YorkU Putnam Training — **Week 9: Integrals and Analysis**

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Important Concepts

1. **Integrals**

- Given a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P = \{t_k\}_{k=0}^n$ where
$$a = t_0 < t_1 < \cdots < t_n = b$$
let $m_k = \inf_{x\in[t_{k-1},t_k]} f(x)$ and $M_k = \sup_{x\in[t_{k-1},t_k]} f(x)$. The *lower and upper Riemann sums of f on $[a, b]$ with respect to P* are
$$L(f, P) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}) \qquad \text{and} \qquad U(f, P) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}).$$
- A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* if
$$\sup(\{L(f, P) \mid P \text{ a partition of } [a, b]\}) = \inf(\{U(f, P) \mid P \text{ a partition of } [a, b]\}).$$
If the above is true, we denote the value of the sup and inf by $\int_a^b f(x)\,dx$. Equivalently, f is Riemann integrable if for all $\epsilon > 0$ there exists a partition P such that $0 \leq U(f, P) - L(f, P) < \epsilon$.
- Note $L(f, P) \leq \int_a^b f(x)\,dx \leq U(f, P)$ for any partition P.
- If $f : [a, b] \to \mathbb{R}$ is continuous, we only need to consider the uniform partitions (i.e. those where $t_k - t_{k-1} = \frac{b-a}{n}$ for all k).
- (Fundamental Theorem of Calculus, Part 1) Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and define $F : [a, b] \to \mathbb{R}$ by
$$F(x) = \int_a^x f(t)\,dt.$$
Then F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.
- (Fundamental Theorem of Calculus, Part 2) Let $f, F : [a, b] \to \mathbb{R}$ be such that f is Riemann integrable on $[a, b]$, F is continuous on $[a, b]$, F is differentiable on (a, b), and $F'(x) = f(x)$ for all $x \in (a, b)$. Then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$
- (Substitution/Chain Rule) If $f : [a, b] \to \mathbb{R}$ is Riemann integrable and $g : [c, d] \to [a, b]$ is differentiable on (a, b) and injective, then
$$\int_a^b f(x)\,dx = \int_c^d f(g(x))g'(x)\,dx.$$
- (Integration by Parts) If $f, g : [a, b] \to \mathbb{R}$ are differentiable on (a, b) and Riemann integrable on $[a, b]$, then
$$\int_a^b f(x)g'(x)\,dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$
- (Comparison Test) Let $f : [1, \infty) \to [0, \infty)$ be a continuous, non-increasing function and define $a_n = f(n)$ for all $n \in \mathbb{N}$. Then $\int_1^\infty f(x)\,dx$ converges if and only if $\sum_{n=1}^\infty a_n$ converges.