Example Problems

Question 1. (2007, B1) Let $f$ be a non-constant polynomial with positive integer coefficients. Prove that if $n$ is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.

Question 2. (2020, A1) How many positive integers $N$ satisfy all of the following three conditions?
(i) $N$ is divisible by 2020.
(ii) $N$ has at most 2020 decimal digits.
(iii) The decimal digits of $N$ are a string of consecutive ones followed by a string of consecutive zeros.

Previous Related Putnam Problems

Question 1. (Modified AB1) Find all ordered pairs $(a, b)$ of positive integers for which $\frac{1}{a} + \frac{1}{b} = \frac{2}{2021}$.

Question 2. (Modified AB1) For a positive integer $n$, define $d(n)$ to be the sum of the digits of $n$ when written in binary (for example, $d(13) = 1 + 1 + 0 + 1 = 3$). Let
$$S = \sum_{k=1}^{2021} (-1)^{d(k)} k^3.$$ Determine $S$ modulo 2021.

Question 3. (AB1) Determine all possible values of the expression
$$A^3 + B^3 + C^3 - 3ABC$$
where $A$, $B$, and $C$ are non-negative integers.

Question 4. (AB2) Let $S$ be the set of all ordered triples $(p, q, r)$ of prime numbers for which at least one rational number $x$ satisfies $px^2 + qx + r = 0$. Which primes appear in seven or more elements of $S$?

Question 5. (AB3) Start with a finite sequence $a_1, a_2, \ldots, a_n$ of positive integers. If possible, choose two indices $j < k$ such that $a_j$ does not divide $a_k$, and replace $a_j$ and $a_k$ with $\gcd(a_j, a_k)$ and $\lcm(a_j, a_k)$, respectively. Prove that if this process is repeated, it must eventually stop and the final sequence does not depend on the choices made. (Note: gcd means the greatest common divisor and lcm means the least common multiple.)

Question 6. (AB3) Find all positive integers $n < 10^{100}$ for which simultaneously $n$ divides $2^n$, $n - 1$ divides $2^n - 1$, and $n - 2$ divides $2^n - 2$.

Question 7. (Modified AB3) Compute
$$\log_2 \left( \prod_{a=1}^{2021} \prod_{b=1}^{2021} \left( 1 + e^{\frac{2\pi i a b}{2021}} \right) \right).$$
Here $i$ is the imaginary unit (that is, $i^2 = -1$).

Question 8. (AB4) Prove that for each positive integer $n$, the number $10^{10^n} + 10^{10^n} + 10^n - 1$ is not prime.