Example Problems

Question 1. Find the number of positive integers $k$ such that $k^2 + 2020$ is a square.

Question 2. Show that there is no non-constant polynomial $p$ with integer coefficient such that $p(n)$ is prime for every positive integer $n$.

Question 3. For positive integers $n$ and $k$, let $\sigma(n,k)$ be the sum of all of the divisors $d$ of $n$ such that $\frac{n}{k} \leq d \leq k$. Find $\sum_{n=1}^{k^2} \sigma(n,k)$.

Previous Related Putnam Problems

Question 1. (AB1) Let $f$ be a non-constant polynomial with positive integer coefficients. Prove that if $n$ is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

Question 2. (Modified AB1) Find all ordered pairs $(a,b)$ of positive integers for which $\frac{1}{a} + \frac{1}{b} = 2^{2019}$.

Question 3. (AB1) Determine all possible values of the expression $A^3 + B^3 + C^3 - 3ABC$ where $A$, $B$, and $C$ are non-negative integers.

Question 4. (AB2) Prove that the expression $\frac{\gcd(m,n)}{n} \binom{n}{m}$ is an integer for all pairs of integers $n \geq m \geq 1$.

Question 5. (AB2) Let $S$ be the set of all ordered triples $(p,q,r)$ of prime numbers for which at least one rational number $x$ satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of $S$?

Question 6. (AB3) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that $a_j$ does not divide $a_k$, and replace $a_j$ and $a_k$ with $\gcd(a_j, a_k)$ and $\text{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means the greatest common divisor and lcm means the least common multiple.)

Question 7. (AB3) Find all positive integers $n < 10^{100}$ for which simultaneously $n$ divides $2^n$, $n - 1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.

Question 8. (Modified AB3) Compute

$$\log_2 \left( \prod_{a=1}^{2019} \prod_{b=1}^{2019} \left( 1 + e^{\frac{2\pi i}{2019}} \right) \right).$$

Here $i$ is the imaginary unit (that is, $i^2 = -1$).

Question 9. (AB4) Prove that for each positive integer $n$, the number $10^{10^n} + 10^{10^n} + 10^n - 1$ is not prime.