IMPORTANT CONCEPTS

1. Induction
   - (Principle of Mathematical Induction) For each $k \in \mathbb{N}$, let $P_k$ be a mathematical statement that is either true or false. Suppose
     (i) (base case) $P_1$ is true, and
     (ii) (inductive step) if $k \in \mathbb{N}$ and $P_k$ is true, then $P_{k+1}$ is true.
   Then $P_n$ is true for all $n \in \mathbb{N}$.
   - Very useful for: showing formulae hold for all $n \in \mathbb{N}$, showing a property is true for all elements of a sequence.
   - Can be useful to include multiple statements in $P_k$ for all $k$.
   - (Principle of Strong Mathematical Induction) For each $k \in \mathbb{N}$, let $P_k$ be a mathematical statement that is either true or false. Suppose
     (a) $P_1$ is true, and
     (b) if $k \in \mathbb{N}$ and $P_m$ is true is true for all $m \leq k$, then $P_{k+1}$ is true.
   Then $P_n$ is true for all $n \in \mathbb{N}$.
   - Strong induction can be used to assume multiple (i.e. all) previous statements are true in order to prove the next is.
   - (Well Ordering Principle) Each non-empty subset $S \subseteq \mathbb{N}$ has a least element.
   - The Well Ordering Principle, Induction, and Strong Induction are logically equivalent.

2. Recursion
   - A recursive process is one in which objects are defined in terms of other objects of the same type.
   - Usually, we can describe a relation between objects based on a number $n$ and the corresponding objects based on the numbers $\{1, \ldots, n\}$. This often lets us reduce a problem to a simpler one.

3. Recurrence Relations
   - Suppose a sequence $(a_n)_{n \geq 0}$ is defined recursively by the equation $a_{n+k} + b_1a_{n+k-1} + \cdots + b_k a_n = 0$ for all $n \geq k$, where $b_1, \ldots, b_k$ are constants. If the equation
     \[ x^k + b_1x^{k-1} + \cdots + b_k = 0 \]
     (called the characteristic equation) has distinct roots $r_1, \ldots, r_k$, then there exists constants $c_1, \ldots, c_k$ such that
     \[ a_n = c_1r_1^n + \cdots + c_k r_k^n \quad \text{for all } n. \]
     The values of $c_1, \ldots, c_k$ can be determined by the values of $a_0, \ldots, a_{k-1}$.
   - Suppose a sequence $(a_n)_{n \geq 0}$ is defined recursively by the equation $a_{n+2} + b_1a_{n+1} + b_2 a_n = 0$ for all $n \geq k$, where $b_1$ and $b_2$ are constants. If the equation
     \[ x^2 + b_1x + b_2 = 0 \]
     (called the characteristic equation) has exactly one root $r$, then there exists constants $c_1$ and $c_2$ such that
     \[ a_n = c_1r^n + c_2nr^n \quad \text{for all } n. \]
     The values of $c_1$ and $c_2$ can be determined by the values of $a_0$ and $a_1$.
   - If a more complicated recurrence relation is given, the only real method for determining a formula for the sequence is via ad hoc methods. Sometimes it is not possible to find a closed form either!