Important Concepts

1. **Fields**
   - Most of basic linear algebra holds over any field: for example, $\mathbb{R}$ and $\mathbb{C}$.
   - The most common other field is $\mathbb{F}_p$ where p is prime. Here, $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ together with the operations of addition and multiplication modulo p.
2. **Vector Spaces**
   - A *vector space* is a set of vectors together with vector addition and scalar multiplication that satisfy certain axioms.
   - A *subspace* of a vector space is contains the zero vector and is closed under addition and scalar multiplication.
   - The *span* of a set of vectors is all linear combinations of the vectors.
   - A set of vectors is *linearly independent* if the only linear combination that produces the zero vector is the zero linear combination.
   - A *basis* for a vector space is a linearly independent set of vectors whose span is the whole vector space.
   - Two bases for a vector space that is spanned by a finite number of elements must have the same number of elements.
   - The *dimension* of a vector space is the number of elements in any basis of the space.
3. **Linear Maps**
   - A map T from a vector space V to a vector space W is *linear* if it preserves addition and scalar multiplication.
   - The image of a linear map $T : V \to W$ is $\{T(\vec{v}) \mid \vec{v} \in V\}$.
   - The kernel of a linear map $T : V \to W$ is $\{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$.
   - (Rank-Nullity Theorem) If $T : V \to W$ is a linear map and V is finite dimensional, then the sum of the dimensions of the kernel and image of T is the dimension of V.
4. **Matrices**
   - If $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are $n \times m$ matrices, then $A + B$ is the $n \times m$ matrix $[a_{i,j} + b_{i,j}]$.
   - If $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are $n \times m$ and $m \times p$ matrices respectively, then AB is the $n \times p$ matrix $[\sum_{k=1}^{m} a_{i,k}b_{k,j}]$.
   - Given a field $\mathbb{F}$, if $\mathbb{F}^n = \{(a_1, \ldots, a_n) \mid a_k \in \mathbb{F}\}$, then every linear map $T : \mathbb{F}^n \to \mathbb{F}^m$ is of the form $T(\vec{x}) = A\vec{v}^T$ where A is an $m \times n$ matrix with entries in $\mathbb{F}$ and $\vec{v}^T$ means write $\vec{v}$ as a column vector.
5. **Determinants**
   - Adding a multiple of one row/column to another does not change the determinant.
   - Swapping two rows/columns multiplies the determinant by -1.
   - Multiplying a row/column by a constant multiplies the determinant by the constant.
   - An $n \times n$ matrix A is invertible if and only if its rows are linearly independent if and only if its columns are linearly independent if and only if $\det(A) \neq 0$.
   - For all $n \times n$ matrices A and B, $\det(AB) = \det(A)\det(B)$.
   - For all invertible $n \times n$ matrices A, $\det(A^{-1}) = \det(A)^{-1}$.
   - For all $n \times n$ matrices A, $\det(A^t) = \det(A)$ where A^t denotes the transpose of A.
   - $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$.

- (Cofactor Expansion) If A is an $n \times n$ matrix and $k \in \{1, \ldots, n\}$, then

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+k} a_{i,k} \det(A_{i,k}) = \sum_{j=1}^{n} (-1)^{k+j} a_{k,j} \det(A_{k,j})$$

where $A_{i,j}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column from A.

- (Permutation Expansion) If A is an $n \times n$ matrix with (i,j)-entry $a_{i,j}$, then

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\text{inv}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$ and, for $\sigma \in S_n$, $\text{inv}(\sigma)$ is the number of inversions of σ (that is, the number of $x, y \in \{1, \ldots, n\}$ such that $x < y$ yet $\sigma(x) > \sigma(y)$). Also note $(-1)^{\text{inv}(\sigma)} = (-1)^m$ for any m such that σ can be written as a composition of m transpositions (a permutation σ is a transposition if there exists $x, y \in \{1, \ldots, n\}$ such that $x \neq y$, $\sigma(x) = y$, $\sigma(y) = x$, and $\sigma(z) = z$ for all $z \neq x, y$).

- Given an $n \times n$ matrix A, an $n \times m$ matrix B, and a $m \times m$ matrix C,

$$\det\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right) = \det(A)\det(C)$$

where 0 is the $m \times n$ zero matrix.

6. **Eigenvalues**

- The *characteristic polynomial* of an $n \times n$ matrix A is $\chi_A(x) = \det(xI_n - A)$.
- A scalar λ is the eigenvalue of an $n \times n$ matrix A if and only if $\chi_A(\lambda) = 0$.
- (Cayley-Hamilton Theorem) If A is an $n \times n$ matrix in an algebraically closed field and if $\chi_A(x) = x^n + b_{n-1}x^{n-1} + \cdots b_1 x + b_0$, then

$$A^n + b_{n-1}A^{n-1} + \cdots + b_1 A + b_0 I_n$$

is the $n \times n$ zero matrix, where I_n is the $n \times n$ identity matrix.

7. **Inner Products**

- An *inner product* on a vector space V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that
  (i) $\langle \vec{v}, \vec{v} \rangle > 0$ for all $\vec{v} \in V \setminus \{\vec{0}\}$,
  (ii) $\langle \alpha\vec{v} + \vec{w}, \vec{x} \rangle = \alpha\langle \vec{v}, \vec{x} \rangle + \langle \vec{w}, \vec{x} \rangle$ for all $\vec{v}, \vec{w}, \vec{x} \in V$ and $\alpha \in \mathbb{C}$, and
  (iii) $\langle \vec{v}, \vec{w} \rangle = \overline{\langle \vec{v}, \vec{w} \rangle}$ for all $\vec{v}, \vec{w} \in V$.
- The most common inner product is the dot product: $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = \sum_{k=1}^{n} a_k \overline{b_k}$.
- The *norm* of a vector in an inner product is $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$.
- (Cauchy-Schwarz Inequality) $|\langle \vec{v}, \vec{w} \rangle| \leq \|\vec{v}\| \, \|\vec{w}\|$.
- Two vectors are *orthogonal* if $\langle \vec{v}, \vec{w} \rangle = 0$. In this case $\|\vec{v} + \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2$.

8. **Canonical Forms**

- (Schur Decomposition) If A is a $n \times n$ matrix with complex entries, then there exists an $n \times n$ unitary matrix with complex entries such that $A = U^*TU$ where U^* is the conjugate transpose of U and T is an $n \times n$ upper triangular matrix with complex entries.
- (Spectral Theorem) If A is a self-adjoint $n \times n$ matrix with real entries, then there exists an $n \times n$ unitary matrix with real entries such that $A = U^tDU$ where U^t is the transpose of U and D is an $n \times n$ diagonal matrix with the eigenvalues of A along the diagonal.
- (Spectral Theorem) If A is a complex $n \times n$ matrix such that $A^*A = AA^*$, where A^* is the conjugate transpose of A, then there exists an $n \times n$ unitary matrix (with complex entries) such that $A = U^*DU$ where D is an $n \times n$ diagonal matrix with the eigenvalues of A along the diagonal.

- If A is a real $n \times n$ matrix that is diagonalizable over $\mathbb{C}$, then there exists an invertible $n \times n$ matrix V such that $V^{-1}AV$ is a block diagonal matrices with each block being a scalar or being of the form

$$\alpha \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

  for some $\alpha, \theta \in \mathbb{R}$.

- (Singular Value Decomposition) If A is an $n \times n$ matrix (with real entries), then there exists $n \times n$ unitary matrices (with real entries) U and V such that $A = UDV$ where D is a diagonal matrix with non-negative entries along the diagonal.

- (Jordan Normal Form) If A is an $n \times n$ matrix with complex entries, then there exists an invertible matrix V with complex entries such that $V^{-1}AV$ is a block diagonal matrix with every block diagonal entries of the form

$$\mathcal{J}_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda & 1 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & \lambda \end{bmatrix}$$

  where the matrix is $m \times m$ and λ is an eigenvalue of A.