1. Sequences

- A sequence \((a_n)_{n \geq 1}\) converges to a number \(L\), denoted \(\lim_{n \to \infty} a_n = L\), if for all \(\epsilon > 0\) there exists a \(N \in \mathbb{N}\) such that \(|a_n - L| < \epsilon\) for all \(n \geq N\).
- (Monotone Convergence Theorem) Every bounded sequence that is either non-decreasing or non-increasing converges.
- (Bolzano-Weierstrass Theorem) Every bounded sequence has a convergent subsequence.
- \(\lim_{n \to \infty} (1 + \frac{1}{n})^n = e\).

2. Series

- A series \(\sum_{n=1}^{\infty} a_n\) converges to a number \(L\), denoted \(\sum_{n=1}^{\infty} a_n = L\), if the sequence \((s_n)_{n \geq 1}\) where \(s_n = \sum_{k=1}^{n} a_k\) converges to \(L\).
- A series \(\sum_{n=1}^{\infty} a_n\) is said to converge absolutely if \(\sum_{n=1}^{\infty} |a_n|\) converges. In this case, any rearrangement of the series \(\sum_{n=1}^{\infty} a_n\) converges to the same value as \(\sum_{n=1}^{\infty} a_n\); that is, \(\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}\) for any bijection \(\sigma : \mathbb{N} \to \mathbb{N}\).
- A series \(\sum_{n=1}^{\infty} a_n\) is said to converge conditionally if it converges but not absolutely. In this case, for all \(r \in \mathbb{R}\) there exists a bijection \(\sigma : \mathbb{N} \to \mathbb{N}\) such that \(\sum_{n=1}^{\infty} a_{\sigma(n)} = r\).
- (Ratio and Root Tests) Given a series \(\sum_{n=1}^{\infty} a_n\), if \(L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|\) exists or \(L = \lim_{n \to \infty} \sqrt[n]{|a_n|}\) exists, then
  (i) if \(L < 1\), the series converges absolutely,
  (ii) if \(L > 1\), the series diverges, and
  (iii) if \(L = 1\), we have no idea.
- Common series:
  (i) For \(r \in \mathbb{C}\) and \(n \in \mathbb{N}\), \(\sum_{k=0}^{\infty} r^k = \frac{r^{n+1} - 1}{r - 1}\).
  (ii) For \(x \in \mathbb{C}\), the sum \(\sum_{k=0}^{\infty} x^k\) converges if and only if \(|x| < 1\) in which case \(\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}\).
  (iii) For all \(x \in \mathbb{R}\), \(\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x\).
  (iv) For \(x \in (-1, 1)\), \(\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = \ln(1 + x)\).
  (v) For \(x \in (-1, 1)\), \(\sum_{k=1}^{\infty} \frac{x^k}{k!} = e^{-x}\).
  (vi) For \(x \in \mathbb{R}\), \(\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k+1}}{(2k+1)!} = \sin(x)\).
  (vii) For \(x \in \mathbb{R}\), \(\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)!} = \cos(x)\).
  (viii) For \(p \in \mathbb{R}\), \(\sum_{n=1}^{\infty} \frac{1}{n^p}\) converges if and only if \(p > 1\).
  (ix) \(\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\).
  (x) \(\sum_{k=1}^{n} \frac{1}{k} = \gamma + \ln(n) + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\).
  (xi) \(\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}\).
  (xii) \(\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}\).

3. Inequalities

- (Hölder’s Inequality) Given \(p, q \in (1, \infty)\) with \(\frac{1}{p} + \frac{1}{q} = 1\), for any sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) we have that
  \[\sum_{n=1}^{\infty} |a_n b_n| \leq \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} |b_n|^q \right)^{\frac{1}{q}}.\]
  The case \(p = q = 2\) is also known as the Cauchy Schwarz Inequality.
- (Arithmetic-Geometric Mean Inequality) For any non-negative numbers \(x_1, x_2, \ldots, x_n\), we have that
  \[\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}.\]