Example Putnam Problem

Question 0. (2009, A3) Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos(1)$, $\cos(2)$, …, $\cos(n^2)$. For example

$$d_3 = \det\left(\begin{bmatrix} \cos(1) & \cos(2) & \cos(3) \\ \cos(4) & \cos(5) & \cos(6) \\ \cos(7) & \cos(8) & \cos(9) \end{bmatrix}\right).$$

The arguments are always in radians and not degrees. Evaluate $\lim_{n\to\infty} d_n$.

Previous Related Putnam Problems

Question 1. (AB2) Let A and B be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Question 2. (Modified AB2) Alan and Barbara play a game in which they take turns filling entries of an initially empty 2020×2020 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Question 3. (AB2) Let A be the $n \times n$ matrix whose entry in the i^{th}-row and j^{th}-column is $\frac{1}{\min(i,j)}$ for $1 \leq i, j \leq n$. Compute $\det(A)$.

Question 4. (AB2) Let $S_1, S_2, \ldots, S_{2^n-1}$ be the non-empty subsets of $\{1, 2, \ldots, n\}$ in some order, and let M be the $(2^n - 1) \times (2^n - 1)$ matrix whose (i, j) entry is

$$m_{i,j} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset \\ 1 & \text{otherwise} \end{cases}.$$

Calculate the determinant of M.

Question 5. (AB3) What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

Question 6. (AB3) Let Q be an $n \times n$ real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the $n \times n$ identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

Question 7. (AB3) Let S be the set of all 2×2 real matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

whose entries a, b, c, d (in that order) form an arithmetic progression. Find all matrices M in S for which there is some integer $k > 1$ such that M^k is also in S.

Question 8. (Modified AB4) Say that a polynomial with real coefficients in two variables x, y, is *balanced* if the average of the polynomial on each circle centred at the origin is 0. The balanced polynomials of degree at most 2021 form a vector space V over $\mathbb{R}$. Find the dimension of V.

Question 9. (AB4) Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Each entry is chosen to be 0 or 1, each with probability $\frac{1}{2}$. Find the expected value of $\det(A - A^t)$ (as a function of n), where A^t is the transpose of A.

Question 10. (AB4) For which positive integers n is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, which every dot product of two different rows is odd?

Question 11. (AB6) Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry of A to the k^{th} power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

Question 12. (AB6) Let n be a positive integer. Suppose that A, B, and M are $n \times n$ matrices with real entries such that $AM = MB$, and such that A and B have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix X with real entries.