Practice Problems

Question 0. (2008, A5) Let $n \geq 3$ be an integer. Let $f(x)$ and $g(x)$ be polynomials with real coefficients such that the points $(f(1), g(1)), (f(2), g(2)), \ldots, (f(n), g(n))$ in $\mathbb{R}^2$ are the vertices of a regular n-gon in counter clockwise order. Prove that at least one of $f(x)$ and $g(x)$ has degree greater than or equal to $n-1$.

Previous Related Putnam Problems

Question 1. (AB1) Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A) + f(B) + f(C) + f(D) = 0$. Does it follow that $f(P) = 0$ for all points P in the plane?

Question 2. (AB1) What is the maximum number of rational points that can lie on a circle in $\mathbb{R}^2$ whose centre is not a rational point? (A *rational point* is a point both of whose coordinates are rational numbers.)

Question 3. (AB2) Given any five points on a sphere, show that some four of them must lie on a closed hemisphere.

Question 4. (AB2) Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Question 5. (AB2) Let n be a positive integer and let $f_n(z) = n + (n-1)z + (n-2)z^2 + \cdots + z^{n-1}$. Prove that f_n has no roots in the closed unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

Question 6. (AB2) In the triangle ABC, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}\left(\frac{1}{3}\right)$. Find α.

Question 7. (AB3) Suppose that S is a finite set of points in the plane such that the area of triangle ΔABC is at most 1 whenever A, B, and C are in S. Show that there exists a triangle of area 4 that (together with its interior) covers the set S.