Example Putnam Problem

Question 0. (2021, A4) Consider a horizontal strip of $N+2$ squares in which the first and the last square are black and the remaining N squares are all white. Choose a white square uniformly at random, choose one of its two neighbours with equal probability, and colour this neighbouring square black if it is not already black. Repeat this process until all the remaining white squares have only black neighbours. Let $w(N)$ be the expected number of white squares remaining. Find

$$\lim_{N\to\infty} \frac{w(N)}{N}.$$

Previous Related Putnam Problems

Question 1. (AB1) Is there an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m?

Question 2. (AB2) Let x_0, x_1, x_2, $\ldots$ be the sequence such that $x_0 = 1$ and for $n \geq 0$,

$$x_{n+1} = \ln(e^{x_n} - x_n)$$

(as usual, the function ln is the natural logarithm). Show that the infinite series

$$x_0 + x_1 + x_2 + \cdots$$

converges and find its sum.

Question 3. (AB2) Given a positive integer n, let $M(n)$ be the largest integer m such that

$$\binom{m}{n-1} > \binom{m-1}{n}.$$

Evaluate $\lim_{n\to\infty} \frac{M(n)}{n}$.

Question 4. (AB2) For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n\to\infty} \frac{a_n}{n^3}.$$

Question 5. (AB2) Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume the sequence $(b_j)_{j\geq 1}$ is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}$$

converges, and evaluate S.

Question 6. (AB3) Let $a_0 = \frac{\pi}{2}$, and let $a_n = \sin(a_{n-1})$ for $n \geq 1$. Determine whether

$$\sum_{n=1}^{\infty} a_n^2$$

converges.

Question 7. (AB4) Let T be the set of all triples (a, b, c) of positive integers for which there exist triangles with side lengths a, b, and c. Express

$$\sum_{(a,b,c)\in T} \frac{2^a}{3^b 5^c}.$$

as a rational number in lowest terms.

Question 8. (AB4) A class of $2N$ students took a quiz on which the possible scores were $0, 1, 2, \dots, 10$. Each of these scores occurred at least once, and the average score was exactly 7.4. Show the class can be divided into two groups of N students in such a way that the average score for each group was exactly 7.4.

Question 9. (AB4) Evaluate the sum

$$\sum_{k=0}^{\infty} \left(3 \cdot \frac{\ln(4k+2)}{4k+2} - \frac{\ln(4k+3)}{4k+3} - \frac{\ln(4k+4)}{4k+4} - \frac{\ln(4k+5)}{4k+5} \right) = 3 \cdot \frac{\ln(2)}{2} - \frac{\ln(3)}{3} - \frac{\ln(4)}{4} - \frac{\ln(5)}{5} + 3 \cdot \frac{\ln(6)}{6} - \frac{\ln(7)}{7} - \frac{\ln(8)}{8} - \frac{\ln(9)}{9} + 3 \cdot \frac{\ln(10)}{10} - \cdots .$$

(As usual, $\ln(x)$ denotes the natural logarithm of x.)

Question 10. Given a real number a, we define a sequence by $x_0 = 1$, $x_1 = x_2 = a$, and $x_{n+1} = 2x_n x_{n-1} - x_{n-2}$ for $n \geq 2$. Prove that if $x_n = 0$ for some $n \geq 1$, then the sequence is periodic.