Example Putnam Problem

Question 0. (2009, B1) Show that ever positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

\[ \frac{10}{9} = \frac{2! \times 5!}{3! \times 3! \times 3!}. \]

Select Previous Related Putnam Problems

Question 1. (Modified AB1) For positive integers \( n \), let the number \( c(n) \) be determined by the rules

\[ c(1) = 1, \quad c(2n) = c(n), \quad \text{and} \quad c(2n+1) = (-1)^n c(n). \]

Find the value of

\[ \sum_{n=1}^{2021} c(n)c(n+2). \]

Question 2. (AB1) Denote by \( \mathbb{Z}^2 \) the set of all points \((x, y)\) in the plane with integer coordinates. For each integer \( n \geq 0 \), let \( P_n \) be the subset of \( \mathbb{Z}^2 \) consisting of the point \((0, 0)\) together with all points \((x, y)\) such that \( x^2 + y^2 = 2^k \) for some integer \( k \leq n \). Determine, as a function of \( n \), the number of four-point subsets of \( P_n \) whose elements are the vertices of a square.

Question 3. (AB1) Show that every positive integer is a sum of one or more numbers of the form \( 2^r 3^s \) where \( r \) and \( s \) are non-negative integers and no summand divides another. (For example, 23 = 9 + 8 + 6).

Question 4. (AB2) Let \( k \) and \( n \) be integers with \( 1 \leq k < n \). Alice and Bob play a game with \( k \) pegs in a line of \( n \) holes. At the beginning of the game, the pegs occupy the \( k \) leftmost holes. A legal move consists of moving a single peg to any vacant hole that is further to the right. The players alternate moves, with Alice playing first. The game ends when the pegs are in the \( k \) rightmost holes, so whoever is next to play cannot move and therefore loses. For what values of \( n \) and \( k \) does Alice have a winning strategy?

Question 5. (AB2) Let \( k \) be a non-negative integer. Evaluate

\[ \sum_{j=0}^{k} 2^{k-j} \binom{k+j}{j}. \]

Question 6. (Modified AB2) Let \( a_0 = 1, \ a_1 = 2, \) and \( a_n = 4a_{n-1} - a_{n-2} \) for \( n \geq 2 \). Find an odd prime factor of \( a_{2020} \).

Question 7. (Modified AB2) Given a list of positive integers 1, 2, 3, 4, . . . , take the first three numbers 1, 2, 3 and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers 4, 5, 7 and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum, and consider the sequence of sums produced: 6, 16, 27, 36, . . . . Prove or disprove that there is some number in the sequence whose base 10 representation ends with 2021.

Question 8. (AB3) Call a subset \( S \) of \( \{1, 2, . . . , n\} \) mediocre if it has the following property: Whenever \( a \) and \( b \) are elements of \( S \) whose average is an integer, that average is also an element of \( S \). Let \( A(n) \) be the number of mediocre subsets of \( \{1, 2, . . . , n\} \). [For instance, every subset of \( \{1, 2, 3\} \) except \( \{1, 3\} \) is mediocre, so \( A(3) = 7 \).] Find all positive integers \( n \) such that \( A(n+2) - 2A(n+1) + A(n) = 1 \).