Important Concepts

1. **Geometry**

   - (Pick's Theorem) Let P be a simple polygon whose vertices are lattice points (points with integer coordinates) and let I and B denote the number of interior and boundary lattice points of the polygon respectively. Then

   $$\text{Area}(P) = I + \frac{B}{2} - 1.$$

   - (Heron's Formula) Let T be a triangle with side lengths a, b, and c and let $s = \frac{a+b+c}{2}$. Then

   $$\text{Area}(T) = \sqrt{s(s-a)(s-b)(s-c)}.$$

   - There are a countless number of possible theorems we might need. Putnam problems related to geometry tend to require very little geometry and more thinking in regards to that geometry. That is, Putnam problems rarely revolve around side length or angle computations.

2. **Complex Numbers**

   - We define i so that $i^2 = -1$. Every complex number has the form $a + bi$ where a and b are real numbers. The number a is called the real part and the number b is called the imaginary part.
   - For $z = a + bi$, the complex conjugate of z is $\overline{z} = a - bi$.
   - The absolute value of a complex number z is $|z| = \sqrt{\overline{z}z} = \sqrt{a^2 + b^2}$.
   - For $z = a + bi$ with $(a, b) \neq (0, 0)$, z is multiplicatively invertible with $z^{-1} = \frac{\overline{z}}{|z|} = \frac{a-bi}{a^2+b^2}$.
   - For $z = a + bi$, $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^a(\cos(b) + i\sin(b))$.
   - For all complex numbers z and w, $e^{z+w} = e^z e^w$.
   - For a natural number n, let $\omega = e^{\frac{2\pi}{n} i}$. Then the polynomial $z^n - 1$ factors as, $\prod_{k=1}^{n}(z - \omega^k)$. That is, the solutions to $z^n = 1$ are the distinct complex numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$. The numbers are called the n^{th} *roots of unity.*
   - (Fundamental Theorem of Algebra) Consider the complex polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ where $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then $p(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_n)$ where $r_1, \ldots, r_n \in \mathbb{C}$.