Example Problem

Question 0. (2019; B4) Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$ and $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

\[
\begin{align*}
xf_x + yf_y &= xy \ln(xy) \\
x^2f_{xx} + y^2f_{yy} &= xy.
\end{align*}
\]

For each $f \in \mathcal{F}$, let

\[
m(f) = \min_{s \geq 1} f(s + 1, s + 1) - f(s + 1, s) - f(s, s + 1) + f(s, s).
\]

Determine $m(f)$, and show that it is independent of the choice of $f$.

Previous Related Putnam Problems

Question 1. (AB1) Let $f$ be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that $f$ has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

Question 2. (AB2) Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(x + n) - f(x)$ for all real numbers $x$ and positive integers $n$.

Question 3. (AB3) Suppose the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

\[
h(x, y) = a \frac{\partial h}{\partial x}(x, y) + b \frac{\partial h}{\partial y}(x, y)
\]

for some constants $a$ and $b$. Prove that if there is a constant $M$ such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then $h$ is identically zero.

Question 4. (AB3) Determine the greatest possible value of $\sum_{i=1}^{10} \cos(3x_i)$ for real numbers $x_1, x_2, \ldots, x_{10}$ satisfying $\sum_{i=1}^{10} \cos(x_i) = 0$.

Question 5. (AB3) Let $f$ and $g$ be (real-valued) functions defined on an open interval containing 0, with $g$ continuous and non-zero at 0. If $fg$ and $f'g$ are differentiable at 0, must $f$ be differentiable at 0?

Question 6. (AB3) Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that

\[
f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0.
\]

Question 7. (AB3) Suppose that the real numbers $a_0, a_1, \ldots, a_n$ and $x$ with $0 < x < 1$ satisfy

\[
\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.
\]

Prove that there exists a real number $y$ with $0 < y < 1$ such that $a_0 + a_1y + \cdots + a_ny^n = 0$.

Question 8. (AB5) Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f(f(x))$ for all $x$?