Example Putnam Problems

Question 1. (2004, B1) Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number with $P(r) = 0$. Show that the n numbers

$$c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \ldots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

Question 2. (2013, A2) Let S be the set of all positive integers that are not perfect squares. For $n \in S$, consider choices of integers $a_1, a_2, \ldots, a_r$ such that $n < a_1 < a_2 < \cdots < a_r$ and $n \cdot a_1 \cdot a_2 \cdots a_r$ is a perfect square, and let $f(n)$ be the minimum of a_r over all such choices. For example, $2 \cdot 3 \cdot 6$ is a perfect square, while $2 \cdot 3$, $2 \cdot 4$, $2 \cdot 5$, $2 \cdot 3 \cdot 4$, $2 \cdot 3 \cdot 5$, $2 \cdot 4 \cdot 5$, and $2 \cdot 3 \cdot 4 \cdot 5$ are not, and so $f(2) = 6$. Show that the function f from S to the integers is one-to-one.

Previous Related Putnam Problems

Question 3. (AB1) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

Question 4. (AB1) Let S be the class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfy:

(i) The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x + 1)$ are in S;

(ii) If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;

(iii) If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Question 5. (AB2) Let $*$ be a commutative and associative binary operation on a set S. Assume that for every x and y in S, there exists z in S such that $x * z = y$ (this z may depend on x and y). Show that if a, b, c are in S and $a * c = b * c$, then $a = b$.

Question 6. (AB2) Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever n is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

Question 7. (Modified AB3) Given real numbers b_0, b_1, $\ldots$, b_{2021} with $b_{2021} \neq 0$, let z_1, z_2, $\ldots$, z_{2021} be the roots in the complex plan of the polynomial

$$P(z) = \sum_{k=0}^{2021} b_k z^k.$$

Let $\mu = \frac{|z_1|+|z_2|+\cdots+|z_{2021}|}{2021}$ be the average of the distance from z_1, z_2, $\ldots$, z_{2020} to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of b_0, b_1, $\ldots$, b_{2021} that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2021} \leq 2021.$$

Question 8. (AB4) Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1.$$

Question 9. (AB5) Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k$$

where

$$a_k = k^{\frac{p-1}{2}} \mod p.$$

Find the greatest non-negative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

Question 10. (Modified AB5) Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2021}?

Question 11. (AB5) Suppose G is a finite group generated by two elements g and h, where the order of g is odd. Show that every element of G can be written in the form

$$g^{m_1} h^{n_1} g^{m_2} h^{n_2} \cdots g^{m_r} h^{n_r}$$

with $1 \leq r \leq |G|$ and $m_1, n_1, m_2, n_2, \ldots, m_r, n_r \in \{1, -1\}$. (Here $|G|$ is the number of elements of G.)