - \Psi_\ell(T)\| \leq \frac{3\|T\|}{m}$ for all $T \in \mathcal{S}_m$ and for all $\ell \in V_m \cap \mathbb{N}$.

For each $\ell \in \mathbb{N} \setminus V_1$, let S_ℓ be a arbitrary, non-unitary isometry in $\mathfrak{A}$ (which clearly exists as $\mathfrak{A}$ is a unital, simple, purely infinite C*-algebra). Since $\mathbb{N} \cap (\bigcap_{m=1}^\infty V_m) = \emptyset$, we have constructed an S_ℓ for each $\ell \in \mathbb{N}$. Let $S := q_\omega(S_1, S_2, S_3, \ldots) \in \mathfrak{A}_\omega$. Clearly S is an isometry in $\mathfrak{A}_\omega$ as $S^*S = q_\omega(I_\mathfrak{A}, I_\mathfrak{A}, \ldots)$. Moreover, S is not a unitary since

$$\|SS^* - I_{\mathfrak{A}_\omega}\| = \lim_{m \to \omega} \|S_m S_m^* - I_\mathfrak{A}\| = 1$$

as each S_m is not a unitary.

Next we claim for each fixed $n \in \mathbb{N}$ that $\lim_{\ell \to \omega} \|S_\ell^* U_n S_\ell - U_n\| = 0$. To see this, fix $n \in \mathbb{N}$. We notice that if $m \geq n$ and $\ell \in V_m \cap \mathbb{N}$ then

$$\|S_\ell^* U_n S_\ell - U_n\| \leq \frac{3}{m} + \left\|\Phi_\ell(I_{C(X)} \otimes U_n) - U_n\right\| + \left\|\Psi_\ell(I_{C(X)} \otimes U_n) - U_n\right\|.$$

Let $\epsilon > 0$. Fix an integer $m \geq \max\{n, \frac{9}{\epsilon}\}$. Since $q_\omega(\Phi(I_{C(X)} \otimes U_n)) = \varphi(U_n) = U_n \in \mathfrak{A} \cap \mathfrak{A}_\omega$ and $q_\omega(\Psi(I_{C(X)} \otimes U_n)) = \psi(U_n) = U_n \in \mathfrak{A} \cap \mathfrak{A}_\omega$, we can find a neighbourhood V of ω such that if $\ell \in V \cap \mathbb{N}$ then

$$\left\|\Phi_\ell(I_{C(X)} \otimes U_n) - U_n\right\|, \left\|\Psi_\ell(I_{C(X)} \otimes U_n) - U_n\right\| < \frac{\epsilon}{3}.$$

Therefore $V_m \cap V$ is a neighbourhood of ω such that if $\ell \in V_m \cap V \cap \mathbb{N}$ then $\|S_\ell^* U_n S_\ell - U_n\| < \epsilon$. Hence $\lim_{\ell \to \omega} \|S_\ell^* U_n S_\ell - U_n\| = 0$ as desired.

Therefore, when we view $U_n \in \mathfrak{A} \cap \mathfrak{A}_\omega$, the limit $\lim_{\ell \to \omega} \|S_\ell^* U_n S_\ell - U_n\| = 0$ implies $S^* U_n S = U_n$ for all $n \in \mathbb{N}$. Since each U_n is a unitary, Lemma 10.7 implies that U_n commutes with SS^*. Hence

$$SU_n = S(S^* U_n S) = (SS^*) U_n S = U_n (SS^*) S = U_n S$$

for all $n \in \mathbb{N}$. Therefore, since the span of $\{U_n\}_{n \geq 1}$ is dense in $\mathfrak{A}$, $S \in \mathfrak{A}' \cap \mathfrak{A}_\omega$.

Next we claim that $\lim_{\ell \to \omega} \|S_\ell^* \Phi_\ell(z \otimes I_\mathfrak{A}) S_\ell - \Psi_\ell(z \otimes I_\mathfrak{A})\| = 0$. To see this, we notice that if $\ell \in V_m \cap \mathbb{N}$ then

$$\|S_\ell^* \Phi_\ell(z \otimes I_\mathfrak{A}) S_\ell - \Psi_\ell(z \otimes I_\mathfrak{A})\| \leq \frac{3}{m}.$$

Thus, by choosing m suitably large, we obtain that $\lim_{\ell \to \omega} \|S_\ell^* \Phi_\ell(z \otimes I_\mathfrak{A}) S_\ell - \Psi_\ell(z \otimes I_\mathfrak{A})\| = 0$ as desired. Hence $S^* W S = e^{iB}$. Therefore, again by Lemma 10.7, SS^* commutes with W.

Since SS^* commutes with W, SS^* commutes with W^*. Therefore, if $\mathfrak{C} := C^*(W) \subseteq \mathfrak{A}_\omega$ (note $A \in \mathfrak{C}$), then SS^* commutes with $\mathfrak{C}$ so the map $T \mapsto S^* T S$ from $\mathfrak{C}$ to $\mathfrak{A}_\omega$ is a *-homomorphism. Therefore $S^* f(V) S = f(S^* V S)$ for all $f \in C(X)$. By letting $f(x) := -i \ln(x)$ for all $x \in X$ (where we choose the principle branch), we obtain that $S^* A S = B$ as desired. □

Proposition 12.5. *Let $\mathfrak{A}$ be a unital, separable, simple, nuclear, purely infinite C*-algebra and let ω be an ultrafilter. Then $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is unital, simple, and purely infinite.*

Proof. Clearly $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is unital. To show that $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is simple and purely infinite, we will show that every hereditary C*-subalgebra $\mathfrak{B}$ of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ has a non-zero projection that is equivalent to the identity. This implies $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is simple since every ideal of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is hereditary and if an ideal of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ contains a projection equivalent to the identity then it contains the identity. This also implies $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is purely infinite as if $V \in \mathfrak{A}' \cap \mathfrak{A}_\omega$ is an isometry such that $P := VV^* \in \mathfrak{B}$ and $P \neq I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$ (as if $P = I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$ then $\mathfrak{B} = \mathfrak{A}' \cap \mathfrak{A}_\omega$) then $W := VP$ is a partial isometry with $W^*W = P$ and $WW^* = VPV^* < VV^* = P$ and $W = VP = PVP \in \mathfrak{B}$ (as $\mathfrak{B}$ is hereditary).

Let $\mathfrak{B}$ be a hereditary C*-subalgebra of $\mathfrak{A}'\cap\mathfrak{A}_\omega$. Let $C \in \mathfrak{B}$ be an arbitrary non-zero self-adjoint operator with $1 \in \sigma(C)$ (which clearly exists). By the previous lemma with $A = C$ and $B = I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$, there exists a non-unitary isometry S in $\mathfrak{A}' \cap \mathfrak{A}_\omega$ such that $S^*CS = I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$ and the projection $P = SS^*$ commutes with C. Therefore P is a non-zero projection that is equivalent to $I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$ in $\mathfrak{A}' \cap \mathfrak{A}_\omega$. Since P commutes with C and since $PCP = S(S^*CS)S^* = SS^* = P$, we obtain that $P = P^2 = (PCP)^2 = PCPCP = CPC \in \mathfrak{B}$ as $\mathfrak{B}$ is hereditary. ☐

The next step in our proof is a slight detour. We need the ability to show that two C*-algebras are isomorphic if there exists certain unital *-homomorphisms between them. Unfortunately Lemma 7.1 is not enough. As it is simpler and clearer to prove the result in the most general setting, we have the following definition.

Definition 12.6. Let $(\mathfrak{A}_n)_{n\geq1}$ and $(\mathfrak{B}_n)_{n\geq1}$ be sequences of unital C*-algebras with injective unital *-homomorphisms $\alpha_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ and $\beta_n : \mathfrak{B}_n \to \mathfrak{B}_{n+1}$. For each $m \geq n$ let $\alpha_{m,n} := \alpha_{m-1} \circ \cdots \circ \alpha_n : \mathfrak{A}_n \to \mathfrak{A}_m$ and $\beta_{m,n} := \beta_{m-1} \circ \cdots \circ \beta_n : \mathfrak{B}_n \to \mathfrak{B}_m$ which are injective unital *-homomorphisms. For each $n \in \mathbb{N}$ let $\varphi_n : \mathfrak{A}_n \to \mathfrak{B}_{n+1}$ and let $\psi_n : \mathfrak{B}_n \to \mathfrak{A}_n$ be *-homomorphisms. We say that the these sequences of *-homomorphisms are approximately intertwining if there exists a sequence $(\delta_n)_{n\geq1}$ of positive numbers and finite subsets $F_n \subseteq \mathfrak{A}_n$ and $G_n \subseteq \mathfrak{B}_n$ such that

1. $\|\psi_{n+1}(\varphi_n(A)) - \alpha_n(A)\| < \delta_n$ for all $A \in F_n$,
2. $\|\varphi_n(\psi_n(B)) - \beta_n(B)\| < \delta_n$ for all $B \in G_n$,
3. $\alpha_n(F_n) \subseteq F_{n+1}$, $\varphi_n(F_n) \subseteq G_{n+1}$, $\beta(G_n) \subseteq G_{n+1}$, and $\psi_n(G_n) \subseteq F_n$ for all $n \in \mathbb{N}$,
4. $\bigcup_{m=n}^\infty \alpha_{m,n}^{-1}(F_m)$ is dense in $\mathfrak{A}_n$ and $\bigcup_{m=n}^\infty \beta_{m,n}^{-1}(G_m)$ is dense in $\mathfrak{B}_n$ for all n, and
5. $\sum_{n=1}^\infty \delta_n < \infty$.

The point of the above definition is that, if we take the direct limits of our sequences of C*-algebras, this approximate intertwining property enables us to conclude that the direct limits are isomorphic.

Proposition 12.7. *With the notation of Definition 12.6, if $\mathfrak{A} := \lim_\to \mathfrak{A}_n$ and $\mathfrak{B} := \lim_\to \mathfrak{B}_n$, then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic. Specifically there exists unital *-isomorphisms $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathfrak{A}$ such that $\psi = \varphi^{-1}$,*

$$\varphi(\alpha_{\infty,n}(A)) = \lim_{m\to\infty} (\beta_{\infty,m+1} \circ \varphi_m \circ \alpha_{m,n})(A)$$

for all $A \in \mathfrak{A}_n$, and

$$\psi(\beta_{\infty,n}(B)) = \lim_{m\to\infty} (\alpha_{\infty,m} \circ \psi_m \circ \beta_{m,n})(B)$$

for all $B \in \mathfrak{B}_n$ (where $\alpha_{\infty,n} : \mathfrak{A}_n \to \mathfrak{A}$ and $\beta_{\infty,n} : \mathfrak{B}_n \to \mathfrak{B}$ are the canonical inclusions).

Proof. First we will show that the two limits illustrated in the proposition exist and define *-homomorphisms on $\mathfrak{A}$. To see this, we will only demonstrate the first. By density, it suffices to check that φ is a *-homomorphism on $\bigcup_{n\geq1} \mathfrak{A}_n$. Fix $A \in \mathfrak{A}_n$. By the third and fourth assumptions from Definition 12.6, it suffices to consider $A \in \mathfrak{A}_n$ where there exists an $m_0 \in \mathbb{N}$ such that $\alpha_{m,n}(A) \in F_m$ for all $m \geq m_0 \geq n$. However, notice by the first three assumptions of Definition 12.6 that for all $T \in F_m \subseteq \mathfrak{A}_m$

$$\begin{aligned}
&\|\varphi_{m+1}(\alpha_m(T)) - (\beta_{m+1}(\varphi_m(T)))\| \\
\leq\ &\|\varphi_{m+1}(\alpha_m(T)) - (\varphi_{m+1} \circ \psi_{m+1})(\varphi_m(T))\| + \|(\varphi_{m+1} \circ \psi_{m+1})(\varphi_m(T)) - (\beta_{m+1}(\varphi_m(T)))\| \\
\leq\ &\|\alpha_m(T) - \psi_{m+1}(\varphi_m(T))\| + \|(\varphi_{m+1} \circ \psi_{m+1})(\varphi_m(T)) - (\beta_{m+1}(\varphi_m(T)))\| \\
\leq\ &\delta_m + \delta_{m+1}
\end{aligned}$$

as $T \in F_m$ and $\varphi_m(T) \in G_{m+1}$. Hence for all $m \geq m_0$

$$\begin{aligned} & \|(\beta_{\infty,m+2} \circ \varphi_{m+1} \circ \alpha_{m+1,n})(A) - (\beta_{\infty,m+1} \circ \varphi_m \circ \alpha_{m,n})(A)\| \\ = \quad & \|\beta_{\infty,m+2}((\varphi_{m+1} \circ \alpha_m)(\alpha_{m,n}(A))) - \beta_{\infty,m+2}((\beta_{m+1} \circ \varphi_m)((\alpha_{m,n})(A)))\| \leq \delta_m + \delta_{m+1} \end{aligned}$$

for all $m \geq m_0$. Therefore, the fifth assumption of Definition 12.6 implies that the sequence under consideration is Cauchy and thus converges. Hence φ is a well-defined map. Since φ is a limit of *-homomorphisms on a dense set, φ is a *-homomorphism. Clearly φ is unital.

By similar arguments, it is clear that ψ is also a well-defined unital *-homomorphism. To verify that $\psi = \varphi^{-1}$, it suffices to verify $\psi(\varphi(\alpha_{\infty,n}(A))) = \alpha_{\infty,n}(A)$ for all $A \in \mathfrak{A}_n$ and all $n \in \mathbb{N}$ and $\varphi(\psi(B)) = B$ for all $B \in \mathfrak{B}_n$ and all $n \in \mathbb{N}$. We will only verify the first as the other will follow by symmetry. To see the first, fix $A \in \mathfrak{A}_n$. Again, by the third and fourth assumptions from Definition 12.6, it suffices to consider $A \in \mathfrak{A}_n$ where there exists an $m_0 \in \mathbb{N}$ such that $\alpha_{m,n}(A) \in F_m$ for all $m \geq m_0 \geq n$. By continuity, we obtain that

$$\psi(\varphi(\alpha_{\infty,n}(A))) = \lim_{m\to\infty} \psi(\beta_{\infty,m+1}(\varphi_m(\alpha_{m,n}(A)))).$$

Restricting to $m \geq m_0$, and since $\varphi_m(\alpha_{m,n}(A)) \in G_{m+1} \subseteq \mathfrak{B}_{m+1}$,

$$\psi(\beta_{\infty,m+1}(\varphi_m(\alpha_{m,n}(A)))) = \lim_{k\to\infty} (\alpha_{\infty,k} \circ \psi_k \circ \beta_{k,m+1})(\varphi_m(\alpha_{m,n}(A))).$$

Thus, by choosing m large enough and then by choosing k large, we can obtain that

$$\|\psi(\varphi(\alpha_{\infty,n}(A))) - (\alpha_{\infty,k} \circ \psi_k \circ \beta_{k,m+1})(\varphi_m(\alpha_{m,n}(A)))\|$$

is small. However, by the above computations, we notice that

$$(\psi_k \circ \beta_{k,m+1})(\varphi_m(\alpha_{m,n}(A))) = (\psi_k \circ \beta_{k-1} \circ \cdots \circ \beta_{m+1})(\varphi_m(\alpha_{m,n}(A)))$$

is within $2\sum_{j=m}^{k+1} \delta_j$ of

$$(\alpha_{k-1} \circ \alpha_{k-2} \circ \cdots \circ \alpha_m)(\alpha_{m,n}(A)) = \alpha_{k,n}(A).$$

Hence $(\alpha_{\infty,k} \circ \psi_k \circ \beta_{k,m+1})(\varphi_m(\alpha_{m,n}(A)))$ is within $2\sum_{j=m}^{k+1} \delta_j$ of $\alpha_{\infty,n}(A)$. Hence the result follows from the fifth condition of Definition 12.6. ☐

With the above technical result out of the way, we can prove the following result which is what we are after. The proof comes down to constructing the desired objects in Definition 12.6.

Lemma 12.8. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital, separable C^*-algebras. Suppose there exists unital *-homomorphisms $\pi : \mathfrak{A} \to \mathfrak{B}$ and $\sigma : \mathfrak{B} \to \mathfrak{A}$ such that $\pi \circ \sigma$ is approximately unitarily equivalent to $Id_{\mathfrak{B}}$ and $\sigma \circ \pi$ is approximately unitarily equivalent to $Id_{\mathfrak{A}}$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic C^*-algebras.*

Proof. To prove this result, we will verify that the conditions of Definition 12.6 hold where $\mathfrak{A}_n := \mathfrak{A}$ for all $n \in \mathbb{N}$, $\mathfrak{B}_n := \mathfrak{B}$ for all $n \in \mathbb{N}$, α_n is an isomorphism for all n, β_n is an isomorphism for all n, $\varphi_j := \pi$ for all $j \in \mathbb{N}$, $\psi_j := \sigma$ for all $j \in \mathbb{N}$, and $\delta_n := \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

To complete the proof, we will define α_n, β_n, $F_n \subseteq \mathfrak{A}$, and $G_n \subseteq \mathfrak{B}$ recursively. Since $\mathfrak{A}$ and $\mathfrak{B}$ are separable, let $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ be dense subsets of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. We will construct sequences of unitaries $(U_n)_{n\geq 1}$ in $\mathfrak{A}$ and $(V_n)_{n\geq 1}$ in $\mathfrak{B}$ such that α_n is conjugation by U_n and β_n is conjugation by V_n for all $n \in \mathbb{N}$. Let $G_1 := \{B_1\}$. Then since $\pi \circ \sigma$ is approximately unitarily equivalent to $Id_{\mathfrak{B}}$, there exists a unitary $V_1 \in \mathfrak{B}$ such that

$$\|\pi(\sigma(B_1)) - V_1 B_1 V_1^*\| < \frac{1}{2}.$$

Thus let $\beta_1(B) := V_1 B V_1^*$ for all $B \in \mathfrak{B}$. Then let $F_1 := \{A_1\} \cup \{\sigma(B_1)\}$. Since $\sigma \circ \pi$ is approximately unitarily equivalent to $Id_{\mathfrak{A}}$, there exists a unitary $U_1 \in \mathfrak{A}$ such that

$$\|\sigma(\pi(A)) - U_1 A U_1^*\| < \frac{1}{2}$$

for all $A \in F_1$. Thus let $\alpha_1(A) := U_1 A U_1^*$ for all $A \in \mathfrak{A}$.

Having defined $G_n, V_n, \beta_n, F_n, U_n, \alpha_n$ recursively in this order, we define

$$G_{n+1} := \beta_n(G_n) \cup \{B_k\}_{k=1}^{n+1} \cup \{\beta_{n+1,k}(B_{n+1}) \mid 1 \leq k \leq n+1\} \cup \pi(F_n).$$

Then, since $\pi \circ \sigma$ is approximately unitarily equivalent to $Id_{\mathfrak{B}}$, there exists a unitary $V_{n+1} \in \mathfrak{B}$ such that

$$\left\|\pi(\sigma(B)) - V_{n+1} B V_{n+1}^*\right\| < \frac{1}{2^{n+1}}$$

for all $B \in G_{n+1}$. Thus let $\beta_{n+1}(B) := V_{n+1} B V_{n+1}^*$ for all $B \in \mathfrak{B}$. Then we define

$$F_{n+1} := \alpha_n(F_n) \cup \{A_k\}_{k=1}^{n+1} \cup \{\alpha_{n+1,k}(A_{n+1}) \mid 1 \leq k \leq n+1\} \cup \sigma(G_{n+1}).$$

Then, since $\sigma \circ \pi$ is approximately unitarily equivalent to $Id_{\mathfrak{A}}$, there exists a unitary $U_{n+1} \in \mathfrak{A}$ such that

$$\left\|\sigma(\pi(A)) - U_{n+1} A U_{n+1}^*\right\| < \frac{1}{2^{n+1}}$$

for all $A \in F_{n+1}$. Thus let $\alpha_{n+1}(A) := U_{n+1} A U_{n+1}^*$ for all $A \in \mathfrak{A}$.

By continuing this construction, it is clear that properties 1), 2), 3), and 5) of Definition 12.6 are satisfied. By construction, $\{B_k\}_{k=1}^n \subseteq G_n$ for all $n \in \mathbb{N}$ and if $m \geq n$, $\beta_{m,n}^{-1}(G_m)$ contains B_m. Hence $\bigcup_{m=n}^\infty \beta_{m,n}^{-1}(G_m)$ is dense in $\mathfrak{B}$ for all $n \in \mathbb{N}$. Similarly, $\bigcup_{m=n}^\infty \alpha_{m,n}^{-1}(F_m)$ is dense in $\mathfrak{A}$ for all $n \in \mathbb{N}$. Therefore, $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic by Proposition 12.7. ☐

With the above technical result out of the way, we continue with the proof of the desired result. In Definition 7.2 we examined sequences of *-homomorphisms with certain central properties. However we now desired to examine a stronger property.

Definition 12.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be unital, separable C*-algebras. An asymptotically central inclusion of $\mathfrak{A}$ into $\mathfrak{B}$ is a sequence of unital, injective *-homomorphisms $(\pi_n)_{n\geq 1}$ from $\mathfrak{A}$ to $\mathfrak{B}$ that are asymptotically central (that is, for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, $\lim_{n\to\infty} \|\pi_n(A)B - B\pi_n(A)\| = 0$).

Our goal is the following is to use approximately central inclusions of the Cuntz algebra to obtain C*-algebras that are isomorphic to $\mathcal{O}_2$. The first step is to show the following.

Lemma 12.10. *Let $\mathfrak{A}$ be a unital, separable, simple C*-algebra and let $\mathfrak{D}$ be a unital, separable, simple, purely infinite C*-algebra. If there exists an asymptotically central inclusion of $\mathfrak{D}$ into $\mathfrak{A}$, then $\mathfrak{A}$ is purely infinite.*

Proof. Let $\mathfrak{A}$ be a unital, separable, simple C*-algebra and let $\mathfrak{D}$ be a unital, separable, simple, purely infinite C*-algebra. By assumption there exists an asymptotically central inclusion of $\mathfrak{D}$ into $\mathfrak{A}$ so there is a unital, isometric inclusion of $\mathfrak{D}$ into $\mathfrak{A}$. Since $\mathfrak{D}$ is purely infinite it is clear that $I_{\mathfrak{A}}$ is an infinite projection in $\mathfrak{A}$.

To show that $\mathfrak{A}$ is purely infinite it suffices by Lemma 2.9 to show that for every non-zero positive element $A \in \mathfrak{A}$ there exists a projection in $\overline{A\mathfrak{A}A}$ that is equivalent to $I_{\mathfrak{A}}$ (as if $V \in \mathfrak{A}$ is an isometry such that $P := VV^* \in \overline{A\mathfrak{A}A}$ and $P \neq I_{\mathfrak{A}}$ (as if $P = I_{\mathfrak{A}}$ then $\overline{A\mathfrak{A}A} = \mathfrak{A}$), then $W := VP$ is a partial isometry with $W^*W = P$, $WW^* = VPV^* < VV^* = P$, and $W = VP = PVP \in \overline{A\mathfrak{A}A}$ (as $\overline{A\mathfrak{A}A}$ is hereditary)).

Fix $A \in \mathfrak{A}$ to be a non-zero positive operator. Since $\mathfrak{A}$ is simple and unital, the closure of the ideal generated by $A \in \mathfrak{A}$ is dense in $\mathfrak{A}$. Hence there exists elements $B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n \in \mathfrak{A}$ such that

$$\left\|I_{\mathfrak{A}} - \sum_{k=1}^n B_k A C_k\right\| < 1.$$

Hence $X := \sum_{k=1}^n B_k A C_k$ is an invertible element of $\mathfrak{A}$. By replacing B_j with $X^{-1}B_j$ for all j, we can assume that $\sum_{k=1}^n B_k A C_k = I_{\mathfrak{A}}$. We desire that $n = 1$ and $B_1 = C_1^*$.

Let
$$M := \max\{\|B_1\|, \|B_2\|, \dots, \|B_n\|, \|C_1\|, \dots, \|C_n\|\}.$$
Since $\mathfrak{D}$ is simple and purely infinite, Lemma 2.3 implies we can find a collection of n isometries $\{S_j\}_{j=1}^n \subseteq \mathfrak{D}$ with orthogonal ranges. Since there exists an asymptotically central inclusion of $\mathfrak{D}$ into $\mathfrak{A}$, there exists an injective, unital *-homomorphism $\pi : \mathfrak{D} \to \mathfrak{A}$ such that $\|\pi(S_j)A - A\pi(S_j)\| < \frac{1}{n^2M^2}$ for all $1 \leq j \leq n$.

Define
$$B := \sum_{k=1}^n B_k\pi(S_k)^* \qquad \text{and} \qquad C := \sum_{k=1}^n \pi(S_k)C_k.$$
Hence, as the S_k's have orthogonal ranges,
$$\begin{array}{rcl} \|BAC - I_{\mathfrak{A}}\| & = & \left\|\sum_{i,j=1}^n B_j\pi(S_j)^*A\pi(S_i)C_i - \sum_{i,j=1}^n B_j\pi(S_j)^*\pi(S_i)AC_i\right\| \\ & \leq & \sum_{i,j=1}^n \|B_j\pi(S_j)^*A\pi(S_i)C_i - B_j\pi(S_j)^*\pi(S_i)AC_i\| \\ & \leq & \sum_{i,j=1}^n \|B_j\| \|A\pi(S_i) - \pi(S_i)A\| \|C_i\| \\ & \leq & \frac{1}{n^2M^2}\sum_{i,j=1}^n \|B_j\| \|C_i\| < 1 \end{array}$$
Hence BAC is an invertible element of $\mathfrak{A}$.

Let $X := BA^{\frac{1}{2}}$ and let $Y := A^{\frac{1}{2}}C(BAC)^{-1}$ which are elements of $\mathfrak{A}$. Clearly $XY = I_{\mathfrak{A}}$ so
$$I_{\mathfrak{A}} = Y^*X^*XY \leq \|X\|^2 Y^*Y.$$
Hence Y^*Y is an invertible element of $\mathfrak{A}$. If we let $V := Y|Y|^{-1}$, then
$$V^*V = (Y^*Y)^{-\frac{1}{2}}(Y^*Y)(Y^*Y)^{-\frac{1}{2}} = I_{\mathfrak{A}}$$
whereas
$$VV^* = Y|Y|^{-1}Y^* = A^{\frac{1}{2}}C(BAC)^{-1}|Y|^{-1}((BAC)^{-1})^*C^*A^{\frac{1}{2}} \in A^{\frac{1}{2}}\mathfrak{A}A^{\frac{1}{2}} \subseteq \overline{A\mathfrak{A}A}$$
(as $A \in \overline{A\mathfrak{A}A}$ by Lemma 2.3 so $A^{\frac{1}{2}}\mathfrak{A}A^{\frac{1}{2}} \subseteq \overline{A\mathfrak{A}A}$ as $\overline{A\mathfrak{A}A}$ is hereditary). Hence the result follows. $\square$

With the above result, we can show that C*-algebras with the same properties as $\mathcal{O}_2$ that contain an asymptotically central inclusion of $\mathcal{O}_2$ must be $\mathcal{O}_2$.

Lemma 12.11. *Let $\mathfrak{A}$ be a unital, separable, simple, nuclear C*-algebra. If $\mathcal{O}_2$ has an asymptotically central inclusion into $\mathfrak{A}$ then $\mathfrak{A} \simeq \mathcal{O}_2$.*

Proof. By assumption there exists an injective, unital *-homomorphism $\pi : \mathcal{O}_2 \to \mathfrak{A}$. Moreover, there exists an injective, unital *-homomorphism $\sigma : \mathfrak{A} \to \mathcal{O}_2$ by Theorem 11.11. However $\sigma \circ \pi : \mathcal{O}_2 \to \mathcal{O}_2$ is approximately unitarily equivalent to the identity on $\mathcal{O}_2$ by Theorem 6.12 (or Theorem 10.10). If $\pi\circ\sigma : \mathfrak{A} \to \mathfrak{A}$ is approximately unitarily equivalent to the identity on $\mathfrak{A}$ then Lemma 12.8 implies that $\mathfrak{A}$ and $\mathcal{O}_2$ are isomorphic. Therefore we desire to use the asymptotically central inclusion of $\mathcal{O}_2$ into $\mathfrak{A}$ to show that any two unital *-homomorphisms from $\mathfrak{A}$ to $\mathfrak{A}$ are approximately unitarily equivalent.

To begin, we note that Lemma 12.10 implies $\mathfrak{A}$ is purely infinite. Fix a unital *-homomorphism $\gamma : \mathfrak{A} \to \mathfrak{A}$. To see that γ is approximately unitarily equivalent to the identity on $\mathfrak{A}$, we notice that since γ is a unital, completely positive map and since $\mathfrak{A}$ is unital, separable, simple, nuclear, and purely infinite, Proposition 10.4 implies that there exists a sequence of isometries $(V_n)_{n\geq 1} \subseteq \mathfrak{A}$ such that $\lim_{n\to\infty} V_n^*AV_n = \gamma(A)$ for all $A \in \mathfrak{A}$. Our goal is to upgrade these partial isometries to unitaries.

Fix an ultrafilter ω and let $V := q_\omega(V_1, V_2, \dots) \in \mathfrak{A}_\omega$. Clearly V is an isometry in $\mathfrak{A}_\omega$. We claim $VV^* \in \mathfrak{A}' \cap \mathfrak{A}_\omega$. Indeed, viewing $\mathfrak{A} \subseteq \mathfrak{A}_\omega$, we obtain that $V^*AV = \gamma(A)$ for all $A \in \mathfrak{A}$. Therefore, if $U \in \mathcal{U}(\mathfrak{A})$ then $V^*UV = \gamma(U)$ is a unitary in $\mathfrak{A}$ so Lemma 10.7 implies that VV^* commutes with U. Hence VV^* commutes with every unitary in $\mathfrak{A}$ so $VV^* \in \mathfrak{A}' \cap \mathfrak{A}_\omega$.

Recall that $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is a unital, simple, purely infinite C*-algebra by Proposition 12.5 (as $\mathfrak{A}$ is unital, separable, simple, nuclear, and purely infinite). We will use K-theory to show that VV^* and $I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$ are equivalent projections in $\mathfrak{A}' \cap \mathfrak{A}_\omega$. Thus will enable us to upgrade V to a unitary.

Fix an increasing sequence $(F_n)_{n\geq 1}$ of finite, self-adjoint subsets of $\mathfrak{A}$ with dense union in $\mathfrak{A}$ such that $V_nV_n^* \in F_n$. Let S_1 and S_2 be the standard generating isometries of $\mathcal{O}_2$. Since $\mathcal{O}_2$ has an asymptotically central inclusion into $\mathfrak{A}$, for each $n \in \mathbb{N}$ there exists a unital, injective *-homomorphism $\sigma_n : \mathcal{O}_2 \to \mathfrak{A}$ such that $\|\sigma_n(T)A - A\sigma_n(T)\| < \frac{1}{n}$ for all $A \in F_n$ and $T \in \{S_1, S_1^*, S_2, S_2^*\} \subset \mathcal{O}_2$. Therefore, if we define $\sigma : \mathcal{O}_2 \to \mathfrak{A}_\omega$ by $\sigma(T) := q_\omega(\sigma_1(T), \sigma_2(T), \ldots)$ for all $T \in \mathcal{O}_2$, then σ is a well-defined, unital, injective *-homomorphism. Moreover, by construction, $\sigma(T)$ commutes with $\bigcup_{n\geq 1} F_n$ for all $T \in \{S_1, S_1^*, S_2, S_2^*\} \subset \mathcal{O}_2$. Therefore, since $\bigcup_{n\geq 1} F_n$ is dense in $\mathfrak{A}$, it is easy to see that $\sigma(T) \in \mathfrak{A}' \cap \mathfrak{A}_\omega$ for all $T \in \mathcal{O}_2$. Moreover, since $V_nV_n^* \in F_n$ for all n, it is easy to see that $\sigma(T)$ commutes with VV^* for all $T \in \{S_1, S_1^*, S_2, S_2^*\} \subseteq \mathcal{O}_2$.

However, in $K_0(\mathfrak{A}' \cap \mathfrak{A}_\omega)$,

$$[VV^*]_0 = [VV^*(\sigma(S_1)\sigma(S_1)^* + \sigma(S_2)\sigma(S_2)^*)]_0 = [\sigma(S_1)VV^*\sigma(S_1)^* + \sigma(S_2)VV^*\sigma(S_2)^*]_0 = 2[VV^*]_0$$

by 6.13. Hence $[VV^*]_0 = 0$. Similarly, since $I_\mathfrak{A} = I_{\mathfrak{A}'\cap\mathfrak{A}_\omega}$, $[I_\mathfrak{A}]_0 = 0$. Therefore, since $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is a unital, simple, purely infinite C*-algebra, VV^* and I are equivalent projections in $\mathfrak{A}' \cap \mathfrak{A}_\omega$ (see Chapter 4). Therefore there exists an isometry $W \in \mathfrak{A}' \cap \mathfrak{A}_\omega$ such that $WW^* = VV^*$.

Let $U := W^*V \in \mathfrak{A}_\omega$. Clearly

$$U^*U = V^*WW^*V = V^*VV^*V = I_\mathfrak{A} \qquad \text{and} \qquad UU^* = W^*VV^*W = W^*WW^*W = I_\mathfrak{A}$$

so U is an unitary operator. Moreover, since $W \in \mathfrak{A}' \cap \mathfrak{A}_\omega$

$$U^*AU = V^*WAW^*V = V^*AV = \gamma(A)$$

for all $A \in \mathfrak{A}$.

Finally, we will use U to obtain that γ is approximately unitarily equivalent to the identity on $\mathfrak{A}$. To begin, let $\mathcal{F}$ be a finite subset of the unit ball of $\mathfrak{A}$ and let $\epsilon > 0$. Clearly we can choose an element $(B_1, B_2, \ldots) \in \ell_\infty(\mathfrak{A})$ such that $q_\infty(B_1, B_2, \ldots) = U$. Since U is a unitary,

$$\lim_{m\to\omega} \|B_m^*B_m - I_\mathfrak{A}\| = 0 = \lim_{m\to\omega} \|B_mB_m^* - I_\mathfrak{A}\|.$$

Hence, since $U^*AU = \gamma(A)$ for all $A \in \mathfrak{A}$, there exists a neighbourhood U' of ω in $\beta\mathbb{N}$ such that for all $n \in U' \cap \mathbb{N}$ $\|B_n^*AB_n - \gamma(A)\| < \frac{\epsilon}{3}$ for all $A \in \mathcal{F}$ and the unitaries $U_n = B_n|B_n|^{-1}$ exist and satisfy $\|U_n - B_n\| \leq \frac{\epsilon}{3}$. By selecting any $n \in U' \cap \mathbb{N}$, we obtain $\|U_n^*AU_n - \gamma(A)\| < \epsilon$ for all $A \in \mathcal{F}$ and thus the result follows. ☐

Our final step in the proof is to show that $\otimes_{n=1}^\infty \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$ and has an approximately central inclusion of $\mathcal{O}_2$.

Theorem 12.12. *Let $\mathfrak{A}$ be a unital, separable, simple, nuclear C*-algebra. Then $\mathfrak{A} \otimes_{\min} \mathcal{O}_2 \simeq \mathcal{O}_2$.*

Proof. Let $\mathfrak{B} := \otimes_{n=1}^\infty \mathcal{O}_2$; that is, $\mathfrak{B}$ is the direct limit of $\mathcal{O}_2^{\otimes_{\min} n}$ with the canonical inclusions $\alpha_n : \mathcal{O}_2^{\otimes_{\min} n} \to \mathcal{O}_2^{\otimes_{\min} n} \otimes_{\min} \mathcal{O}_2$ defined by $\sigma(T) = T \otimes I_{\mathcal{O}_2}$. Clearly $\mathfrak{B}$ is a unital. Moreover $\mathfrak{B}$ is separable and nuclear being the direct limit of separable, nuclear C*-algebras. To see that $\mathfrak{B}$ is simple, let $\mathfrak{J}$ be an ideal of $\mathfrak{B}$. Then $\mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} n}$ is an ideal of $\mathcal{O}_2^{\otimes_{\min} n} \simeq \mathcal{O}_2$ and thus $\mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} n}$ is either $\{0\}$ or $\mathcal{O}_2^{\otimes_{\min} n}$ as $\mathcal{O}_2$ is simple. Since $\mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} n} \subseteq \mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} n+1}$ for all $n \in \mathbb{N}$, either $\mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} m} = \{0\}$ for all $m \in \mathbb{N}$ or there exists an $n \in \mathbb{N}$ such that $\mathfrak{J} \cap \mathcal{O}_2^{\otimes_{\min} m} = \mathcal{O}_2^{\otimes_{\min} m}$ for all $m \geq n$. In the first case we obtain that $\mathfrak{J} = \{0\}$ and in the second we obtain that $\mathfrak{J} = \mathfrak{B}$. Hence $\mathfrak{B}$ is simple.

To see that $\mathfrak{B}$ is purely infinite, we notice that the inclusions $\pi_n : \mathcal{O}_2 \to \mathcal{O}_2^{\otimes_{\min} n-1} \otimes_{\min} \mathcal{O}_2 \subseteq \mathfrak{B}$ defined by $\pi_n(T) = I_{\mathcal{O}_2^{\otimes_{\min} n-1}} \otimes_{\min} T$ are an asymptotically central inclusion of $\mathcal{O}_2$ into $\mathfrak{B}$. Hence $\mathfrak{B}$ is purely infinite by Lemma 12.10. Clearly there then exists an asymptotically central inclusion of $\mathcal{O}_2$ into $\mathfrak{B} \otimes_{\min} \mathfrak{A}$ (by embedding into $\mathfrak{B} \otimes_{\min} I_\mathfrak{A}$). Since $\mathfrak{A}$ be a unital, separable, simple, nuclear C*-algebra, $\mathfrak{B} \otimes_{\min} \mathfrak{A}$ is a unital, simple and therefore purely infinite C*-algebra by Lemma 12.10. Moreover $\mathfrak{B} \otimes_{\min} \mathfrak{A}$ is separable and nuclear as $\mathfrak{A}$ and $\mathfrak{B}$ are separable and nuclear.

By Lemma 12.11, $\mathfrak{B} \simeq \mathcal{O}_2$ and $\mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathcal{O}_2$. Hence $\mathcal{O}_2 \otimes_{\min} \mathfrak{A} \simeq \mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathcal{O}_2$ as desired. ☐

13 $\mathcal{O}_\infty \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$

In this chapter we will prove that if $\mathfrak{A}$ is a unital, separable, simple, nuclear, purely infinite C*-algebra then $\mathcal{O}_\infty \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$. The idea of the result is to show that if $\mathfrak{B}$ is a unital C*-subalgebra contained in $\mathfrak{A}' \cap \mathfrak{A}_\omega$ such that certain inclusions of $\mathfrak{B}$ into $\mathfrak{B} \otimes_{\min} \mathfrak{B}$ are approximately unitarily equivalent then $\mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$. The majority of this result is proved in Lemma 13.3. We will the use the fact that $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is a unital, simple, purely infinite C*-algebra to embed $\mathcal{O}_\infty$ inside by Proposition 12.5. Knowledge of approximate unitary equivalence of *-homomorphisms from $\mathcal{O}_\infty$ will complete the proof (although this will be our only complete omission in this document as the proof requires a significant amount of K-Theory).

Most of the results for this chapter were developed from the paper [KP].

We begin with the following lemma. Note that we do not actually need the following lemma to prove the result (as we will alway take $\mathfrak{B} = \mathcal{O}_\infty$) but we record the proof as it is simple.

Lemma 13.1. *Let $\mathfrak{B}$ be a unital, separable C*-algebra. Suppose that the two unital *-homomorphisms $\pi, \sigma : \mathfrak{B} \to \mathfrak{B} \otimes_{\min} \mathfrak{B}$ given by $\pi(B) = B \otimes I_{\mathfrak{B}}$ and $\sigma(B) = I_{\mathfrak{B}} \otimes B$ for all $B \in \mathfrak{B}$ are approximately unitarily equivalent. Then $\mathfrak{B}$ is simple and nuclear.*

Proof. To see that $\mathfrak{B}$ is simple, suppose to the contrary that there exists a non-trivial ideal $\mathfrak{J}$ of $\mathfrak{B}$. Hence $\mathfrak{B} \otimes_{\min} \mathfrak{J}$ and $\mathfrak{J} \otimes_{\min} \mathfrak{B}$ are non-trivial ideals of $\mathfrak{B} \otimes_{\min} \mathfrak{B}$. Moreover, if $B \in \mathfrak{J}$ then $B \otimes I_{\mathfrak{B}} \in \mathfrak{J} \otimes_{\min} \mathfrak{B}$ yet $I_{\mathfrak{B}} \otimes B \notin \mathfrak{J} \otimes_{\min} \mathfrak{B}$ (or else $I_{\mathfrak{B}} \in \mathfrak{J}$; see the end of the first argument in Remarks 12.1). However, since π and σ are approximately unitarily equivalent, there exists a sequence $(U_n)_{n\geq 1}$ of unitaries in $\mathfrak{B} \otimes_{\min} \mathfrak{B}$ such that

$$I_{\mathfrak{B}} \otimes B = \sigma(B) = \lim_{n\to\infty} U_n(\pi(B))U_n^* = \lim_{n\to\infty} U_n(B \otimes I_{\mathfrak{B}})U_n^* \in \mathfrak{J} \otimes_{\min} \mathfrak{B}$$

which is a contradiction. Hence $\mathfrak{B}$ is simple.

To see that $\mathfrak{B}$ is nuclear, since σ and π are approximately unitarily equivalent and $\mathfrak{B}$ is separable, there exists a sequence $(U_n)_{n\geq 1}$ of unitaries in $\mathfrak{B} \otimes_{\min} \mathfrak{B}$ such that $\lim_{n\to\infty} U_n(B \otimes I_{\mathfrak{B}})U_n^* = I_{\mathfrak{B}} \otimes B$ for all $B \in \mathfrak{B}$. Fix $C_n \in \mathfrak{B} \odot \mathfrak{B}$ such that $\|C_n\| \leq 1$ and $\lim_{n\to\infty} \|C_n - U_n\| = 0$. Clearly we obtain that $\lim_{n\to\infty} C_n(B \otimes I_{\mathfrak{B}})C_n^* = I_{\mathfrak{B}} \otimes B$ for all $B \in \mathfrak{B}$.

Let ϕ be any state on $\mathfrak{B}$. For each $n \in \mathbb{N}$ define $\psi_n : \mathfrak{B} \to \mathfrak{B}$ by

$$\psi_n(B) := (\phi \otimes Id_{\mathfrak{B}})(C_n(B \otimes I_{\mathfrak{B}})C_n^*).$$

Since the tensor product of contractive, completely positive maps on the minimal tensor product are contractive, completely positive maps, each ψ_n is a completely positive map that is contractive as $\|C_n\| \leq 1$.

For each $n \in \mathbb{N}$ write $C_n = \sum_{i=1}^m X_j \otimes Y_j$ with $X_j, Y_j \in \mathfrak{B}$. Then

$$\psi_n(B) = (\phi \otimes Id_{\mathfrak{B}}) \left(\sum_{i,j=1}^m X_i B X_j^* \otimes Y_i Y_j^* \right) = \sum_{i,j=1}^m \phi(X_i B X_j^*) Y_i Y_j^*$$

for all $B \in \mathfrak{B}$. Hence ψ_n is a finite rank map. Moreover

$$\|\psi_n(B) - B\| = \|(\phi \otimes Id_{\mathfrak{B}})(C_n(B \otimes I_{\mathfrak{B}})C_n^*) - (\phi \otimes Id_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes B)\| \leq \|C_n(B \otimes I_{\mathfrak{B}})C_n^* - I_{\mathfrak{B}} \otimes B\|$$

for all $B \in \mathfrak{B}$ and thus converges to zero as $n \to \infty$. Hence $Id_{\mathfrak{B}}$ is a pointwise norm limit of completely positive contractions of finite rank. It follows from the papers [CE] and [La] that $\mathfrak{B}$ is nuclear. $\square$

Next we have the following trivial lemma that we record for convenience in the proof of the subsequent lemma.

Lemma 13.2. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be unital, separable C*-algebras and let ω be an ultrafilter. Let $\Phi : \mathfrak{B} \to \ell_\infty(\mathfrak{A})$ and $\Psi : \mathfrak{C} \to \ell_\infty(\mathfrak{A})$ be unital completely positive maps given by*

$$\Phi(B) = (\varphi_1(B), \varphi_2(B), \ldots) \quad \textit{and} \quad \Psi(B) = (\psi_1(C), \psi_2(C), \ldots)$$

where $\varphi_j : \mathfrak{B} \to \mathfrak{A}$ and $\psi_j : \mathfrak{C} \to \mathfrak{A}$ are unital, completely positive maps. Suppose that $q_{\omega,\mathfrak{A}} \circ \Phi$ and $q_{\omega,\mathfrak{A}} \circ \Psi$ are unital *-homomorphism with range inside $\mathfrak{A}' \cap \mathfrak{A}_\omega$. Then for any finite subsets $F \subseteq \mathfrak{A}$, $G \subseteq \mathfrak{B}$, and $H \subseteq \mathfrak{C}$, any $k \in \mathbb{N}$, and any $\epsilon > 0$ there exists a neighbourhood U of ω such that

$$\|\psi_n(C)\varphi_k(B) - \varphi_k(B)\psi_n(C)\| < \epsilon, \quad \|\psi_n(C)A - A\psi_n(C)\| < \epsilon, \quad \text{and} \quad \|\psi_n(C_1C_2) - \psi_n(C_1)\psi_n(C_2)\| < \epsilon$$

for every $n \in U \cap \mathbb{N}$, $A \in F$, $B \in G$, and $C, C_1, C_2 \in H$.

Proof. Fix finite subsets $F \subseteq \mathfrak{A}$, $G \subseteq \mathfrak{B}$, and $H \subseteq \mathfrak{C}$, fix $k \in \mathbb{N}$, and fix $\epsilon > 0$. Then $F \cup \{\varphi_k(G)\}$ is a finite subset of $\mathfrak{A}$. Since $q_{\omega,\mathfrak{A}} \circ \Psi$ is a unital *-homomorphism with range inside $\mathfrak{A}' \cap \mathfrak{A}_\omega$ and since H is a finite subset, the result clearly follows. ☐

With the above trivial lemma out of the way, we have the following technical result.

Lemma 13.3. *Let $\mathfrak{A}$ be a unital, separable, C*-algebra, let ω be an ultrafilter, and let $\mathfrak{B}$ be a separable C*-algebra of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ such that $I_{\mathfrak{A}_\omega} \in \mathfrak{B}$ and such that the two unital *-homomorphisms $\pi, \sigma : \mathfrak{B} \to \mathfrak{B} \otimes_{\min} \mathfrak{B}$ given by $\pi(B) = B \otimes I_{\mathfrak{B}}$ and $\sigma(B) = I_{\mathfrak{B}} \otimes B$ for all $B \in \mathfrak{B}$ are approximately unitarily equivalent (in $\mathfrak{B} \otimes_{\min} \mathfrak{B}$). Then there exists a unital *-homomorphism $\varphi : \mathfrak{B} \otimes_{\min} \mathfrak{A} \to \mathfrak{A}$ such that the map $\psi : \mathfrak{A} \to \mathfrak{A}$ defined by $\psi(A) = \varphi(I_{\mathfrak{B}} \otimes A)$ for all $A \in \mathfrak{A}$ is approximately unitarily equivalent to $Id_{\mathfrak{A}}$.*

Proof. The idea of the proof is to use the nuclearity of $\mathfrak{B}$ given by Lemma 13.1 to construct a sequence of asymptotically multiplicative, unital, completely positive maps from $\mathfrak{B}$ into $\mathfrak{A}$. Using the unitaries that make the *-homomorphisms σ and π approximately unitarily equivalent, we will be able to construct unitary elements of $\mathfrak{A}$ that asymptotically commute with each fixed element of $\mathfrak{A}$ and intertwine the above sequence of unital, completely positive maps. Conjugation by these unitaries will define *-homomorphisms from $\mathfrak{A}$ and $\mathfrak{B}$ into $\mathfrak{A}$ whose ranges commute which will enable us to obtain the desired *-homomorphism.

By Lemma 13.1 $\mathfrak{B}$ is a unital, separable, simple, nuclear C*-algebra. Hence Theorem 9.12 implies the inclusion of $\mathfrak{B}$ inside $\mathfrak{A}' \cap \mathfrak{A}_\omega$ lifts to a sequence of unital, completely positive maps $\varphi_i : \mathfrak{B} \to \mathfrak{A}$ such that

$$q_{\omega,\mathfrak{A}}(\varphi_1(B), \varphi_2(B), \ldots) = B$$

for all $B \in \mathfrak{B}$. Thus the φ_i's are asymptotically multiplicative.

Since $\mathfrak{A}$ and $\mathfrak{B}$ are separable, we can choose increasing sequences $(F_n)_{n\geq 1}$ and $(G_n)_{n\geq 1}$ of finite, self-adjoint subsets of $\mathfrak{A}$ and $\mathfrak{B}$ respectively whose unions are dense in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Finally, by the hypotheses on $\mathfrak{B}$, there exists a sequence of unitaries $(U_k)_{k\geq 1} \subseteq \mathfrak{B} \otimes_{\min} \mathfrak{B}$ such that

$$\|U_k(B \otimes I_{\mathfrak{B}})U_k^* - I_{\mathfrak{B}} \otimes B\| < \frac{1}{2^k}$$

for all $B \in G_k$.

We claim for each $k \geq 1$ there exists a finite set $G_k' \subseteq \mathfrak{B}$ containing G_k and an $\epsilon_k \in \left(0, \frac{1}{k}\right)$ such that whenever $\Phi, \Psi : \mathfrak{B} \to \mathfrak{A}$ are unital, completely positive maps such that

$$\|\Phi(BC) - \Phi(B)\Phi(C)\| < \epsilon_k, \|\Psi(BC) - \Psi(B)\Psi(C)\| < \epsilon_k, \|\Psi(B)\Phi(C) - \Phi(C)\Psi(B)\| < \epsilon_k,$$
$$\|\Phi(B)A - A\Phi(B)\| < \epsilon_k, \text{ and } \|\Psi(B)A - A\Psi(B)\| < \epsilon_k$$

for all $B, C \in G_k'$ and $A \in F_k$, then there exists a unitary $V \in \mathfrak{A}$ such that

$$\|V\Phi(B)V^* - \Psi(B)\| < \frac{1}{2^{k-1}} \quad \text{and} \quad \|VAV^* - A\| < \frac{1}{2^k}$$

for all $A \in F_k$ and $B \in G_k$.

To prove the claim, we will proceed by contradiction. Fix $k \geq 1$. Clearly, if the result fails, it fails without the assumption that $G_k \subseteq G_k'$. Therefore, if $\{T_n\}_{n=1}^\infty$ is a dense subsets of $\mathfrak{B}$, then for every $n \in \mathbb{N}$ there exists unital, completely positive maps $\Phi_n, \Psi_n : \mathfrak{B} \to \mathfrak{A}$ such that

$$\|\Phi_n(BC) - \Phi_n(B)\Phi_n(C)\| < \tfrac{1}{n}, \|\Psi_n(BC) - \Psi_n(B)\Psi_n(C)\| < \tfrac{1}{n}, \|\Psi_n(B)\Phi_n(C) - \Phi_n(C)\Psi_n(B)\| < \tfrac{1}{n},$$
$$\|\Phi_n(B)A - A\Phi_n(B)\| < \tfrac{1}{n}, \text{ and } \|\Psi_n(B)A - A\Psi_n(B)\| < \tfrac{1}{n}$$

for all $A \in F_k$ and $B, C \in \{T_1, \ldots, T_n\}$ and yet there does not exists a unitary $V \in \mathfrak{A}$ such that

$$\|V\Phi_n(B)V^* - \Psi_n(B)\| < \frac{1}{2^{k-1}} \qquad \text{and} \qquad \|VAV^* - A\| < \frac{1}{2^k}$$

for all $A \in F_k$ and $B \in G_k$.

Since each Φ_n and Ψ_n is a unital, completely positive map and therefore contractive, we may define the maps $\Phi, \Psi : \mathfrak{B} \to \mathfrak{A}_\infty$ by

$$\Phi(B) = q_{\infty,\mathfrak{A}}(\Phi_1(B), \Phi_2(B), \ldots) \qquad \text{and} \qquad \Psi(B) = q_{\infty,\mathfrak{A}}(\Psi_1(B), \Psi_2(B), \ldots)$$

for all $B \in \mathfrak{B}$. Since

$$\|\Phi_n(BC) - \Phi_n(B)\Phi_n(C)\| < \frac{1}{n} \qquad \text{and} \qquad \|\Psi_n(BC) - \Psi_n(B)\Psi_n(C)\| < \frac{1}{n}$$

for all $B, C \in \{T_1, \ldots, T_n\}$, we obtain by the density of $\{T_n\}_{n=1}^\infty$ in $\mathfrak{B}$ that Φ and Ψ are unital *-homomorphisms. Since $\|\Psi_n(B)\Phi_n(C) - \Phi_n(C)\Psi_n(B)\| < \frac{1}{n}$ for all $B, C \in \{T_1, \ldots, T_n\}$, we again obtain by the density of $\{T_n\}_{n=1}^\infty$ in $\mathfrak{B}$ that Φ and Ψ have commuting ranges. Finally, since

$$\|\Phi_n(B)A - A\Phi_n(B)\| < \frac{1}{n} \qquad \text{and} \qquad \|\Psi_n(B)A - A\Psi_n(B)\| < \frac{1}{n}$$

for all $A \in F_k$ and $B, C \in \{T_1, \ldots, T_n\}$, we again obtain by the density of $\{T_n\}_{n=1}^\infty$ in $\mathfrak{B}$ that Φ and Ψ commute with

$$\mathcal{F} := \{q_{\infty,\mathfrak{A}}(A, A, A, \ldots) \mid A \in F_k\}.$$

Since $\mathfrak{B}$ is nuclear, by the universal property of the maximal tensor product and by the facts illustrated above, there exists a unital *-homomorphism $\theta := \Phi \otimes \Psi : \mathfrak{B} \otimes_{\min} \mathfrak{B} \to \mathfrak{A}_\infty$ whose range commutes with $\mathcal{F}$.

Using the unitaries $(U_n)_{n\geq 1}$ defined earlier, we can write $\theta(U_k) = q_{\infty,\mathfrak{A}}(S_1, S_2, \ldots)$ where $\{S_n\}_{n\geq 1} \subseteq \mathfrak{A}$ are such that $\sup_{n\geq 1} \|S_n\| < \infty$. Since θ is a unital *-homomorphism, $\theta(U_k)$ is a unitary and thus $\lim_{n\to\infty} S_n^* S_n = \lim_{n\to\infty} S_n S_n^* = I_{\mathfrak{A}}$. For each $n \in \mathbb{N}$ define

$$V_n := \left\{ \begin{array}{ll} S_n|S_n|^{-1} & \text{whenever } S_n^*S_n \text{ and } S_nS_n^* \text{ are invertible} \\ I_{\mathfrak{A}} & \text{otherwise} \end{array} \right. .$$

Therefore, as $\lim_{n\to\infty} S_n^* S_n = \lim_{n\to\infty} S_n S_n^* = I_{\mathfrak{A}}$, we obtain that $q_{\infty,\mathfrak{A}}(V_1, V_2, \ldots) = \theta(U_k)$. However, this implies for all $B \in G_k$ that

$$\begin{aligned} \limsup_{n\to\infty} \|V_n\Phi_n(B)V_n^* - \Psi_n(B)\| &\leq \|\theta(U_k)\theta(B \otimes I_{\mathfrak{B}})\theta(U_k)^* - \theta(I_{\mathfrak{B}} \otimes B)\| \\ &\leq \|U_k(B \otimes I_{\mathfrak{B}})U_k^* - I_{\mathfrak{B}} \otimes B\| < \tfrac{1}{2^k} \end{aligned}$$

and for all $A \in F_k$ that

$$\limsup_{n\to\infty} \|V_nAV_n^* - A\| \leq \|\theta(U_k)q_{\infty,\mathfrak{A}}(A, A, A, \ldots)\theta(U_k)^* - q_{\infty,\mathfrak{A}}(A, A, A, \ldots)\| = 0$$

as the range of θ commutes with $\mathcal{F}$. Therefore, since G_k and F_k are finite, by choosing n large enough, we clearly get a contradiction to the fact that there does not exists a unitary $V \in \mathfrak{A}$ such that

$$\|V\Phi_n(B)V^* - \Psi_n(B)\| < \frac{1}{2^{k-1}} \qquad \text{and} \qquad \|VAV^* - A\| < \frac{1}{2^k}$$

for all $A \in F_k$ and $B \in G_k$. Hence we have obtained our contradiction so the claim has be proven.

By constructing the sets G'_k and ϵ_k recursively, we may assume that $G'_k \subseteq G'_{k+1}$ for all k, $\epsilon_k > \epsilon_{k+1}$ for all k, and $\lim_{k\to\infty} \epsilon_k = 0$. By the fact that

$$q_{\omega,\mathfrak{A}}(\varphi_1(B), \varphi_2(B), \ldots) = B$$

for all $B \in \mathfrak{B}$ and by the trivial Lemma 13.2, we can selected $n_1 < n_2 < \dots$ recursively such that

$$\|\varphi_{n_1}(BC) - \varphi_{n_1}(B)\varphi_{n_1}(C)\| < \epsilon_1 \qquad \text{and} \qquad \|\varphi_{n_1}(B)A - A\varphi_{n_1}(B)\| < \epsilon_1$$

for all $A \in F_1$ and $B, C \in G_1'$, and

$$\begin{gathered}\left\|\varphi_{n_{k+1}}(BC) - \varphi_{n_{k+1}}(B)\varphi_{n_{k+1}}(C)\right\| < \epsilon_{k+1}, \quad \left\|\varphi_{n_{k+1}}(B)A - A\varphi_{n_{k+1}}(B)\right\| < \epsilon_{k+1}, \\ \text{and} \quad \left\|\varphi_{n_{k+1}}(B)\varphi_{n_k}(C) - \varphi_{n_k}(C)\varphi_{n_{k+1}}(B)\right\| < \epsilon_{k+1} < \epsilon_k\end{gathered}$$

for all $A \in F_k$ and $B, C \in G_{k+1}'$. Therefore, the claim implies there exists unitaries $\{V_k\}_{k\geq 1} \subseteq \mathfrak{A}$ such that

$$\left\|V_k\varphi_{n_{k+1}}(B)V_k^* - \varphi_{n_k}(B)\right\| < \frac{1}{2^{k-1}} \qquad \text{and} \qquad \|V_kAV_k^* - A\| < \frac{1}{2^k}$$

for all $A \in F_k$ and $B \in G_k$.

For each $n \in \mathbb{N}$, let $W_n := V_1V_2\cdots V_{n-1} \in \mathfrak{A}$ (so $W_1 = I_{\mathfrak{A}}$). Clearly each W_n is a unitary. Since $\sum_{k=1}^\infty \frac{1}{2^k} < \infty$, $\|V_kAV_k^* - A\| < \frac{1}{2^k}$ for all $A \in F_k$, and the union of the F_k's is dense in $\mathfrak{A}$, $\lim_{n\to\infty} W_kAW_k^*$ exists for all $A \in \mathfrak{A}$. Define $\alpha : \mathfrak{A} \to \mathfrak{A}$ by $\alpha(A) := \lim_{n\to\infty} W_kAW_k^*$ for all $A \in \mathfrak{A}$. Clearly α is a unital *-homomorphism that is approximately unitarily equivalent to the identity map on $\mathfrak{A}$. Moreover notice that

$$\left\|W_{k+1}\varphi_{n_{k+1}}(B)W_{k+1}^* - W_k\varphi_{n_k}(B)W_k^*\right\| = \left\|V_k\varphi_{n_{k+1}}(B)V_k^* - \varphi_{n_k}(B)\right\| < \frac{1}{2^{k-1}}$$

for all $k \in \mathbb{N}$ and $B \in G_k$. Hence, since the union of the G_k's is dense in $\mathfrak{B}$, $\lim_{k\to\infty} W_k\varphi_{n_k}(B)W_k^*$ exists for all $B \in \mathfrak{B}$. Therefore we can define $\beta : \mathfrak{B} \to \mathfrak{A}$ by $\beta(B) := \lim_{k\to\infty} W_k\varphi_{n_k}(B)W_k^*$ for all $B \in \mathfrak{B}$. Since $\left\|\varphi_{n_{k+1}}(BC) - \varphi_{n_{k+1}}(B)\varphi_{n_{k+1}}(C)\right\| < \epsilon_{k+1}$ for all $B, C \in G_{k+1}$, $\lim_{k\to\infty} \epsilon_k = 0$, and the union of the G_k's is dense in $\mathfrak{B}$, β is a unital *-homomorphism. Finally, since

$$\|(W_kAW_k^*)(W_k\varphi_{n_k}(B)W_k^*) - (W_k\varphi_{n_k}(B)W_k^*)(W_kAW_k^*)\| = \|\varphi_{n_k}(B)A - A\varphi_{n_k}(B)\| < \epsilon_k$$

for all $A \in F_k$, $B \in G_k$, and $k \geq 0$, and since $\lim_{k\to\infty} \epsilon_k = 0$, the ranges of α and β commute. Therefore, since $\mathfrak{B}$ is nuclear, by the universal property of the maximal tensor product there exists a unital *-homomorphism $\varphi := \beta \otimes \alpha : \mathfrak{B} \otimes_{\min} \mathfrak{A} \to \mathfrak{A}$. Therefore, if $\psi : \mathfrak{A} \to \mathfrak{A}$ is defined by $\psi(A) = \varphi(I_{\mathfrak{B}} \otimes A) = \alpha(A)$ for all $A \in \mathfrak{A}$, then ψ is approximately unitarily equivalent to the identity map on $\mathfrak{A}$ as desired. ☐

The above result allows us to prove, with the conditions of the above result, that $\mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$. The result will be proved using Lemma 12.8.

Proposition 13.4. *Let $\mathfrak{A}$ be a unital, separable, C*-algebra, let ω be an ultrafilter, and let $\mathfrak{B}$ be a separable C*-algebra of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ such that $I_{\mathfrak{A}_\omega} \in \mathfrak{B}$ and such that the two unital *-homomorphisms $\pi, \sigma : \mathfrak{B} \to \mathfrak{B} \otimes_{\min} \mathfrak{B}$ given by $\pi(B) = B \otimes I_{\mathfrak{B}}$ and $\sigma(B) = I_{\mathfrak{B}} \otimes B$ for all $B \in \mathfrak{B}$ are approximately unitarily equivalent. Then $\mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$.*

Proof. This result follows from Lemma 13.3 and Lemma 12.8. Let $\varphi : \mathfrak{B} \otimes_{\min} \mathfrak{A} \to \mathfrak{A}$ be the unital *-homomorphism from Lemma 13.3 and let $\theta : \mathfrak{A} \to \mathfrak{B} \otimes_{\min} \mathfrak{A}$ by the unital *-homomorphism defined by $\theta(A) = I_{\mathfrak{B}} \otimes A$ for all $A \in \mathfrak{A}$. Then for all $A \in \mathfrak{A}$

$$(\varphi \circ \theta)(A) = \varphi(I_{\mathfrak{B}} \otimes A).$$

Therefore, by Lemma 13.3, $\varphi \circ \theta$ is approximately unitarily equivalent to the identity map on $\mathfrak{A}$. Therefore, if $\theta \circ \varphi$ is approximately unitarily equivalent to the identity map on $\mathfrak{B} \otimes_{\min} \mathfrak{A}$, $\mathfrak{B} \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$ by Lemma 12.8.

Since $\varphi \circ \theta$ is approximately unitarily equivalent to the identity on $\mathfrak{A}$, there exists a sequence of unitaries $(W_n)_{n\geq 1}$ of $\mathfrak{A}$ such that $\lim_{n\to\infty} \|W_n\varphi(\theta(A))W_n^* - A\| = 0$ for all $A \in \mathfrak{A}$. Moreover, by the assumptions on $\mathfrak{B}$, there exists a sequence of unitaries $(V_n)_{n\geq 1}$ in $\mathfrak{B} \otimes_{\min} \mathfrak{B}$ such that $\lim_{n\to\infty} \|V_n(B \otimes I_{\mathfrak{B}})V_n^* - I_{\mathfrak{B}} \otimes B\| = 0$ for all $B \in \mathfrak{B}$. However, for all $B_1, B_2 \in \mathfrak{B}$, we notice that $B_1 \otimes I_{\mathfrak{A}}$ and $\theta(\varphi(B_2 \otimes I_{\mathfrak{A}})) = I_{\mathfrak{B}} \otimes \varphi(B_2)$ commute.

Hence, since $\mathfrak{B}$ is nuclear by Lemma 13.1, the universal property of the maximal tensor product implies that there exists a unital *-homomorphism $\Psi : \mathfrak{B} \otimes_{\min} \mathfrak{B} \to \mathfrak{B} \otimes_{\min} \mathfrak{A}$ such that $\Psi(B \otimes I_{\mathfrak{B}}) = \theta(\varphi(B \otimes I_{\mathfrak{A}}))$ and $\Psi(I_{\mathfrak{B}} \otimes B) = B \otimes I_{\mathfrak{B}}$ for all $B \in \mathfrak{B}$. In addition, notice

$$\begin{aligned}
\Psi(B_1 \otimes B_2)\theta(\varphi(I_{\mathfrak{B}} \otimes A)) &= \Psi(B_1 \otimes I_{\mathfrak{B}})\Psi(I_{\mathfrak{B}} \otimes B_2)(I_{\mathfrak{B}} \otimes \varphi(I_{\mathfrak{B}} \otimes A)) \\
&= \theta(\varphi(B_2 \otimes I_{\mathfrak{A}}))(B_2 \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes \varphi(I_{\mathfrak{B}} \otimes A)) \\
&= \theta(\varphi(B_2 \otimes I_{\mathfrak{A}}))(I_{\mathfrak{B}} \otimes \varphi(I_{\mathfrak{B}} \otimes A))(B_2 \otimes I_{\mathfrak{B}}) \\
&= \theta(\varphi(B_2 \otimes I_{\mathfrak{A}}))\theta(\varphi(I_{\mathfrak{B}} \otimes A))(B_2 \otimes I_{\mathfrak{B}}) \\
&= \theta(\varphi(I_{\mathfrak{B}} \otimes A))\theta(\varphi(B_2 \otimes I_{\mathfrak{A}}))(B_2 \otimes I_{\mathfrak{B}}) \\
&= \theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(B_1 \otimes B_2)
\end{aligned}$$

for all $B_1, B_2 \in \mathfrak{B}$ and for all $A \in \mathfrak{A}$. Hence, by linearity and density, $\Psi(V_n)\theta(\varphi(I_{\mathfrak{B}} \otimes A)) = \theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(V_n)$ for all $A \in \mathfrak{A}$ and all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ define $U_n = (I_{\mathfrak{B}} \otimes W_n)\Psi(V_n) \in \mathfrak{B} \otimes_{\min} \mathfrak{A}$. Therefore each U_n is the product of two unitary elements of $\mathfrak{B} \otimes_{\min} \mathfrak{A}$ and thus is a unitary element. Moreover, for all $B \in \mathfrak{B}$ and $A \in \mathfrak{A}$

$$\begin{aligned}
\lim_{n\to\infty} U_n\theta(\varphi(B \otimes A))U_n^* &= \lim_{n\to\infty} U_n\theta(\varphi(B \otimes I_{\mathfrak{A}}))\theta(\varphi(I_{\mathfrak{B}} \otimes A))U_n^* \\
&= \lim_{n\to\infty}(I_{\mathfrak{B}} \otimes W_n)\Psi(V_n)\Psi(B \otimes I_{\mathfrak{B}})\theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(V_n)^*(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(I_{\mathfrak{B}} \otimes W_n)\Psi(I_{\mathfrak{B}} \otimes B)\Psi(V_n)\theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(V_n)^*(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(I_{\mathfrak{B}} \otimes W_n)(B \otimes I_{\mathfrak{B}})\Psi(V_n)\theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(V_n)^*(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(B \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes W_n)\Psi(V_n)\theta(\varphi(I_{\mathfrak{B}} \otimes A))\Psi(V_n)^*(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(B \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes W_n)\theta(\varphi(I_{\mathfrak{B}} \otimes A))(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(B \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes W_n)(I_{\mathfrak{B}} \otimes (\varphi(I_{\mathfrak{B}} \otimes A)))(I_{\mathfrak{B}} \otimes W_n^*) \\
&= \lim_{n\to\infty}(B \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes W_n(\varphi(\theta(A)))W_n^*) \\
&= (B \otimes I_{\mathfrak{B}})(I_{\mathfrak{B}} \otimes A) = B \otimes A.
\end{aligned}$$

Hence, by linearity and density, $\lim_{n\to\infty} U_n\theta(\varphi(T))U_n^* = T$ for all $T \in \mathfrak{B} \otimes_{\min} \mathfrak{A}$. Therefore $\theta \circ \varphi$ is approximately unitarily equivalent to the identity on $\mathfrak{B} \otimes_{\min} \mathfrak{A}$ as desired. ☐

Our last technical result is the following.

Proposition 13.5. *Let $\mathfrak{A}$ be a unital, simple, purely infinite C*-algebra. Any two unital *-homomorphisms from $\mathcal{O}_\infty$ into $\mathfrak{A}$ are approximately unitarily equivalent.*

Proof. The proof of this result takes a significant amount of K-theory and therefore is omitted. An interested reader many consult [LP]. ☐

Now that we have the above, we are finally able to prove the main result of this chapter. The following proof is not the original proof from [KP], but is much simpler.

Theorem 13.6. *Let $\mathfrak{A}$ be a unital, separable, simple, nuclear, purely infinite C*-algebra. Then $\mathcal{O}_\infty \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$.*

Proof. Let $\mathfrak{A}$ be a unital, separable, simple, nuclear, purely infinite C*-algebra and let ω be any ultrafilter. Recall that $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is a unital, simple, purely infinite C*-algebra by Proposition 12.5. Therefore Lemma 2.3 implies that the identity of $\mathfrak{A}' \cap \mathfrak{A}_\omega$ is a properly infinite projection. Therefore, by the universal property of $\mathcal{O}_\infty$, there exists a unital *-homomorphism $\Phi : \mathcal{O}_\infty \to \mathfrak{A}' \cap \mathfrak{A}_\omega$. Since $\mathcal{O}_\infty$ is simple by Theorem 1.13, Φ is injective.

Recall that $\mathcal{O}_\infty$ is unital, separable, simple (Theorem 1.13), nuclear (Theorem 1.20), and purely infinite (Corollary 2.12). Hence $\mathcal{O}_\infty \otimes_{\min} \mathcal{O}_\infty$ is unital, separable, simple, nuclear, and purely infinite (Theorem 3.11). Hence Proposition 13.5 implies that the two *-homomorphisms $\pi, \sigma : \mathcal{O}_\infty \to \mathcal{O}_\infty \otimes_{\min} \mathcal{O}_\infty$ are approximately unitarily equivalent. Hence Proposition 13.4 implies that $\mathcal{O}_\infty \otimes_{\min} \mathfrak{A} \simeq \mathfrak{A}$ as desired. ☐

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