1. Integrals

- Given a bounded function \( f : [a, b] \rightarrow \mathbb{R} \) and a partition \( P = \{t_k\}_{k=0}^{n} \) where 
  \[ a = t_0 < t_1 < \cdots < t_n = b \]

  let \( m_k = \inf_{x \in [t_{k-1}, t_k]} f(x) \) and \( M_k = \sup_{x \in [t_{k-1}, t_k]} f(x) \). The lower and upper Riemann sums of \( f \) on \([a, b]\) with respect to \( P \) are
  \[ L(f, P) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}) \quad \text{and} \quad U(f, P) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}). \]

- A bounded function \( f : [a, b] \rightarrow \mathbb{R} \) is said to be \textit{Riemann integrable} if
  \[ \sup\{(L(f, P) \mid P \text{ a partition of } [a, b])\} = \inf\{(U(f, P) \mid P \text{ a partition of } [a, b])\}. \]

  If the above is true, we denote the value of the sup and inf by \( \int_{a}^{b} f(x) \, dx \). Equivalently, \( f \) is Riemann integrable if for all \( \epsilon > 0 \) there exists a partition \( P \) such that \( 0 \leq U(f, P) - L(f, P) < \epsilon \).

- Note \( L(f, P) \leq \int_{a}^{b} f(x) \, dx \leq U(f, P) \) for any partition \( P \).

- If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, we only need to consider the uniform partitions (i.e. those where \( t_k - t_{k-1} = \frac{b-a}{n} \) for all \( k \)).

- (Fundamental Theorem of Calculus, Part 1) Let \( f : [a, b] \rightarrow \mathbb{R} \) be continuous on \([a, b]\) and define \( F : [a, b] \rightarrow \mathbb{R} \) by
  \[ F(x) = \int_{a}^{x} f(t) \, dt. \]

  Then \( F \) is differentiable on \((a, b)\) and \( F'(x) = f(x) \) for all \( x \in (a, b) \).

- (Fundamental Theorem of Calculus, Part 2) Let \( f, F : [a, b] \rightarrow \mathbb{R} \) be such that \( f \) is Riemann integrable on \([a, b]\), \( F \) is continuous on \([a, b]\), \( F \) is differentiable on \((a, b)\), and \( F'(x) = f(x) \) for all \( x \in (a, b) \). Then
  \[ \int_{a}^{b} f(x) \, dx = F(b) - F(a). \]

- (Substitution/Chain Rule) If \( f : [a, b] \rightarrow \mathbb{R} \) is Riemann integrable and \( g : [c, d] \rightarrow [a, b] \) is differentiable on \((a, b)\) and injective, then
  \[ \int_{a}^{b} f(x) \, dx = \int_{c}^{d} f(g(x))g'(x) \, dx. \]

- (Integration by Parts) If \( f, g : [a, b] \rightarrow \mathbb{R} \) are differentiable on \((a, b)\) and Riemann integrable on \([a, b]\), then
  \[ \int_{a}^{b} f(x)g'(x) \, dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f'(x)g(x) \, dx. \]

- (Comparison Test) Let \( f : [1, \infty) \rightarrow [0, \infty) \) be a continuous, non-increasing function and define \( a_n = f(n) \) for all \( n \in \mathbb{N} \). Then \( \int_{1}^{\infty} f(x) \, dx \) converges if and only if \( \sum_{n=1}^{\infty} a_n \) converges.