Important Concepts

1. Continuity
   - A function \( f : (a, b) \to \mathbb{R} \) is continuous at a point \( c \in (a, b) \) if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if \( x \in (a, b) \) and \( |x - c| < \delta \) then \( |f(x) - f(c)| < \varepsilon \).
   - If \( f \) is continuous at a point \( x_0 \) and \( f(x_0) > 0 \), there exists an \( \varepsilon > 0 \) such that \( f(x) > 0 \) for all \( x \in (x_0 - \varepsilon, x_0 + \varepsilon) \).

2. Derivatives
   - A function \( f : (a, b) \to \mathbb{R} \) is said to be differentiable at a point \( c \in (a, b) \) if \( \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \) exists. If the limit exists, the value of the limit is denoted \( f'(c) \).
   - If a function \( f \) is differentiable at a point \( c \), then \( f \) is continuous at \( c \).
   - (Product Rule) \( (fg)'(x) = f'(x)g(x) + f(x)g'(x) \) provided the derivatives make sense.
   - (Quotient Rule) \( \left( \frac{f}{g} \right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \) provided this makes sense.
   - (Chain Rule) \( (f \circ g)'(x) = f'(g(x))g'(x) \) provided this makes sense.

3. The Value Theorems
   - (Intermediate Value Theorem) If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and either \( f(a) < c < f(b) \) or \( f(b) < c < f(a) \), then there exists a \( d \in (a, b) \) such that \( f(d) = c \).
   - (Extreme Value Theorem) If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\), then there exists \( x_1, x_2 \in [a, b] \) such that \( f(x_1) \leq f(x) \leq f(x_2) \) for all \( x \in [a, b] \). Furthermore, if \( f \) is differentiable on \([a, b]\) and for \( k = 1 \) or \( k = 2 \) we have that \( x_k \in (a, b) \), then \( f'(x_k) = 0 \).
   - (Mean Value Theorem) If \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists a \( c \in (a, b) \) such that \( f'(c) = \frac{f(b) - f(a)}{b - a} \).

4. Taylor’s Theorem
   - Let \( k \geq 1 \) be an integer and let \( f : \mathbb{R} \to \mathbb{R} \) be \( k \) times differentiable at a point \( a \in \mathbb{R} \). Then there exists a function \( h_k : \mathbb{R} \to \mathbb{R} \) such that
     \[
     f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k + h_k(x)(x - a)^k.
     \]
     where \( \lim_{x \to a} h_k(x) = 0 \).
   - Let \( k \geq 1 \) be an integer and let \( f : \mathbb{R} \to \mathbb{R} \) be \( k \) times continuously differentiable at a point \( a \in \mathbb{R} \). Then there exists a point \( c \) in the open interval between \( x \) and \( a \) so that
     \[
     f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}(x - a)^{k-1} + \frac{f^{(k)}(c)}{k!}(x - a)^k.
     \]

5. Convexity
   - A function \( f : [a, b] \to \mathbb{R} \) is said to be convex if \( tf(x) + (1 - t)y \leq f(tx + (1 - t)y) \) for all \( x, y \in [a, b] \) and \( t \in [0, 1] \) (that is, the graph of \( f \) from \( x \) to \( y \) lies below the line from \( (x, f(x)) \) to \( (y, f(y)) \)).
   - If \( f : [a, b] \to \mathbb{R} \) is convex, then for all \( x_1, \ldots, x_n \in [a, b] \) and \( t_1, \ldots, t_n \in [0, 1] \) with \( t_1 + t_2 + \cdots + t_n = 1 \), we have that \( f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n) \). The case \( t_k = \frac{1}{n} \) for all \( n \) is known as Jessen’s Inequality.
   - A continuous function \( f : [a, b] \to \mathbb{R} \) that is differentiable on \((a, b)\) is convex if and only if \( f''(x) \geq 0 \) for all \( x \in (a, b) \).