Example Putnam Problem

Question 0. (2009, B1) Show that ever positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2! \times 5!}{3! \times 3! \times 3!}.$$

Select Previous Related Putnam Problems

Question 1. (Modified AB1) For positive integers n, let the number $c(n)$ be determined by the rules $c(1) = 1$, $c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2021} c(n)c(n+2).$$

Question 2. (AB1) Denote by $\mathbb{Z}^2$ the set of all points (x, y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0, 0)$ together with all points (x, y) such that $x^2 + y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

Question 3. (AB1) Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$ where r and s are non-negative integers and no summand divides another. (For example, $23 = 9 + 8 + 6$).

Question 4. (AB2) Let k and n be integers with $1 \leq k < n$. Alice and Bob play a game with k pegs in a line of n holes. At the beginning of the game, the pegs occupy the k leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the k rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of n and k does Alice have a winning strategy?

Question 5. (AB2) Let k be a non-negative integer. Evaluate

$$\sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}$$

Question 6. (Modified AB2) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2020}.

Question 7. (Modified AB2) Given a list of positive integers 1, 2, 3, 4, ..., take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: 6, 16, 27, 36, ... Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2021.

Question 8. (AB3) Call a subset S of $\{1, 2, \ldots, n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1, 2, \ldots n\}$. [For instance, every subset of $\{1, 2, 3\}$ except $\{1, 3\}$ is mediocre, so $A(3) = 7$.] Find all positive integers n such that $A(n+2) - 2A(n+1) + A(n) = 1$.