Linear Algebra, Quantum Information Theory, and Operator Algebras

Paul Skoufranis

York University

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Simplified Quantum Mechanics via Linear Algebra

- The *states of a quantum system* are the vectors

$$\mathcal{S} = \{\vec{v} \in \mathbb{C}^n \mid \|\vec{v}\|_2 = 1\}$$

  for some fixed n.
- Two states $\vec{v}$ and $\vec{w}$ are considered the same (modulo a phase shift) if there exists a $z \in \mathbb{C}$ such that $|z| = 1$ and $\vec{v} = z\vec{w}$.
- Quantum mechanics was derived around the idea that observables of a quantum system are *quantized* in the sense that there will exists a basis of eigenstates $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ where the value of the observable for each state is a fixed number $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, and all other states are superpositions of the eigenstates.
- Thus observables are modelled by $A \in M_{n,n}(\mathbb{C})$ where $\vec{v}_k$ is an eigenvector with eigenvalue λ_k for all k.
- Matrices do not necessarily commute $\implies$ uncertainty!

Valid Observables

- As
$$\langle A\vec{v}_k, \vec{v}_k\rangle = \langle \lambda_k \vec{v}_k, \vec{v}_k\rangle = \lambda_k \left\|\vec{v}_k\right\|^2 = \lambda_k,$$
it is extrapolated that the value of the observable A on the state $\vec{v} \in \mathcal{S}$ is $\langle A\vec{v}, \vec{v}\rangle$.
- As observed values should be real, we require A to be self-adjoint:

$$\begin{aligned}
\langle A^* \vec{v}, \vec{w}\rangle &= \frac{1}{4}\sum_{k=1}^{4} i^k \langle A^*(\vec{v} + i^k \vec{w}), \vec{v} + i^k \vec{w}\rangle \\
&= \frac{1}{4}\sum_{k=1}^{4} i^k \langle \vec{v} + i^k \vec{w}, A(\vec{v} + i^k \vec{w})\rangle \\
&= \frac{1}{4}\sum_{k=1}^{4} i^k \overline{\langle A(\vec{v} + i^k \vec{w}), \vec{v} + i^k \vec{w}\rangle} \\
&= \frac{1}{4}\sum_{k=1}^{4} i^k \langle A(\vec{v} + i^k \vec{w}), \vec{v} + i^k \vec{w}\rangle = \langle A\vec{v}, \vec{w}\rangle.
\end{aligned}$$

Spectral Theorem for Self-Adjoint Matrices

Conversely, if A is self-adjoint, then

$$\lambda_k = \lambda_k \langle \vec{v}_k, \vec{v}_k \rangle = \langle \lambda_k \vec{v}_k, \vec{v}_k \rangle = \langle A\vec{v}_k, \vec{v}_k \rangle = \langle \vec{v}_k, A\vec{v}_k \rangle = \langle \vec{v}_k, \lambda_k \vec{v}_k \rangle = \overline{\lambda_k}$$

so $\lambda_k \in \mathbb{R}$ for all k.

Theorem (Spectral Theorem for Self-Adjoint Matrices)

Let $A \in M_{n,n}(\mathbb{C})$. Then the following are equivalent:

1. *A is self-adjoint.*
2. *There exists an orthonormal basis of eigenvectors of A corresponding to real eigenvalues.*
3. *There exists a unitary matrix $U \in M_{n,n}(\mathbb{C})$ and a diagonal matrix $D \in M_{n,n}(\mathbb{R})$ such that $A = UDU^*$.*

For observables, the eigenstates are an orthonormal basis for $\mathbb{C}^n$.

Superposition of Eigenstates

If $\vec{v} \in \mathcal{S}$, then there exists $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n.$$

Note

$$\langle A\vec{v}, \vec{v}\rangle = \sum_{k=1}^{n}\sum_{j=1}^{n} a_k\overline{a_j}\langle A\vec{v}_k, \vec{v}_j\rangle = \sum_{k=1}^{n}\sum_{j=1}^{n} a_k\overline{a_j}\lambda_k\langle \vec{v}_k, \vec{v}_j\rangle = \sum_{k=1}^{n} |a_k|^2\lambda_k.$$

As $\|\vec{v}_2\| = 1$, we have by a similar computation that

$$\sum_{k=1}^{n} |a_k|^2 = 1.$$

Thus we view $\vec{v}$ to be the state which is in the eigenstate $\vec{v}_k$ with probability $p_k = |a_k|^2$ for all k.

Traces and Density Matrices

Definition

Given a matrix $B = [b_{i,j}] \in M_{m,m}(\mathbb{C})$, the *trace of B* is

$$\mathrm{Tr}_m(B) = \sum_{k=1}^{m} b_{k,k}.$$

- Note if $B \in M_{m,n}(\mathbb{C})$ and $C \in M_{n,m}(\mathbb{C})$, then $\mathrm{Tr}_m(BC) = \mathrm{Tr}_n(CB)$.
- Thinking of $\vec{v} \in \mathcal{S}$ as a column vector

$$\langle A\vec{v}, \vec{v}\rangle = \mathrm{Tr}_1(\vec{v}^* A\vec{v}) = \mathrm{Tr}_n(A(\vec{v}\vec{v}^*)).$$

- We call $\vec{v}\vec{v}^* \in M_{n,n}(\mathbb{C})$ the *density matrix of the state* $\vec{v}$.
- Note $\mathrm{Tr}_n(\vec{v}\vec{v}^*) = \mathrm{Tr}_1(\vec{v}^*\vec{v}) = \|\vec{v}\|_2^2 = 1$, $(\vec{v}\vec{v}^*)^* = \vec{v}\vec{v}^*$, and if $\vec{w}$ is an eigenvector for $\vec{v}\vec{v}^*$ with eigenvalue λ,

$$\lambda\langle \vec{w}, \vec{w}\rangle = \langle \vec{v}\vec{v}^*\vec{w}, \vec{w}\rangle = \langle\langle \vec{w}, \vec{v}\rangle\vec{v}, \vec{w}\rangle = |\langle \vec{w}, \vec{v}\rangle|^2 \geq 0.$$

  Hence $\vec{v}\vec{v}^*$ is a positive matrix (i.e. positive semi-definite) of trace 1.

Quantum Channels

A *quantum channel* is a communication channel that can transmit quantum information. Such channels are important in quantum computing where a *qubit* (a two-dimensional quantum system) is used in place of a bit.

To proceed mathematically, quantum information is encoded via the density matrices of quantum states. Then quantum channels are precisely the functions $\Phi : M_{n,n}(\mathbb{C}) \to M_{m,m}(\mathbb{C})$ such that

- Φ preserves the trace: $\mathrm{Tr}_m(\Phi(A)) = \mathrm{Tr}_n(A)$ for all $A \in M_{n,n}(\mathbb{C})$,
- Φ is positive: $\Phi(A)$ is positive for all positive $A \in M_{n,n}(\mathbb{C})$,
- Φ is linear, and
- Φ is *completely positive*.

To summarize, quantum channels are completely positive, trace-preserving linear maps.

Completely Positive Maps

Note for all $d, n \in \mathbb{N}$ that $M_{d,d}(M_{n,n}(\mathbb{C})) \cong M_{dn,dn}(\mathbb{C})$.

Definition

A linear map $\Phi : M_{n,n}(\mathbb{C}) \to M_{m,m}(\mathbb{C})$ is said to be *completely positive* if for every $d \in \mathbb{N}$ the map $\Phi_d : M_{d,d}(M_{n,n}(\mathbb{C})) \to M_{d,d}(M_{m,m}(\mathbb{C}))$ defined by

$$\Phi_d([A_{i,j}]) = [\Phi(A_{i,j})]$$

is a positive map.

Examples of Positive Maps

- Define $\Phi : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ by $\Phi(A) = A^T$. As

$$\Phi(A^*A) = (A^*A)^T = A^T(A^*)^T = A^T(A^T)^*$$

  we see that Φ is positive.
- Define $\Phi : M_{n,n}(\mathbb{C}) \to \mathbb{C}$ by $\Phi(A) = \mathrm{Tr}_n(A)$. Then Φ is positive as $\mathrm{Tr}_n(A)$ will be the sum of the eigenvalues of A (counting algebraic multiplicity).
- Define $\Phi : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ by

$$\Phi([a_{i,j}]) = \mathrm{diag}(a_1, a_2, \ldots, a_n).$$

  Then Φ is positive as a positive matrix must have positive entries along the diagonal and a diagonal matrix is positive if and only if the entries along the diagonal are positive.

Transpose and Complete Positivity

Define $\Phi : M_{2,2}(\mathbb{C}) \to M_{2,2}(\mathbb{C})$ by $\Phi(A) = A^T$. Then Φ is not completely positive. Indeed consider

$$A = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly A is self-adjoint with eigenvalues 0, 0, 0, and 2, so A is positive. However

$$\Phi_2(A) = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which has eigenvalues 1, 1, 1, and -1 and thus is not positive.

Trace and Complete Positivity

Define $\Phi : M_{n,n}(\mathbb{C}) \to \mathbb{C}$ by $\Phi(A) = \mathrm{Tr}_n(A)$. Then Φ is completely positive. Indeed let $A = [A_{i,j}] \in M_{d,d}(M_{n,n}(\mathbb{C}))$ be positive. Then for all $\vec{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{C}^d$ we have

$$\langle \Phi_d(A)\vec{x}, \vec{x}\rangle = \sum_{i,j=1}^{d} \mathrm{Tr}_n(A_{i,j}) x_j \overline{x_i} = \mathrm{Tr}_n\left(\sum_{i,j=1}^{d} \overline{x_i} A_{i,j} x_j\right).$$

However, writing $A = B^*B$ we see that

$$\sum_{i,j=1}^{d} \overline{x_i} A_{i,j} x_j = \begin{bmatrix} x_1 I_n \\ \vdots \\ x_d I_n \end{bmatrix}^* A \begin{bmatrix} x_1 I_n \\ \vdots \\ x_d I_n \end{bmatrix} = \left(B \begin{bmatrix} x_1 I_n \\ \vdots \\ x_d I_n \end{bmatrix}\right)^* \left(B \begin{bmatrix} x_1 I_n \\ \vdots \\ x_d I_n \end{bmatrix}\right)$$

is a positive matrix. Therefore, as Tr_n is positive, $\langle \Phi_d(A)\vec{x}, \vec{x}\rangle \geq 0$. Hence $\Phi_d(A)$ is positive.

Trace and Complete Positivity

For a fixed positive matrix $B \in M_{n,n}(\mathbb{C})$, define $\Phi_B : M_{n,n}(\mathbb{C}) \to \mathbb{C}$ by $\Phi_B(A) = \mathrm{Tr}_n(AB)$. Then Φ_B is completely positive. Indeed, as B is positive, we can write $B = C^*C$ for some $C \in M_{n,n}(\mathbb{C})$. Then for all positive $A = [A_{i,j}] \in M_{d,d}(M_{n,n}(\mathbb{C}))$ we have that

$$(\Phi_B)_d(A) = [\mathrm{Tr}_n(C^*CA_{i,j})] = [\mathrm{Tr}_n(CA_{i,j}C^*)] = (\mathrm{Tr}_n)_d(SAS^*)$$

where

$$S = \mathrm{diag}(C, C, \dots, C).$$

As A is positive, we can write $A = T^*T$ for some $T \in M_{d,d}(M_{n,n}(\mathbb{C}))$ so

$$SAS^* = (ST)(ST)^*$$

is positive. Therefore, as Tr_n is completely positive, $(\Phi_B)_d(A)$ is positive.

Thus 'positive' observables define completely positive maps on the density matrices.

Diagonals and Complete Positivity

Define $\Phi : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ by

$$\Phi([a_{i,j}]) = \mathrm{diag}(a_1, a_2, \ldots, a_n).$$

Then Φ is completely positive. Indeed let $A = [A_{i,j}]_{i,j} \in M_{d,d}(M_{n,n}(\mathbb{C}))$ be positive. Write $A_{i,j} = [a_{i,j,p,q}]_{p,q}$. Then

$$\Phi_d(A) = \begin{bmatrix} \begin{bmatrix} a_{1,1,1,1} & & 0 \\ & \ddots & \\ 0 & & a_{1,1,n,n} \end{bmatrix} & \cdots & \begin{bmatrix} a_{1,d,1,1} & & 0 \\ & \ddots & \\ 0 & & a_{1,d,n,n} \end{bmatrix} \\ \vdots & & \vdots \\ \begin{bmatrix} a_{d,1,1,1} & & 0 \\ & \ddots & \\ 0 & & a_{d,1,n,n} \end{bmatrix} & \cdots & \begin{bmatrix} a_{d,d,1,1} & & 0 \\ & \ddots & \\ 0 & & a_{d,d,n,n} \end{bmatrix} \end{bmatrix}$$

Diagonals and Complete Positivity

Reordering the basis (which is conjugating by a unitary and thus preserves positivity), we obtain that $\Phi_d(A)$ is positive if and only if

$$C = \mathrm{diag}\left(\begin{bmatrix} a_{1,1,1,1} & \cdots & a_{1,d,1,1} \\ \vdots & & \vdots \\ a_{d,1,1,1} & \cdots & a_{d,d,1,1} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1,1,n,n} & \cdots & a_{1,d,n,n} \\ \vdots & & \vdots \\ a_{d,1,n,n} & \cdots & a_{d,d,n,n} \end{bmatrix} \right)$$

is positive. By conjugating A by the same unitary matrix, we get that $T = [[a_{i,j,p,q}]_{p,q}]_{i,j}$ is positive. As each diagonal entry in C is a diagonal minor of T and thus positive, we quickly check that C is positive.

Quantum to Classical

For the remainder of the talk, we will use E to denote the *expectation onto the diagonal map*; that is, $E : M_{n,n}(\mathbb{C}) \to M_{n,n}(\mathbb{C})$ is defined by

$$E([a_{i,j}]) = \mathrm{diag}(a_1, a_2, \ldots, a_n).$$

The expectation map is of interest as it takes matrices (quantum information) and produces commuting matrices (classical information).

Question

Given a self-adjoint $A \in M_{n,n}(\mathbb{R})$, can we determine the possible values of $E(A)$ based on the eigenvalues of A?

To be specific, if $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$ with $x_1, x_2, \ldots x_n \in \mathbb{R}$, then every self-adjoint matrix with eigenvalues $x_1, x_2, \ldots x_n$ is of the form UDU^* for some unitary matrix $U \in M_{n,n}(\mathbb{C})$. So what is

$$\{E(UDU^*) \mid U \in M_{n,n}(\mathbb{C}) \text{ a unitary}\}?$$

Expectations to Doubly Stochastic Matrices

Fix $x_1, x_2, \ldots x_n \in \mathbb{R}$ with $x_1 \geq x_2 \geq \cdots \geq x_n$ and $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$. Let $U = [u_{i,j}]$ and suppose $E(UDU^*) = \mathrm{diag}(y_1, y_2, \ldots, y_n)$. As

$$UDU^* = \left[\sum_{k=1}^{n} u_{i,k} x_k \overline{u_{j,k}}\right],$$

we see that

$$y_i = \sum_{k=1}^{n} |u_{i,k}|^2 x_k$$

for all i. Thus if

$$S = [|u_{i,j}|^2], \quad \vec{x} = (x_1, x_2, \ldots, x_n)^T, \quad \text{and} \quad \vec{y} = (y_1, y_2, \ldots, y_n)^T$$

we have $S\vec{x} = \vec{y}$. Moreover, the sum of any row or column of S is 1 as U is a unitary matrix. Such a matrix is an example of a *doubly stochastic matrix*.

Doubly Stochastic Matrices to Majorization

Note

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n}\sum_{k=1}^{n} |u_{i,k}|^2 x_k = \sum_{k=1}^{n}\sum_{i=1}^{n} |u_{i,k}|^2 x_k = \sum_{k=1}^{n} x_k$$

(i.e. $\mathrm{Tr}(E(UDU^*)) = \mathrm{Tr}(D)$). For distinct $i_1, i_2, \ldots, i_\ell \in \{1, 2, \ldots, n\}$,

$$\sum_{p=1}^{\ell} y_{i_p} = \sum_{p=1}^{\ell}\sum_{k=1}^{n} |u_{i_p,k}|^2 x_k \leq \sum_{k=1}^{\ell} x_k.$$

Definition

Given $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, it is said that $\vec{x}$ *majorizes* $\vec{y}$, denoted $\vec{y} \prec \vec{x}$, if

- $\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} x_k$, and
- for all l,

$$\max\left\{ \sum_{p=1}^{\ell} y_{i_p} \,\middle|\, i_1,\ldots,i_\ell \in \{1,2,\ldots,n\} \text{ distinct}\right\} \leq \max\left\{ \sum_{p=1}^{\ell} x_{i_p} \,\middle|\, i_1,\ldots,i_\ell \in \{1,2,\ldots,n\} \text{ distinct}\right\}.$$

Majorization to Permutation Majorization

Suppose $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ with

$$\vec{y} \prec \vec{x}, \qquad x_1 \geq x_2 \geq \cdots \geq x_n \qquad \text{and} \qquad y_1 \geq y_2 \geq \cdots \geq y_n.$$

As $\vec{y} \prec \vec{x}$, $x_1 \geq y_1$ and if $x_k = y_k$ for all $1 \leq k < \ell$, then $x_\ell \geq y_\ell$. Let p be the first index where $x_p > y_p$. As $\sum_{k=1}^n x_k = \sum_{k=1}^n y_k$, there exists a first index q where $x_q < y_q$. So $x_p > y_p \geq y_q > x_q$. Choose $\theta \in [0, 2\pi]$ such that if

$$x_p' = \cos^2(\theta)x_p + \sin^2(\theta)x_q \qquad \text{and} \qquad x_q' = \sin^2(\theta)x_p + \cos^2(\theta)x_q$$

then either $x_p' = y_p$ and $x_q' < y_q$, or $x_p' > y_p$ and $x_q' = y_q$. As

$$x_p' + x_q' = x_p + x_q$$

and $x_k \geq y_k$ for all $p < k < q$, if $x_k' = x_k$ for all $k \neq p, q$, then

$$(y_1, y_2, \ldots, y_n) \prec (x_1', x_2', \ldots, x_n') \prec (x_1, x_2, \ldots, x_n).$$

Permutation Majorization to Expectations

Repeating, we get a chain

$$\vec{y} = \vec{x}_r \prec \vec{x}_{r-1} \prec \cdots \prec \vec{x}_1 \prec \vec{x}_0 = \vec{x}$$

where $\vec{x}_k$ and $\vec{x}_{k+1}$ differ in only two entries in the above way and the pairs of entries differ from all other pairs of $(\vec{x}_\ell, \vec{x}_{\ell+1})$. With the notation above, if

$$U_1 = \cos(\theta)E_{p,p} - \sin(\theta)E_{p,q} + \sin(\theta)E_{q,p} + \cos(\theta)E_{q,q} + \sum_{\substack{k=1 \\ k\neq p,q}}^{n} E_{k,k},$$

then the diagonal entries of $U_1 \mathrm{diag}(x_1, x_2, \ldots, x_n)U_1^*$ are $\vec{x}_1$. This can be repeated with each progressive unitary U_k not disturbing the previously 'corrected' diagonal entries in lieu of new off-diagonal entries so that when we take the product of the U_k we get a unitary U such that

$$E(UDU^*) = \mathrm{diag}(y_1, y_2, \ldots, y_n).$$

Schur-Horn Theorem

Theorem (Schur-Horn Theorem; 1923, 1954, etc.)

Let $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ *and* $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. *The following are equivalent:*

- *If* $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$, *there exists a unitary* $U \in M_{n,n}(\mathbb{R})$ *such that*
$$E(UDU^*) = \mathrm{diag}(y_1, y_2, \ldots, y_n).$$
- *There exists a doubly stochastic matrix* $A \in M_{n,n}(\mathbb{R})$ *such that* $A\vec{x} = \vec{y}$.
- $\vec{y} \prec \vec{x}$.
- *There exists a chain*
$$\vec{y} = \vec{x}_r \prec \vec{x}_{r-1} \prec \cdots \prec \vec{x}_1 \prec \vec{x}_0 = \vec{x}$$
where $\vec{x}_k$ *and* $\vec{x}_{k+1}$ *differ in only two entries in the above way and the pairs of entries differ from all other pairs of* $(\vec{x}_\ell, \vec{x}_{\ell+1})$.

Convex Hull of Unitary Orbit

Convexity is important in functional analysis. Given $A \in M_{n,n}(\mathbb{C})$, let

$$\mathrm{conv}(\mathcal{U}(A)) = \left\{ \sum_{k=1}^{m} t_k U_k A U_k^* \,\middle|\, \begin{array}{c} U_1, U_2, \ldots, U_m \in M_{n,n}(\mathbb{C}) \text{ unitaries,} \\ t_1, t_2, \ldots, t_m \in (0,1), \text{ and} \\ t_1 + t_2 + \cdots + t_m = 1 \end{array} \right\}.$$

If $A = \mathrm{diag}(a_1, a_2, \ldots, a_n) \in M_{n,n}(\mathbb{R})$, if $B = \mathrm{diag}(b_1, b_2, \ldots, b_n)$ is the above matrix, and $U_k = [u_{i,j,k}]$ then

$$\vec{b} = \left(\sum_{k=1}^{m} t_k [|u_{i,j,k}|^2] \right) \vec{a},$$

so $\vec{b} \prec \vec{a}$. Conversely, by using the chain of majorizations, and

$$\begin{bmatrix} x'_p & 0 \\ 0 & x'_q \end{bmatrix} = \cos^2(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_p & 0 \\ 0 & x_q \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin^2(\theta) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_p & 0 \\ 0 & x_q \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

if $\vec{b} \prec \vec{a}$ then $\mathrm{diag}(b_1, b_2, \ldots, b_n) \in \mathrm{conv}(\mathcal{U}(\mathrm{diag}(a_1, a_2, \ldots, a_n)))$.

Convex Hulls and Density Matrices

Theorem

Let $A, B \in M_{n,n}(\mathbb{C})$ be self-adjoint. Then the following are equivalent:

- *The eigenvalues of B are majorized by the eigenvalues of A.*
- *$B \in \mathrm{conv}(\mathcal{U}(A))$.*

Let $P = \mathrm{diag}(1, 0, 0, \ldots, 0) \in M_{n,n}(\mathbb{R})$. Note if $B = M_{n,n}(\mathbb{C})$, then $B \prec P$ if and only if $\mathrm{Tr}(B) = 1$ and B is positive. Thus for any positive matrix B of trace 1 we can write

$$B = \sum_{k=1}^{m} t_k U_k P U_k^*.$$

If $\vec{u}_1, \ldots, \vec{u}_m$ are the first rows of $U_1, \ldots, U_m$ respectively, then

$$B = \sum_{k=1}^{m} t_k \vec{u}_k \vec{u}_k^*$$

Thus B represents the density matrix of a superposition of states!

To Infinity and Beyond

- A *Hilbert space* is an inner product space that is complete (i.e. every Cauchy sequence converges) with respect to the norm

$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}.$$

- Given a Hilbert space $\mathcal{H}$, an *orthonormal basis* is a maximal orthonormal set.
- Every Hilbert space with an infinitely countable orthonormal basis is isomorphic to

$$\ell_2 = \left\{ (x_n)_{n\geq 1} \,\middle|\, \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\}.$$

- If $L_2[0,1]$ is the square-integrable 'functions' on $[0,1]$, then $L_2[0,1] \cong \ell_2$.

Bounded Linear Maps

- Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the *bounded linear maps from $\mathcal{H}$ to $\mathcal{K}$*, denoted $\mathcal{B}(\mathcal{H}, \mathcal{K})$, are all linear maps $T : \mathcal{H} \to \mathcal{K}$ such that

$$\|T\| = \sup \{\|T(\vec{x})\|_{\mathcal{K}} \mid \vec{x} \in \mathcal{H}, \|\vec{x}\|_{\mathcal{H}} \leq 1\} < \infty.$$

- If $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then there exists a $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$\langle T^*(\vec{x}), \vec{y}\rangle_{\mathcal{H}} = \langle \vec{x}, T(\vec{y})\rangle_{\mathcal{K}}$$

  for all $\vec{x} \in \mathcal{K}$ and $\vec{y} \in \mathcal{H}$.
- A *C^*-algebra* is a norm-closed subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ such that if $A \in \mathfrak{A}$ then $A^* \in \mathfrak{A}$.
- An $A \in \mathfrak{A}$ is *positive* if $A = B^*B$ for some $B \in \mathfrak{A}$; equivalently $\langle A\vec{x}, \vec{x}\rangle_{\mathcal{H}} \geq 0$ for all $\vec{x} \in \mathcal{H}$.
- A *II_1 factor* is a unital C*-algebra with a faithful tracial state, trivial centre, and is closed in the weak operator topology.

Schur-Horn Theorem in II_1 Factors

Theorem (Ravichandran; preprint 2012)

Let $\mathfrak{M}$ be a type II_1 factor, let $\mathcal{A}$ be a MASA in $\mathfrak{M}$, and let $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ denote the normal conditional expectation. Given self-adjoint elements $T \in \mathfrak{M}$ and $S \in \mathcal{A}$, the following are equivalent:

1. $S \prec T$,
2. *there exists an element $R \in \overline{\mathcal{U}}(T)$ such that $E_{\mathcal{A}}(R) = S$.*

Theorem (Kennedy, Skoufranis; 2015)

Let $\mathfrak{M}$ be a type II_1 factor, let $\mathcal{A}$ be a MASA in $\mathfrak{M}$, and let $E_{\mathcal{A}} : \mathfrak{M} \to \mathcal{A}$ denote the normal conditional expectation. Given elements $T \in \mathfrak{M}$ and $S \in \mathcal{A}$, the following are equivalent:

1. $S \prec_w T$,
2. *there exists an element $R \in \mathfrak{M}$ with $\sigma_R = \sigma_T$ such that $E_{\mathcal{A}}(R) = S$.*

In addition, the above is logically equivalent to Ravichandran's result.

Closed Convex Hull of Unitary Orbits in C*-Algebras

Let $\mathfrak{A}$ be a unital C*-algebra and let $\mathcal{T}(\mathfrak{A})$ denote all 'unbounded traces'; that is, all maps $\tau : \mathfrak{A}_+ \to [0, \infty]$ such that

- $\tau(T + S) = \tau(T) + \tau(S)$ for all $T, S \in \mathfrak{A}_+$,
- $\tau(\alpha T) = \alpha\tau(T)$ for all $T \in \mathfrak{A}_+$ and $\alpha \in \mathbb{R}_+$ $(0 \cdot \infty = 0)$,
- $\tau(X^*X) = \tau(XX^*)$ for all $X \in \mathfrak{A}$, and
- τ is lower semicontinuous.

Theorem (Ng, Robert, Skoufranis; 2018)

Let $\mathfrak{A}$ be a unital C-algebra and let $T, S \in \mathfrak{A}$ be self-adjoint. The following are equivalent:*

- $S \in \overline{\mathrm{conv}}(\mathcal{U}(T))$.
- $\tau((S - \alpha)_+) \leq \tau((T - \alpha)_+)$ *and* $\tau((-S - \alpha)_+) \leq \tau((-T - \alpha)_+)$ *for all* $\tau \in \mathcal{T}(\mathfrak{A})$ *and* $\alpha \in \mathbb{R}$.

Courses to Take

Undergraduate:

- MATH 2022: Linear Algebra II
- MATH 3001: Analysis II
- MATH 4011: Metric Spaces
- MATH 4012: Lebesgue Measure Theory
- PHYS 4010: Quantum Mechanics
- EECS 4141: Introduction to Quantum Computing?

Graduate:

- MATH 6280: Measure Theory
- MATH 6450: Topology
- MATH 6461: Functional Analysis I
- MATH 6462: Functional Analysis II
- PHYS 5000: Quantum Mechanics I

Thanks for Listening!