Example Putnam Problem

Question 0. (2009, A3) Let $d_n$ be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos(1), \cos(2), \ldots, \cos(n^2)$. For example

$$d_3 = \det \begin{pmatrix} \cos(1) & \cos(2) & \cos(3) \\ \cos(4) & \cos(5) & \cos(6) \\ \cos(7) & \cos(8) & \cos(9) \end{pmatrix}.$$ 

The arguments are always in radians and not degrees. Evaluate $\lim_{n \to \infty} d_n$.

Previous Related Putnam Problems

Question 1. (AB2) Let $A$ and $B$ be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

Question 2. (Modified AB2) Alan and Barbara play a game in which they take turns filling entries of an initially empty $2020 \times 2020$ array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

Question 3. (AB2) Let $A$ be the $n \times n$ matrix whose entry in the $i^{th}$-row and $j^{th}$-column is $\frac{1}{\min(i,j)}$ for $1 \leq i, j \leq n$. Compute $\det(A)$.

Question 4. (AB2) Let $S_1, S_2, \ldots, S_{2^n-1}$ be the non-empty subsets of $\{1, 2, \ldots, n\}$ in some order, and let $M$ be the $(2^n - 1) \times (2^n - 1)$ matrix whose $(i,j)$ entry is

$$m_{i,j} = \begin{cases} 0 & \text{if } S_i \cap S_j = \emptyset \\ 1 & \text{otherwise} \end{cases}.$$ 

Calculate the determinant of $M$.

Question 5. (AB3) What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

Question 6. (AB3) Let $Q$ be an $n \times n$ real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where $I$ is the $n \times n$ identity matrix. Show that if 1 is not an eigenvalue of $Q$, then 1 is an eigenvalue of $PQ$.

Question 7. (AB3) Let $S$ be the set of all $2 \times 2$ real matrices

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

whose entries $a, b, c, d$ (in that order) form an arithmetic progression. Find all matrices $M$ in $S$ for which there is some integer $k > 1$ such that $M^k$ is also in $S$.

Question 8. (Modified AB4) Say that a polynomial with real coefficients in two variables $x, y$, is balanced if the average of the polynomial on each circle centred at the origin is 0. The balanced polynomials of degree at most 2021 form a vector space $V$ over $\mathbb{R}$. Find the dimension of $V$. 
**Question 9.** (AB4) Let $A$ be a $2n \times 2n$ matrix, with entries chosen independently at random. Each entry is chosen to be 0 or 1, each with probability $\frac{1}{2}$. Find the expected value of $\det(A - A^t)$ (as a function of $n$), where $A^t$ is the transpose of $A$.

**Question 10.** (AB4) For which positive integers $n$ is there an $n \times n$ matrix with integer entries such that every dot product of a row with itself is even, which every dot product of two different rows is odd?

**Question 11.** (AB6) Let $A$ be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer $k$, let $A^{[k]}$ be the matrix obtained by raising each entry of $A$ to the $k^{th}$ power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n + 1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

**Question 12.** (AB6) Let $n$ be a positive integer. Suppose that $A$, $B$, and $M$ are $n \times n$ matrices with real entries such that $AM = MB$, and such that $A$ and $B$ have the same characteristic polynomial. Prove that $\det(A - MX) = \det(B - XM)$ for every $n \times n$ matrix $X$ with real entries.