**Example Putnam Problems**

**Question 1.** (2001, A1) Consider the set $S$ and a binary operation $*$, i.e. for each $a, b \in S$, $a * b \in S$. Assume $(a * b) * a = b$ for all $a, b \in S$. Prove that $a * (b * a) = b$ for all $a, b \in S$.

**Question 2.** (2004, B1) Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients. Suppose that $r$ is a rational number with $P(r) = 0$. Show that the $n$ numbers

$$c_n r, c_n r^2, c_n r^3, c_{n-1} r^2, \ldots, c_1 r$$

are integers.

**Previous Related Putnam Problems**

**Question 3.** (AB1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers $x$, $y$, and $z$. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers $x$ and $y$.

**Question 4.** (AB1) Let $S$ be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfy:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in $S$;

(ii) If $f(x)$ and $g(x)$ are in $S$, the functions $f(x) + g(x)$ and $f(g(x))$ are in $S$;

(iii) If $f(x)$ and $g(x)$ are in $S$ and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in $S$.

Prove that if $f(x)$ and $g(x)$ are in $S$, then the function $f(x)g(x)$ is also in $S$.

**Question 5.** (AB2) Let $*$ be a commutative and associative binary operation on a set $S$. Assume that for every $x$ and $y$ in $S$, there exists $z$ in $S$ such that $x * z = y$ (this $z$ may depend on $x$ and $y$). Show that if $a, b, c$ are in $S$ and $a * c = b * c$, then $a = b$.

**Question 6.** (AB2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever $n$ is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

**Question 7.** (AB2) Let $p(x)$ be a polynomial that is non-negative for all real $x$. Prove that for some $k$ there are polynomials $f_1(x), \ldots, f_k(x)$ such that

$$p(x) = \sum_{j=1}^{k} (f_j(x))^2.$$  

**Question 8.** (AB2) Let $S$ be the set of all positive integers that are not perfect squares. For $n \in S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of $a_r$ over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3, 2 \cdot 4, 2 \cdot 5,$ $2 \cdot 3 \cdot 4,$ $2 \cdot 3 \cdot 5,$ $2 \cdot 4 \cdot 5,$ and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function $f$ from $S$ to the integers is one-to-one.
Question 9. (AB3) For each integer $m$, consider the polynomial

$$P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.$$  

For what values of $m$ is $P_m(x)$ the product of two non-constant polynomials with integer coefficients?

Question 10. (Modified AB3) Given real numbers $b_0, b_1, \ldots, b_{2020}$ with $b_{2020} \neq 0$, let $z_1, z_2, \ldots, z_{2020}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2020} b_k z^k.$$  

Let $\mu = \frac{\sum |z_1| + |z_2| + \cdots + |z_{2020}|}{2020}$ be the average of the distance from $z_1, z_2, \ldots, z_{2020}$ to the origin. Determine the largest constant $M$ such that $\mu \geq M$ for all choices of $b_0, b_1, \ldots, b_{2020}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2020} \leq 2020.$$  

Question 11. (AB4) Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x + 1) - p(x + 1)q(x) = 1.$$  

Question 12. (AB5) Let $p$ be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo $p$. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k$$  

where

$$a_k = k^{p-1} \mod p.$$  

Find the greatest non-negative integer $n$ such that $(x - 1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Question 13. (Modified AB5) Is there a finite abelian group $G$ such that the product of the orders of all its elements is $2^{2020}$?

Question 14. (AB5) Suppose $G$ is a finite group generated by two elements $g$ and $h$, where the order of $g$ is odd. Show that every element of $G$ can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$$  

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of $G$.)