

EXAMPLE PUTNAM PROBLEM

Question 0. (2017, A3) Let a and b be real numbers with $a < b$, and let f and g be continuous functions from $[a, b]$ to $(0, \infty)$ such that $\int_a^b f(x) dx = \int_a^b g(x) dx$ but $f \neq g$. For every positive integer n , define

$$I_n = \int_a^b \frac{(f(x))^{n+1}}{(g(x))^n} dx.$$

Show that I_1, I_2, I_3, \dots is an increasing sequence with $\lim_{n \rightarrow \infty} I_n = \infty$.

PREVIOUS RELATED PUTNAM PROBLEMS

Question 1. (AB1) Let A and B be points on the same branch of the hyperbola $xy = 1$. Suppose that P is a point lying between A and B on this hyperbola, such that the area of the triangle APB is as large as possible. Show that the region bounded by the hyperbola and the chord AP has the same area as the region bounded by the hyperbola and the chord PB .

Question 2. (AB2) Functions f, g, h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, & f(0) &= 1, \\ g' &= fg^2h + \frac{4}{fh}, & g(0) &= 1, \\ h' &= 3fgh^2 + \frac{1}{fg}, & h(0) &= 1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

Question 3. (AB2) Let $F_0(x) = \ln(x)$. For $n \geq 0$ and $x > 0$, let $F_{n+1}(x) = \int_0^x F_n(t) dt$. Evaluate

$$\lim_{n \rightarrow \infty} \frac{n!F_n(1)}{\ln(n)}.$$

Question 4. (AB2) A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c ?

Question 5. (AB2) Suppose $f : [0, 1] \rightarrow \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x) dx = 0$. Prove that for every $\alpha \in (0, 1)$,

$$\left| \int_0^\alpha f(x) dx \right| \leq \frac{1}{8} \max_{0 \leq x \leq 1} |f'(x)|.$$

Question 6. (AB3) Suppose that f is a function from \mathbb{R} to \mathbb{R} such that

$$f(x) + f\left(1 - \frac{1}{x}\right) = \arctan(x)$$

for all real $x \neq 0$. (As usual, $y = \arctan(x)$ means $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $\tan(y) = x$.) Find

$$\int_0^1 f(x) dx.$$

Question 7. (AB3) Find a real number c and a positive number L for which

$$\lim_{r \rightarrow \infty} \frac{r^c \int_0^{\frac{\pi}{2}} x^r \sin(x) dx}{\int_0^{\frac{\pi}{2}} x^r \cos(x) dx} = L.$$

Question 8. (AB4) Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ xf(\ln(x)) & \text{if } x > e \end{cases}.$$

Does $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converge?

Question 9. (AB4) Show that the improper integral

$$\lim_{B \rightarrow \infty} \int_0^B \sin(x) \sin(x^2) dx.$$

converges.

Question 10. (AB4) Let f be a continuous real-valued function on \mathbb{R}^3 . Suppose that for every sphere S of radius 1, the integral of $f(x, y, z)$ over the surface of S equals 0. Must $f(x, y, z)$ be identically 0?

Question 11. (AB5) Evaluate $\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$.