IMPORTANT CONCEPTS

Main tricks: Define new sequences/series!

1. **Sequences**
   - A sequence $(a_n)_{n\geq 1}$ *converges to a number* L, denoted $\lim_{n\to\infty} a_n = L$, if for all $\epsilon > 0$ there exists a $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$.
   - (Monotone Convergence Theorem) Every bounded sequence that is either non-decreasing or non-increasing converges.
   - (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.
   - $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.
2. **Series**
   - A series $\sum_{n=1}^\infty a_n$ *converges to a number* L, denoted $\sum_{n=1}^\infty a_n = L$, if the sequence $(s_n)_{n\geq 1}$ where $s_n = \sum_{k=1}^n a_k$ converges to L.
   - A series $\sum_{n=1}^\infty a_n$ is said to *converges absolutely* if $\sum_{n=1}^\infty |a_n|$ converges. In this case, any rearrangement of the series $\sum_{n=1}^\infty a_n$ converges to the same value as $\sum_{n=1}^\infty a_n$; that is, $\sum_{n=1}^\infty a_n = \sum_{n=1}^\infty a_{\sigma(n)}$ for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$.
   - A series $\sum_{n=1}^\infty a_n$ is said to *converge conditionally* if it converges but not absolutely. In this case, for all $r \in \mathbb{R}$ there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^\infty a_{\sigma(n)} = r$.
   - (Ratio and Root Tests) Given a series $\sum_{n=1}^\infty a_n$, if $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then
     (i) if $L < 1$, the series converges absolutely,
     (ii) if $L > 1$, the series diverges, and
     (iii) if $L = 1$, we have no idea.
   - Common series:
     (i) For $r \in \mathbb{C}$ and $n \in \mathbb{N}$, $\sum_{k=0}^n r^k = \frac{r^{n+1}-1}{r-1}$.
     (ii) For $x \in \mathbb{C}$, the sum $\sum_{k=0}^\infty x^k$ converges if and only if $|x| < 1$ in which case $\sum_{k=0}^\infty x^k = \frac{1}{1-x}$.
     (iii) For all $x \in \mathbb{R}$, $\sum_{k=0}^\infty \frac{x^k}{k!} = e^x$.
     (iv) For $x \in (-1, 1)$, $\sum_{k=1}^\infty \frac{(-1)^{k+1}x}{k} = \ln(1 + x)$.
     (v) For $x \in (-1, 1)$, $\sum_{k=1}^\infty kx^k = \frac{x}{(1-x)^2}$.
     (vi) For $x \in \mathbb{R}$, $\sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sin(x)$.
     (vii) For $x \in \mathbb{R}$, $\sum_{k=0}^\infty \frac{(-1)^{k+1} x^{2k}}{(2k)!} = \cos(x)$.
     (viii) For $p \in \mathbb{R}$, $\sum_{n=1}^\infty \frac{1}{n^p}$ converges if and only if $p > 1$.
     (ix) $\sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$.
     (x) $\sum_{k=1}^n k = \frac{n(n+1)}{2}$.
     (xi) $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.
     (xii) $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$.
3. **Inequalities**
   - (Hölder's Inequality) Given $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, for any sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ we have that
     $$\sum_{n=1}^\infty |a_n b_n| \leq \left(\sum_{n=1}^\infty |a_n|^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty |b_n|^q\right)^{\frac{1}{q}}.$$
     The case $p = q = 2$ is also known as the Cauchy Schwarz Inequality.
   - (Arithmetic-Geometric Mean Inequality) For any non-negative numbers $x_1, x_2, \ldots, x_n$, we have that
     $$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}.$$