YorkU Putnam Training **Week 4: Number Theory**

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Example Problems

Question 1. (2007, B1) Let f be a non-constant polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Question 2. (2020, A1) How many positive integers N satisfy all of the following three conditions?

(i) N is divisible by 2020.

(ii) N has at most 2020 decimal digits.

(iii) The decimal digits of N are a string of consecutive ones followed by a string of consecutive zeros.

Previous Related Putnam Problems

Question 1. (Modified AB1) Find all ordered pairs (a, b) of positive integers for which $\frac{1}{a} + \frac{1}{b} = \frac{2}{2021}$.

Question 2. (Modified AB1) For a positive integer n, define $d(n)$ to be the sum of the digits of n when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let

$$S = \sum_{k=1}^{2021} (-1)^{d(k)} k^3.$$

Determine S modulo 2021.

Question 3. (AB1) Determine all possible values of the expression

$$A^3 + B^3 + C^3 - 3ABC$$

where A, B, and C are non-negative integers.

Question 4. (AB2) Let S be the set of all ordered triples (p, q, r) of prime numbers for which at least one rational number x satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of S?

Question 5. (AB3) Start with a finite sequence $a_1, a_2, \ldots a_n$ of positive integers. If possible, choose two indices $j < k$ such that a_j does not divide a_k, and replace a_j and a_k with $\gcd(a_j, a_k)$ and $\operatorname{lcm}(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means the greatest common divisor and lcm means the least common multiple.)

Question 6. (AB3) Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n - 1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.

Question 7. (Modified AB3) Compute

$$\log_2\left(\prod_{a=1}^{2021}\prod_{b=1}^{2021}\left(1 + e^{\frac{2\pi ab}{2021} i}\right)\right).$$

Here i is the imaginary unit (that is, $i^2 = -1$).

Question 8. (AB4) Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.