YorkU Putnam Training **Week 8: Derivatives and Analysis**

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Important Concepts

1. **Continuity**

   - A function $f : (a, b) \to \mathbb{R}$ is *continuous at a point* $c \in (a, b)$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in (a, b)$ and $|x - c| < \delta$ then $|f(x) - f(c)| < \epsilon$.
   - If f is continuous at a point x_0 and $f(x_0) > 0$, there exists an $\epsilon > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$.

2. **Derivatives**

   - A function $f : (a, b) \to \mathbb{R}$ is said to be *differentiable at a point* $c \in (a, b)$ if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists. If the limit exists, the value of the limit is denoted $f'(c)$.
   - If a function f is differentiable at a point c, then f is continuous at c.
   - (Product Rule) $(fg)'(x) = f'(x)g(g) + f(x)g'(x)$ provided the derivatives make sense.
   - (Quotient Rule) $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$ provided this makes sense.
   - (Chain Rule) $(f \circ g)'(x) = f'(g(x))g'(x)$ provided this makes sense.

3. **The Value Theorems**

   - (Intermediate Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and either $f(a) < c < f(b)$ or $f(b) < c < f(a)$, then there exists a $d \in (a, b)$ such that $f(d) = c$.
   - (Extreme Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$. Furthermore, if f is differentiable on $[a, b]$ and if for $k = 1$ or $k = 2$ we have that $x_k \in (a, b)$, then $f'(x_k) = 0$.
   - (Mean Value Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

4. **Taylor's Theorem**

   - Let $k \geq 1$ be an integer and let $f : \mathbb{R} \to \mathbb{R}$ be k times differentiable at a point $a \in \mathbb{R}$. Then there exists a function $h_k : \mathbb{R} \to \mathbb{R}$ such that

   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + h_k(x)(x-a)^k$$

   where $\lim_{x\to a} h_k(x) = 0$.
   - Let $k \geq 1$ be an integer and let $f : \mathbb{R} \to \mathbb{R}$ be k times continuously differentiable at a point $a \in \mathbb{R}$. Then there exists a point c in the open interval between x and a so that

   $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}(x-a)^{k-1} + \frac{f^{(k)}(c)}{k!}(x-a)^k.$$

5. **Convexity**

   - A function $f : [a, b] \to \mathbb{R}$ is said to be *convex* if $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ for all $x, y \in [a, b]$ and $t \in [0, 1]$ (that is, the graph of f from x to y lies below the line from $(x, f(x))$ to $(y, f(y))$).
   - If $f : [a, b] \to \mathbb{R}$ is convex, then for all $x_1, \ldots, x_n \in [a, b]$ and $t_1, \ldots, t_n \in [0, 1]$ with $t_1 + t_2 + \cdots + t_n = 1$, we have that $f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots t_nf(x_n)$. The case $t_k = \frac{1}{n}$ for all n is known as Jessen's Inequality.
   - A continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on (a, b) is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.