## EXAMPLE PROBLEM

**Question 0.** (2019; B4) Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$ and $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$xf_x + yf_y = xy\ln(xy)$$
$$x^2 f_{xx} + y^2 f_{yy} = xy.$$

For each $f \in \mathcal{F}$, let

$$m(f) = \min_{s\geq 1} f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s).$$

Determine $m(f)$, and show that it is independent of the choice of $f$.

## PREVIOUS RELATED PUTNAM PROBLEMS

**Question 1.** (AB1) Let $f$ be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that $f$ has at least five distinct real zeros. Prove that $f+6f'+12f''+8f'''$ has at least two distinct real zeros.

**Question 2.** (AB2) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers $x$ and positive integers $n$.

**Question 3.** (AB3) Suppose the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants $a$ and $b$. Prove that if there is a constant $M$ such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then $h$ is identically zero.

**Question 4.** (AB3) Determine the greatest possible value of $\sum_{i=1}^{10}\cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10}\cos(x_i) = 0$.

**Question 5.** (AB3) Let $f$ and $g$ be (real-valued) functions defined on an open interval containing 0, with $g$ continuous and non-zero at 0. If $fg$ and $\frac{f}{g}$ are differentiable at 0, must $f$ be differentiable at 0?

**Question 6.** (AB3) Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

$$f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.$$

**Question 7.** (AB3) Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$ with $0 < x < 1$ satisfy

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

Prove that there exists a real number $y$ with $0 < y < 1$ such that $a_0 + a_1y + \cdots + a_ny^n = 0$.

**Question 8.** (AB5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all $x$?