YorkU Putnam Training | **Week 1: Induction and Recursion**

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Important Concepts

1. **Induction**

- (Principle of Mathematical Induction) For each $k \in \mathbb{N}$, let P_k be a mathematical statement that is either true or false. Suppose
  (i) (base case) P_1 is true, and
  (ii) (inductive step) if $k \in \mathbb{N}$ and P_k is true, then P_{k+1} is true.

  Then P_n is true for all $n \in \mathbb{N}$.
- Very useful for: showing formulae hold for all $n \in \mathbb{N}$, showing a property is true for all elements of a sequence.
- Can be useful to include multiple statements in P_k for all k.
- (Principle of Strong Mathematical Induction) For each $k \in \mathbb{N}$, let P_k be a mathematical statement that is either true or false. Suppose
  (a) P_1 is true, and
  (b) if $k \in \mathbb{N}$ and P_m is true is true for all $m \leq k$, then P_{k+1} is true.

  Then P_n is true for all $n \in \mathbb{N}$.
- Strong induction can be used to assume multiple (i.e. all) previous statements are true in order to prove the next is.
- (Well Ordering Principle) Each non-empty subset $S \subseteq \mathbb{N}$ has a least element.
- The Well Ordering Principle, Induction, and Strong Induction are logically equivalent.

2. **Recursion**

- A recursive process is one in which objects are defined in terms of other objects of the same type.
- Usually, we can describe a relation between objects based on a number n and the corresponding objects based on the numbers $\{1, \dots, n\}$. This often lets us reduce a problem to a simpler one.

3. **Recurrence Relations**

- Suppose a sequence $(a_n)_{n\geq 0}$ is defined recursively by the equation $a_{n+k}+b_1a_{n+k-1}+\cdots+b_ka_n = 0$ for all $n \geq k$, where $b_1, \dots, b_k$ are constants. If the equation

  $$x^k + b_1x^{k-1} + \cdots + b_{k-1}x + b_k = 0$$

  (called the characteristic equation) has distinct roots $r_1, \dots, r_k$, then there exists constants $c_1, \dots, c_k$ such that

  $$a_n = c_1r_1^n + \cdots + c_kr_k^n \qquad \text{for all } n.$$

  The values of $c_1, \dots, c_k$ can be determined by the values of $a_0, \dots, a_{k-1}$.
- Suppose a sequence $(a_n)_{n\geq 0}$ is defined recursively by the equation $a_{n+2} + b_1a_{n+1} + b_2a_n = 0$ for all $n \geq k$, where b_1 and b_2 are constants. If the equation

  $$x^2 + b_1x + b_2 = 0$$

  (called the characteristic equation) has exactly one root r, then there exists constants c_1 and c_2 such that

  $$a_n = c_1r^n + c_2nr^n \qquad \text{for all } n.$$

  The values of c_1 and c_2 can be determined by the values of a_0 and a_1.
- If a more complicated recurrence relation is given, the only real method for determining a formula for the sequence is via ad hoc methods. Sometimes it is not possible to find a closed form either!