MATH 4011
Real Analysis: Metric Spaces

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Preface:

These are the first edition of these lecture notes for MATH 4011. Consequently, there may be several typographical errors. Not every result in these notes will be covered in class. For example, some results will be covered through assignments. However, these notes should be fairly self-contained. If you come across any typos, errors, omissions, or unclear explanations, please feel free to contact me so that I may continually improve these notes.

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Chapter 1

Metric Spaces

One way or another, analysis is the mathematical study of limits thereby obtaining the ability to approximate quantities. This is very different from say algebra where everything has to be computed exactly. In algebra, one can only add a finite number of elements in a ring, field, or vector space. However, as seen in previous courses, analysis allows for infinite series of numbers or functions by considering finite series and taking a limit. It is this flexibility that enables the analysis of real numbers to be such a useful tool in mathematics.

At this point we have mainly studied analysis for real numbers and continuous functions on the real line. However, if one takes a moment and thinks back to how arguments worked in previous analysis courses, everything revolves around taking limits of sequences. Furthermore, to take a limit of a sequence, one needs only a notion of what it means for one element to be close to another. This works well for real numbers by asking for the absolute value of the difference between two numbers to be small. However, there are only certain properties required of this distance in order to make our arguments work.

In this course, we will study what happens when we extend our basic analytic tools and techniques to a wider variety of spaces. In particular, this chapter will develop the basic analytic concepts with respect to distance functions, known as metrics. The only properties all metrics share are those that one would absolutely require for a well-defined notion of distance and for our basic analytic arguments to work. After defining and providing examples of metrics, we will study convergence of sequences, topology, and continuous functions in this metric space setting. These basic concepts are truly the doorway into a vast new realm of analysis.

1.1 Metric and Normed Linear Spaces

To begin, we must start with the correct notion of a distance function. Note all of the properties we require of a distance function are those that one would expect a proper distance function to have.

Definition 1.1.1. Let X be a non-empty set. A *metric* on X is a function $d : X \times X \to [0, \infty)$ such that

1. for $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$, and
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We already know of one example of a metric.

Example 1.1.2. For any $c > 0$, define $d : \mathbb{R}\times\mathbb{R} \to [0, \infty)$ by $d(x, y) = c|x-y|$. Then $(\mathbb{R}, d)$ is a metric space. Indeed it is trivial to verify the three properties of d being a metric.

Note often we will desire to work with the complex numbers.

Example 1.1.3. Define $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ by $d(x, y) = |x - y|$. Then $(\mathbb{C}, d)$ is a metric space. Indeed it is trivial to verify the three properties of d being a metric.

Note there are a diverse collection of metrics one can place on set.

Example 1.1.4. Define $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x, y) = |e^{-x} - e^{-y}|$. Then $(\mathbb{R}, d)$ is a metric space. To see this, we note that since the function $x \mapsto e^{-x}$ is injective, $d(x, y) = 0$ if and only if $e^{-x} = e^{-y}$ if and only if $x = y$. Moreover, the fact that $d(x, y) = d(y, x)$ and the triangle inequality holds follow trivially. Hence $(\mathbb{R}, d)$ is a metric space.

Consequently, as we desire to study a space together with a pre-described fixed metric, we define the following.

Definition 1.1.5. A *metric space* is a pair $(\mathcal{X}, d)$ where $\mathcal{X}$ is a non-empty set and d is a metric on $\mathcal{X}$.

Note we may on occasion abuse notation by saying that $\mathcal{X}$ is a metric space without specifying the metric d.

Remark 1.1.6. Due to the fact that there are many possible metrics on a given set, unless otherwise specified, we will use the metric $d(x, y) = |x - y|$ as the canonical metric on both $\mathbb{R}$ and $\mathbb{C}$.

Of course, there are many more metric spaces we can consider.

Example 1.1.7. Let $\mathcal{C}[a, b]$ denote the set of all real-valued continuous functions on a closed interval $[a, b]$. Define $d_\infty : \mathcal{C}[a, b] \times \mathcal{C}[a, b] \to [0, \infty)$ by

$$d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

for all $f, g \in \mathcal{C}[a, b]$. Then d is a metric on $\mathcal{C}[a, b]$. Indeed first notice that $d(f, g) < \infty$ as $f - g$ is a continuous function so the Extreme Value Theorem (see Theorem 4.1.15 for example) implies that the supremum is obtained and thus finite. The remaining three properties of a metric are then trivial to verify. We call d_∞ the *uniform metric* on $\mathcal{C}[a, b]$.

However, there are more exotic metrics. In particular, the following can be used as a metric on any set!

Example 1.1.8. Let X be a non-empty set. Define $d : X \times X \to [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

It is elementary to verify that d is a metric. We call d the *discrete metric* on X.

Moreover, there are ways to construct new metrics from other metrics. For some examples of this, consider the following.

Example 1.1.9. Given $n \in \mathbb{N}$, define $d_1 : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)$ by

$$d_1((z_1, \ldots, z_n), (w_1, \ldots, w_n)) = \sum_{k=1}^{n} |z_k - w_k|$$

for all $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$. Then it is easy to verify that $(\mathbb{C}^n, d_1)$ is a metric space.

Example 1.1.10. Given $n \in \mathbb{N}$, define $d_\infty : \mathbb{C}^n \times \mathbb{C}^n \to [0, \infty)$ by

$$d_\infty((z_1, \ldots, z_n), (w_1, \ldots, w_n)) = \sup_{1 \leq k \leq n} |z_k - w_k|$$

for all $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{C}^n$. Then it is easy to verify that $(\mathbb{C}^n, d_\infty)$ is a metric space.

Of course, the above can be modified to consider metric d_1 and d_∞ on n-tuples from arbitrary metric spaces. More generally, we can place a metric on sequences instead of n-tuples.

Example 1.1.11. Let $\{(\mathcal{X}_n, d_n)\}_{n=1}^\infty$ be a countable collection of metric spaces. Let

$$\mathcal{X} = \{(x_n)_{n \geq 1} \mid x_n \in \mathcal{X}_n \text{ for all } n \in \mathbb{N}\}$$

be the set of all sequences whose n^{th} term comes from $\mathcal{X}_n$. Define $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ by

$$d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))}$$

for all $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \mathcal{X}$. We claim that d is a metric on $\mathcal{X}$.

To see that d is a metric on $\mathcal{X}$, we first notice for all $n \in \mathbb{N}$, $x_n \in \mathcal{X}_n$, and $y_n \in \mathcal{X}_n$ that

$$0 \leq \frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))} \leq \frac{1}{2^n}.$$

Hence for all $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \mathcal{X}$ we have that

$$0 \leq d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

so $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$.

Fix $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \in \mathcal{X}$. Then clearly $d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = 0$ if and only if

$$\frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))} = 0$$

for all $n \in \mathbb{N}$ if and only if $d_n(x_n, y_n) = 0$ if and only if $x_n = y_n$ for all $n \in \mathbb{N}$ (as $(\mathcal{X}_n, d_n)$ is a metric space) if and only if $(x_n)_{n\geq 1} = (y_n)_{n\geq 1}$. Hence d satisfies the first property of a metric. Furthermore, since $d_n(x_n, y_n) = d_n(y_n, x_n)$ for all $n \in \mathbb{N}$ as $(\mathcal{X}_n, d_n)$ is a metric space, we clearly obtain that

$$d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = d((y_n)_{n\geq 1}, (x_n)_{n\geq 1}).$$

Thus d satisfies the second property of a metric. Finally, to see that d satisfies the triangle inequality, notice for all $a, b, c \in \mathbb{R}$ with $a, b, c \geq 0$ that

$$\begin{aligned} & c \leq a + b \\ \Rightarrow\ & c \leq a + b + 2ab + abc \\ \Rightarrow\ & c + ac + bc + abc \leq a + ab + ac + abc + b + ab + bc + abc \\ \Rightarrow\ & c(1 + a)(1 + b) \leq a(1 + b)(1 + c) + b(1 + a)(1 + c) \\ \Rightarrow\ & \frac{c}{1 + c} \leq \frac{a}{1 + a} + \frac{b}{1 + b}. \end{aligned}$$

Thus for all $(x_n)_{n\geq 1}, (y_n)_{n\geq 1}, (z_n)_{n\geq 1} \in \mathcal{X}$, by substituting $a = d_n(x_n, z_n)$, $b = d_n(z_n, y_n)$, and $c = d_n(x_y, y_n)$ for all $n \in \mathbb{N}$ we obtain that

$$\frac{d_n(x_n, y_n)}{2^n(1 + d_n(x_n, y_n))} \leq \frac{d_n(x_n, z_n)}{2^n(1 + d_n(x_n, z_n))} + \frac{d_n(z_n, y_n)}{2^n(1 + d_n(z_n, y_n))}$$

for all $n \in \mathbb{N}$, and thus

$$d((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) \leq d((x_n)_{n\geq 1}, (z_n)_{n\geq 1}) + d((z_n)_{n\geq 1}, (y_n)_{n\geq 1}).$$

Hence d satisfies the triangle inequality so d is a metric on $\mathcal{X}$.

We call d the *product metric* on $\{(\mathcal{X}_n, d_n)\}_{n=1}^{\infty}$.

Of course, $\mathbb{R}^n$ can also be made into a metric space by restricting the definition of d_1 and d_∞ to $\mathbb{R}^n \times \mathbb{R}^n$ in Examples 1.1.9 and 1.1.10. This may be generalized as follows.

Example 1.1.12. Let $(\mathcal{X}, d)$ be a metric space and let Y be a non-empty subset of $\mathcal{X}$. Define $d|_Y : Y \times Y \to [0, \infty)$ by $d|_Y(y_1, y_2) = d(y_1, y_2)$ for all $y_1, y_2 \in Y$. Then $(Y, d|_Y)$ is a metric space. We call $d|_Y$ *the metric on Y induced by* $(\mathcal{X}, d)$.

Of course, we haven't even touched on the usual metric we use to measure distance in $\mathbb{R}^n$.

Example 1.1.13. Define $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ by

$$d_2((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, d_2)$ is a metric space and the metric d_2 is called the *Euclidean metric*.

Of course, we must ask how do we know the Euclidean metric is a metric? Verifying all but the triangle inequality is trivial. However, verifying the triangle inequality is non-trivial. So how can we see the triangle inequality holds for the Euclidean metric?

Well, it turns out that not all metrics were created equal. In particular, we desire to study special types of metric spaces. These metric spaces come from specific functions on vector spaces that behave like the absolute value does on $\mathbb{R}$ and $\mathbb{C}$. Consequently, we will restrict to vector spaces where the scalars are either the real or the complex numbers. Consequently, it will be convenient to use $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

The following is our generalization of the absolute value to vector spaces.

Definition 1.1.14. Let $\mathcal{V}$ be a vector space over $\mathbb{K}$. A *norm* on $\mathcal{V}$ is a function $\|\cdot\| : \mathcal{V} \to [0, \infty)$ such that

1. for $\vec{v} \in \mathcal{V}$, $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$,
2. $\|\alpha\vec{v}\| = |\alpha|\, \|\vec{v}\|$ for all $\alpha \in \mathbb{K}$ and $\vec{v} \in \mathcal{V}$, and
3. (triangle inequality) $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$ for all $\vec{v}, \vec{w} \in \mathcal{V}$.

Of course, we desire to study vector spaces with a fixed pre-described norm, so we make the following definition.

Definition 1.1.15. A *normed linear space* is a pair $(\mathcal{V}, \|\cdot\|)$ where $\mathcal{V}$ is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is a norm on $\mathcal{V}$.

Again we may abuse notation by saying that $\mathcal{V}$ is a normed linear space without specifying the norm.

As our motivation for generalizing the absolute value was to induce a metric, we note the following.

Proposition 1.1.16. *If* $(\mathcal{V}, \|\cdot\|)$ *is a normed linear space, then* $\mathcal{V}$ *is a metric space with the metric* $d : \mathcal{V} \times \mathcal{V} \to [0, \infty)$ *defined by* $d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|$. *We call* d *the metric induced by* $\|\cdot\|$.

Proof. It suffices to show that d is a metric. Clearly $d : \mathcal{V} \times \mathcal{V} \to [0, \infty)$. Furthermore notice $d(\vec{v}, \vec{w}) = 0$ if and only if $\|\vec{v} - \vec{w}\| = 0$ if and only if $\vec{v} - \vec{w} = \vec{0}$ if and only if $\vec{v} = \vec{w}$.

Next notice for all $\vec{v}, \vec{w} \in \mathcal{V}$ that

$$d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\| = \|(-1)(\vec{w} - \vec{v})\| = |-1| \, \|\vec{w} - \vec{v}\| = d(\vec{w}, \vec{v}).$$

Finally, to see that d satisfies the triangle inequality, notice for all $\vec{v}, \vec{w}, \vec{z} \in \mathcal{V}$ that

$$\begin{aligned} d(\vec{v}, \vec{z}) + d(\vec{z}, \vec{w}) &= \|\vec{v} - \vec{z}\| + \|\vec{z} - \vec{w}\| \\ &\geq \|(\vec{v} - \vec{z}) + (\vec{z} - \vec{w})\| \\ &= \|\vec{v} - \vec{w}\| = d(\vec{v}, \vec{w}). \end{aligned}$$

Hence d is a metric. ∎

Remark 1.1.17. Notice in the proof of the triangle inequality in Proposition 1.1.16 that using $\vec{w} = \vec{0}$ produced $\|\vec{v} - \vec{z}\| + \|\vec{z}\| \geq \|\vec{v}\|$ for all $\vec{v}, \vec{z} \in \mathcal{V}$. Hence

$$\|\vec{v}\| - \|\vec{z}\| \leq \|\vec{v} - \vec{z}\|$$

for all $\vec{v}, \vec{z} \in \mathcal{V}$. Thus, by interchanging $\vec{v}$ and $\vec{z}$, we obtain that

$$\|\vec{z}\| - \|\vec{v}\| \leq \|\vec{v} - \vec{z}\|$$

so

$$|\|\vec{v}\| - \|\vec{z}\|| \leq \|\vec{v} - \vec{z}\|$$

for all $\vec{v}, \vec{z} \in \mathcal{V}$. This potentially useful inequality is often called the *reverse triangle inequality*.

Clearly the absolute value on $\mathbb{K}$ is an norm on $\mathbb{K}$. Furthermore, the metric induced by this norm is exactly the metric introduced in Examples 1.1.2 and 1.1.3. In fact, some of the other metrics we have seen come from norms.

Example 1.1.18. For $n \in \mathbb{N}$, define $\|\cdot\|_1 : \mathbb{K}^n \to [0, \infty)$ by

$$\|(z_1, \ldots, z_n)\|_1 = \sum_{k=1}^{n} |z_k|$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$. It is elementary to verify that $\|\cdot\|_1$ is a norm on $\mathbb{K}^n$ that induced the metric d_1 as in Example 1.1.9. We call $\|\cdot\|_1$ the 1-*norm*.

Example 1.1.19. For $n \in \mathbb{N}$, define $\|\cdot\|_\infty : \mathbb{K}^n \to [0, \infty)$ by

$$\|(z_1, \ldots, z_n)\|_\infty = \sup_{1 \leq k \leq n} |z_k|$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$. It is elementary to verify that $\|\cdot\|_\infty$ is a norm on $\mathbb{K}^n$ that induced the metric d_∞ as in Example 1.1.10. We call $\|\cdot\|_\infty$ the sup-*norm* or the ∞-*norm*.

Example 1.1.20. Let $\mathcal{C}[a, b]$ denote the set of all real-valued continuous functions on a closed interval $[a, b]$. Then $\mathcal{C}[a, b]$ is a vector space over $\mathbb{R}$ under the operations of pointwise addition and scalar multiplication. Define $\|\cdot\|_\infty : \mathcal{C}[a, b] \to [0, \infty)$ by

$$\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$$

for all $f \in \mathcal{C}[a, b]$. Note $\|\cdot\|_\infty$ does take values in $[0, \infty)$ by the Extreme Value Theorem (see Theorem 4.1.15). It is elementary to see that $\|\cdot\|_\infty$ is a norm on $\mathcal{C}[a, b]$. We call $\|\cdot\|_\infty$ the sup-*norm*.

Of course, the sup-norm works perfectly well if we restrict the set of continuous functions to, for example, the polynomials. In particular, this holds true in more generality.

Proposition 1.1.21. *Let $(\mathcal{V}, \|\cdot\|)$ be a normed linear space and let W be a subspace of $\mathcal{V}$. The restriction of $\|\cdot\|$ to W is a norm on W.*

Remark 1.1.22. However, some of the metrics we have seen are not norms. For example, if $\mathcal{V}$ is a vector space over $\mathbb{K}$, the discrete metric cannot be induced by a norm since if a norm (and thus its induced metric) takes the value 1, then it takes all values in $[0, \infty)$.

Of course, returning to the problem at hand of showing the Euclidean metric is indeed a metric, we can instead ask whether the following is a norm.

Example 1.1.23. For $n \in \mathbb{N}$, define $\|\cdot\|_2 : \mathbb{K}^n \to [0, \infty)$ by

$$\|(z_1, \ldots, z_n)\|_2 = \left(\sum_{1 \leq k \leq n} |z_k|^2 \right)^{\frac{1}{2}}$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$. Then $\|\cdot\|_\infty$ is a norm on $\mathbb{K}^n$ called the *Euclidean norm* or the 2-*norm*.

Clearly the Euclidean norm will induce the Euclidean metric provided the Euclidean norm is in fact a norm. The Euclidean norm is in fact a very special norm as it is induced by a structure on a vector space that is even superior to a norm, namely an inner product. In particular, the dot product on $\mathbb{C}^n$ can be shown to satisfy the Cauchy-Schwarz Inequality (Theorem 6.1.11) and the fact that the triangle inequality holds for the Euclidean norm then easily follows. Instead of introducing inner products here and providing the plethora of examples, we will be postponing the discussion of inner products until Section 6.1 as it serves as an excellent introduction to an important topic, as we do not want to focus our attention to too specific a structure at this time, and as we do not want to digress into specifics for inner product spaces at every point throughout this course.

Instead we note there are many more norms we can place on $\mathbb{K}^n$ that generalize the Euclidean norm.

Example 1.1.24. For $n \in \mathbb{N}$ and a fixed $p \in (1, \infty)$, define $\|\cdot\|_p : \mathbb{K}^n \to [0, \infty)$ by

$$\|(z_1, \ldots, z_n)\|_p = \left(\sum_{k=1}^{n} |z_k|^p\right)^{\frac{1}{p}}$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$. Then $\|\cdot\|_p$ is a norm on $\mathbb{K}^n$ called the *p-norm*.

It is not difficult to see that $\|\cdot\|_p$ satisfies the first two properties of Definition 1.1.14. Indeed $\|(z_1, \ldots, z_n)\|_p \geq 0$ with equality if and only if $z_k = 0$ for all k. Furthermore, for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$ and $\alpha \in \mathbb{K}$, we see that

$$\begin{aligned}
\|\alpha(z_1, \ldots, z_n)\|_p &= \left(\sum_{k=1}^{n} |\alpha z_k|^p\right)^{\frac{1}{p}} \\
&= \left(\sum_{k=1}^{n} |\alpha|^p |z_k|^p\right)^{\frac{1}{p}} \\
&= \left(|\alpha|^p \sum_{k=1}^{n} |z_k|^p\right)^{\frac{1}{p}} \\
&= |\alpha| \left(\sum_{k=1}^{n} |z_k|^p\right)^{\frac{1}{p}} \\
&= |\alpha| \, \|(z_1, \ldots, z_n)\|_p .
\end{aligned}$$

However, it is difficult to see whether $\|\cdot\|_p$ satisfies the triangle inequality. Thus how can we see that the p-norm is indeed a norm?

1.2 The p-Norms

To see that the p-norm satisfies the triangle inequality, we will need to develop some additional inequalities. First consider the function $f : (1, \infty) \to (1, \infty)$ defined by $f(x) = \frac{x}{x-1}$. Using elementary calculus, f is a bijection. In particular, for each $p \in (1, \infty)$ there exists a unique $q \in (1, \infty)$ such that $p = \frac{q}{q-1}$. Thus

$$\frac{1}{p} = \frac{q-1}{q} = 1 - \frac{1}{q}.$$

Hence for each $p \in (1, \infty)$ there exists a unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1.2.1. Let $p \in (1, \infty)$. The unique $q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$ is called the *conjugate* of p. Note we consider ∞ to be the conjugate of 1 and 1 to be the conjugate of ∞.

The following inequality related to dual pairs is a key step towards verifying the triangle inequality for the p-norms.

Lemma 1.2.2 (Young's Inequality). *Let $a, b \geq 0$ and let $p, q \in (1, \infty)$ be conjugates. Then $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.*

Proof. Notice $1 = \frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq}$ implies $p + q - pq = 0$. Hence $q = \frac{p}{p-1}$.

Fix $b \geq 0$. Notice if $b = 0$, the inequality easily holds. Thus we will assume $b > 0$.

Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \frac{1}{p}x^p + \frac{1}{q}b^q - bx$. Clearly $f(0) > 0$ and $\lim_{x\to\infty} f(x) = \infty$ as $p > 1$ so x^p grows faster than x. We claim that $f(x) \geq 0$ for all $x \in [0, \infty)$ thereby proving the inequality. Notice f is differentiable on $[0, \infty)$ with

$$f'(x) = x^{p-1} - b.$$

Therefore $f'(x) = 0$ if and only if $x = b^{\frac{1}{p-1}}$. Moreover, it is elementary to see from the derivative that f has a local minimum at $b^{\frac{1}{p-1}}$ and thus f has a global minimum at $b^{\frac{1}{p-1}}$ due to the boundary conditions. Therefore, since

$$f\left(b^{\frac{1}{p-1}}\right) = \frac{1}{p}b^{\frac{p}{p-1}} + \frac{1}{q}b^q - b^{1+\frac{1}{p-1}} = \frac{1}{p}b^q + \frac{1}{q}b^q - b^q = 0,$$

we obtain that $f(x) \geq 0$ for all $x \in [0, \infty)$ as desired. ∎

Using Young's Inequality, we have a stepping stone towards the triangle inequality.

Theorem 1.2.3 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$,*

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}.$$

Proof. Let $\alpha = (\sum_{i=1}^n |a_i|^p)^{\frac{1}{p}}$ and let $\beta = (\sum_{i=1}^n |b_i|^q)^{\frac{1}{q}}$. It is clear that $\alpha = 0$ implies $a_i = 0$ for all i which implies $\sum_{i=1}^n |a_i b_i| = 0$ and thus the inequality will hold in this case. Similarly if $\beta = 0$, then the inequality holds. Hence we may assume that $\alpha, \beta > 0$.

Since $\alpha, \beta > 0$, we obtain that

$$\begin{aligned}
\sum_{i=1}^n |a_i b_i| &= \alpha\beta \sum_{i=1}^n \left|\frac{a_i}{\alpha}\right| \left|\frac{b_i}{\beta}\right| \\
&\leq \alpha\beta \left(\sum_{i=1}^n \frac{1}{p}\left|\frac{a_i}{\alpha}\right|^p + \frac{1}{q}\left|\frac{b_i}{\beta}\right|^q \right) \qquad \text{by Lemma 1.2.2} \\
&= \alpha\beta \left(\frac{1}{p\alpha^p} \sum_{i=1}^n |a_i|^p + \frac{1}{q\beta^q} \sum_{i=1}^n |b_i|^q \right) \\
&= \alpha\beta \left(\frac{1}{p} + \frac{1}{q} \right) \\
&= \alpha\beta
\end{aligned}$$

as desired. ∎

Note Hölder's Inequality has the following trivial extension.

Theorem 1.2.4. *For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$,*

$$\sum_{i=1}^n |a_i b_i| \leq \|(a_1, \ldots, a_n)\|_1 \, \|(b_1, \ldots, b_n)\|_\infty .$$

Finally Hölder's Inequality enables us to prove the triangle inequality for the p-norm.

Theorem 1.2.5 (Minkowski's Inequality). *Let $p \in (1, \infty)$. For any $n \in \mathbb{N}$ and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$,*

$$\left(\sum_{i=1}^n |a_i + b_i|^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |a_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |b_i|^p \right)^{\frac{1}{p}} .$$

Proof. Choose $q \in (1, \infty)$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Thus $q = \frac{p}{p-1}$. Since $p \in (1, \infty)$, notice by Hölder's Inequality (Theorem 1.2.3) that

$$\begin{aligned}
&\sum_{i=1}^{n} |a_i + b_i|^p \\
&= \sum_{i=1}^{n} (|a_i + b_i|)(|a_i + b_i|)^{p-1} \\
&\leq \sum_{i=1}^{n} (|a_i| + |b_i|)(|a_i + b_i|)^{p-1} \\
&= \sum_{i=1}^{n} |a_i|(|a_i + b_i|)^{p-1} + \sum_{i=1}^{n} |b_i|(|a_i + b_i|)^{p-1} \\
&\leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (|a_i + b_i|^{p-1})^q\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} (|a_i + b_i|^{p-1})^q\right)^{\frac{1}{q}} \\
&= \left(\left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{q}}.
\end{aligned}$$

If $\sum_{i=1}^{n} |a_i + b_i|^p = 0$, the result follows trivially. Otherwise, we may divide both sides of the equation by $(\sum_{i=1}^{n} |a_i + b_i|^p)^{\frac{1}{q}}$ to obtain that

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |a_i + b_i|^p\right)^{1-\frac{1}{q}} \leq \left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{\frac{1}{p}}$$

as desired. ■

Most of the above examples of normed linear spaces have been finite dimensional. As we will see later that finite dimensional normed linear spaces are particularly nice, it is useful to have some examples of infinite dimensional normed linear spaces. Using the inequalities developed above, we can extend the notion of p-norms to sequences and continuous functions.

Example 1.2.6. Let $p \in [1, \infty)$. Let $\ell_p(\mathbb{N})$ (or to specify the field, $\ell_p(\mathbb{N}, \mathbb{K})$) denote all sequences $(z_n)_{n \geq 1}$ of elements of $\mathbb{K}$ such that

$$\sum_{k=1}^{\infty} |z_k|^p < \infty.$$

Then $\ell_p(\mathbb{N})$ is a normed linear space with norm $\|\cdot\|_p : \ell_p(\mathbb{N}) \to [0, \infty)$ defined by

$$\|(z_n)_{n\geq 1}\|_p = \left(\sum_{k=1}^{\infty} |z_k|^p\right)^{\frac{1}{p}}.$$

It is elementary to see that $\|\cdot\|_p$ is well-defined (an infinite sum of non-negative numbers is non-negative) and satisfies the first two properties of a norm (and is closed under scalar multiplication) as defined in Definition 1.1.14. To see that $\|\cdot\|_p$ satisfies the triangle inequality (and that $\ell_p(\mathbb{N})$ is indeed closed under addition), we note that Minkowski's Inequality (Theorem 1.2.5) implies

$$\left(\sum_{k=1}^{m} |z_k + w_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{m} |z_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{m} |w_k|^p\right)^{\frac{1}{p}}$$

for all $m \in \mathbb{N}$ and $(z_n)_{n\geq 1}, (w_n)_{n\geq 1} \in \ell_p(\mathbb{N})$. Therefore, taking a limit as m tends to infinite yields the triangle inequality and the fact that if $\vec{z}, \vec{w} \in \ell_p(\mathbb{N})$, then $\vec{z} + \vec{w} \in \ell_p(\mathbb{N})$. We call $\|\cdot\|_p$ the *p-norm*.

Example 1.2.7. Let $\ell_\infty(\mathbb{N})$ (or to specify the field, $\ell_\infty(\mathbb{N}, \mathbb{K})$) denote all sequences $(z_n)_{n\geq 1}$ of elements of $\mathbb{K}$ such that

$$\sup_{k\in\mathbb{N}} |z_k| < \infty.$$

Then $\ell_\infty(\mathbb{N})$ is a normed linear space with norm $\|\cdot\|_\infty : \ell_\infty(\mathbb{N}) \to [0, \infty)$ defined by

$$\|(z_n)_{n\geq 1}\|_\infty = \sup_{k\in\mathbb{N}} |z_k|.$$

It is elementary to see that $\|\cdot\|_\infty$ is well-defined norm, which we call the *sup-norm* or the ∞*-norm*.

Remark 1.2.8. It is not difficult to see that if $p, q \in [1, \infty]$ and $p < q$, then $\ell_p(\mathbb{N}) \subsetneq \ell_q(\mathbb{N})$. Indeed, if $(z_n)_{n\geq 1} \in \ell_p(\mathbb{N})$, then $\sum_{n=1}^\infty |z_n|^p < \infty$ so $(z_n)_{n\geq 1}$ is bounded (i.e. $(z_n)_{n\geq 1} \in \ell_\infty(\mathbb{N})$) and $\sum_{n=1}^\infty |z_n|^q < \infty$ for all $q \in (p, \infty)$. To see the inclusion is strict, notice that $(\frac{1}{n^{\frac{1}{p}}})_{n\geq 1}$ is not in $\ell_p(\mathbb{N})$ but is in $\ell_q(\mathbb{N})$ for all $q > p$.

Using similar arguments to those used in Example 1.2.6, we obtain the following versions of Hölder's Inequality.

Theorem 1.2.9 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (or $p = 1$ and $q = \infty$). If $(a_n)_{n\geq 1} \in \ell_p(\mathbb{N})$ and $(b_n)_{n\geq 1} \in \ell_q(\mathbb{N})$, then $(a_n b_n)_{n\geq 1} \in \ell_1(\mathbb{N})$ and*

$$\|(a_n b_n)_{n\geq 1}\|_1 \leq \|(a_n)_{n\geq 1}\|_p \|(b_n)_{n\geq 1}\|_q.$$

Proof. For each $m \in \mathbb{N}$, we obtain by Hölder's Inequality (Theorem 1.2.3) that

$$\sum_{k=1}^{m} |a_k b_k| \leq \left(\sum_{k=1}^{m} |a_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{m} |b_k|^q\right)^{\frac{1}{q}}.$$

Taking the limit at m tends to infinity yields the result. ∎

We have already developed an ∞-norm for $C[a, b]$. Like with infinite sequences, we can define a p-norm on the continuous functions. To do this, we replace sums with their generalization; namely integrals.

Definition 1.2.10. For $p \in [1, \infty)$ define $\|\cdot\|_p : \mathcal{C}[a, b] \to [0, \infty)$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for all $f \in \mathcal{C}[a, b]$.

To see that $\|\cdot\|_p$ is indeed a norm on $\mathcal{C}[a, b]$, first notice that if $f \in \mathcal{C}[a, b]$ then $|f|^p \in \mathcal{C}[a, b]$ and thus

$$0 \leq \int_a^b |f(x)|^p \, dx < \infty.$$

Hence $0 \leq \|f\|_p < \infty$. Next, notice that $\|f\|_p = 0$ if and only if

$$\int_a^b |f(x)|^p \, dx = 0.$$

As $|f|^p$ is continuous, the above occurs if and only if $|f(x)|^p = 0$ for all $x \in [a, b]$ which is equivalent to $f = 0$. Furthermore, if $\alpha \in \mathbb{R}$ then

$$\begin{aligned} \|\alpha f\|_p &= \left(\int_a^b |\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &= \left(\int_a^b |\alpha|^p |f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &= \left(|\alpha|^p \int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} \\ &= |\alpha| \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}} = |\alpha| \, \|f\|_p . \end{aligned}$$

Finally, to see that the triangle inequality holds and thus $\|\cdot\|_p$ is indeed a norm, we prove the following versions of Hölder's and Minkowski's Inequality.

Theorem 1.2.11 (Hölder's Inequality). *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ (or $p = 1$ and $q = \infty$). If $f, g \in \mathcal{C}[a, b]$, then*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q .$$

Proof. Let $f, g \in \mathcal{C}[a, b]$ be arbitrary. First notice

$$\|fg\|_1 = \int_a^b |f(x)g(x)| \, dx \leq \int_a^b |f(x)| \, \|g\|_\infty \, dx = \|f\|_1 \, \|g\|_\infty .$$

Otherwise suppose $p, q \in (1, \infty)$. Let

$$\alpha = \|f\|_p \qquad \text{and} \qquad \beta = \|g\|_q .$$

If $\alpha = 0$, then $|f|^p = 0$ and thus $|f| = 0$. This implies $fg = 0$ almost everywhere and hence the inequality holds. Similarly, if $\beta = 0$ then the inequality holds. Hence we may assume that $\alpha, \beta > 0$.

Since $\alpha, \beta > 0$, we obtain that

$$\begin{aligned}
\int_a^b |f(x)g(x)|\, dx &= \alpha\beta \int_a^b \frac{|f(x)|}{\alpha}\frac{|g(x)|}{\beta}\, dx \\
&\leq \alpha\beta \int_a^b \frac{|f(x)|^p}{p\alpha^p} + \frac{|g(x)|^q}{q\beta^q}\, dx \qquad \text{by Lemma 1.2.2} \\
&= \alpha\beta \left(\frac{1}{p\alpha^p} \int_a^b |f(x)|^p \, d\mu + \frac{1}{q\beta^q} \int_a^b |g(x)|^q \, dx \right) \\
&= \alpha\beta \left(\frac{1}{p} + \frac{1}{q} \right) = \alpha\beta
\end{aligned}$$

as desired. ∎

Theorem 1.2.12 (Minkowski's Inequality). *Let $p \in [1, \infty)$. If $f, g \in \mathcal{C}[a, b]$, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

Proof. Fix $p \in [1, \infty)$ and $f, g \in \mathcal{C}[a, b]$. If $p = 1$, we notice that

$$\begin{aligned}
\|f + g\|_1 = \int_a^b |f(x) + g(x)|\, dx &\leq \int_a^b |f(x)| + |g(x)|\, dx \\
&= \int_a^b |f(x)|\, dx + \int_a^b |g(x)|\, dx \\
&= \|f\|_1 + \|g\|_1 .
\end{aligned}$$

Otherwise suppose $p \in (1, \infty)$. Choose $q \in (1, \infty)$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Thus $q = \frac{p}{p-1}$. Since $p \in (1, \infty)$, notice by Hölder's inequality (Theorem

1.2.11) that

$$\begin{aligned}
&\int_a^b |f(x)+g(x)|^p\,dx \\
&= \int_a^b |f(x)+g(x)||f(x)+g(x)|^{p-1}\,dx \\
&\leq \int_a^b (|f(x)|+|g(x)|)|f(x)+g(x)|^{p-1}\,dx \\
&= \int_a^b |f(x)||f(x)+g(x)|^{p-1}\,dx + \int_a^b |g(x)||f(x)+g(x)|^{p-1}\,dx \\
&\leq \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (|f(x)+g(x)|^{p-1})^q\,dx\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_a^b |g(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_a^b (|f(x)+g(x)|^{p-1})^q\,dx\right)^{\frac{1}{q}} \\
&= \left(\|f\|_p + \|g\|_p\right)\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{\frac{1}{q}}.
\end{aligned}$$

If $\int_a^b |f(x)+g(x)|^p\,dx = 0$, the result follows trivially. Otherwise, we may divide both sides of the equation by $\left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{\frac{1}{q}}$ to obtain that

$$\|f+g\|_p = \left(\int_a^b |f(x)+g(x)|^p\,dx\right)^{1-\frac{1}{q}} \leq \|f\|_p + \|g\|_p$$

as desired. ∎

Although there are many other norms, metric, normed linear spaces, and metric spaces we could discuss, for now we move on to studying the common properties and notions for these spaces, such as convergence of sequences.

1.3 The Metric Topology

In this section, we will analyze the notion of convergent sequences in metric spaces. Of course we could jump right in and define the convergence of a sequence using our distance function. However, in doing so we would miss out on obtaining some important information about the structure of metric spaces and of the subsets of our spaces. Thus we will begin with another view of what it means for a sequence to converge and thereby permit a deeper discussion of types and properties of subsets of metric spaces.

One way to interpret the notion of a convergence sequence of real numbers $(a_n)_{n\geq 1}$ to converges to a number L without a notion of distance is to say that $a_n \in (L-\epsilon, L+\epsilon)$ for all $n \geq N$. Thus for $(a_n)_{n\geq 1}$ to be 'close' to L

means that each element in $(a_n)_{n\geq 1}$ must eventually be in any fixed open interval containing L. Thus if we can analyze the essential properties of open intervals and generalize these to metric spaces, we may generalize the notion of a convergent sequence. If fact, we want a concept slightly more general than an open interval.

Definition 1.3.1. Let X be a non-empty set. A collection $\mathcal{T} \subseteq \mathcal{P}(X)$ is said to be a *topology* on X if

1. $\emptyset, X \in \mathcal{T}$,
2. if $\{U_\alpha\}_{\alpha\in I} \subseteq \mathcal{T}$, then $\bigcup_{\alpha\in I} U_\alpha \in \mathcal{T}$, and
3. if $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called the *open sets* of the topology.

There are many examples of topologies we may place on a set.

Example 1.3.2. Let X be a non-empty set. The set $\mathcal{T} = \{X, \emptyset\}$ is a topology on X known as the *trivial topology*.

Example 1.3.3. Let X be a non-empty set. The set $\mathcal{T} = \mathcal{P}(X)$ is a topology on X known as the *discrete topology*.

Of course, the above topologies may not be the best topologies for a metric space as we desire a topology related to the metric. Thus we define the following which are motivated by the Euclidean metric.

Definition 1.3.4. Let $(\mathcal{X}, d)$ be a metric space. Given an $x \in \mathcal{X}$ and an $r > 0$, the *open ball* of radius r centred at x, denoted $B(x, r)$, is the set

$$B(x, r) = \{y \in \mathcal{X} \mid d(x, y) < r\}.$$

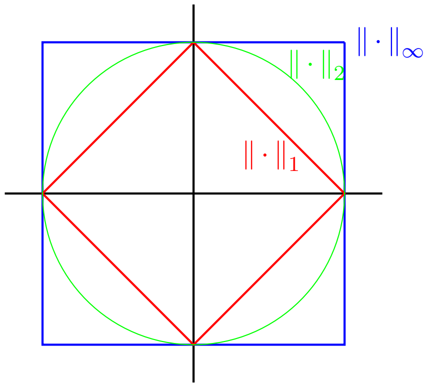
Similarly, given an $x \in \mathcal{X}$ and an $r \geq 0$, the *closed ball* of radius r centred at x, denoted $B[x, r]$, is the set

$$B[x, r] = \{y \in \mathcal{X} \mid d(x, y) \leq r\}.$$

Example 1.3.5. In $\mathbb{R}$ with the absolute value metric, $B(x, r) = (x - r, x + r)$ and $B[x, r] = [x - r, x + r]$ for all $x \in \mathbb{R}$ and $r > 0$.

Example 1.3.6. For $\mathbb{R}^2$, the following diagram illustrates $B(0, 1)$ for various

p-norms:



Example 1.3.7. Let X be a non-empty set and let d be the discrete metric on X. Then, for all $x \in X$,

$$\begin{aligned} B(x, r_1) = B[x, r_2] = \{x\} &\quad \text{if } r_1 \leq 1 \text{ and } r_2 < 1, \text{ and} \\ B(x, r_1) = B[x, r_2] = X &\quad \text{if } r_1 > 1 \text{ and } r_2 \leq 1. \end{aligned}$$

Unsurprisingly, to obtain a desirably topology on a metric space, we will use our open balls to construct the open sets.

Theorem 1.3.8. *Let $(\mathcal{X}, d)$ be a metric space. Let $\mathcal{T}$ be the set of all subsets U of $\mathcal{X}$ such that for each $x \in U$ there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Then $\mathcal{T}$ is a topology on $\mathcal{X}$.*

Proof. To see that $\mathcal{T}$ is a topology, we must verify the three properties in Definition 1.3.1. It is clear by definition that $\emptyset, \mathcal{X} \in \mathcal{T}$.

Suppose $\{U_\alpha\}_{\alpha \in I}$ is a set of elements of $\mathcal{T}$. To see that $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$, let $x \in \bigcup_{\alpha \in I} U_\alpha$ be arbitrary. Then there must be an $i \in I$ such that $x \in U_i$. Since $U_i \in \mathcal{T}$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_i$. Hence $B(x, \epsilon) \subseteq U_i \subseteq \bigcup_{\alpha \in I} U_\alpha$. As $x \in \bigcup_{\alpha \in I} U_\alpha$ was arbitrary, $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$.

Finally, suppose $U_1, \ldots, U_n \in \mathcal{T}$. To see that $\bigcap_{i=1}^n U_i \in \mathcal{T}$, suppose $x \in \bigcap_{i=1}^n U_i$ be arbitrary. Hence $x \in U_i$ for all $i \in \{1, \ldots, n\}$. Since each $U_i \in \mathcal{T}$, there exists an $\epsilon_i > 0$ such that $B(x, \epsilon_i) \subseteq U_i$ for all $i \in \{1, \ldots, n\}$. Let $\epsilon = \min_{1 \leq i \leq n} \epsilon_i > 0$. Notice for each $i \in \{1, \ldots, n\}$ that

$$B(x, \epsilon) \subseteq B(x, \epsilon_i) \subseteq U_i.$$

Hence $B(x, \epsilon) \subseteq \bigcap_{i=1}^n U_i$. As $x \in \bigcap_{i=1}^n U_i$ was arbitrary, $\bigcap_{i=1}^n U_i \in \mathcal{T}$ as desired. ∎

Definition 1.3.9. Let $(\mathcal{X}, d)$ be a metric space. The topology $\mathcal{T}$ from Theorem 1.3.8 is called the *metric space topology* on $(\mathcal{X}, d)$. Unless otherwise specified, given a metric space $(\mathcal{X}, d)$ the topology on $\mathcal{X}$ will always be the metric space topology and the elements of $\mathcal{T}$ will be referred to as open sets.

Of course, it is useful to be able to determine which sets are open. It should not be a surprise that our open balls are indeed open sets. In fact, it is not difficult to see that the metric topology is the smallest topology where every open ball is an open set.

Proposition 1.3.10. *Let $(\mathcal{X}, d)$ be a metric space. Every open ball in $\mathcal{X}$ is an open set.*

Proof. Consider the open ball $B(x, \epsilon)$ for some $x \in \mathcal{X}$ and $\epsilon > 0$. To see that $B(x, \epsilon)$ is open, let $y \in B(x, \epsilon)$ be arbitrary. Thus $d(x, y) < \epsilon$.

Let $\delta = \epsilon - d(x, y) > 0$. We claim that $B(y, \delta) \subseteq B(x, \epsilon)$. To see this, let $z \in B(y, \delta)$ be arbitrary. Then $d(z, y) < \delta$ so, by the triangle inequality,

$$d(z, x) \leq d(z, y) + d(y, x) < \delta + d(y, x) = \epsilon.$$

Therefore, since $z \in B(y, \delta)$ was arbitrary, $B(y, \delta) \subseteq B(x, \epsilon)$. Hence $B(x, \epsilon)$ is open as $y \in B(x, \epsilon)$ was arbitrary. ∎

We also note the following complete description of open subsets of $\mathbb{R}$.

Proposition 1.3.11. *Every open subset of $\mathbb{R}$ is a countable union of open intervals.*

Proof. Let U be an arbitrary non-empty open subset of $\mathbb{R}$. Define a relation $\sim$ on U by $x \sim y$ if and only if whenever $x < z < y$ or $y < z < x$ then $z \in U$. We claim that $\sim$ is an equivalence relation on U.

To see that $\sim$ is an equivalence relation, first notice that if $x \in U$, then $x \sim x$ trivially. Furthermore, clearly if $x \sim y$ then $z \in U$ whenever $x < z < y$ or $y < z < x$, and thus $y \sim x$. Finally, suppose $x, y, w \in U$ are such that $x \sim y$ and $y \sim w$. To see that $x \sim w$, we divide the discussion into five cases:

<u>Case 1: $x \leq y \leq w$.</u> In this case, we have $x < z < y$ implies $z \in U$ and $y < z < w$ implies $z \in U$. If z is such that $x < z < w$, then either $x < z < y$, $y < z < w$, or $y = z$. As all of these imply $z \in U$, we have $x \sim w$ in this case.

<u>Case 2: $w \leq y \leq x$.</u> This case follows from Case 1 by interchanging x and w.

<u>Case 3: $y \leq x \leq w$.</u> In this case, we have $y < z < w$ implies $z \in U$. Thus if $x < z < w$ then $y < z < w$ so $z \in U$. Hence $z \sim x$ in this case.

<u>Case 4: $y \leq w \leq x$.</u> This case follows from Case 3 by interchanging x and w.

<u>Case 5: $x \leq w \leq y$ or $w \leq x \leq y$.</u> This case follows from Cases 3 and 4 by reversing the inequalities.

Thus, in any case $x \sim w$. Thus $\sim$ is an equivalence relation.

Next we claim that each equivalence class is an open interval. To see this let $x \in U$ be arbitrary and let E_x denote the equivalence class of x with respect to $\sim$. To see that E_x is an open interval, let

$$\alpha_x = \inf(E_x) \qquad \text{and} \qquad \beta_x = \sup(E_x).$$

We claim that $E_x = (\alpha_x, \beta_x)$.

First, we claim that $\alpha_x < \beta_x$. To see this, notice that $x \in E_x \subseteq U$. Hence, as U is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. Clearly $y \sim x$ for all $y \in (x - \epsilon, x + \epsilon)$ so

$$\alpha_x \leq x - \epsilon < x + \epsilon \leq \beta_x.$$

To see that $(\alpha_x, \beta_x) \subseteq E_x$, let $y \in (\alpha_x, \beta_x)$ be arbitrary. Since $\alpha_x < y < \beta_x$, by the definition of inf and sup there exists $z_1, z_2 \in E_x$ such that

$$\alpha_x \leq z_1 < y < z_2 \leq \beta_x.$$

Since $z_1, z_2 \in E_x$, we have $z_1 \sim x$ and $z_2 \sim x$. Thus $z_1 \sim z_2$ so $[z_1, z_2] \subseteq U$. Hence $y \in [z_1, z_2] \subseteq U$. Therefore, as $y \in (\alpha_x, \beta_x)$ was arbitrary, $(\alpha_x, \beta_x) \subseteq E_x$.

To see that $E_x \subseteq (\alpha_x, \beta_x)$, note that $E_x \subseteq (\alpha_x, \beta_x) \cup \{\alpha_x, \beta_x\}$ by the definition of α_x and β_x. Thus it suffices to show that $\alpha_x, \beta_x \notin E_x$. Suppose $\beta_x \in E_x$ (this implies $\beta_x \neq \infty$). Then $\beta_x \in U$ so there exists an $\epsilon > 0$ so that $(\beta_x - \epsilon, \beta_x + \epsilon) \subseteq U$. Hence $\beta_x + \frac{1}{2}\epsilon \sim \beta_x \sim x$ (as $\beta_x \in E_x$). Hence $\beta_x + \frac{1}{2}\epsilon \in E_x$. However $\beta_x + \frac{1}{2}\epsilon > \beta_x$ so $\beta_x + \frac{1}{2}\epsilon \in E_x$ contradicts the fact that $\beta_x = \sup(E_x)$. Hence we have obtained a contradiction so $\beta_x \notin E_x$. Similar arguments show that $\alpha_x \notin E_x$. Hence $E_x = (\alpha_x, \beta_x)$ as desired.

To complete the proof, first notice that clearly

$$U = \bigcup_{x \in U} E_x$$

so U is a union of open intervals. It remains to be verified that the above union can be made countable. Since each E_x is an open interval, $E_x \cap \mathbb{Q} \neq \emptyset$. Hence, as each $E_x \cap \mathbb{Q}$ is non-empty, by the Axiom of Choice there exists a function $f : \{E_x \mid x \in U\} \to \mathbb{Q}$ such that $f(E_x) \in E_x$ for all $x \in U$. Hence, as $E_x \cap E_y = \emptyset$ if $E_x \neq E_y$, f is an injective function. Hence $\{E_x \mid x \in U\}$ is countable. Thus the union $U = \bigcup_{x \in U} E_x$ can be made into a countable union by choosing one representative from each equivalence class (or, alternatively, $U = \bigcup_{q \in \mathbb{Q}} f^{-1}(\{q\})$). ∎

Remark 1.3.12. Note that Definition 1.3.1 only requires that a finite intersection of open sets is open. To see why this is required, note that in $\mathbb{R}$ that $U_n = (-\frac{1}{n}, \frac{1}{n})$ is an open subset of $\mathbb{R}$ for all $n \in \mathbb{N}$ yet $\bigcap_{n=1}^{\infty} U_n = \{0\}$ is not an open set.

Example 1.3.13. Let X be an arbitrary set and let d be the discrete metric on X. Then the metric topology on (X, d) is the discrete topology. To see this, note by Example 1.3.7 that $\{x\}$ is an open set in the metric topology on (X, d) for all $x \in X$. Therefore, for all $A \subseteq X$ we see that $A = \bigcup_{a \in A} \{a\}$ is open in (X, d). Hence, as $A \subseteq X$ was arbitrary, every subset of X is open in (X, d).

Remark 1.3.14. As we have many p-norms on $\mathbb{K}^n$, it is natural to ask how their topologies compare. In turns out that each p-norm yields the same topology! To see this, fix $n \in \mathbb{N}$. If $p \in [1, \infty]$ let $\mathcal{T}_p$ denoted the topology on $\mathbb{K}^n$ induced by the p-norm.

To see that $\mathcal{T}_p = \mathcal{T}_\infty$ for all $p \in [1, \infty)$ (and thus $\mathcal{T}_p = \mathcal{T}_q$ for all $p, q \in [1, \infty]$), first notice for an arbitrary $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$ that

$$\begin{aligned}\|\vec{x}\|_\infty^p &= \sup_{1 \leq k \leq n} |x_k|^p \\ &\leq \sum_{k=1}^n |x_k|^p \\ &= \|\vec{x}\|^p \\ &\leq \sum_{k=1}^n \|\vec{x}\|_\infty^p \\ &= n \|\vec{x}\|_\infty^p .\end{aligned}$$

Hence $\|\vec{x}\|_\infty \leq \|\vec{x}\|_p \leq n^{\frac{1}{p}} \|\vec{x}\|_\infty$ for all $\vec{x} \in \mathbb{K}^n$.

To show that $\mathcal{T}_p = \mathcal{T}_\infty$ we must show that every open subset of $\mathbb{K}^n$ with respect to either norm is open with respect to the other norm. For notational simplicity, we will use $B^p(\vec{x}, r)$ to denote the open ball centred at $\vec{x}$ of radius r with respect to the p-norm and we will use $B^\infty(\vec{x}, r)$ to denote the open ball centred at $\vec{x}$ of radius r with respect to the ∞-norm

To begin, let $U \in \mathcal{T}_p$ be arbitrary. To see that $U \in \mathcal{T}_\infty$, let $x \in U$ be arbitrary. Since $U \in \mathcal{T}_p$ there exists an $r > 0$ such that $B^p(\vec{x}, r) \subseteq U$. As $B^\infty\left(\vec{x}, \frac{1}{n^p} r\right) \subseteq B^p(\vec{x}, r) \subseteq U$ by the above norm estimates, and as $x \in U$ was arbitrary, we obtain that $U \in \mathcal{T}_\infty$. Hence $\mathcal{T}_p \subseteq \mathcal{T}_\infty$.

For the other inclusion, let $U \in \mathcal{T}_\infty$ be arbitrary. To see that $U \in \mathcal{T}_p$, let $x \in U$ be arbitrary. Since $U \in \mathcal{T}_\infty$ there exists an $r > 0$ such that $B^\infty(\vec{x}, r) \subseteq U$. As $B^p(\vec{x}, r) \subseteq B^\infty(\vec{x}, r) \subseteq U$ by the above norm estimates, we obtain that $U \in \mathcal{T}_p$. Hence $\mathcal{T}_\infty \subseteq \mathcal{T}_p$. Thus $\mathcal{T}_\infty = \mathcal{T}_p$ as desired.

While we are comparing topologies from different metrics, we can describe the topologies on subsets.

Proposition 1.3.15. *Let $(\mathcal{X}, d)$ be a metric space and let $Y \subseteq \mathcal{X}$ be non-empty. Recall $(Y, d|_Y)$ is a metric space. A subset $A \subseteq Y$ is open in $(Y, d|_Y)$ if and only if $A = Y \cap U$ for some open subset U of $(\mathcal{X}, d)$.*

Proof. For notational clarity, for $y \in Y$ and $r > 0$, we will use $B_{\mathcal{X}}(y, r)$ to denote the open ball centred at y of radius r in $(\mathcal{X}, d)$ and we will use $B_Y(y, r)$ to denote the open ball centred at y of radius r in $(y, d|_Y)$.

First, suppose $A \subseteq Y$ is such that $A = Y \cap U$ for some open subset U of (X, d). To see that A is open, let $a \in A$ be arbitrary (if $A = \emptyset$, then clearly

A is open). Thus $a \in Y \cap U \subseteq U$ so, as U is open in $(\mathcal{X}, d)$, there exists a $r > 0$ such that $B_{\mathcal{X}}(a, r) \subseteq U$. Therefore

$$B_Y(a, r) = Y \cap B_{\mathcal{X}}(a, r) \subseteq Y \cap U = A.$$

Therefore, since $a \in A$ was arbitrary, A is open in $(Y, d|_Y)$.

Conversely, suppose that $A \subseteq Y$ is open in $(Y, d|_Y)$. If $A = \emptyset$, then clearly we may take $U = \emptyset$. Otherwise, if $A \neq \emptyset$, then since A is open in $(Y, d|_Y)$, for all $a \in A$ there exists an $r_a > 0$ such that $B_Y(a, r_a) \subseteq A$. Hence we clearly have that

$$A = \bigcup_{a \in A} B_Y(a, r_a)$$

by construction. Let

$$U = \bigcup_{a \in A} B_{\mathcal{X}}(a, r_a).$$

Clearly U is open in $(\mathcal{X}, d)$. Moreover

$$Y \cap U = \bigcup_{a \in A} Y \cap B_{\mathcal{X}}(a, r_a) = \bigcup_{a \in A} B_Y(a, r_a) = A.$$

Therefore, as $A \subseteq Y$ was arbitrary, the result is complete. ∎

Example 1.3.16. Let $Y = [0, 1] \subseteq \mathbb{R}$. Then the interval $[0, \frac{1}{2})$ is open in $([0, 1], |\cdot|)$ by Proposition 1.3.15 since $[0, \frac{1}{2}) = [0, 1] \cap (-\frac{1}{2}, \frac{1}{2})$ and $(-\frac{1}{2}, \frac{1}{2})$ is open in $(\mathbb{R}, |\cdot|)$.

Although we are mainly interested in open sets in relation to convergent sequences at this time, the complements of open sets will be of incredibly interest.

Definition 1.3.17. Let $\mathcal{T}$ be a topology on a set X. A subset $F \subseteq X$ is said to be *closed* if F^c is open.

Example 1.3.18. Let $(\mathcal{X}, d)$ be a metric space. Then $\emptyset$ and $\mathcal{X}$ are both closed and open sets.

Example 1.3.19. In $\mathbb{R}$ with the absolute value metric, $(a, b]$ is neither open nor closed. Indeed $(a, b]$ is not open as there is no open ball around b contained in $(a, b]$, and $(a, b]$ is not closed as $(a, b]^c = (-\infty, a] \cup (b, \infty)$ is not open since there is no open ball around a contained in $(a, b]^c$.

Example 1.3.20. In $\mathbb{R}$ with the absolute value metric, $[a, b]$ is closed for all $a, b \in \mathbb{R}$ since $[a, b]^c = (-\infty, a) \cup (b, \infty)$ is a union of open sets and thus open.

Proposition 1.3.21. *Every closed ball and singleton in a metric space* $(\mathcal{X}, d)$ *is a closed set.*

Proof. Let $x \in \mathcal{X}$ and $r \geq 0$. We will abuse notation and consider $B[x, 0] = \{x\}$.

We claim that $B[x, r]^c$ is open. To see this, let $y \in B[x, r]^c$ be arbitrary. Then $d(x, y) > r$. Let $\epsilon = d(x, y) - r > 0$. Notice if $z \in B(y, \epsilon)$ then

$$d(x, y) \leq d(x, z) + d(z, y) < d(x, z) + \epsilon = d(x, z) + d(x, y) - r$$

which implies $r < d(x, z)$. Hence $B(y, \epsilon) \subseteq B[x, r]^c$. Therefore, as $y \in B[x, r]^c$ was arbitrary, $B[x, r]^c$ is an open set. Whence $B[x, r]$ is closed. ∎

Example 1.3.22. Let d be the discrete metric on a non-empty set X. Then, as the metric topology on (X, d) is the discrete topology by Example 1.3.13, every subset of (X, d) is closed.

Like with open sets, there are set operations we may perform on closed sets.

Proposition 1.3.23. *Let $\mathcal{T}$ be a topology on a set X, let I be an non-empty set, and for each $\alpha \in I$ let F_α be a closed subset of X. Then*

- $\bigcap_{\alpha \in I} F_\alpha$ *is closed in X, and*
- $\bigcup_{\alpha \in I} F_\alpha$ *is open in X provided I has a finite number of element.*

Proof. Since De Morgan's Laws imply

$$\left(\bigcap_{\alpha \in I} F_\alpha\right)^c = \bigcup_{\alpha \in I} F_\alpha^c \qquad \text{and} \qquad \left(\bigcup_{\alpha \in I} F_\alpha\right)^c = \bigcap_{\alpha \in I} F_\alpha^c,$$

the result follows by the definition of a closed set along with the definition of a topology. ∎

Remark 1.3.24. Complementing the fact that a countable intersection of open sets need not be open, a countable union of closed sets need not be closed. Indeed $A = \bigcup_{n=1}^\infty \{\frac{1}{n}\}$ is a countable union of closed sets in $\mathbb{R}$ that is not closed since $0 \in A^c$ yet $(-\epsilon, \epsilon) \nsubseteq A^c$ for all $\epsilon > 0$ (we will see later that $A \cup \{0\}$ is a closed set). Furthermore, there exist closed subsets of $\mathbb{R}$ that are not countable unions of closed intervals.

Finally, similar to Proposition 1.3.15, we can describe closed subsets of induced metrics on subsets.

Corollary 1.3.25. *Let $(\mathcal{X}, d)$ be a metric space and let $Y \subseteq \mathcal{X}$ be non-empty. Recall $(Y, d|_Y)$ is a metric space. A subset $A \subseteq Y$ is closed in $(Y, d|_Y)$ if and only if $A = Y \cap F$ for some closed subset F of $(\mathcal{X}, d)$.*

Proof. Notice $A \subseteq Y$ is closed in $(Y, d|_Y)$ if and only if $Y \setminus A$ is open in $(Y, d|_Y)$ if and only if $Y \setminus A = Y \cap U$ for some open subset U of $(\mathcal{X}, d)$ by Proposition 1.3.15 if and only if $A = Y \cap (\mathcal{X} \setminus U)$ for some open subset U of $(\mathcal{X}, d)$ if and only if $A = Y \cap F$ for some closed subset F of $(\mathcal{X}, d)$ as desired. ∎

1.4 Converging Sequences

Now that we have developed the notion of the metric topology, we have finally arrived at defining when a sequence in a metric spaces converges. First we will model the standard definition for convergent sequences of real numbers, and then we will see the connection to the metric topology.

Definition 1.4.1. Let $(\mathcal{X}, d)$ be a metric space and let $(x_n)_{n\geq 1}$ be a sequence in $\mathcal{X}$. The sequence $(x_n)_{n\geq 1}$ is said to *converge* in $\mathcal{X}$ to an element $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ for all $n \geq N$. In this case x_0 is said to be a *limit* of the sequence $(x_n)_{n\geq 1}$ and we write $x_0 = \lim_{n\to\infty} x_n$.

Remark 1.4.2. Of course, like in previous courses, the '$< \epsilon$' in Definition 1.4.1 can be replaced with '$\leq \epsilon$' without changing the definition.

As the statement "$d(x_n, x_0) < \epsilon$" is equivalent to saying that $x_n \in B(x_0, \epsilon)$ and as every open set containing x_0 contains an open ball centred at x_0, we directly have a connection between convergence of sequences and topology.

Proposition 1.4.3. *Let $(\mathcal{X}, d)$ be a metric space. A sequence $(x_n)_{n\geq 1}$ converges to an element $x_0 \in \mathcal{X}$ if and only if for every open set U of $\mathcal{X}$ such that $x_0 \in U$ there exists an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

Remark 1.4.4. For general topological spaces (i.e. a space with a topology), the notion of convergence is defined via Proposition 1.4.3 as we need not have a metric. One thinks of each open set as a 'neighbourhood' around a point and for a sequence to converge to a point, it must eventually inside every open set.

Example 1.4.5. Let $m \in \mathbb{N}$. Note Proposition 1.4.3 implies that convergent sequences are completely determined by the open sets, and thus the topology of a metric space. In particular, since all $(\mathbb{K}^m, \|\cdot\|_p)$ for $p \in [1, \infty]$ have the same topology by Remark 1.3.14, a sequence converges in $(\mathbb{K}^m, \|\cdot\|_p)$ if and only if it converges in $(\mathbb{K}^m, \|\cdot\|_\infty)$.

To see what it means for a sequence to converge in $(\mathbb{K}^m, \|\cdot\|_\infty)$, for each $n \in \mathbb{N}$, let $\vec{x}_n = (z_{1,n}, \ldots, z_{m,n}) \in \mathbb{K}^m$. Given $\vec{x} = (z_1, \ldots, z_m) \in \mathbb{K}^m$, the following are equivalent:

(1) $(\vec{x}_n)_{n\geq 1}$ converges to $\vec{x}$ with respect to the ∞-norm.

(2) $\lim_{n\to\infty} |z_{k,n} - z_k| = 0$ for all $k \in \{1, \ldots, m\}$.

To see that (1) implies (2), let $\epsilon > 0$ be arbitrary. Since $(\vec{x}_n)_{n\geq 1}$ converges to $\vec{x}$ with respect to the ∞-norm, there exists an $N \in \mathbb{N}$ such that $\|\vec{x}_n - \vec{x}\|_\infty < \epsilon$ for all $n \geq N$. Since that $|z_{k,n} - z_k| \leq \|\vec{x}_n - \vec{x}\|_\infty$ for all $k \in \{1, \ldots, m\}$, we see that $|z_{k,n} - z_k| < \epsilon$ for all $n \geq N$ and $k \in \{1, \ldots, m\}$. Hence, as $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} |z_{k,n} - z_k| = 0$ for all $k \in \{1, \ldots, m\}$.

For the other direction, suppose that (2) holds and let $\epsilon > 0$ be arbitrary. Hence for all $k \in \{1, \ldots, m\}$ there exists an $N_k \in \mathbb{N}$ such that $|z_{k,n} - z_k| < \epsilon$ for all $n \geq N_k$. Thus if $N = \max_{1\leq k\leq m} N_k$, the for all $n \geq N$ we have that

$$\|\vec{x}_n - \vec{x}\|_\infty = \sup_{1\leq k\leq m} |z_{k,n} - z_k| < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $(\vec{x}_n)_{n\geq 1}$ converges to $\vec{x}$ with respect to the ∞-norm.

Example 1.4.6. By using similar arguments to those used in Example 1.4.5 for $\mathbb{K} = \mathbb{R}$ and $p = 2$, if $(z_n)_{n\geq 1}$ is a sequence in $\mathbb{C}$, $z \in \mathbb{C}$, and $a_n, b_n, a, b \in \mathbb{R}$ are such that $z = a + bi$ and $z_n = a_n + b_n i$ for all $n \in \mathbb{N}$, then $z = \lim_{n\to\infty} z_n$ if and only if $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$.

Remark 1.4.7. A careful analysis of the above examples reveals that $(\mathbb{C}, |\cdot|)$ and $(\mathbb{R}^2, \|\cdot\|_2)$ behave very similarly as metric spaces. We will later analyze what it means for two normed linear spaces to be 'the same'.

As we have seen examples of convergent sequences in $\mathbb{R}$ in previous courses, we will examine some more exotic examples.

Example 1.4.8. Given a sequence $(f_n)_{n\geq 1}$ of elements of $\mathcal{C}[a, b]$, notice that $(f_n)_{n\geq 1}$ converges to an element $f \in \mathcal{C}[a, b]$ with respect to $\|\cdot\|_\infty$ if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ and $n \geq N$. This is precisely the notion of *uniform convergence of functions* discussed in previous analysis courses.

Example 1.4.9. Let d be the discrete metric on a non-empty set X. If $(x_n)_{n\geq 1}$ is a sequence in X, then $(x_n)_{n\geq 1}$ converges to a point $x_0 \in X$ if and only if there exists an $N \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq N$; that is, the sequence is eventually constant. This is due to the fact that $d(x_n, x_0) < 1$ if and only if $x_n = x_0$.

Example 1.4.10. Given a $p \in [1, \infty]$, it is not difficult to see that if $\vec{x}_n = (x_{n,k})_{k\geq 1} \in \ell_p(\mathbb{N}, \mathbb{K})$ for all $n \in \mathbb{N}$, $\vec{y} = (y_k)_{k\geq 1}$, and $\lim_{n\to\infty} \vec{x}_n = \vec{y}$ in $\ell_p(\mathbb{N}, \mathbb{K})$, then $\lim_{n\to\infty} x_{n,k} = y_k$ for all $k \in \mathbb{N}$ as

$$|x_{n,k} - y_k| \leq \|\vec{x}_n - \vec{y}\|_p$$

for all $n, k \in \mathbb{N}$. However, the converse need not hold. To see this, for each $n \in \mathbb{N}$, let $\vec{x}_n = (x_{n,k})_{k\geq 1}$ where

$$x_{n,k} = \begin{cases} 1 & k \leq n \\ 0 & k > n \end{cases}.$$

Then it is elementary to see that $\lim_{n\to\infty} x_{n,k} = 1$ for all k, yet $(\vec{x}_n)_{n\geq 1}$ does not converge in $(\ell_p(\mathbb{N}, \mathbb{K}), \|\cdot\|_p)$. Indeed this is clear if $p \neq \infty$ as the constant sequence $\vec{x} = (1)_{n\geq 1}$ is not an element of $\ell_p(\mathbb{N})$ and is the only option for the limit by the first part of this example. If $p = \infty$, notice $\|\vec{x} - \vec{x}_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Thus clearly $(\vec{x}_n)_{n\geq 1}$ does not converge to $\vec{x}$ in $(\ell_\infty(\mathbb{N}, \mathbb{K}), \|\cdot\|_\infty)$.

In the case of normed linear spaces, the notion of convergent sequences behaves well with respect to the vector space operations.

Proposition 1.4.11. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed linear space over $\mathbb{K}$. If $(\vec{x}_n)_{n\geq 1}$ and $(\vec{y}_n)_{n\geq 1}$ are sequences that converge to $\vec{x}$ and $\vec{y}$ respectively, then*

- *$(\vec{x}_n + \vec{y}_n)_{n\geq 1}$ converges to $\vec{x} + \vec{y}$, and*
- *$(\alpha\vec{x}_n)_{n\geq 1}$ converges to $\alpha\vec{x}$ for all $\alpha \in \mathbb{K}$.*

Proof. Let $\epsilon > 0$. Since

$$\|(\vec{x}_n + \vec{y}_n) - (\vec{x} - \vec{y})\| \leq \|\vec{x}_n - \vec{x}\| + \|\vec{y}_n - \vec{y}\| \text{ and}$$
$$\|\alpha\vec{x}_n - \alpha\vec{x}\| \leq |\alpha| \, \|\vec{x}_n - \vec{x}\|$$

for all n and since we may chose N sufficiently large so that the right-hand sides of both inequalities is less than ϵ, the result follows. ∎

As we have mentioned, the notion of convergent sequences makes sense in any topological space by using Proposition 1.4.3 as the definition of a convergent sequence. However, this raises a problem in that, under this definition, it is possible for a sequence to converge to multiple points. Indeed, if we consider the trivial topology, then every sequence converges to every point in the space since the only open sets are the empty set and the full set. This is why in Definition 1.4.1 we only defined 'a' limit of a sequence instead of 'the' limit of a sequence. However, for our metric topologies, we can prove that limits are unique.

Proposition 1.4.12. *Let $(\mathcal{X}, d)$ be a metric space and let $(x_n)_{n\geq 1}$ be a sequence in $\mathcal{X}$. If $x_0 = \lim_{n\to\infty} x_n$ and $y_0 = \lim_{n\to\infty} x_n$, then $x_0 = y_0$.*

Proof. Suppose $x_0 = \lim_{n\to\infty} x_n$ and $y_0 = \lim_{n\to\infty} x_n$. Let $\epsilon > 0$ be arbitrary. Since $x_0 = \lim_{n\to\infty} x_n$ there exists an $N_1 \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ for all $n \geq N_1$. Similarly, since $y_0 = \lim_{n\to\infty} x_n$ there exists an $N_2 \in \mathbb{N}$ such that $d(x_n, y_0) < \epsilon$ for all $n \geq N_2$. Therefore, if $N = \max\{N_1, N_2\}$, we obtain that

$$0 \leq d(x_0, y_0) \leq d(x_0, x_N) + d(x_N, y_0) < 2\epsilon.$$

Since the above inequality holds for all $\epsilon > 0$, we obtain that $d(x_0, y_0) = 0$. Hence $x_0 = y_0$ by property (1) of Definition 1.1.1 ∎

To finish off our initial discussion of convergent sequences, we note that given a sequence it is often useful to be able to construct other sequences by removing elements. This leads to the following notion.

Definition 1.4.13. Let $(\mathcal{X}, d)$ be a metric space. A *subsequence* of a sequence $(x_n)_{n\geq 1}$ of elements of $\mathcal{X}$ is any sequence $(y_n)_{n\geq 1}$ such that there exists an increasing sequence of natural numbers $(k_n)_{n\geq 1}$ so that $y_n = x_{k_n}$ for all $n \in \mathbb{N}$.

Unsurprisingly, if a sequence converges to a point, so does every subsequence.

Proposition 1.4.14. *Let $(\mathcal{X}, d)$ be a metric space and let $(x_n)_{n\geq 1}$ be a sequence that converges to $x \in \mathcal{X}$. Every subsequence of $(x_n)_{n\geq 1}$ converges to x.*

Proof. Let $(x_{k_n})_{n\geq 1}$ be a subsequence of $(x_n)_{n\geq 1}$. Let $\epsilon > 0$. Since $x = \lim_{n\to\infty} x_n$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq N$. Since $(k_n)_{n\geq 1}$ is an increasing sequence of natural numbers, there exists an $N_0 \in \mathbb{N}$ such that $k_n \geq N$ for all $n \geq N_0$. Hence $d(x_{k_n}, x) < \epsilon$ for all $n \geq N_0$. Therefore, as $\epsilon > 0$ was arbitrary, we obtain that $\lim_{n\to\infty} x_{k_n} = x$ by the definition of the limit. ∎

1.5 Points and Sets

There are many useful notions related to the convergence of sequences in metric spaces. In this section, we will analyze specific types of points and sets in relation to sequences. These notions have useful theoretical applications in later in the course and give us a greater understanding of the structural aspects of a metric space.

We begin with the following types of elements we may be interested in studying.

Definition 1.5.1. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. An element $x_0 \in \mathcal{X}$ is said to be a:

- *limit point* of A if there exists a sequence $(a_n)_{n\geq 1}$ of points in A that converges to x_0.
- *cluster point* of A if there exists a sequence $(a_n)_{n\geq 1}$ of points in $A \setminus \{x_0\}$ that converges to x_0.
- *boundary point* of A if x_0 is a limit point of both A and A^c.
- *interior point* of A if there exists an $\epsilon > 0$ such that $B(x_0, \epsilon) \subseteq A$.

The set of limit, cluster, boundary, and interior points of are denoted $\lim(A)$, $\mathrm{cluster}(A)$, $\mathrm{bdy}(A)$, and $\mathrm{int}(A)$ respectively.

Before we get to examples, we note the following which will give us an alternative characterization of $\lim(A)$, $\mathrm{cluster}(A)$, and $\mathrm{bdy}(A)$ which look more like the characterization of $\mathrm{int}(A)$.

Lemma 1.5.2. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. Then $x \in \lim(A)$ if and only if $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$. Consequently $x \in \mathrm{bdy}(A)$ if and only if for all $\epsilon > 0$ we have $A \cap B(x, \epsilon) \neq \emptyset$ and $A^c \cap B(x, \epsilon) \neq \emptyset$.*

Similarly $x \in \mathrm{cluster}(A)$ if and only if $A \cap B(x, \epsilon) \setminus \{x\} \neq \emptyset$ for all $\epsilon > 0$.

Proof. First, suppose $x \in \lim(A)$. Hence there exists a sequence $(a_n)_{n\geq 1}$ of points in A that converges to x. Therefore, if $\epsilon > 0$ then there exists an $N \in \mathbb{N}$ such that $a_n \in B(x, \epsilon)$ for all $n \geq N$. Hence $a_N \in A \cap B(x, \epsilon)$ so the claim follows.

For the other direction, suppose that $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$. Hence for each $n \in \mathbb{N}$ there exists an $a_n \in A \cap B(x, \frac{1}{n})$. We claim that $(a_n)_{n\geq 1}$ converges to x thereby proving $x \in \lim(A)$. To see this, let $\epsilon > 0$ be arbitrary and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Hence for all $n \geq N$ we have $d(x, a_n) < \frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore, as $\epsilon > 0$ was arbitrary, we obtain that $x \in \lim(A)$ by definition.

The proof of the claim for $\mathrm{bdy}(A)$ is trivial and the proof of the claim for $\mathrm{cluster}(A)$ is nearly identical. ∎

The following examples demonstrate that no two of these sets need to be equal in general.

Example 1.5.3. Given $a, b \in \mathbb{R}$ with $a < b$, it is easy to see that if $A \in \{[a, b], (a, b], [a, b), (a, b)\}$, then

$$\lim(A) = \mathrm{cluster}(A) = [a, b], \quad \mathrm{bdy}(A) = \{a, b\}, \quad \text{and} \quad \mathrm{int}(A) = (a, b).$$

Example 1.5.4. Let $X = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ and let $Y = X \cup \{0\}$. Then

$$\lim(X) = \mathrm{bdy}(X) = Y, \quad \mathrm{cluster}(X) = \{0\}, \quad \text{and} \quad \mathrm{int}(X) = \emptyset.$$

Furthermore

$$\lim(Y) = \mathrm{bdy}(Y) = Y, \quad \mathrm{cluster}(Y) = \{0\}, \quad \text{and} \quad \mathrm{int}(Y) = \emptyset.$$

Example 1.5.5. Let $Z = \{(0, y) \mid y \in [0, 1]\}$ viewed as a subset of $(\mathbb{R}^2, \|\cdot\|_2)$. Then

$$\lim(Z) = \mathrm{cluster}(Z) = \mathrm{bdy}(Z) = Z \quad \text{and} \quad \mathrm{int}(Z) = \emptyset.$$

Example 1.5.6. Let

$$W = \{x \in \mathbb{Q} \mid 0 < x < \sqrt{2}\}.$$

Then

$$\lim(W) = \mathrm{cluster}(W) = [0, \sqrt{2}].$$

However

$$\mathrm{bdy}(W) = [0, \sqrt{2}], \qquad \text{and} \qquad \mathrm{int}(W) = \emptyset.$$

Looking at the above examples and giving a moments thought, one can see that if A is a set then $\lim(A) = \mathrm{cluster}(A) \cup A$. Furthermore, in the above examples, notice only $[a, b]$, Y, Z are closed. The fact that these are exactly the sets that contain their limit, cluster, and boundary points is no coincidence.

Theorem 1.5.7. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. The following are equivalent:*

(1) A is a closed set.

(2) $\lim(A) \subseteq A$.

(3) $\mathrm{cluster}(A) \subseteq A$.

(4) $\mathrm{bdy}(A) \subseteq A$.

Proof. First notice (2) and (3) are equivalent as $\lim(A) = A \cup \mathrm{cluster}(A)$. In addition, since $\mathrm{bdy}(A) \subseteq \lim(A)$, clearly (2) implies (4). To see that (4) implies (2), suppose $\mathrm{bdy}(A) \subseteq A$ but there exists an $x \in \lim(A) \setminus A$. Since $x \in \lim(A)$ and $x \in A^c \subseteq \lim(A^c)$, x is a boundary point of A by definition. Hence $x \in \mathrm{bdy}(A) \subseteq A$ which is a contradiction. Hence (4) implies (2).

To see that (1) implies (2), suppose that A is a closed set and that there exists an $x \in \lim(A) \setminus A$. Hence there exists a sequence $(a_n)_{n\geq 1}$ of elements from A such that $x = \lim_{n\to\infty} a_n$, and $x \in A^c$. Since A is closed, A^c is open so there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A^c$. However, since $x = \lim_{n\to\infty} a_n$, there exists an $N \in \mathbb{N}$ such that $a_n \in B(x, \epsilon) \subseteq A^c$ for all $n \geq N$. Notice this is contradiction as $a_n \in A$ for all $n \in \mathbb{N}$. Hence $\lim(A) \subseteq A$.

To see that (2) implies (1), suppose that A is not closed. Therefore A^c is not open. Thus there exists an $x \in A^c$ such that $B(x, \epsilon) \cap A \neq \emptyset$ for all $\epsilon > 0$. Hence $x \in \lim(A) \setminus A$ by Lemma 1.5.2. Thus $\lim(A) \setminus A$ is non-empty whenever A is not closed. ∎

Example 1.5.8. It is not difficult to see that if $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then $\mathrm{cluster}(A) = \{0\}$. Hence A is a closed set.

Remark 1.5.9. Notice that $A = \mathrm{int}(A) \cup (\mathrm{bdy}(A) \cap A)$. Consequently, if A is closed, Theorem 1.5.7 shows $A = \mathrm{int}(A) \cup \mathrm{bdy}(A)$.

One of the most important examples to consider in analysis of subset of $\mathbb{R}$ is the following set.

Definition 1.5.10. Let $P_0 = [0, 1]$. Construct P_1 from P_0 by removing the open interval of length $\frac{1}{3}$ from the middle of P_0 (i.e. $P_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$). Then construct P_2 from P_1 by removing the open intervals of length $\frac{1}{3^2}$ from the middle of each closed subinterval of P_1. Subsequently, having constructed P_n, construct P_{n+1} by removing the open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of P_n. The set

$$\mathcal{C} = \bigcap_{n\geq 1} P_n$$

is known as the *Cantor set*.

Remark 1.5.11. The Cantor set has many interesting properties. Firstly, we note that the Cantor set is closed being the intersection of closed sets.

The following gives another characterization of the Cantor sets.

Lemma 1.5.12. *Let $x \in \mathbb{R}$. Then $x \in \mathcal{C}$ if and only if there is a sequence $(a_n)_{n\geq 1}$ with $a_n \in \{0,2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ (i.e. $x \in [0,1]$ and x has a ternary expansion using only 0s and 2s).*

Proof. Suppose $x \in \mathcal{C}$. Hence $x \in P_n$ for all $n \in \mathbb{N}$. Hence, by the recursive construction of the P_n, there exists numbers $a_1, a_2, a_3, \ldots \in \{0,2\}$ such that

$$x \in \left[\sum_{k=1}^{n} \frac{a_k}{3^k}, \frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right] \subseteq P_n$$

for all $n \in \mathbb{N}$. To see that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, we notice that

$$\left|x - \sum_{k=1}^{n} \frac{a_k}{3^k}\right| \leq \left|\left(\frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right) - \sum_{k=1}^{n} \frac{a_k}{3^k}\right| = \frac{1}{3^n}.$$

Therefore, since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, we obtain that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ as desired.

Conversely, suppose that $x \in \mathbb{R}$ is such that there exists a sequence $(a_n)_{n\geq 1}$ with $a_n \in \{0,2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$. For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$. Hence, by the description of P_n, we obtain that $s_n \in P_n$ for all n. In fact, upon closer examination, we see that $s_m \in P_n$ whenever $m \geq n$. Indeed if $m \geq n$ then

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k}{3^k} \leq \sum_{k=1}^{m} \frac{a_k}{3^k} = s_m &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \sum_{k=n+1}^{m} \frac{2}{3^k} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{2}{3^{n+1}} \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{1 - \frac{1}{3}} \\
&= \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{3^n} \\
&\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1}{3^n}.
\end{aligned}$$

Since each P_n is a closed set, since $x = \lim_{m\to\infty} s_m$, and since $s_m \in P_n$ whenever $m \geq n$, we obtain that $x \in P_n$ for each $n \in \mathbb{N}$ by the sequential description of closed sets. Hence $x \in \bigcap_{n\geq 1} P_n = \mathcal{C}$. ∎

Lemma 1.5.12 enables us to demonstrate the following two results.

Corollary 1.5.13. int$(\mathcal{C}) = \emptyset$ *and* cluster$(\mathcal{C}) = \mathcal{C}$.

Proof. To see that $\text{int}(\mathcal{C})$ is empty, suppose to the contrary that there exists an $x \in \text{int}(\mathcal{C})$. Hence there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq \mathcal{C} = \bigcap_{n\geq 1} P_n$. Hence $(x-\epsilon, x+\epsilon) \subseteq P_n$ for all $n \in \mathbb{N}$. Choose $N \in \mathbb{N}$ such that $\frac{1}{3^N} < \epsilon$. By the construction of P_N we see that P_N cannot contain an open interval of length more that $\frac{1}{3^N}$. Hence it is impossible for $(x-\epsilon, x+\epsilon) \subseteq P_N$. Thus we have obtained a contradiction so it must be the case that $\text{int}(\mathcal{C}) = \emptyset$.

For the other equality, notice since $\mathcal{C}$ is closed, $\text{cluster}(\mathcal{C}) \subseteq \mathcal{C}$. We claim that $\text{cluster}(\mathcal{C}) = \mathcal{C}$. To see this, let $x \in \mathcal{C}$ be arbitrary. By Lemma 1.5.12 there exists a sequence $(a_n)_{n\geq 1}$ with $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^n \frac{a_k}{3^k}$. For each $m \in \mathbb{N}$, let $x_m = \lim_{n\to\infty} \sum_{k=1}^n \frac{a_{m,k}}{3^k}$ where

$$a_{m,k} = \begin{cases} a_k & \text{if } k \neq m \\ 0 & \text{if } k = m \text{ and } a_k = 2 \\ 2 & \text{if } k = m \text{ and } a_k = 0 \end{cases}.$$

By Lemma 1.5.12 we see that $x_m \in \mathcal{C}$ for all $m \in \mathbb{N}$ and that $x_m \neq x$ for all $m \in \mathbb{N}$ by construction. Furthermore, as $|x - x_m| = \frac{2}{3^m}$, we see that $x = \lim_{m\to\infty} x_m$. Hence $x \in \text{cluster}(\mathcal{C})$. Therefore, as x was arbitrary, $\text{cluster}(\mathcal{C}) = \mathcal{C}$. ∎

Corollary 1.5.14. $|\mathcal{C}| = |\mathbb{R}|$.

Proof. To see that $\mathcal{C}$ is uncountable, define $f : \prod_{n=1}^\infty \{0, 1\} \to \mathcal{C}$ by

$$f((b_n)_{n\geq 1}) = \lim_{n\to\infty} \sum_{k=1}^n \frac{2b_k}{3^k}.$$

Clearly f is a well-defined injection so $|\mathcal{C}| \geq 2^{|\mathbb{N}|} = |\mathbb{R}|$. Since $\mathcal{C} \subseteq \mathbb{R}$, we obtain that $|\mathcal{C}| = |\mathbb{R}|$ as desired. ∎

As the limit, boundary, and cluster points are related to the notion of 'closedness', it is unsurprising that the interior points are related to 'openness'.

Proposition 1.5.15. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. The interior of A is an open set.*

Proof. Let $x \in \text{int}(A)$ be arbitrary. Therefore there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$. We claim that $B\left(x, \frac{\epsilon}{2}\right) \subseteq \text{int}(A)$. To see this, let $y \in B\left(x, \frac{\epsilon}{2}\right)$ be arbitrary. Notice if $z \in B\left(y, \frac{\epsilon}{2}\right)$ then

$$d(z, x) \leq d(z, y) + d(y, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $B\left(y, \frac{\epsilon}{2}\right) \subseteq B(x, \epsilon) \subseteq A$. Hence $y \in \text{int}(A)$ by the definition of the interior. Therefore, as $y \in B\left(x, \frac{\epsilon}{2}\right)$ was arbitrary, $B\left(x, \frac{\epsilon}{2}\right) \subseteq \text{int}(A)$. Hence $\text{int}(A)$ is open by the definition of an open set. ∎

In particular, the interior of a set is the largest open subset of a given set. To see this, we first note the following.

Proposition 1.5.16. *Let* $(\mathcal{X}, d)$ *be a metric space and let* $A \subseteq \mathcal{X}$*. If* $U \subseteq A$ *is an open set in* $(\mathcal{X}, d)$*, then* $U \subseteq \mathrm{int}(A)$*.*

Proof. Suppose $U \subseteq A$ is an open set in $(\mathcal{X}, d)$. Since U is open, for each $x \in U$ then there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U \subseteq A$. Hence, by the definition of the interior, $x \in \mathrm{int}(A)$ for all $x \in U$. ∎

Corollary 1.5.17. *Let* $(\mathcal{X}, d)$ *be a metric space and let* $A \subseteq \mathcal{X}$*. Then*

$$\mathrm{int}(A) = \bigcup_{U \in \Gamma} U \qquad \textit{where } \Gamma = \{U \subseteq A \mid U \textit{ is an open subset of } \mathcal{X}\}.$$

Hence $\mathrm{int}(A)$ *is the largest open subset of* A*.*

Proof. If $U \in \Gamma$ then $U \subseteq \mathrm{int}(A)$ by Proposition 1.5.16. Hence $\bigcup_{U \in \Gamma} U \subseteq \mathrm{int}(A)$.

For the other inclusion, let $x \in \mathrm{int}(A)$ be arbitrary. By the definition of the interior there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$. As $B(x, \epsilon) \in \Gamma$, we obtain that $x \in B(x, \epsilon) \subseteq \bigcup_{U \in \Gamma} U$. Therefore, as $x \in \mathrm{int}(A)$ was arbitrary, we obtain that $\mathrm{int}(A) \subseteq \bigcup_{U \in \Gamma} U$. ∎

The above shows that if a point is in the interior of a set A, then it is 'far' away from elements of A^c. These types of sets are useful when it comes to the notion of convergence.

Definition 1.5.18. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. It is said that A is a *neighbourhood* of an element $x \in \mathcal{X}$ if $x \in \mathrm{int}(A)$; that is, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A$.

In particular, using Proposition 1.4.3, we have another characterization of convergent sequences.

Proposition 1.5.19. *Let* $(\mathcal{X}, d)$ *be a metric space. A sequence* $(x_n)_{n \geq 1}$ *converges to an element* $x_0 \in \mathcal{X}$ *if and only if for every neighbourhood* A *of* x_0 *there exists an* $N \in \mathbb{N}$ *such that* $x_n \in A$ *for all* $n \geq N$*.*

As a complement to finding the largest open subset of a set, we may desire to find the smallest closed set containing a set. This serves the important operation of adding the minimal amount of points to a set to make it closed. This may be performed as follows.

Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. As $\mathcal{X}$ is a closed set,

$$\mathcal{F} = \{F \subseteq \mathcal{X} \mid F \text{ is closed and } A \subseteq F\}$$

is a non-empty set. Consequently, by Proposition 1.3.23, $\overline{A} = \bigcap_{F \in \mathcal{F}} F$ is a closed set in $\mathcal{X}$ that contains A. Furthermore, $\overline{A}$ is the smallest closed set (under inclusion) containing A.

Definition 1.5.20. The set $\overline{A}$ described above is called the *closure* of A in $\mathcal{X}$.

Of course, as unions change to intersections and open sets change to closed sets under complementation, the closure of a set is related to the interior of the complement.

Proposition 1.5.21. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. Then* $\mathrm{int}(A^c) = (\overline{A})^c$.

Proof. Notice that $x \in \mathrm{int}(A^c)$ if and only if there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq A^c$. If $\epsilon > 0$ and $B(x, \epsilon) \subseteq A^c$ then $A \subseteq B(x, \epsilon)^c$. Since $B(x, \epsilon)^c$ is a closed set and $x \notin B(x, \epsilon)^c$, we obtain that $x \notin \overline{A}$ so $x \in (\overline{A})^c$. Similarly, if $x \in (\overline{A})^c$ then $x \notin \overline{A}$. As $\overline{A}$ is closed, $(\overline{A})^c$ is open so there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq (\overline{A})^c \subseteq A^c$. Thus the proof is complete. ∎

Furthermore, it is unsurprising that to make a set closed, one need only add certain points we have seen before.

Proposition 1.5.22. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. The set* $A \cup \lim(A) = \lim(A)$ *is a closed set. In particular* $\lim(A) = \overline{A}$. *Similarly* $\overline{A} = A \cup \mathrm{cluster}(A) = A \cup \mathrm{bdy}(A)$.

Proof. To see that $\lim(A)$ is a closed set, we will show that

$$\lim(\lim(A)) \subseteq \lim(A)$$

and apply Theorem 1.5.7. Let $x \in \lim(\lim(A))$ be arbitrary. Therefore for each $n \in \mathbb{N}$ there exists an $y_n \in \lim(A)$ such that $d(x, y_n) < \frac{1}{n}$. Since $y_n \in \lim(A)$, there exists an $x_n \in A$ such that $d(x_n, y_n) < \frac{1}{n}$. Hence $d(x, x_n) < \frac{2}{n}$ for each $n \in \mathbb{N}$. Thus $(x_n)_{n \geq 1}$ is a sequence of elements of A that converges to x. Hence $x \in \lim(A)$. Therefore, as x was arbitrary, $A \cup \lim(A)$ is closed.

To see that $\overline{A} = A \cup \lim(A)$, we obtain by the definition of the closure that $\overline{A} \subseteq A \cup \lim(A)$. Since any closed set containing A must also contain $\lim(A)$ by Theorem 1.5.7, the other inclusion is apparent. Furthermore, clearly $A \cup \mathrm{cluster}(A) = A \cup \lim(A) = \overline{A}$. Finally, a near identical proof to the above shows that $A \cup \mathrm{bdy}(A)$ is closed and since $A \cup \mathrm{bdy}(A) \subseteq A \cup \lim(A) = \overline{A}$, the proof is complete. ∎

Corollary 1.5.23. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. Then $x \in \overline{A}$ if and only if $A \cap B(x, \epsilon) \neq \emptyset$ for all $\epsilon > 0$.*

Proof. As Proposition 1.5.22 implies $\overline{A} = \lim(A)$, the result trivially follows from Lemma 1.5.2. ∎

Remark 1.5.24. Given a metric space $(\mathcal{X}, d)$, it is possible that $\overline{B(x, r)} \neq B[x, r]$. For example, let d be the discrete metric on a non-empty set $\mathcal{X}$. Then for all $x \in \mathcal{X}$, $\overline{B(x, 1)} = \{x\}$ whereas $B[x, 1] = \mathcal{X}$.

Recall that every real number is a limit of rational numbers; that is, $\overline{\mathbb{Q}} = \mathbb{R}$. In general, given a metric space $(\mathcal{X}, d)$, it is incredibly useful to have a set $A \subseteq \mathcal{X}$ such that $\overline{A} = \mathcal{X}$ as this permits us to obtain a great deal of information about $\mathcal{X}$ from the potentially simpler set A. Thus we make the following definition.

Definition 1.5.25. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq B \subseteq \mathcal{X}$. It is said that A is *dense* in B if $B \subseteq \overline{A}$. Equivalently, A is dense in B if for every $b \in B$ there exists a sequence $(a_n)_{n\geq 1}$ of elements from A such that $b = \lim_{n\to\infty} a_n$.

It is particularly useful to have the smallest possible dense set. The following is, in general, the smallest possible type of set.

Definition 1.5.26. A metric space $(\mathcal{X}, d)$ is said to be *separable* if there exists a countable dense subset of $\mathcal{X}$.

Example 1.5.27. Since $\overline{\mathbb{Q}} = \mathbb{R}$, clearly $\mathbb{R}$ is separable even though $\mathbb{R}$ is uncountable. Similarly, by Example 1.4.5, it is not difficult to see using $\mathbb{Q}$ that $\mathbb{K}^n$ is separable with respect to any p-norm.

It is not difficult to see if $\mathcal{X}$ is a metric space with the discrete metric, then $\mathcal{X}$ is separable if and only if $\mathcal{X}$ is countable.

Example 1.5.28. The space $\ell_1(\mathbb{N}, \mathbb{R})$ is separable. To see this, we claim that

$$A = \{(a_n)_{n\geq 1} \mid a_n \in \mathbb{Q} \text{ for all } n, a_n = 0 \text{ for all but finitely many } n\}$$

is a countable dense subset of $\ell_1(\mathbb{N}, \mathbb{R})$. To see this, we first notice that $\mathbb{Q}^n$ is countable for all $n \in \mathbb{N}$. Since A may be viewed as an increasing union of copies of $\mathbb{Q}^n$, A can be viewed as a countable union of countable sets and thus A is countable.

To see that $\overline{A} = \ell_1(\mathbb{N}, \mathbb{R})$, let $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N}, \mathbb{R})$ be arbitrary. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} |x_n| < \infty$, there exists an $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} |x_n| < \frac{1}{2}\epsilon$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, for each $n \leq N$ there exists an $a_n \in \mathbb{Q}$ such that $|x_n - a_n| < \frac{1}{2N}\epsilon$. For $n > N$, define $a_n = 0$. Hence $(a_n)_{n\geq 1} \in A$ and

$$\begin{aligned}
\|(x_n)_{n\geq 1} - (a_n)_{n\geq 1}\|_1 &= \sum_{n=1}^{\infty} |x_n - a_n| \\
&= \sum_{n=1}^{N} |x_n - a_n| + \sum_{n=N+1}^{\infty} |x_n| \\
&\leq \sum_{n=1}^{N} \frac{1}{2N}\epsilon + \sum_{n=N+1}^{\infty} |x_n| \\
&\leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $(x_n)_{n\geq 1} \in \overline{A}$. Therefore, as $(x_n)_{n\geq 1} \in \ell_1(\mathbb{N}, \mathbb{R})$ was arbitrary, $\overline{A} = \ell_1(\mathbb{N}, \mathbb{R})$ so $\ell_1(\mathbb{N}, \mathbb{R})$ is separable.

Example 1.5.29. The space $\ell_\infty(\mathbb{N}, \mathbb{R})$ is not separable. To see this, suppose to the contrary that there exists a countable dense subset C of $\ell_\infty(\mathbb{N}, \mathbb{R})$. Consider the set

$$B = \{(b_n)_{n\geq 1} \mid b_n \in \{0, 1\}\} \subseteq \ell_\infty(\mathbb{N}, \mathbb{R}).$$

By Assignment 1, B is uncountable. Further notice if $\vec{x}, \vec{y} \in B$ are distinct, then $\|\vec{x} - \vec{y}\|_\infty = 1$. Therefore, $B(\vec{x}, \frac{1}{2}) \cap B(\vec{y}, \frac{1}{2}) = \emptyset$ for all distinct $\vec{x}, \vec{y} \in B$. However, since C is dense in $\ell_\infty(\mathbb{N}, \mathbb{R})$, there must be an element of C in $B(\vec{x}, \frac{1}{2})$ for each $\vec{x} \in B$ which is impossible as C has a countable number of points, these balls are disjoint, and there are an uncountable number of balls. Hence $\ell_\infty(\mathbb{N}, \mathbb{R})$ is not separable.

1.6 Continuity

As with everything in mathematics, once one has defined the main objects one desires to study, one then defines the morphisms or functions related to ones' central object. As with every previous analysis course, these morphisms are the continuous functions. In particular, continuous functions are those that preserve convergent sequences and topological properties.

To generalize the notion of a continuous function on $\mathbb{R}$ to a function between metric spaces, we simply generalize the ϵ-δ notion of continuity.

Definition 1.6.1. Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be metric spaces. It is said that a function $f : \mathcal{X} \to \mathcal{Y}$ is *continuous* at a point $x_0 \in \mathcal{X}$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $d_\mathcal{X}(x, x_0) < \delta$ then $d_\mathcal{Y}(f(x), f(x_0)) < \epsilon$. Otherwise it is said that f is *discontinuous* at x_0.

Remark 1.6.2. Note that the '$<$' in both the '$< \delta$' and '$< \epsilon$' portions of Definition 1.6.1 may be replaced by '$\leq$'. Indeed this follows since for all $x_0 \in \mathcal{X}$ and $r > 0$,

$$B\left(x_0, \frac{1}{2}r\right) \subseteq B\left[x_0, \frac{1}{2}r\right] \subseteq B\left(x_0, r\right).$$

Of course, our real desire is functions that are continuous everywhere.

Definition 1.6.3. Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be metric spaces. It is said that a function $f : \mathcal{X} \to \mathcal{Y}$ is *continuous* (on $\mathcal{X}$) if f is continuous at each point in $\mathcal{X}$. The set of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $\mathcal{C}(\mathcal{X}, \mathcal{Y})$.

We have already seen several continuous functions on $\mathbb{R}$ in previous courses (e.g. polynomials, trigonometric functions, exponentials, etc.). Here are some more unusual examples.

Example 1.6.4. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces.

(1) If $d_{\mathcal{X}}$ is the discrete metric, then any function $f : \mathcal{X} \to \mathcal{Y}$ is continuous.

(2) If $d_{\mathcal{Y}}$ is the discrete metric, then a function $f : \mathcal{X} \to \mathcal{Y}$ is continuous at x_0 if and only if there exists a neighbourhood U of x_0 such that f is constant on U. In particular, if $\mathcal{X} = \mathbb{R}$ and $d_{\mathcal{Y}}$ is the discrete metric, $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if f is constant.

Example 1.6.5. Let $(\mathcal{X}, d)$ be a metric space, let $x_0 \in \mathcal{X}$ be fixed, and define $f : \mathcal{X} \to \mathbb{R}$ by $f(x) = d(x, x_0)$ for all $x \in \mathcal{X}$. Then f is a continuous function (when $\mathbb{R}$ is equipped with the absolute value metric as always). To see this, let $y_0 \in \mathcal{X}$ be a fixed point. To see that f is continuous at y_0, notice for all $x \in \mathcal{X}$ that

$$d(x, x_0) \leq d(x, y_0) + d(y_0, x_0) \qquad \text{and} \qquad d(y_0, x_0) \leq d(x, y_0) + d(x, x_0)$$

so

$$d(x, x_0) - d(y_0, x_0) \leq d(x, y_0) \qquad \text{and} \qquad d(y_0, x_0) - d(x, x_0) \leq d(x, y_0).$$

Hence

$$|f(x) - f(y_0)| \leq |d(x, x_0) - d(y_0, x_0)| \leq d(x, y_0)$$

(in particular, a reverse triangle inequality holds in metric spaces too). Hence, if $\epsilon > 0$ is arbitrary, then by taking $\delta = \epsilon > 0$ we see that if $x \in B(y_0, \delta)$ then $|f(x) - f(y_0)| < \epsilon$. Hence, as $\epsilon > 0$ was arbitrary, f is continuous at y_0. Therefore, as $y_0 \in \mathcal{X}$ was arbitrary, f is continuous on $(\mathcal{X}, d)$.

Similarly, if $(\mathcal{V}, \|\cdot\|)$ is a normed linear space, if $\vec{v}_0 \in \mathcal{V}$, and $f : \mathcal{V} \to \mathbb{R}$ is defined by $f(\vec{v}) = \|\vec{v} - \vec{v}_0\| = d(\vec{v}, \vec{v}_0)$, then f is continuous. In particular, by taking $\vec{v}_0 = \vec{0}$, we see that $\vec{v} \mapsto \|\vec{v}\|$ is a continuous function.

As with continuous functions on $\mathbb{R}$, continuity of functions between metric spaces may be characterized via preservation of convergent sequences. Furthermore, continuity can also be characterized using topological properties.

Theorem 1.6.6. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces, let $f : \mathcal{X} \to \mathcal{Y}$, and let $x_0 \in \mathcal{X}$. The following are equivalent:*

(1) f is continuous at x_0.

(2) For every sequence $(x_n)_{n\geq 1}$ in $\mathcal{X}$ that converges to x_0, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x_0)$.

(3) For every neighbourhood V of $f(x_0)$, $f^{-1}(V)$ is a neighbourhood of x_0.

Proof. To see that (1) implies (2), suppose f is continuous at x_0 and that $(x_n)_{n\geq 1}$ is a sequence in $\mathcal{X}$ that converges to x_0. To see that $(f(x_n))_{n\geq 1}$ converges to $f(x_0)$, let $\epsilon > 0$. Since f is continuous at x_0, there exists a $\delta > 0$ such that if $d_{\mathcal{X}}(x, x_0) < \delta$ then $d_{\mathcal{Y}}(f(x), f(x_0)) < \epsilon$. Since $x_0 = \lim_{n\to\infty} x_n$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \delta$ for all $n \geq N$. Hence $d(f(x_n), f(x_0)) < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we obtain that $f(x_0) = \lim_{n\to\infty} f(x_n)$ as desired.

To see that (2) implies (3), suppose to the contrary that there exists a neighbourhood V of $f(x_0)$ such that $f^{-1}(V)$ is not a neighbourhood of x_0. Since $x_0 \in f^{-1}(V)$ this implies that $B(x_0, \frac{1}{n}) \cap (f^{-1}(V))^c \neq \emptyset$ for all $n \in \mathbb{N}$. For each n choose an element

$$x_n \in B\left(x_0, \frac{1}{n}\right) \cap (f^{-1}(V))^c.$$

Hence $(x_n)_{n\geq 1}$ converges to x_0. Therefore, by the assumption of (2), we obtain that $f(x_0) = \lim_{n\to\infty} f(x_n)$. Since V is a neighbourhood of $f(x_0)$, this implies $f(x_n) \in V$ for some $n \in \mathbb{N}$ which implies $x_n \in f^{-1}(V)$. As $x_n \in (f^{-1}(V))^c$, we have obtained a contradiction. Hence (2) implies (3).

To see that (3) implies (1), let $\epsilon > 0$ be arbitrary. Since $B(f(x_0), \epsilon)$ is a neighbourhood of $f(x_0)$, $f^{-1}(B(f(x_0), \epsilon))$ is a neighbourhood of x_0 by the assumption of (3). Hence there exists a $\delta > 0$ such that

$$B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon)).$$

Thus, if $d(x, x_0) < \delta$ then

$$x \in B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \epsilon))$$

so $f(x) \in B(f(x_0), \epsilon)$ and thus $d(f(x), f(x_0)) < \epsilon$. Hence f is continuous by definition. ∎

In addition to the above we obtain the following characterization of continuity using open sets. As the following characterization makes no use of the metric, one may generalize this result to obtain a definition of continuous functions between any two topological spaces.

Theorem 1.6.7. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(V)$ is open in $\mathcal{X}$ for every open subset V of $\mathcal{Y}$.*

Proof. First, suppose that $f^{-1}(V)$ is open in $\mathcal{X}$ for every open subset V of $\mathcal{Y}$. To see that f is continuous at every point in $\mathcal{X}$, we note we may simply repeat the proof of (3) implies (1) in Theorem 1.6.6 at each point in $\mathcal{X}$.

Conversely, suppose $f : \mathcal{X} \to \mathcal{Y}$ is continuous and let V be an arbitrary open subset of $\mathcal{Y}$. Let $U = f^{-1}(V)$. To see that U is open, let $x \in U$ be arbitrary. Since $f(x) \in V$ and since V is open, V is a neighbourhood of

$f(x)$. Hence, by Theorem 1.6.6 we obtain that U is a neighbourhood of x. Thus $x \in \int(U)$. Therefore, as $x \in U$ was arbitrary, $\int(U) = U$. Hence U is open. ∎

If one desires, one may used closed sets in place of open sets.

Corollary 1.6.8. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if $f^{-1}(F)$ is closed (in $\mathcal{X}$) for every closed subset F of $\mathcal{Y}$.*

Proof. Since $f^{-1}(A^c) = (f^{-1}(A))^c$, the result follows trivially from Theorem 1.6.7. ∎

As with continuous functions on $\mathbb{R}$, composition continuous functions preserves continuity.

Proposition 1.6.9. *Let $(\mathcal{X}, d_{\mathcal{X}})$, $(\mathcal{Y}, d_{\mathcal{Y}})$, and $(\mathcal{Z}, d_{\mathcal{Z}})$ be metric spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be continuous functions. Then $g \circ f : \mathcal{X} \to \mathcal{Z}$ is continuous.*

Proof. To see that $g \circ f$ is continuous, let U be an arbitrary open subset of $\mathcal{Z}$. Notice $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since $g : \mathcal{Y} \to \mathcal{Z}$ is continuous and $U \subseteq \mathcal{Z}$ is open, $g^{-1}(U)$ open in $\mathcal{Y}$ by Theorem 1.6.7. Hence, since $f : \mathcal{X} \to \mathcal{Y}$ is continuous and $g^{-1}(U)$ is open in $\mathcal{Y}$, $f^{-1}(g^{-1}(U))$ open in $\mathcal{X}$ by Theorem 1.6.7. Therefore, as U was arbitrary, $g \circ f : \mathcal{X} \to \mathcal{Z}$ is continuous by Theorem 1.6.7. ∎

Of course, there are many examples continuous functions that may be of use in this course. To study some of these functions, we begin with the following notion.

Definition 1.6.10. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$ be a non-empty set. Given $x \in \mathcal{X}$, the *distance from x to A*, denoted $\mathrm{dist}(x, A)$, is defined to be

$$\mathrm{dist}(x, A) = \inf\{d(x, a) \mid a \in A\}.$$

Example 1.6.11. If $A = \{a\}$ then clearly $d(x, A) = d(x, a)$. Furthermore, if $\mathcal{X}$ is a normed linear space, then $d(x, A) = \|x - a\|$.

As a further example and to exhibit some important properties of $\mathrm{dist}(x, A)$, we note the following.

Lemma 1.6.12. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$ be a non-empty set. For each $x \in \mathcal{X}$, $\mathrm{dist}(x, A) = 0$ if and only if $x \in \overline{A}$. Consequently $\mathrm{dist}(x, A) = \mathrm{dist}(x, \overline{A})$ for all $x \in \mathcal{X}$.*

Proof. Suppose that $\text{dist}(x, A) = 0$. Therefore for all $n \in \mathbb{N}$ there exists an $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. Hence $x = \lim_{n\to\infty} a_n$ so $x \in \lim(A) = \overline{A}$.

Conversely, suppose $x \in \overline{A}$. Hence $x \in \lim(A)$ so there exists a sequence $(a_n)_{n\geq 1}$ of elements of A such that $x = \lim_{n\to\infty} a_n$. Thus $\lim_{n\to\infty} d(x, a_n) = 0$ so $\text{dist}(x, A) = 0$.

As $A \subseteq \overline{A}$, we clearly obtain that $\text{dist}(x, \overline{A}) \leq \text{dist}(x, A)$ for all $x \in \mathcal{X}$. To see the other inequality, fix $x \in \mathcal{X}$. Let $\epsilon > 0$ be arbitrary. By the definition of the distance, there exists an $y \in \overline{A}$ such that $d(x, y) \leq \text{dist}(x, \overline{A}) + \epsilon$. However, since $y \in \overline{A}$ there exists an $a \in A$ such that $d(y, a) < \epsilon$. Hence

$$d(x, a) \leq d(x, y) + d(y, a) \leq \text{dist}(x, \overline{A}) + 2\epsilon.$$

Thus, as $a \in A$,

$$\text{dist}(x, A) \leq \text{dist}(x, \overline{A}) + 2\epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\text{dist}(x, A) \leq \text{dist}(x, \overline{A})$ thereby completing the proof. ∎

Next we demonstrate the continuity of the distance function to a set. In particular, by applying the following to the examples contained in Example 1.6.11, we generalize Example 1.6.5.

Theorem 1.6.13. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$ be a non-empty set. The function $F : \mathcal{X} \to \mathbb{R}$ defined by $F(x) = \text{dist}(x, A)$ for all $x \in X$ is continuous.*

Proof. To see that F is continuous, let $x, y \in \mathcal{X}$ be arbitrary. If $\delta > 0$, then by the definition of the distance there exists an $a \in A$ such that $d(x, a) \leq \text{dist}(x, A) + \delta$. Therefore

$$\text{dist}(y, A) \leq d(y, a) \leq d(x, y) + d(x, a) \leq d(x, y) + \text{dist}(x, A) + \delta.$$

As the above inequality holds for all $\delta > 0$, we obtain that $F(y) \leq F(x) + d(x, y)$. By reversing the roles of x and y, we obtain that $F(x) \leq F(y) + d(x, y)$ and hence $|F(x) - F(y)| \leq d(x, y)$.

To see now that F is continuous, fix $x_0 \in \mathcal{X}$ and let $\epsilon > 0$ be arbitrary. Let $\delta = \epsilon > 0$. Therefore, if $y \in \mathcal{X}$ is such that $d(x_0, y) < \delta$ then $|F(x_0) - F(y)| \leq d(x_0, y) < \delta = \epsilon$. Hence F is continuous at x_0. Therefore, as x_0 was arbitrary, F is continuous as desired. ∎

Using the functions from Theorem 1.6.13, we can prove the metric space version of the following theorem quite easily.

Theorem 1.6.14 (Urysohn's Lemma). *Let $(\mathcal{X}, d)$ be a metric space and let B and C be two non-empty disjoint closed subsets of $\mathcal{X}$. There exists a continuous function $f : \mathcal{X} \to [0, 1]$ such that $f(x) = 0$ if $x \in B$, $f(x) = 1$ if $x \in C$, and $0 < f(x) < 1$ if $x \notin B \cup C$.*

Proof. Consider the function $f : \mathcal{X} \to \mathbb{R}$ defined by

$$f(x) = \frac{\mathrm{dist}(x, B)}{\mathrm{dist}(x, B) + \mathrm{dist}(x, C)}.$$

for all $x \in \mathcal{X}$. We claim that f is well-defined; that is, the denominator never vanishes. To see this, suppose to the contrary that there exists an $x \in \mathcal{X}$ such that $\mathrm{dist}(x, B) + \mathrm{dist}(x, C) = 0$. Thus $\mathrm{dist}(x, B) = \mathrm{dist}(x, C) = 0$ so by part (a), $x \in \overline{B} = B$ and $x \in \overline{C} = C$ as B and C are closed. Therefore, as $B \cap C = \emptyset$, we have obtained a contradiction. Hence f is well-defined.

Next, clearly $f(x) \geq 0$ for all $x \in \mathcal{X}$. Since

$$0 \leq \mathrm{dist}(x, B) \leq \mathrm{dist}(x, B) + \mathrm{dist}(x, C)$$

we see that $f : \mathcal{X} \to [0, 1]$. Furthermore, by Theorem 1.6.13 and elementary properties of continuous functions, f is continuous (i.e. if $(x_n)_{n\geq1}$ converges to x in $\mathcal{X}$, then $(\mathrm{dist}(x_n, B))_{n\geq1}$ converges to $\mathrm{dist}(x, B)$ by Theorem 1.6.13 and $(\mathrm{dist}(x_n, C))_{n\geq1}$ converges to $\mathrm{dist}(x, C)$ by Theorem 1.6.13. Since $\mathrm{dist}(x, B)+ \mathrm{dist}(x, C) \neq 0$, we obtain by elementary properties of convergent sequences of real numbers that $(f(x_n))_{n\geq1}$ converges to $f(x)$. Hence f is continuous).

To complete the proof, first notice that $f(x) = 0$ if and only if $\mathrm{dist}(x, B) = 0$ if and only if $x \in B$ by Lemma 1.6.12. Similarly $f(x) = 1$ if and only if $\mathrm{dist}(x, B) = \mathrm{dist}(x, B) + \mathrm{dist}(x, C)$ if and only if $\mathrm{dist}(x, C) = 0$ if and only if $x \in C$ Lemma 1.6.12. Since $f : \mathcal{X} \to [0, 1]$, we obtain that $0 < f(x) < 1$ for all $x \notin B \cup C$ thereby completing the proof. ∎

In addition to having continuous functions, it is also quite often useful to have the following explicit description of the points of discontinuity of a function.

Theorem 1.6.15. *Let $f : \mathbb{R} \to \mathbb{R}$ and let*

$$D(f) = \{x \in \mathbb{R} \mid f \text{ is discontinuous at } x\}.$$

For each $n \in \mathbb{N}$ let

$$D_n(f) = \left\{ x \in \mathbb{R} \,\middle|\, \begin{array}{c} \text{for every } \delta > 0 \text{ there exists } y, z \text{ such that} \\ |x - y| < \delta, |x - z| < \delta, \text{ and } |f(y) - f(z)| \geq \frac{1}{n} \end{array} \right\}.$$

Then $D_n(f)$ is closed for each $n \in \mathbb{N}$ and $D(f) = \bigcup_{n=1}^{\infty} D_n(f)$. Hence the discontinuities of f is a countable union of closed sets.

Proof. Fix $m \in \mathbb{N}$. To see that $D_m(f)$ is closed, let $(x_n)_{n\geq1}$ be an arbitrary sequence of elements of $D_m(f)$ that converges to some $x \in \mathbb{R}$. To see that $x \in D_m(f)$, let $\delta > 0$ be arbitrary. Since $x = \lim_{n\to\infty} x_n$, there exists an $N \in \mathbb{N}$ such that $|x - x_N| < \frac{1}{2}\delta$. Furthermore, since $x_N \in D_m(f)$, there exists $y, z \in \mathbb{R}$ such that $|x_N - y| < \frac{1}{2}\delta$, $|x_N - z| < \frac{1}{2}\delta$, and $|f(y) - f(z)| \geq \frac{1}{m}$. As

$|x-y| < \delta$ and $|x-z| < \delta$ by the triangle inequality, and as $|f(y)-f(z)| \geq \frac{1}{m}$, we obtain that $x \in D_m(f)$ as $\delta > 0$ was arbitrary. Hence, as $(x_n)_{n\geq 1}$ was arbitrary, $D_m(f)$ is closed.

To see that $D(f) = \bigcup_{n=1}^{\infty} D_n(f)$, first suppose $x \in \bigcup_{n=1}^{\infty} D_n(f)$. Hence $x \in D_m(f)$ for some $m \in \mathbb{N}$. To see that f is discontinuous at x, suppose to the contrary that f is continuous at x. Notice by the definition of $D_m(f)$ that for each $n \in \mathbb{N}$ there exists points $y_n, z_n \in \mathbb{R}$ such that $|x - y_n| < \frac{1}{n}$, $|x - z_n| < \frac{1}{n}$, and $|f(y_n) - f(z_n)| \geq \frac{1}{m}$. Since $x = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$, the continuity of f implies $f(x) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(z_n)$, which contradicts the fact that $|f(y_n) - f(z_n)| \geq \frac{1}{m}$ for all n. Hence we have obtained a contradiction so $x \in D(f)$. Hence $\bigcup_{n=1}^{\infty} D_n(f) \subseteq D(f)$.

For the other inclusion, notice if $x \in D(f)$ then f is discontinuous at x. Therefore there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists a $y \in \mathbb{R}$ such that $|x - y| < \delta$ yet $|f(x) - f(y)| \geq \epsilon$. Choose $m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$. By taking $z = x$ in the definition of $D_m(f)$, we see that $x \in D_m(f)$. Hence, as x was arbitrary, $D(f) \subseteq \bigcup_{n=1}^{\infty} D_n(f)$ thereby completing the proof. ∎

1.7 Metric Spaces of Continuous Functions

As we have seen that $\mathcal{C}[a, b]$ is a metric space with respect to the uniform metric, it is natural to ask whether the same holds for the set of continuous functions between two metric spaces. Unfortunately, the set of continuous functions between two metric spaces need not be a 'nice' metric space. Of course we may place the discrete metric on any set, but for continuous functions we would like a non-trivial metric such that the distance between two functions is related to the pointwise distance between the functions. The issue with generalizing the uniform metric from $\mathcal{C}[a, b]$ to this context is that the supremum used need not be finite. In particular, we will need some notion of boundedness for our functions in order to generalize the uniform metric.

Definition 1.7.1. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. It is said that A is *bounded* if there exists an $x \in \mathcal{X}$ such that

$$\sup\{d(x, a) \mid a \in A\} < \infty.$$

Remark 1.7.2. Since for all $y \in \mathcal{X}$ we have

$$d(y, a) \leq d(y, x) + d(x, a),$$

the choice of x does not matter in Definition 1.7.1. Hence, if $\mathcal{X}$ is a normed linear space, we may choose $x = \vec{0}$ to obtain that A is bounded if and only if

$$\sup\{\|a\|_{\mathcal{X}} \mid a \in A\} < \infty.$$

Thus we can truly see that this is a good notion of a bounded set in a metric space.

To obtain a good metric space of continuous functions, we will restrict ourselves to the following continuous functions.

Definition 1.7.3. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *bounded* if $f(\mathcal{X})$ is a bounded set in $\mathcal{Y}$. The set of all bounded continuous functions $f : \mathcal{X} \to \mathcal{Y}$ is denoted $\mathcal{C}_b(\mathcal{X}, \mathcal{Y})$.

Example 1.7.4. If $\mathcal{X} = \mathbb{N}$ and $\mathcal{Y} = \mathbb{K}$ equipped with the discrete metric and absolute value metric respectively, then $\mathcal{C}_b(\mathcal{X}, \mathcal{Y}) = \ell_\infty(\mathbb{N}, \mathbb{K})$ via the map $f \mapsto (f(n))_{n \geq 1}$.

Of course, once we restrict to bounded functions, we can easily generalize the uniform metric.

Theorem 1.7.5. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. Then $\mathcal{C}_b(\mathcal{X}, \mathcal{Y})$ is a metric space with the metric*

$$d(f, g) = \sup\{d_{\mathcal{Y}}(f(x), g(x)) \mid x \in \mathcal{X}\}.$$

We call d the uniform metric*.*

Proof. First, given $f, g \in \mathcal{C}_b(\mathcal{X}, \mathcal{Y})$, to see that $d(f, g) < \infty$, we note there exists an $a \in Y$ such that

$$\sup\{d_{\mathcal{Y}}(f(x), a) \mid x \in \mathcal{X}\} < \infty \quad \text{and} \quad \sup\{d_{\mathcal{Y}}(g(x), a) \mid x \in \mathcal{X}\} < \infty.$$

From this it clearly follows from the triangle inequality on $d_{\mathcal{Y}}$ that $d(f, g) < \infty$. The remaining properties of a metric are trivial to verify. ∎

Of course, with continuous functions on $\mathbb{R}$, the sum of continuous functions is continuous and a scalar multiple of continuous functions is continuous. This means that continuous functions on $\mathbb{R}$ are a vector space. To repeat these ideas for $\mathcal{C}_b(\mathcal{X}, \mathcal{Y})$ is only possible if $\mathcal{Y}$ is a normed linear space. This yields the following thereby generalizing the sup norm on $\mathcal{C}[a, b]$.

Theorem 1.7.6. *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space and let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a normed linear space over $\mathbb{K}$. Then $\mathcal{C}_b(\mathcal{X}, \mathcal{Y})$ is a normed linear space over $\mathbb{K}$ with the operations of pointwise addition and scalar multiplication, and the norm*

$$\|f\|_\infty = \sup\{\|f(x)\|_{\mathcal{Y}} \mid x \in \mathcal{X}\}.$$

The norm $\|\cdot\|_\infty$ is called the supremum norm*.*

Proof. If $f, g : \mathcal{X} \to \mathcal{Y}$ are continuous functions, then one can verify that $f + g$ and αf are continuous for all $\alpha \in \mathbb{K}$ by using part (2) of Theorem 1.6.6 together with Proposition 1.4.11. If f and g are bounded, the properties of $\|\cdot\|_{\mathcal{Y}}$ easily imply that $f + g$ and αf are bounded. Hence $\mathcal{C}_b(\mathcal{X}, \mathcal{Y})$ is a vector space over $\mathbb{K}$. The fact that $\|\cdot\|_\infty$ is a norm easily follows (with the proof that it is finite following as in Theorem 1.7.5). ∎

To complete this section, we desire to analyze continuity in the context of normed linear spaces. In particular, the 'nice' maps between vector spaces are the linear maps as these are precisely the functions that preserve the vector space operations. Thus we desire to study when a linear map between normed linear spaces is continuous. To do this, as linear maps will clearly not be bounded as defined above (i.e. if their range contains a non-zero vector, then by linearity we can scale that vector in the range to have arbitrarily large norm), we need to modify the definition of boundedness.

Definition 1.7.7. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces over $\mathbb{K}$. A linear map $T : \mathcal{X} \to \mathcal{Y}$ is said to be *bounded* if

$$\sup\left\{\|T(\vec{x})\|_{\mathcal{Y}} \mid \vec{x} \in \mathcal{X}, \|\vec{x}\|_{\mathcal{X}} \leq 1\right\} < \infty.$$

If T is bounded, then we write

$$\|T\| = \sup\{\|T(\vec{x})\|_{\mathcal{Y}} \mid \vec{x} \in \mathcal{X}, \|\vec{x}\|_{\mathcal{X}} \leq 1\}.$$

The quantity $\|T\|$ is called the *operator norm* of T. Furthermore, the set of bounded linear maps from $\mathcal{X}$ to $\mathcal{Y}$ is denoted $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Remark 1.7.8. Note we can only discuss bounded linear maps between normed linear spaces over the same field. Thus throughout these notes, this will be a standing assumption when discussing bounded linear maps.

In addition, note that $\|T\|$ is a measure of how large the unit ball (the ball of radius 1 centred at $\vec{0}$) in $\mathcal{X}$ is scaled by applying T.

To see that the operator norm is indeed a norm, we note that the only non-trivial property of Definition 1.1.14 to verify is that if $\|T\| = 0$, then T is the zero linear map. Note the following lemma yields the result.

Lemma 1.7.9. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces over $\mathbb{K}$ and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then*

$$\|T(\vec{x})\|_{\mathcal{Y}} \leq \|T\| \, \|\vec{x}\|_{\mathcal{X}}$$

for all $\vec{x} \in \mathcal{X}$.

Proof. Since $\left\|T(\vec{0})\right\|_{\mathcal{Y}} = \left\|\vec{0}\right\|_{\mathcal{X}} = 0$, the result holds when $\vec{x} = \vec{0}$. If $\vec{x} \neq \vec{0}$, then $\|\vec{x}\|_{\mathcal{X}} \neq 0$. Consequently, as

$$\left\|\frac{1}{\|\vec{x}\|_{\mathcal{X}}}\vec{x}\right\|_{\mathcal{X}} = \frac{1}{\|\vec{x}\|_{\mathcal{X}}} \|\vec{x}\|_{\mathcal{X}} = 1,$$

we obtain from the definition of the operator norm that

$$\frac{1}{\|\vec{x}\|_{\mathcal{X}}} \|T(\vec{x})\|_{\mathcal{Y}} = \left\|\frac{1}{\|\vec{x}\|_{\mathcal{X}}} T(\vec{x})\right\|_{\mathcal{Y}} = \left\|T\left(\frac{1}{\|\vec{x}\|_{\mathcal{X}}}\vec{x}\right)\right\|_{\mathcal{Y}} \leq \|T\|.$$

Therefore $\|T(\vec{x})\|_{\mathcal{Y}} \leq \|T\| \, \|\vec{x}\|_{\mathcal{X}}$ as desired. ∎

Corollary 1.7.10. *Let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *and* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *be normed linear spaces over* $\mathbb{K}$. *Then* $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ *is a normed linear space over* $\mathbb{K}$ *with the operator norm as defined in Definition 1.7.7.*

The reason we have been analyzing bounded linear maps in reference to continuous function is that $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is all continuous linear functions from $\mathcal{X}$ to $\mathcal{Y}$.

Theorem 1.7.11. *Let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *and* $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ *be normed linear spaces over* $\mathbb{K}$ *and let* $T : \mathcal{X} \to \mathcal{Y}$ *be linear. The following are equivalent:*

(1) T *is continuous.*

(2) T *is continuous at* 0.

(3) T *is bounded.*

Proof. Clearly (1) implies (2). To see that (2) implies (3), let $\epsilon = 1$. Since T is continuous at 0, there exists a $\delta > 0$ such that if $\|\vec{x}\|_{\mathcal{X}} \leq \delta$ then $\|T(\vec{x})\|_{\mathcal{Y}} \leq 1$. Therefore, if $\vec{x} \in \mathcal{X}$ is such that $\|\vec{x}\|_{\mathcal{X}} \leq 1$, then $\|\delta\vec{x}\|_{\mathcal{X}} \leq \delta$ so

$$\delta \|T(\vec{x})\|_{\mathcal{Y}} = \|\delta T(\vec{x})\|_{\mathcal{Y}} = \|T(\delta\vec{x})\|_{\mathcal{Y}} \leq 1.$$

Hence $\|\vec{x}\|_{\mathcal{X}} \leq 1$ implies $\|T(\vec{x})\|_{\mathcal{Y}} \leq \delta^{-1}$ so T is bounded with $\|T\| \leq \delta^{-1}$ by definition.

To see that (3) implies (1), let $\vec{x}_0 \in \mathcal{X}$ be arbitrary. To see that T is continuous at x, let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{\|T\|+1} > 0$. If $\vec{x} \in \mathcal{X}$ is such that $\|\vec{x} - \vec{x}_0\|_{\mathcal{X}} < \delta$, then Lemma 1.7.9 implies that

$$\|T(\vec{x}) - T(\vec{x}_0)\|_{\mathcal{Y}} = \|T(\vec{x} - \vec{x}_0)\|_{\mathcal{Y}} \leq \|T\| \, \|\vec{x} - \vec{x}_0\|_{\mathcal{X}} < \|T\| \, \frac{\epsilon}{\|T\| + 1} < \epsilon.$$

Therefore T is continuous at $\vec{x}_0$ as $\epsilon > 0$ was arbitrary. Therefore, as $\vec{x}_0 \in \mathcal{X}$ was arbitrary, T is continuous on $\mathcal{X}$. ∎

Perhaps it is surprising at this point in the course, but $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is one of the most important normed linear spaces! Thus it is perhaps useful to include some examples. Note it is often quite difficult to actually compute the operator norm of a bounded linear map.

Example 1.7.12. Let $x_0 \in [0, 1]$. Define $T_{x_0} : C[0, 1] \to \mathbb{R}$ by

$$T_{x_0}(f) = f(x_0)$$

for all $f \in C[0, 1]$. Clearly T_{x_0} is a linear map. If $C[0, 1]$ is equipped with the ∞-norm, then T_{x_0} is bounded with $\|T_{x_0}\| = 1$. To see this, notice for all $f \in C[0, 1]$ that

$$|T_{x_0}(f)| = |f(x_0)| \leq \|f\|_\infty$$

by definition of the ∞-norm. Hence T_{x_0} is bounded with $\|T_{x_0}\| \leq 1$. To see that $\|T_{x_0}\| = 1$, notice the function $g(x) = 1$ for all $x \in [0,1]$ is an element of $C[0,1]$ with $\|g\|_\infty = 1$. Since

$$|T_{x_0}(g)| = |g(x_0)| = 1,$$

However, if $C[0,1]$ is equipped with the 1-norm, then T_{x_0} is not bounded. To see this, for each $n \in \mathbb{N}$, define $f_n \in C[0,1]$ by

$$f_n(x) = \begin{cases} 2n\left(x - \left(x_0 - \frac{1}{2n}\right)\right) & \text{if } x \in \left[x_0 - \frac{1}{2n}, x_0\right] \\ -2n\left(x - \left(x_0 + \frac{1}{2n}\right)\right) & \text{if } x \in \left[x_0, x_0 + \frac{1}{2n}\right] \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in [0,1]$. It is not difficult to see that

$$\|f_n\|_1 = \int_0^1 |f(x)|\, dx \leq 1$$

regardless of the value of x_0 (in fact, if $x_0 \in \{0,1\}$ then $\|f_n\|_1 = \frac{1}{2}$ for all $n \in \mathbb{N}$, and if $x_0 \notin \{0,1\}$ then $\|f_n\|_1 = 1$ for sufficiently large n). Furthermore, as

$$|T_{x_0}(f_n)| = |f_n(x_0)| = 2n$$

we obtain that

$$\sup\{|T_{x_0}(f)| \mid f \in C[0,1], \|f\|_1 \leq 1\} = \infty$$

so T_{x_0} is unbounded.

Example 1.7.13. Let $p, q \in [1,\infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $\vec{y} = (y_n)_{n\geq 1} \in \ell_p(\mathbb{N}, \mathbb{R})$. Define $T : \ell_q(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$ by

$$T((x_n)_{n\geq 1}) = \sum_{n=1}^{\infty} x_n y_n$$

for all $(x_n)_{n\geq 1} \in \ell_q(\mathbb{N}, \mathbb{R})$. We claim that T is a well-defined bounded linear map with $\|T\| = \|\vec{y}\|_p$. To see this, first, note as $\frac{1}{p} + \frac{1}{q} = 1$ that $p = \frac{q}{q-1}$. To see that T is well-defined, note since $\vec{y} = (y_n)_{n\geq 1} \in \ell_p(\mathbb{N}, \mathbb{R})$ we have for all $(x_n)_{n\geq 1} \in \ell_q(\mathbb{N}, \mathbb{R})$ that $(x_n y_n)_{n\geq 1} \in \ell_1(\mathbb{N}, \mathbb{R})$ by Hölders' inequality so that

$$\sum_{n=1}^{\infty} |x_n y_n| < \infty$$

and thus, as $\mathbb{R}$ is complete and thus absolutely summable series converge, we have that

$$T((x_n)_{n\geq 1}) = \sum_{n=1}^{\infty} x_n y_n$$

is a well-defined element of $\mathbb{R}$. Hence T is well-defined. Furthermore, the fact that T is linear follows from basic properties of convergent series.

To see that T is bounded, notice for all $(x_n)_{n\geq 1} \in \ell_q(\mathbb{N}, \mathbb{R})$ that

$$|T((x_n)_{n\geq 1})| = \left|\sum_{n=1}^{\infty} x_n y_n\right| \leq \sum_{n=1}^{\infty} |x_n y_n| \leq \left\|(x_n)_{n\geq 1}\right\|_q \left\|\vec{y}\right\|_p$$

by Hölder's inequality. Hence we easily see that T is bounded and $\|T\| \leq \|\vec{y}\|_p$.

To obtain equality, we first notice if $\vec{y} = \vec{0}$, then clearly the inequality holds. Thus we may assume that $\|\vec{y}\|_p > 0$. For each $n \in \mathbb{N}$ let

$$x_n = \begin{cases} 0 & \text{if } y_n = 0 \\ y_n(|y_n|)^{\frac{q}{q-1}-2} & \text{if } y_n \neq 0 \end{cases}.$$

Clearly $(x_n)_{n\geq 1}$ is a well-defined sequence. We claim that $(x_n)_{n\geq 1} \in \ell_q(\mathbb{N})$. To see this, we notice that

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n|^q &= \sum_{n=1}^{\infty} |y_n|^{q+\frac{q^2}{q-1}-2q} \\ &= \sum_{n=1}^{\infty} |y_n|^{\frac{q}{q-1}} \\ &= \sum_{n=1}^{\infty} |y_n|^p < \infty \end{aligned}$$

as $\frac{1}{p} + \frac{1}{q} = 1$. Hence $(x_n)_{n\geq 1} \in \ell_q(\mathbb{N})$. Moreover, we see that

$$\left\|(x_n)_{n\geq 1}\right\|_q = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{q}} = \|\vec{y}\|_p^{\frac{p}{q}}.$$

Let $\vec{z} = \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}}(x_n)_{n\geq 1}$. Hence $\vec{z} \in \ell_q(\mathbb{N}, \mathbb{R})$ and $\|\vec{z}\|_q = 1$ by construction.

Moreover

$$\begin{aligned}
T(\vec{z}) &= \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}} \sum_{n=1}^{\infty} y_n x_n \\
&= \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}} \sum_{n=1}^{\infty} y_n^2 (|y_n|)^{\frac{q}{q-1}-2} \\
&= \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}} \sum_{n=1}^{\infty} |y_n|^{\frac{q}{q-1}} \\
&= \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}} \sum_{n=1}^{\infty} |y_n|^p \\
&= \frac{1}{\|\vec{y}\|_p^{\frac{p}{q}}} \|\vec{y}\|_p^p \\
&= \|\vec{y}\|_p^{p\left(1-\frac{1}{q}\right)} = \|\vec{y}\|_p .
\end{aligned}$$

Hence $\|T\| \geq \|\vec{y}\|_p$ thereby completing the question.

1.8 Connected Sets

To complete this chapter, we will generalize an essential result from a previous analysis course: the Intermediate Value Theorem. Recall the Intermediate Value Theorem states that if $f : [a, b] \to \mathbb{R}$ is a continuous function and $f(a) < y < f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = y$. This theorem can be rephrased using a specific type of set.

Definition 1.8.1. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. It is said that A is *disconnected* if there exists two open disjoint subsets U and V of $\mathcal{X}$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, and $A \subseteq U \cup V$.

It is said that A is *connected* if A is not disconnected.

Notice that the empty set is vacuously not disconnected and thus is connected. For some more concrete examples, we prove the following.

Lemma 1.8.2. *A subset A of $\mathbb{R}$ is connected if and only if whenever $x, y \in A$ are such that $x < y$, then $[x, y] \subseteq A$. In particular, a subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Proof. Suppose there exists $x, y \in A$ such that $x < y$ yet $[x, y] \not\subseteq A$. Hence there exists a $z \in (x, y)$ such that $z \notin A$. Let $U = (-\infty, z)$ and $V = (z, \infty)$. Clearly U and V are non-empty open subsets of of $\mathbb{R}$ such that $x \in U \cap A$, $y \in V \cap A$, and $U \cup V = \mathbb{R} \setminus \{z\} \supseteq A$. Hence A is disconnected.

Conversely, suppose A is disconnected. Therefore there exists non-empty disjoint open subsets U and V of $\mathcal{X}$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, and

$A \subseteq U \cup V$. Let $x \in U \cap A$ and $y \in V \cap A$. By interchanging the roles of U and V if necessary, we may assume that $x < y$. We desire to show that $[x, y] \not\subseteq A$.

Let

$$z = \sup\{u \in U \mid x < u < y\}.$$

Notice that the set in the above supremum is non-empty since $x \in U$ and U is an open set so there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$.

We claim that $z \notin U$. To see this, suppose to the contrary that $z \in U$. Since $y \in V$ and $U \cap V = \emptyset$, it must be the case that $x < z < y$. However, as $z < y$, $z \in U$, and U is open, there exists a $\delta > 0$ such that $(z - \delta, z + \delta) \subseteq U$ so

$$z < \min\left\{y, z + \frac{1}{2}\delta\right\} \leq \sup\{u \in U \mid x < u < y\} = z$$

which is a contraction. Hence $z \notin U$.

Next, we claim that $z \notin V$. To see this, suppose to the contrary that $z \in V$. Then there exists an $r > 0$ such that $(z - r, z + r) \subseteq V$. However, by the definition of z there must exists a $u \in U$ such that $z - \frac{1}{2}r \leq u$, which implies $u \in (z - r, z + r) \subseteq V$. As this contradicts the fact that $U \cap V = \emptyset$, we obtain that $z \notin V$.

By the above, we see that $z \notin U \cup V$. As $A \subseteq U \cup V$, we obtain that $z \notin A$. Therefore, as $z \in [x, y]$, we obtain that $[x, y] \not\subseteq A$ as desired. ∎

To relate this to the Intermediate Value Theorem, we need to consider the notion of connectedness in the metric space $([a, b], |\cdot|)$. We note the following which implies that $[a, b]$ is connected since $[a, b]$ is a connected subset of $\mathbb{R}$.

Lemma 1.8.3. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. Then A is connected as a subset of $(\mathcal{X}, d)$ if and only if A is connected as a subset of $(A, d|A)$.*

Proof. Suppose A is a disconnected subset of $(\mathcal{X}, d)$. Therefore there exists there exists two open disjoint subsets U and V of $(\mathcal{X}, d)$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, and $A \subseteq U \cup V$. Since $U \cap A$ and $V \cap A$ are open subsets of $(A, d|_A)$ by Proposition 1.3.15, clearly A is a disconnected subset of $(\mathcal{X}, d)$.

Suppose A is a disconnected subset of $(A, d|_A)$. Therefore there exists two non-empty open disjoint subsets U and V of $(A, d|_A)$ such that $U \cup V = A$. The issue with simply using Proposition 1.3.15 is that the open subsets of $(\mathcal{X}, d)$ obtained may not be disjoint. Thus we need to carefully analyze the proof of Proposition 1.3.15.

Since U is open in $(A, d|_A)$ for each $u \in U$ there exists an $\epsilon_u > 0$ such that $B_A(u, \epsilon_u) = B_{\mathcal{X}}(u, \epsilon_u) \cap A \subseteq U$. Similarly, for each $v \in V$ there exists an $\epsilon_v > 0$ such that $B_{\mathcal{X}}(v, \epsilon_v) \cap A \subseteq V$. We claim that $B_{\mathcal{X}}\left(u, \frac{\epsilon_u}{2}\right) \cap B_{\mathcal{X}}\left(v, \frac{\epsilon_v}{2}\right) = \emptyset$.

To see this, suppose to the contrary that $x \in B_{\mathcal{X}}\left(u, \frac{\epsilon_u}{2}\right) \cap B_{\mathcal{X}}\left(v, \frac{\epsilon_v}{2}\right)$. Hence

$$d(u,v) \leq d(u,x) + d(x,v) \leq \frac{\epsilon_u}{2} + \frac{\epsilon_v}{2} \leq \min\{\epsilon_u, \epsilon_v\}.$$

Therefore, either $u \in V$ (when $\epsilon_u \leq \epsilon_v$) or $v \in U$ (when $\epsilon_v \leq \epsilon_u$), which contradicts the fact that U and V are disjoint. Hence the claim has been show.

Thus

$$U' = \bigcup_{u \in U} B_{\mathcal{X}}\left(u, \frac{\epsilon_u}{2}\right) \qquad \text{and} \qquad V' = \bigcup_{v \in V} B_{\mathcal{X}}\left(v, \frac{\epsilon_v}{2}\right)$$

are disjoint open subsets of $(\mathcal{X}, d)$ such that $U' \cap A = U \neq \emptyset$, $V' \cap A = V \neq \emptyset$, and $A \subseteq U \cup V \subseteq U' \cup V'$. Hence A is a disconnected subset in $(\mathcal{X}, d)$ as desired. ∎

Finally we arrive at the true version of the Intermediate Value Theorem.

Theorem 1.8.4 (Intermediate Value Theorem). *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. If $f : \mathcal{X} \to \mathcal{Y}$ is continuous and $\mathcal{X}$ is connected, then $f(\mathcal{X})$ is connected.*

Proof. Let $f : \mathcal{X} \to \mathcal{Y}$ be continuous. Suppose $f(\mathcal{X})$ is disconnected. Then there exists two non-empty disjoint open subsets U and V of $\mathcal{Y}$ such that $f(\mathcal{X}) \cap U \neq \emptyset$, $f(\mathcal{X}) \cap V \neq \emptyset$, and $f(\mathcal{X}) \subseteq U \cap V$. Let $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$. Since f is continuous, U' and V' are open subsets of $\mathcal{X}$. Furthermore, since $f(\mathcal{X}) \cap U \neq \emptyset$ and $f(\mathcal{X}) \cap V \neq \emptyset$, U' and V' are non-empty. Finally, since $f(\mathcal{X}) \subseteq U \cap V$, we obtain that $U' \cup V' = \mathcal{X}$. Hence $\mathcal{X}$ is disconnected as desired. ∎

Chapter 2

Completeness

As we have seen in the previous chapter, there are many exotic examples of metric and normed linear spaces. Consequently, there are a diverse collection of behaviours in metric spaces. For example, we have seen via Example 1.4.9 that the only sequences that converge in the discrete metric are eventually constant. This is possibly a undesirable behaviour as it limits the ability for us to approximate quantities; that is, we are back in the setting of having to compute things exact. Thus, it is natural to ask what properties do we wish to impose on metric spaces in order for there to be a rich analytic theory.

Perhaps unsurprisingly, the most powerful analysis will come from restricting to spaces that have similar properties to those observed in $\mathbb{R}$. One interesting properties that $\mathbb{R}$ has relates to determining when sequences converge. A priori, in order to determine when a sequence converges, one must first know the limit and prove that the sequence converges to the limit. In this chapter, we will examine when we may deduce a sequence converges without knowing its limit. This 'completeness' of a metric space will enable several results related to series and continuous functions to be developed and will be a major assumption require of most important results in this course.

2.1 Cauchy Sequences

As seen in previous analysis courses, one major obstruction in verifying a sequence converges is that one needs a prior to know the limit of the sequence as only then can one verify Definition 1.4.1. Consequently, it is useful to have an alternate method of determining a sequence converges without knowing the limit.

This leads us to a previously seen concept for sequences in $\mathbb{R}$. In order for a sequence to converge, given any $\epsilon > 0$ all the elements of the sequence must be within ϵ of their limit. In particular, this means that the terms in the sequence must eventually be within 2ϵ of each other. Thus we define the following.

Definition 2.1.1. Let $(\mathcal{X}, d)$ be a metric space. A sequence $(x_n)_{n\geq 1}$ is said to be *Cauchy* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Remark 2.1.2. There exists sequences $(x_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

that are not Cauchy. Indeed let $x_n = \sum_{k=1}^{n} \frac{1}{k}$ for all $n \in \mathbb{N}$. Clearly $d(x_n, x_{n+1}) = \frac{1}{n+1}$ yet $(x_n)_{n\geq 1}$ is not Cauchy as for all $m \in \mathbb{N}$

$$\sup_{m\to\infty} d(x_n, x_m) = \sup_{m\to\infty} \sum_{k=n}^{m} \frac{1}{k} = \infty.$$

As with Cauchy sequences in $\mathbb{R}$, there are immediately sequences we can deduce are not Cauchy.

Lemma 2.1.3. *Every Cauchy sequence in a metric space is bounded.*

Proof. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in a metric space $(\mathcal{X}, d)$. Since $(x_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for all $n, m \geq N$. Let

$$M = \max\{d(x_1, x_N), \ldots, d(x_{N-1}, x_N), 1\}.$$

Using the above paragraph, we see that $d(x_n, x_N) \leq M$ for all $n \in \mathbb{N}$. Hence $(x_n)_{n\geq 1}$ is bounded. ∎

Furthermore, we have already seen several examples of Cauchy sequences.

Lemma 2.1.4. *Every convergent sequence in a metric space is Cauchy.*

Proof. Let $(x_n)_{n\geq 1}$ be a convergent sequence in a metric space $(\mathcal{X}, d)$ and let $x_0 = \lim_{n\to\infty} x_n$. To see that $(x_n)_{n\geq 1}$ is Cauchy, let $\epsilon > 0$ be arbitrary. Since $x_0 = \lim_{n\to\infty} x_n$, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_0) < \frac{\epsilon}{2}$ for all $n \geq N$. Therefore, for all $n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x_0) + d(x_0, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, as $\epsilon > 0$ was arbitrary, $(x_n)_{n\geq 1}$ is Cauchy by definition. ∎

Corollary 2.1.5. *Every convergent sequence in a metric spaces is bounded.*

Of course, it would be nice if the converse Lemma 2.1.4 were true as it would enable us to determine that sequences converge without knowing their limits as we would simply need to verify that they are Cauchy. As perhaps this is not true in every metric space, we make the following definition.

Definition 2.1.6. A metric space $(\mathcal{X}, d)$ is said to be *complete* if every Cauchy sequence converges.

Any metric space with the discrete metric is complete as any Cauchy sequence with respect to the discrete metric is eventually constant. Furthermore $\mathbb{R}$ is complete. We will quickly recall the proof that $\mathbb{R}$ is complete by beginning with the following result which holds in any metric space.

Lemma 2.1.7. *Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in a metric space $(\mathcal{X}, d)$. If a subsequence of $(x_n)_{n\geq 1}$ converges, then $(x_n)_{n\geq 1}$ converges.*

Proof. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence with a convergent subsequence $(x_{k_n})_{n\geq 1}$ and let $x_0 = \lim_{n\to\infty} x_{k_n}$. We claim that $\lim_{n\to\infty} x_n = x_0$. To see this, let $\epsilon > 0$ be arbitrary. Since $(x_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq N$. Furthermore, since $x_0 = \lim_{n\to\infty} x_{k_n}$, there exists an $k_j \geq N$ such that $d(x_{k_j}, x_0) < \frac{\epsilon}{2}$. Hence, if $n \geq N$ then

$$d(x_n, x_0) \leq d(x_n, x_{k_j}) + d(x_{k_j}, x_0) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, as $\epsilon > 0$ was arbitrary, $(x_n)_{n\geq 1}$ is converges to x_0 by definition. ∎

In addition, recall the following theorem.

Theorem 2.1.8 (Bolzano-Weierstrass Theorem). *Every bounded sequence of real numbers has a convergent subsequence with respect to the absolute value metric.*

Theorem 2.1.9 (Completeness of the Real Numbers). *Every Cauchy sequence of real numbers converges with respect to the absolute value metric.*

Proof. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence of real numbers. Thus $(x_n)_{n\geq 1}$ is bounded by Lemma 2.1.3. Therefore $(x_n)_{n\geq 1}$ has a convergent sequence by the Bolzano-Weierstrass Theorem. Hence $(x_n)_{n\geq 1}$ converges by Lemma 2.1.7. ∎

Example 2.1.10. If we use a different metric on the real numbers, it is possible that Cauchy sequences need not converge. To see this, recall from Example 1.1.4 if $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined by

$$d(x, y) = |e^{-x} - e^{-y}|$$

for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a metric space.

We claim that $(\mathbb{R}, d)$ is not complete. To see this, consider the sequence of natural numbers $(n)_{n\geq 1}$. We claim that $(n)_{n\geq 1}$ is Cauchy in $(\mathbb{R}, d)$. To see this, let $\epsilon > 0$. Since $\lim_{n\to\infty} e^{-n} = 0$, there exists an $N \in \mathbb{N}$ such that $0 < e^{-n} < \frac{\epsilon}{2}$ for all $n \geq N$. Hence for all $n, m \geq N$ we have that

$$d(n, m) = \left|e^{-n} - e^{-m}\right| \leq e^{-n} + e^{-m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $(n)_{n\geq 1}$ is Cauchy in $(\mathbb{R}, d)$.

However, we claim that $(n)_{n\geq 1}$ does not converge in $(\mathbb{R}, d)$. To see this, we note that a sequence $(x_n)_{n\geq 1}$ converges in $(\mathbb{R}, d)$ to $x_0 \in \mathbb{R}$ if and only if $e^{-x_0} = \lim_{n\to\infty} e^{-x_n}$ in the absolute value metric on $\mathbb{R}$ if and only if $x_0 = \lim_{n\to\infty} x_n$ in the absolute value metric on $\mathbb{R}$. Therefore, as $(n)_{n\geq 1}$ does not converge in the absolute value metric on $\mathbb{R}$, $(n)_{n\geq 1}$ does not converge in $(\mathbb{R}, d)$ as claimed. Hence $(\mathbb{R}, d)$ is not complete.

In addition, note the above argument shows that a sequence $(x_n)_{n\geq 1}$ converges in $(\mathbb{R}, d)$ to $x_0 \in \mathbb{R}$ if and only if $(x_n)_{n\geq 1}$ converges in $(\mathbb{R}, |\cdot|)$ to x_0. Thus although $(\mathbb{R}, d)$ and $(\mathbb{R}, |\cdot|)$ have the same convergent sequences, only one is complete. Thus completeness is truly a property that is reliant on the metric, not just the topology!

For other examples of complete metric spaces, we turn to the following.

Corollary 2.1.11. *For every $p \in [1, \infty]$ and $n \in \mathbb{N}$, $(\mathbb{K}^n, \|\cdot\|_p)$ is complete.*

Proof. To see that $(\mathbb{R}^n, \|\cdot\|_p)$ is complete, let $(\vec{x}_k)_{k\geq 1}$ be an arbitrary Cauchy sequence in $(\mathbb{R}^n, \|\cdot\|_p)$. Write $\vec{x}_k = (x_{k,1}, \ldots, x_{k,n})$. Since for all $k, m \in \mathbb{N}$ we have

$$|x_{k,j} - x_{m,j}| \leq \|\vec{x}_k - \vec{x}_m\|_p \,,$$

it is elementary to see that $(x_{k,j})_{k\geq 1}$ is a Cauchy sequence in $\mathbb{R}$ for all $j \in \{1, \ldots, n\}$. Since $\mathbb{R}$ is complete, for each $j \in \{1, \ldots, n\}$ there exists an $x_j \in \mathbb{R}$ such that $x_j = \lim_{k\to\infty} x_{k,j}$. If $\vec{x} = (x_1, \ldots, x_n)$, then $\vec{x} = \lim_{k\to\infty} \vec{x}_k$ in $(\mathbb{R}^n, \|\cdot\|_p)$ by Example 1.4.5. Therefore, as $(\vec{x}_k)_{k\geq 1}$ was arbitrary, $(\mathbb{R}^n, \|\cdot\|_p)$ is complete.

To see that $(\mathbb{C}^n, \|\cdot\|_p)$, it suffices by the same arguments to show that $(\mathbb{C}, |\cdot|)$ is complete. To see that $(\mathbb{C}, |\cdot|)$ is complete, let $(z_k)_{k\geq 1}$ be an arbitrary Cauchy sequence in $\mathbb{C}$. For each k, write $z_k = a_k + ib_k$ where $a_k, b_k \in \mathbb{R}$. Since for all $k, m \in \mathbb{N}$ we have

$$|a_k - a_m|, |b_k - b_m| \leq |z_k - z_m|,$$

it is elementary to see that $(a_k)_{k\geq 1}$ and $(b_k)_{k\geq 1}$ are Cauchy sequences in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $a = \lim_{k\to\infty} a_k$ and $b = \lim_{k\to\infty} b_k$ exist. Hence $z = a + bi$, then $z = \lim_{k\to\infty} z_k$ by Example 1.4.6. Hence, as $(z_k)_{k\geq 1}$ was arbitrary, $(\mathbb{C}, |\cdot|)$ is complete. ∎

Once we have a complete metric space, the following shows that we have many other complete metric spaces.

Theorem 2.1.12. *Let $(\mathcal{X}, d)$ be a complete metric space and let $A \subseteq \mathcal{X}$ be non-empty. Then $(A, d|_A)$ is complete if and only if A is closed in $\mathcal{X}$.*

Proof. Suppose $(A, d|_A)$ is complete. To see that A is closed, let $(a_n)_{n\geq 1}$ be an arbitrary sequence of elements from A that converges to some element

$x \in \mathcal{X}$. Since $(a_n)_{n\geq 1}$ converges in $\mathcal{X}$, $(a_n)_{n\geq 1}$ is Cauchy in $\mathcal{X}$ by Lemma 2.1.4 and therefore is Cauchy in $(A, d|_A)$. Hence $(a_n)_{n\geq 1}$ converges in A to some element $a \in A$ as $(A, d|_A)$ is complete. Since limits in metric spaces are unique by Proposition 1.4.12, $a = x$. Hence $x \in A$ so A is closed by Theorem 1.5.7.

For the converse, suppose A is closed in $\mathcal{X}$. To see that $(A, d|_A)$ is complete, let $(a_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $(A, d|_A)$. Hence $(a_n)_{n\geq 1}$ is a Cauchy sequence in $(\mathcal{X}, d)$. Since $(\mathcal{X}, d)$ is complete, $(a_n)_{n\geq 1}$ converges to some element $x \in \mathcal{X}$. Since A is closed in $\mathcal{X}$, Theorem 1.5.7 implies that $x \in A$. Hence as $(a_n)_{n\geq 1}$ was an arbitrary Cauchy sequence, (A, d) is complete. ∎

Note the following gives us examples of complete metric spaces that are not complete normed linear spaces.

Corollary 2.1.13. *Every closed subset of* $(\mathbb{K}^n, \|\cdot\|_p)$ *is a complete metric space for all* $p \in [1, \infty]$*.*

Notice that one direction of the proof of Theorem 2.1.12 did not require $(\mathcal{X}, d)$ to be complete. Thus we obtain the following.

Corollary 2.1.14. *Let* $(\mathcal{X}, d)$ *be a metric space and let* $A \subseteq \mathcal{X}$ *be non-empty. If* (A, d) *is complete, then* A *is closed in* $\mathcal{X}$*.*

2.2 Banach Spaces

The above produced several examples of complete metric spaces including many that were not normed linear spaces. As complete normed linear spaces are incredibly nice and important for the remainder of the course, and as saying/typing complete normed linear spaces is rather cumbersome, we make the following definition.

Definition 2.2.1. A *Banach space* is a complete normed linear space.

Corollary 2.1.11 produced for us a collection of Banach spaces. For the remainder of this subsection, we will note several of the normed linear spaces we have seen previously are Banach spaces. Furthermore, via Theorem 2.1.12, we obtain any closed vector subspace of these Banach spaces is also a Banach space (and any closed subset is a complete metric space).

As we go through the following, note there is a similar theme to the proofs.

Proposition 2.2.2. *For each* $p \in [1, \infty]$, $(\ell_p(\mathbb{N}, \mathbb{K}), \|\cdot\|_p)$ *is a Banach space.*

Proof. Note the proof of this proposition is very similar to that of Proposition 1.4.12 except for the complication that arose in Example 1.4.10 (i.e.

convergences entrywise need not imply convergence in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$). To bi-pass this problem, we will invoke a technique that will be used repeatedly in this section.

Fix $p \in [1,\infty]$ and let $(\vec{x}_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$. For each $n \in \mathbb{N}$, write $\vec{x}_n = (x_{n,k})_{k\geq 1}$. Since for all $m, j, k \in \mathbb{N}$,

$$|x_{m,k} - x_{j,k}| \leq \|\vec{x}_m - \vec{x}_j\|_p,$$

we see that for each $k \in \mathbb{N}$ the sequence $(x_{n,k})_{n\geq 1}$ is Cauchy in $(\mathbb{K}, |\cdot|)$. Therefore, as $(\mathbb{K}, |\cdot|)$ is complete, $y_k = \lim_{n\to\infty} x_{n,k}$ exists in $(\mathbb{K}, |\cdot|)$ for each $k \in \mathbb{N}$.

Let $\vec{y} = (y_n)_{n\geq 1}$. To complete the proof, it suffices to verify two things: that $\vec{y} \in \ell_p(\mathbb{N},\mathbb{K})$, and that $\lim_{n\to\infty} \|\vec{y} - \vec{x}_n\|_p = 0$. We will only discuss the case $p \neq \infty$ and the case $p = \infty$ is similar. For $p \neq \infty$ notice for all $m \in \mathbb{N}$ that

$$\left(\sum_{k=1}^{m} |y_k - x_{1,k}|^p\right)^{\frac{1}{p}} = \lim_{n\to\infty} \left(\sum_{k=1}^{m} |x_{n,k} - x_{1,k}|^p\right)^{\frac{1}{p}} \leq \limsup_{n\to\infty} \|\vec{x}_n - \vec{x}_1\|_p.$$

Since $(\vec{x}_n)_{n\geq 1}$ is Cauchy in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$, we also have that $(\vec{x}_n)_{n\geq 1}$ is bounded in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$ by Lemma 2.1.3. Hence $\limsup_{n\to\infty} \|\vec{x}_n - \vec{x}_1\|_p$ is finite. Therefore, by taking the limit as m tends to infinity, we obtain that

$$\left(\sum_{k=1}^{\infty} |y_k - x_{1,k}|^p\right)^{\frac{1}{p}} \leq \limsup_{n\to\infty} \|\vec{x}_n - \vec{x}_1\|_p.$$

Hence $\vec{z} = (y_k - x_{1,k})_{k\geq 1} \in \ell_p(\mathbb{N},\mathbb{K})$. Therefore, as $\vec{y} = \vec{z} + \vec{x}_1$, we obtain that $\vec{y} \in \ell_p(\mathbb{N},\mathbb{K})$ by the triangle inequality.

To see that $\lim_{n\to\infty} \|\vec{y} - \vec{x}_n\|_p = 0$, let $\epsilon > 0$ be arbitrary. Note the above proof also shows for all $j \in \mathbb{N}$ that

$$\|\vec{y} - \vec{x}_j\|_p \leq \limsup_{n\to\infty} \|\vec{x}_n - \vec{x}_j\|_p.$$

Since $(\vec{x}_n)_{n\geq 1}$ is Cauchy in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$, there exists an $N \in \mathbb{N}$ such that $\|\vec{x}_m - \vec{x}_j\|_p \leq \epsilon$ for all $m, j \geq N$. Hence if $j \geq N$, the above implies $\|\vec{y} - \vec{x}_j\|_p \leq \epsilon$. Therefore, as $\epsilon > 0$ was arbitrary, we obtain that $\lim_{n\to\infty} \|\vec{y} - \vec{x}_n\|_p = 0$. Hence $(\vec{x}_n)_{n\geq 1}$ converges in $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$ so, as $(\vec{x}_n)_{n\geq 1}$ was arbitrary, $(\ell_p(\mathbb{N},\mathbb{K}), \|\cdot\|_p)$ is complete. ∎

To discuss Banach spaces consisting of functions, we first note the following types of convergence and a lemma which guarantees certain limits are continuous. This lemma is the generalization to metric spaces of a result that is a cornerstone of any first course in analysis.

Definition 2.2.3. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. For each $n \in \mathbb{N}$ let $f_n : \mathcal{X} \to \mathcal{Y}$. Given $f : \mathcal{X} \to \mathcal{Y}$, it is said that the sequence $(f_n)_{n\geq 1}$

- *converges pointwise* to f if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathcal{X}$.
- *converges uniformly* to f if $(f_n)_{n\geq 1}$ converges to f with respect to the uniform metric (provided it makes sense); that is, for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d_{\mathcal{Y}}(f(x), f_n(x)) < \epsilon$ for all $n \geq N$ and for all $x \in \mathcal{X}$.

Theorem 2.2.4. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$. If $(f_n)_{n\geq 1}$ is a sequence of continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ that converge to f uniformly, then f is continuous.*

Proof. To see that f is continuous, let $x_0 \in \mathcal{X}$ be arbitrary. To see that f is continuous at x_0 let $\epsilon > 0$ be arbitrary. Since $(f_n)_{n\geq 1}$ converges to f uniformly, there exists an $N \in \mathbb{N}$ such that $d_{\mathcal{Y}}(f(x), f_N(x)) < \frac{\epsilon}{3}$ for all $x \in \mathcal{X}$. Since f_N is continuous at x_0, there exists a $\delta > 0$ such that if $d_{\mathcal{X}}(x, x_0) < \delta$ then $d_{\mathcal{Y}}(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$. Hence if $x \in \mathcal{X}$ and $d_{\mathcal{X}}(x, x_0) < \delta$, then, by the triangle inequality,

$$\begin{aligned} d_{\mathcal{Y}}(f(x), f(x_0)) &\leq d_{\mathcal{Y}}(f(x), f_N(x)) + d_{\mathcal{Y}}(f_N(x), f_N(x_0)) + d_{\mathcal{Y}}(f_N(x_0), f(x_0)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Hence, as $\epsilon > 0$ was arbitrary, f is continuous at x_0. Thus, as x_0 was arbitrary, f is continuous on $\mathcal{X}$. ∎

Using the above, we obtain the following result for metric spaces.

Theorem 2.2.5. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. If $\mathcal{Y}$ is complete, then $(\mathcal{C}_b(\mathcal{X}, \mathcal{Y}), d_\infty)$ is a complete metric space.*

Proof. Let $(f_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $(\mathcal{C}_b(\mathcal{X}, \mathcal{Y}), d_\infty)$. For each $x \in \mathcal{X}$, notice

$$d_{\mathcal{Y}}(f_n(x), f_m(x)) \leq d_\infty(f_n, f_m)$$

for all $n, m \in \mathbb{N}$. Hence it is elementary to see that $(f_n(x))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{Y}$ for all $x \in \mathcal{X}$. Therefore, since $\mathcal{Y}$ is complete, for each $x \in \mathcal{X}$ there exists an $f(x) \in \mathcal{Y}$ such that $f(x) = \lim_{n\to\infty} f_n(x)$. Clearly the function $x \mapsto f(x)$ defines a function $f : \mathcal{X} \to \mathcal{Y}$.

To complete the proof, it suffices to verify three things: that $f : \mathcal{X} \to \mathcal{Y}$ is continuous, that f is bounded, and that $\lim_{n\to\infty} d_\infty(f, f_n) = 0$. For the first, we claim that $(f_n)_{n\geq 1}$ converges to f uniformly on $\mathcal{X}$. To see this, first notice for all $x \in \mathcal{X}$ and $m \in \mathbb{N}$ that

$$d_{\mathcal{Y}}(f(x), f_m(x)) = \lim_{n\to\infty} d_{\mathcal{Y}}(f_n(x), f_m(x)) \leq \limsup_{n\to\infty} d_\infty(f_n, f_m).$$

Since $(f_n)_{n\geq 1}$ is Cauchy in $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), d_\infty)$, we also have that $(f_n)_{n\geq 1}$ is bounded in $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), d_\infty)$ by Lemma 2.1.3. Hence $\limsup_{n\to\infty} d_\infty(f_n, f_m)$ is finite. Therefore, by taking the supremum over all $x \in \mathcal{X}$, we obtain that

$$\sup\{d_{\mathcal{Y}}(f(x), f_m(x)) \mid x \in \mathcal{X}\} \leq \limsup_{n\to\infty} d_\infty(f_n, f_m)$$

for all $m \in \mathbb{N}$. Thus, by taking $m = 1$ and using the fact that f_1 is bounded, we easily see that f is bounded.

To see that f is continuous, we will show that $(f_n)_{n\geq 1}$ converges uniformly to f using the above. Thus let $\epsilon > 0$ be arbitrary. Since $(f_n)_{n\geq 1}$ is Cauchy in $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), d_\infty)$, there exists an $N \in \mathbb{N}$ such that $d_\infty(f_j, f_m) \leq \epsilon$ for all $m, j \geq N$. Hence if $m \geq N$, the above implies

$$\sup\{d_{\mathcal{Y}}(f(x), f_m(x)) \mid x \in \mathcal{X}\} < \epsilon.$$

Thus $(f_n)_{n\geq 1}$ converges to f uniformly on $\mathcal{X}$. Hence f is continuous by Theorem 2.2.4.

As the above shows that $\lim_{m\to\infty} d_\infty(f, f_m) = 0$, $(f_n)_{n\geq 1}$ converges to f in $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), d_\infty)$. Thus, as $(f_n)_{n\geq 1}$ was an arbitrary Cauchy sequence, $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), d_\infty)$ is complete. ∎

Since $\mathcal{C}_b(\mathcal{X},\mathcal{Y})$ is a normed linear space provided $\mathcal{Y}$ is, we obtain the following.

Corollary 2.2.6. *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space and let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a Banach space. Then $(\mathcal{C}_b(\mathcal{X},\mathcal{Y}), \|\cdot\|_\infty)$ is a Banach space.*

Corollary 2.2.7. *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space. Then $(\mathcal{C}_b(\mathcal{X},\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.*

Finally, returning to bounded linear maps between normed linear spaces, we obtain the following.

Theorem 2.2.8. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces. If $\mathcal{Y}$ is a Banach space, then $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$ is a Banach space (where $\|\cdot\|$ is the operator norm).*

Proof. Let $(T_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$. For each $\vec{x} \in \mathcal{X}$, notice

$$\|T_n(\vec{x}) - T_m(\vec{x})\|_{\mathcal{Y}} \leq \|T_n - T_m\| \, \|\vec{x}\|_{\mathcal{X}}$$

for all $n, m \in \mathbb{N}$. Hence it is elementary to see that $(T_n(\vec{x}))_{n\geq 1}$ is a Cauchy sequence in $\mathcal{Y}$ for all $\vec{x} \in \mathcal{X}$. Therefore, since $\mathcal{Y}$ is complete, for each $\vec{x} \in \mathcal{X}$ there exists an $T(\vec{x}) \in \mathcal{Y}$ such that $T(\vec{x}) = \lim_{n\to\infty} T_n(\vec{x})$.

To complete the proof, it suffices to verify three things: that $T : \mathcal{X} \to \mathcal{Y}$ is linear, that T is bounded, and that $\lim_{n\to\infty} \|T - T_n\| = 0$. To see that T is linear, notice for all $\vec{x}_1, \vec{x}_2 \in \mathcal{X}$ and $\alpha \in \mathbb{K}$ that

$$T(\alpha\vec{x}_1+\vec{x}_2) = \lim_{n\to\infty} T_n(\alpha\vec{x}_1+\vec{x}_2) = \lim_{n\to\infty} \alpha T_n(\vec{x}_1)+T_n(\vec{x}_2) = \alpha T(\vec{x}_1)+T(\vec{x}_2).$$

Hence T is linear.

To see that T is bounded, notice for all $\vec{x} \in \mathcal{X}$ with $\|\vec{x}\|_{\mathcal{X}} \leq 1$ and $m \in \mathbb{N}$ that

$$\|T(\vec{x}) - T_m(\vec{x})\|_{\mathcal{Y}} = \lim_{n\to\infty} \|T_n(\vec{x}) - T_m(\vec{x})\|_{\mathcal{Y}} \leq \limsup_{n\to\infty} \|T_n - T_m\|$$

Since $(T_n)_{n\geq 1}$ is Cauchy in $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$, we also have that $(T_n)_{n\geq 1}$ is bounded in $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$ by Lemma 2.1.3. Hence $\limsup_{n\to\infty} \|T_n - T_m\|$ is finite. In particular, we obtain that there exists a constant K such that

$$\|T(\vec{x})\|_{\mathcal{Y}} \leq \|T_1(\vec{x})\|_{\mathcal{Y}} + K \leq \|T_1\| + K$$

for all $\vec{x} \in \mathcal{X}$ with $\|\vec{x}\|_{\mathcal{X}} \leq 1$. Hence T is bounded with $\|T\| \leq \|T_1\| + K$.

To see that $\lim_{n\to\infty} \|T - T_n\| = 0$, let $\epsilon > 0$ be arbitrary. Since $(T_n)_{n\geq 1}$ is Cauchy in $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$, there exists an $N \in \mathbb{N}$ such that $\|T_m - T_j\| \leq \epsilon$ for all $m, j \geq N$. Hence if $j \geq N$, the above implies $\|T(\vec{x}) - T_j(\vec{x})\| \leq \epsilon$ for all $\vec{x} \in \mathcal{X}$ with $\|\vec{x}\|_{\mathcal{X}} \leq 1$. Therefore, as $\epsilon > 0$ was arbitrary, we obtain that $\lim_{n\to\infty} \|T - T_n\| = 0$. Hence $(T_n)_{n\geq 1}$ converges in $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$ so, as $(T_n)_{n\geq 1}$ was arbitrary, $(\mathcal{B}(\mathcal{X},\mathcal{Y}), \|\cdot\|)$ is complete. ∎

To finish this section, we demonstrate that there are normed linear spaces we have seen that are not Banach spaces.

Example 2.2.9. Let $p \in [1, \infty)$ and consider the p-norm on $\mathcal{C}[0,1]$ from Definition 1.2.10. We claim that $(\mathcal{C}[0,1], \|\cdot\|_p)$ is not complete. To see this, for each $n \in \mathbb{N}$ let $f_n \in \mathcal{C}[0,1]$ be the function defined by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right] \\ 1 - n\left(x - \frac{1}{2}\right) & \text{if } x \in \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{n}\right] \\ 0 & \text{otherwise} \end{cases}.$$

We claim that $(f_n)_{n\geq 1}$ is a Cauchy sequence that does not converge. To see that $(f_n)_{n\geq 1}$ is Cauchy, notice if $n, m \in \mathbb{N}$ with $n > m$ then

$$\begin{aligned}
\|f_n - f_m\|_p &= \left(\int_0^1 |f_n(x) - f_m(x)|^p \, dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |f_n(x) - f_m(x)|^p \, dx\right)^{\frac{1}{p}} \\
&\leq \left(\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} 1 \, dx\right)^{\frac{1}{p}} \\
&\leq \frac{1}{m^{\frac{1}{p}}}
\end{aligned}$$

as $|f_n(x) - f_m(x)| \leq 1$ for all $x \in [0,1]$. Therefore, as $\lim_{m\to\infty} \frac{1}{m^{\frac{1}{p}}} = 0$, we obtain that $(f_n)_{n\geq 1}$ is Cauchy in $(\mathcal{C}[0,1], \|\cdot\|_p)$.

To see that $(f_n)_{n\geq 1}$ does not have a limit in $(\mathcal{C}[0,1], \|\cdot\|_p)$, suppose to the contrary that $f \in \mathcal{C}[0,1]$ is a limit of $(f_n)_{n\geq 1}$. Then for all $a, b \in [0,1]$ with $a < b$, we have that

$$\begin{aligned}
\limsup_{n\to\infty} \left(\int_a^b |f_n(x) - f(x)|^p \, dx\right)^{\frac{1}{p}} &\leq \limsup_{n\to\infty} \left(\int_0^1 |f_n(x) - f(x)|^p \, dx\right)^{\frac{1}{p}} \\
&= \limsup_{n\to\infty} \|f_n - f\|_p = 0
\end{aligned}$$

as the integral of a positive function is positive and the function $x \mapsto x^{\frac{1}{p}}$ is increasing on $[0,\infty)$. Thus for each $a, b \in \left[0, \frac{1}{2}\right]$ with $a < b$ we obtain that

$$0 = \limsup_{n\to\infty} \left(\int_a^b |f_n(x) - f(x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_a^b |1 - f(x)|^p \, dx\right)^{\frac{1}{p}}$$

However, as f is continuous on $[0,1]$, this implies that $f(x) = 1$ for all $x \in \left[0, \frac{1}{2}\right]$. Similarly, if $\frac{1}{2} < a < b \leq 1$, we obtain by selecting n large enough so that $\frac{1}{2} + \frac{1}{n} < a$ that

$$0 = \limsup_{n\to\infty} \left(\int_a^b |f_n(x)|^p \, dx\right)^{\frac{1}{p}} = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Hence, the same arguments imply that $f(x) = 0$ for all $x \in \left(\frac{1}{2}, 1\right]$. Thus, as f is continuous at $\frac{1}{2}$, we have obtained that $0 = f\left(\frac{1}{2}\right) = 1$ which is a contradiction. Thus $(f_n)_{n\geq 1}$ does not have a limit in $(\mathcal{C}[0,1], \|\cdot\|_p)$ so $(\mathcal{C}[0,1], \|\cdot\|_p)$ is not complete.

Note we had to exclude $p = \infty$ from Example 2.2.9 as $(\mathcal{C}[0,1], \|\cdot\|_\infty) = (\mathcal{C}_b[0,1], \|\cdot\|_\infty)$ is complete by Theorem 2.2.5.

2.3 Verifying Completeness

The above has demonstrated that several of the space we naturally desire to consider are Banach spaces. Thus, as we have several Banach spaces and complete metric spaces, it is nice to determine what additional properties these spaces have beyond the convergence of all Cauchy sequences. In particular, we will demonstrate additional properties that are equivalent to the convergence of all Cauchy sequences. Each of these additional properties has their own particular use.

For an alternate description of completeness, we need the following notion for how wide a set is.

Definition 2.3.1. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$ be non-empty. The *diameter* of A, denoted $\mathrm{diam}(A)$, is defined to be

$$\mathrm{diam}(A) = \sup\{d(x,y) \mid x, y \in A\} \in [0, \infty].$$

Example 2.3.2. In $\mathbb{R}$,

$$\mathrm{diam}((0,1)) = \mathrm{diam}([0,1]) = 1$$

whereas $\mathrm{diam}(\mathbb{R}) = \infty$.

Example 2.3.3. In any metric space $(\mathcal{X}, d)$, it is elementary to see that

$$\mathrm{diam}(B(x,r)) \leq \mathrm{diam}(B[x,r]) \leq 2r$$

for all $x \in \mathcal{X}$ and $r > 0$. However, it is possible that these inequalities are strict. Indeed if d is the discrete metric, then

$$\mathrm{diam}(B(x,r)) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } r > 1 \end{cases} \qquad \text{and} \qquad \mathrm{diam}(B[x,r]) = \begin{cases} 0 & \text{if } r < 1 \\ 1 & \text{if } r \geq 1 \end{cases}.$$

Using the notion of the diameter of a set, we can describe completeness using small closed sets instead of Cauchy sequences. This adds to the validity of the term 'completeness' in that it shows we do not have any holes in our complete metric spaces.

Theorem 2.3.4 (Cantor's Theorem). *Let $(\mathcal{X}, d)$ be a metric space. Then the following are equivalent:*

(1) $(\mathcal{X}, d)$ is a complete metric space.

(2) If $(F_n)_{n\geq 1}$ is a sequence of non-empty closed subsets of $\mathcal{X}$ such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mathrm{diam}(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof. Suppose $(\mathcal{X}, d)$ is a complete metric space. Let $(F_n)_{n\geq 1}$ be an arbitrary sequence of non-empty closed subsets of $\mathcal{X}$ such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$. To see that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, for each $n \in \mathbb{N}$ choose $x_n \in F_n$. We claim that $(x_n)_{n\geq 1}$ is a Cauchy sequence. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, there exists an $N \in \mathbb{N}$ such that $\operatorname{diam}(F_N) < \epsilon$. As $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$, we obtain that $x_n \in F_n$ for all $n \geq N$. Hence $d(x_n, x_m) \leq \operatorname{diam}(F_N) < \epsilon$ for all $n, m \geq N$. Hence, as $\epsilon > 0$ was arbitrary, $(x_n)_{n\geq 1}$ is a Cauchy sequence.

Since $(\mathcal{X}, d)$ is complete, $x = \lim_{n\to\infty} x_n$ exists. Since for each $m \in \mathbb{N}$ we have $x_n \in F_m$ for all $n \geq m$, we obtain from Theorem 1.5.7 together with the fact that F_m is closed that $x \in F_m$ for all $m \in \mathbb{N}$. Hence $x \in \bigcap_{n=1}^{\infty} F_n$ so $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

For the converse direction, suppose $(\mathcal{X}, d)$ has property (2). To see that $(\mathcal{X}, d)$ is complete, let $(x_n)_{n\geq 1}$ be an arbitrary Cauchy sequence. For each $n \in \mathbb{N}$, let

$$F_n = \overline{\{x_k \mid k \geq n\}}.$$

Clearly each F_n is a non-empty closed subset of $\mathcal{X}$ such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$.

We claim that $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$. To see this, let $\epsilon > 0$ be arbitrary. Since $(x_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\epsilon}{3}$ for all $n, m \geq N$. We claim that $\operatorname{diam}(F_n) \leq \epsilon$ whenever $n \geq N$. To see this, fix $n \geq N$ and let $x, y \in F_n$ be arbitrary. By the definition of F_n, there exists $k, j \geq n \geq N$ such that

$$d(x, x_j) < \frac{\epsilon}{3} \qquad \text{and} \qquad d(y, x_k) < \frac{\epsilon}{3}.$$

Hence

$$d(x, y) \leq d(x, x_j) + d(x_j, x_k) + d(x_k, y) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

as $k, j \geq N$ and by our choice of N. Hence $\operatorname{diam}(F_n) \leq \epsilon$ whenever $n \geq N$ by the definition of the diameter of a set. Thus the claim is complete.

As we are assuming property (2), the above implies that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. Let $x \in \bigcap_{n=1}^{\infty} F_n$. We claim that $(x_n)_{n\geq 1}$ converges to x. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, there exists an $N \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \epsilon$ for all $n \geq N$. Since $x, x_n \in F_n$ for all $n \in \mathbb{N}$, we obtain that

$$d(x, x_n) \leq \operatorname{diam}(F_n) < \epsilon$$

for all $n \geq N$. Therefore, as $\epsilon > 0$ was arbitrary, $x = \lim_{n\to\infty} x_n$. Hence, as $(x_n)_{n\geq 1}$ was an arbitrary Cauchy sequence, $(\mathcal{X}, d)$ is complete. ∎

We will see several uses of Cantor's Theorem (Theorem 2.3.4) later in the course. For now we turn to another important property of the real numbers, namely the convergence of specific types of series. In particular, every 'absolutely summable' series converges. We can generalize these concepts to metric spaces as follows.

Definition 2.3.5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. A series $\sum_{n=1}^{\infty} \vec{x}_n$ is said to be *summable* if the sequence of partial sums $(s_n)_{n\geq 1}$ converges (where $s_n = \sum_{k=1}^{n} \vec{x}_k$).

A series $\sum_{n=1}^{\infty} \vec{x}_n$ is said to be *absolutely summable* if $\sum_{n=1}^{\infty} \|\vec{x}_n\| < \infty$.

Theorem 2.3.6. *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. Then $\mathcal{X}$ is complete (i.e. a Banach space) if and only if every absolutely summable series is summable.*

Proof. Suppose $(\mathcal{X}, \|\cdot\|)$ is complete. Let $\sum_{n=1}^{\infty} \vec{x}_n$ be an arbitrary absolutely summable series in $(\mathcal{X}, \|\cdot\|)$. To see that $\sum_{n=1}^{\infty} \vec{x}_n$ is summable, let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} \|\vec{x}_n\| < \infty$, there exists an $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \|\vec{x}_n\| < \epsilon$. Therefore, if $k, m \geq N$ and, without loss of generality, $m \geq k$, then

$$\begin{aligned} \|s_m - s_k\| &= \left\| \sum_{n=1}^{m} \vec{x}_n - \sum_{n=1}^{k} \vec{x}_n \right\| \\ &= \left\| \sum_{n=k+1}^{m} \vec{x}_n \right\| \\ &\leq \sum_{n=k+1}^{m} \|\vec{x}_n\| \\ &\leq \sum_{n=N}^{\infty} \|\vec{x}_n\| < \epsilon. \end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, the sequence of partial sums $(s_n)_{n\geq 1}$ is Cauchy. Hence $(s_n)_{n\geq 1}$ converges as $\mathcal{X}$ is complete. Thus, as $\sum_{n=1}^{\infty} \vec{x}_n$ was arbitrary, every absolutely summable series in $\mathcal{X}$ is summable.

For the converse, suppose every absolutely summable sequence in $\mathcal{X}$ is summable. To see that $\mathcal{X}$ is complete, let $(\vec{x}_n)_{n\geq 1}$ be an arbitrary Cauchy sequence. Since $(\vec{x}_n)_{n\geq 1}$ is Cauchy, there exists an $n_1 \in \mathbb{N}$ such that $\|\vec{x}_m - \vec{x}_j\| < \frac{1}{2}$ for all $m, j \geq n_1$. Similarly, since $(\vec{x}_n)_{n\geq 1}$ is Cauchy, there exists an $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $\|\vec{x}_m - \vec{x}_j\| < \frac{1}{2^2}$ for all $m, j \geq n_2$. By repeating the above process, for each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ such that $n_k < n_{k+1}$ for all k and $\|\vec{x}_m - \vec{x}_j\| < \frac{1}{2^k}$ for all $m, j \geq n_k$.

For each $k \in \mathbb{N}$ let $\vec{y}_k = \vec{x}_{n_{k+1}} - \vec{x}_{n_k}$. By the above paragraph, we see that

$$\sum_{k=1}^{\infty} \|\vec{y}_k\| \leq \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

Hence $\sum_{k=1}^{\infty} \vec{y}_k$ is an absolutely summable series in $\mathcal{X}$. Therefore, by the assumptions on $\mathcal{X}$, $\sum_{k=1}^{\infty} \vec{y}_k$ is summable in $\mathcal{X}$.

Let $\vec{x} = \vec{x}_{n_1} + \sum_{k=1}^\infty \vec{y}_k$. We claim that $(\vec{x}_{n_k})_{k\geq 1}$ converges to $\vec{x}$. To see this, let $\epsilon > 0$ be arbitrary. Then there exists a $M \in \mathbb{N}$ such that if $m \geq M$ then

$$\left\| \sum_{k=1}^\infty \vec{y}_k - \sum_{k=1}^m \vec{y}_k \right\| < \epsilon.$$

Therefore, if $m \geq M$,

$$\begin{aligned}
\|\vec{x} - \vec{x}_{n_{m+1}}\| &\leq \left\| \sum_{k=1}^\infty \vec{y}_k - \sum_{k=1}^m \vec{y}_k \right\| + \left\| \vec{x}_{n_1} - \vec{x}_{n_{m+1}} + \sum_{k=1}^m \vec{y}_k \right\| \\
&< \epsilon + \left\| \vec{x}_{n_1} - \vec{x}_{n_{m+1}} + \sum_{k=1}^m \vec{x}_{n_{k+1}} - \vec{x}_{n_k} \right\| \\
&= \epsilon.
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $(\vec{x}_{n_k})_{k\geq 1}$ converges to $\vec{x}$. Hence $(\vec{x}_n)_{n\geq 1}$ converges to $\vec{x}$ by Lemma 2.1.7. Therefore, as $(\vec{x}_n)_{n\geq 1}$ was an arbitrary Cauchy sequence, $\mathcal{X}$ is complete. ∎

As an immediate corollary, we obtain the following result pertaining to convergence of series of continuous functions.

Corollary 2.3.7 (Weierstrass M-Test). *Let $(\mathcal{X}, d)$ be a metric space and let $(f_n)_{n\geq 1}$ be a sequence of functions from $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$. Suppose there exists an $M \in \mathbb{R}$ such that $\sum_{n=1}^\infty \|f_n\|_\infty < M$. Then $\sum_{n=1}^\infty f_n$ converges uniformly on $\mathcal{X}$ to a continuous function.*

2.4 Tietz Extension Theorem

Using Theorem 2.3.6 and Urysohn's Lemma (Theorem 1.6.14), we can prove an important result about extending continuous functions on closed sets. To begin, we note there exists an elementary proof in the case we are considering closed subsets of $(\mathbb{R}, |\cdot|)$.

Theorem 2.4.1 (Tietze's Extension Theorem on $\mathbb{R}$). *Let $F \subseteq \mathbb{R}$ be closed and let $f : F \to \mathbb{C}$ be continuous. There exists a continuous function $g : \mathbb{R} \to \mathbb{C}$ such that $g(x) = f(x)$ for all $x \in F$ and*

$$\sup(\{|g(x)| \mid x \in \mathbb{R}\}) \leq \sup(\{|f(x)| \mid x \in F\}).$$

Proof. Since F^c is open, by Proposition 1.3.11 we may write F^c as a countable union of disjoint non-empty open intervals, say $\bigcup_{n=1}^\infty (a_n, b_n)$. Define $g : \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in F \\ f(a_n) & \text{if } x \in (a_n, b_n) \text{ and } b_n = \infty \\ f(b_n) & \text{if } x \in (a_n, b_n) \text{ and } a_n = -\infty \\ \frac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) + h(a_n) & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{R}$; that is g agrees with f on F and is linear on each (a_n, b_n). It is not difficult to see that g is continuous and

$$\sup(\{|g(x)| \mid x \in \mathbb{R}\}) \leq \sup(\{|f(x)| \mid x \in F\}). \qquad \blacksquare$$

However, if we had a continuous function on a closed subsets of $\mathbb{R}^n$ or a general metric space, it is far more difficult to know how to extend the function to a continuous function on $\mathbb{R}^n$. The following theorem solves this problem for us.

Theorem 2.4.2 (Tietz's Extension Theorem - Bounded Version). *Let $(\mathcal{X}, d)$ be a metric space, let F be a closed subset of $(\mathcal{X}, d)$, and let $f \in \mathcal{C}_b(F, \mathbb{R})$ be continuous (where F is equipped with the metric $d|_F$). There exists a $g \in \mathcal{C}_b(X, \mathbb{R})$ such that $g|_F = f$ and $\|g\|_\infty = \|f\|_\infty$.*

Proof. Since $f \in \mathcal{C}_b(F, \mathbb{R})$, we know that

$$\|f\|_\infty = \sup(\{|f(x)| \mid x \in F\}) < \infty.$$

Clearly if $\|f\|_\infty = 0$, we can take g to be the zero function thereby completing the claim. Hence we may assume that $\|f\|_\infty > 0$. Therefore, by scaling f if necessary, we can assume without loss of generality that $\|f\|_\infty = 1$. Thus $|f(x)| \leq 1$ for all $x \in F$.

To proceed with the proof, our goal is to Urysohn's Lemma (Theorem 1.6.14) to get several elements of $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$. We will construct these functions in a specific way so that their sum closer and closer approximates f on F. We will then take a limit of these functions to obtain the desired extension of f.

To begin, let

$$A_1 = \left\{x \in F \,\middle|\, f(x) \in \left[-1, -\frac{1}{3}\right]\right\} = f^{-1}\left(\left[-1, -\frac{1}{3}\right]\right) \text{ and}$$
$$B_1 = \left\{x \in F \,\middle|\, f(x) \in \left[\frac{1}{3}, 1\right]\right\} = f^{-1}\left(\left[\frac{1}{3}, 1\right]\right).$$

Therefore, since f is continuous on F, A_1 and B_1 disjoint closed subsets of F by Corollary 1.6.8. Therefore, since F is closed in $(\mathcal{X}, d)$ and since closed subsets of F are the intersection of F with a closed subset of $(\mathcal{X}, d)$ by Corollary 1.3.25 and therefore closed in $(\mathcal{X}, d)$, we see that A_1 and B_1 disjoint closed subsets of $(\mathcal{X}, d)$. Thus Urysohn's Lemma (Theorem 1.6.14) implies there exists a continuous function $h_1 : \mathcal{X} \to \left[-\frac{1}{3}, \frac{1}{3}\right]$ such that $h_1(a) = -\frac{1}{3}$ for all $a \in A_1$ and $h_1(b) = \frac{1}{3}$ for all $b \in B_1$ [Note Urysohn's Lemma only applies if $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. If $A_1 = B_1 = \emptyset$, use the zero function for h_1. Otherwise, if $A_1 = \emptyset$, use the constant function $\frac{1}{3}$ for h_1, and if $B_1 = \emptyset$, use the constant function $-\frac{1}{3}$ for h_1.]

We claim that $|f(x) - h_1(x)| \leq \frac{2}{3}$ for all $x \in F$. To see this, notice if $x \in A_1$ then $-1 \leq f(x) \leq -\frac{1}{3}$ so the fact that $h_1(x) = -\frac{1}{3}$ as $x \in A_1$ implies

$|f(x) - h_1(x)| \leq \frac{2}{3}$. Similarly, if $x \in B_1$ then $\frac{1}{3} \leq f(x) \leq 1$ so the fact that $h_1(x) = \frac{1}{3}$ as $x \in B_1$ implies $|f(x) - h_1(x)| \leq \frac{2}{3}$. Finally, if $x \in F \setminus (A_1 \cup B_1)$, then the definitions of A_1 and B_1 imply that $-\frac{1}{3} < f(x) < \frac{1}{3}$ so, as $h_1 : X \to \left[-\frac{1}{3}, \frac{1}{3}\right]$, we obtain that $|f(x) - h_1(x)| \leq \frac{2}{3}$. Hence $|f(x) - h_1(x)| \leq \frac{2}{3}$ for all $x \in F$.

Let $\alpha = \frac{2}{3}$. We claim that there exists a sequence $(h_n)_{n\geq 1}$ in $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$ such that

$$\|h_n\|_\infty \leq \frac{1}{3}\alpha^{n-1} \qquad \text{and} \qquad \left| f(x) - \sum_{k=1}^{n} h_k(x) \right| \leq \alpha^n \text{ for all } x \in F$$

for all $n \in \mathbb{N}$. To see this, we proceed by induction on n with the base case $n = 1$ completed by the above arguments. Thus, to proceed with the inductive step, suppose there exist $(h_k)_{k=1}^n$ in $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$ such that

$$\|h_m\|_\infty \leq \frac{1}{3}\alpha^{k-1} \qquad \text{and} \qquad \left| f(x) - \sum_{k=1}^{m} h_k(x) \right| \leq \alpha^m \text{ for all } x \in F$$

for all $m \in \{1, \ldots, n\}$. Let

$$A_{n+1} = \left\{ x \in F \,\middle|\, f(x) - \sum_{k=1}^{n} h_k(x) \in \left[-\alpha^n, -\frac{1}{3}\alpha^n\right] \right\} \text{ and}$$
$$B_{n+1} = \left\{ x \in F \,\middle|\, f(x) - \sum_{k=1}^{n} h_k(x) \in \left[\frac{1}{3}\alpha^n, \alpha^n\right] \right\}.$$

Since $h_k \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$ for all $k \in \{1, \ldots, n\}$ and since F is a closed subspace of $(\mathcal{X}, d)$, we see that $x \mapsto f(x) - \sum_{k=1}^n h_k(x)$ is a continuous function on F and thus A_{n+1} and B_{n+1} are disjoint closed subsets of F. Therefore, since F is closed in $(\mathcal{X}, d)$ and since closed subsets of F are the intersection of F with a closed subset of $(\mathcal{X}, d)$ by Corollary 1.3.25 and therefore closed in $(\mathcal{X}, d)$, we see that A_{n+1} and B_{n+1} disjoint closed subsets of $(\mathcal{X}, d)$. Thus, Urysohn's Lemma (Theorem 1.6.14) implies there exists a continuous function $h_{n+1} : \mathcal{X} \to \left[-\frac{1}{3}\alpha^n, \frac{1}{3}\alpha^n\right]$ such that $h_{n+1}(a) = -\frac{1}{3}\alpha^n$ for all $a \in A_{n+1}$ and $h_{n+1}(b) = \frac{1}{3}\alpha^n$ for all $b \in B_{n+1}$.

Clearly $\|h_{n+1}\|_\infty \leq \frac{1}{3}\alpha^n$ by construction. To see that

$$\left| f(x) - \sum_{k=1}^{n+1} h_k(x) \right| \leq \alpha^{n+1}$$

for all $x \in F$, we will proceed as we did in the $n = 1$ case. Indeed, if $x \in A_{n+1}$ then

$$h_{n+1}(x) = -\frac{1}{3}\alpha^n \qquad \text{and} \qquad f(x) - \sum_{k=1}^{n} h_k(x) \in \left[-\alpha^n, -\frac{1}{3}\alpha^n\right]$$

so

$$\left|f(x) - \sum_{k=1}^{n+1} h_k(x)\right| \leq \alpha^n - \frac{1}{3}\alpha^n = \frac{2}{3}\alpha^n = \alpha^{n+1}.$$

Similarly, if $x \in B_{n+1}$ then

$$h_{n+1}(x) = \frac{1}{3}\alpha^n \qquad \text{and} \qquad f(x) - \sum_{k=1}^{n} h_k(x) \in \left[\frac{1}{3}\alpha^n, \alpha^n\right]$$

so

$$\left|f(x) - \sum_{k=1}^{n+1} h_k(x)\right| \leq \alpha^n - \frac{1}{3}\alpha^n = \frac{2}{3}\alpha^n = \alpha^{n+1}.$$

Finally, if $x \in F \setminus (A_{n+1} \cup B_{n+1})$, then the definitions of A_{n+1} and B_{n+1} imply that

$$\left|f(x) - \sum_{k=1}^{n} h_k(x)\right| < \frac{1}{3}\alpha^n \qquad \text{and} \qquad |h_{n+1}(x)| \leq \frac{1}{3}\alpha^n$$

so

$$\left|f(x) - \sum_{k=1}^{n+1} h_k(x)\right| \leq \frac{1}{3}\alpha^n + \frac{1}{3}\alpha^n = \frac{2}{3}\alpha^n = \alpha^{n+1}.$$

Hence

$$\left|f(x) - \sum_{k=1}^{n+1} h_k(x)\right| \leq \alpha^{n+1}$$

for all $x \in F$. Therefore, the inductive step is complete so there exist $(h_n)_{n\geq 1}$ in $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$ with the desired properties.

Of course, by construction we know that

$$\sum_{n=1}^{\infty} \|h_n\|_\infty \leq \sum_{n=1}^{\infty} \frac{1}{3}\alpha^{n-1} = \frac{1}{3}\frac{1}{1-\alpha} = 1 < \infty.$$

Hence $\sum_{n=1}^{\infty} h_n$ is an absolutely summable series in $(\mathcal{C}_b(\mathcal{X}, \mathbb{R}), \|\cdot\|_\infty)$. Therefore, since $(\mathcal{C}_b(X, \mathbb{R}), \|\cdot\|_\infty)$ is a Banach space by Corollary 2.2.6, Theorem 2.3.6 implies that $\sum_{n=1}^{\infty} h_n$ is summable. Hence

$$g = \sum_{n=1}^{\infty} h_n$$

is a well-defined element of $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$. We claim that g is the function we seek.

To begin to see that g has the desired properties, we note since the norm is a continuous function on any normed linear space by Example 1.6.5 that

$$\begin{aligned}\|g\|_\infty &= \lim_{n\to\infty}\left\|\sum_{k=1}^{n} h_k\right\|_\infty \\ &\leq \limsup_{n\to\infty}\sum_{k=1}^{n}\|h_k\|_\infty \\ &\leq \limsup_{n\to\infty}\sum_{k=1}^{n}\frac{1}{3}\alpha^{k-1} \\ &= \frac{1}{3}\frac{1}{1-\alpha} = 1\end{aligned}$$

so $\|g\|_\infty \leq 1$. Hence, provided we can show that $g|_F = f$, we will then be able to use the fact that $\|f\|_\infty = 1$ to obtain that $\|g\|_\infty = 1 = \|f\|_\infty$ as desired. Hence, all that remains to be shown is that $g|_F = f$.

To see that $g|_F = f$, let $x \in F$ be arbitrary. Then the definition of g implies that

$$|f(x) - g(x)| = \lim_{n\to\infty}\left|f(x) - \sum_{k=1}^{n} h_k(x)\right| \leq \limsup_{n\to\infty}\alpha^n = 0$$

as $\alpha = \frac{2}{3}$. Hence $g(x) = f(x)$. Therefore, since $x \in F$ was arbitrary, $g|_F = f$ as desired. ∎

With Theorem 2.4.2, it is not too difficult to extend these results to unbounded functions.

Theorem 2.4.3 (Tietz's Extension Theorem - Unbounded Version). *Let $(\mathcal{X}, d)$ be a metric space, let F be a closed subset of $\mathcal{X}$, and let $f : F \to \mathbb{R}$ be continuous (with respect to the metric $d|_F$ on F). There exists a continuous function $g : X \to \mathbb{R}$ such that $g|_F = f$.*

Proof. Our goal in this proof is to use a homeomorphism (a continuous map with a continuous inverse) to reduce the result to the bounded case studied in Theorem 2.4.2. Indeed consider the function $\varphi : \mathbb{R} \to (-1, 1)$ defined by

$$\varphi(x) = \frac{x}{1 + |x|}$$

for all $x \in \mathbb{R}$. It is elementary to see that φ is continuous with continuous inverse $\varphi^{-1} : (-1, 1) \to \mathbb{R}$ defined by

$$\varphi^{-1}(y) = \frac{y}{1 - |y|}$$

for all $y \in (-1, 1)$. Hence, if $f_0 : F \to (-1, 1)$ is defined by $f_0 = \varphi \circ f$, then $f_0 \in \mathcal{C}_b(F, \mathbb{R})$ is such that $\|f_0\|_\infty \leq 1$. Hence the bounded version of Tietz

Extension Theorem (Theorem 2.4.2) implies there exists an $h_0 \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$ such that $\|h_0\|_\infty = \|f_0\|_\infty \leq 1$ and $h_0|_F = f_0$.

Of course, if $h_0(x) \neq \pm 1$ for all $x \in \mathcal{X}$, then one can immediately take $g = \varphi^{-1} \circ h_0$ thereby completing the proof. Therefore, as we only know that $\|h_0\|_\infty \leq 1$ so it is possible that $h_0(x) = \pm 1$ for some $x \in \mathcal{X}$, we must correct h_0.

Let $C = h_0^{-1}(\{-1, 1\})$. Since $h_0 \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$, C is a closed (possibly empty subset) of $\mathcal{X}$ by Corollary 1.6.8. We claim that $C \cap F = \emptyset$. To see this, notice if $x \in F$ then $h_0(x) = f_0(x) \in (-1, 1)$ so $x \notin C$ by definition. Hence $C \cap F = \emptyset$. Thus Urysohn's Lemma (Theorem 1.6.14) implies there exists a continuous function $h : \mathcal{X} \to [0, 1]$ such that $h(x) = 0$ for all $x \in C$ and $h(x) = 1$ for all $x \in \mathcal{F}$. Define $g_0 : \mathcal{X} \to \mathbb{R}$ by

$$g_0(x) = h_0(x)h(x)$$

for all $x \in \mathcal{X}$. Since g_0 is a product of elements of $\mathcal{C}_b(\mathcal{X}, \mathbb{R})$, it is elementary to see that $g_0 \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$. Furthermore, we claim that $g(X) \subseteq (-1, 1)$. To see this, notice if $x \in C$ then $|h_0(x)| = 1$ and $h(x) = 0$ so $g_0(x) = 0 \in (-1, 1)$. Furthermore, if $x \in \mathcal{X} \setminus C$ then $|h_0(x)| < 1$ and $h(x) \in [0, 1]$ so $g_0(x) \in (-1, 1)$. Hence $g(\mathcal{X}) \subseteq (-1, 1)$ as claimed.

Define $g : \mathcal{X} \to \mathbb{R}$ by

$$g(x) = \varphi^{-1}(g_0(x))$$

for all $x \in \mathcal{X}$, which is well-defined as $g(X) \subseteq (-1, 1)$. Furthermore $g \in \mathcal{C}(\mathcal{X}, \mathbb{R})$ as g is the composition of two continuous functions and thus continuous. Finally, to see that $g|_F = f$, let $x \in F$ be arbitrary. Then

$$g(x) = \varphi^{-1}(g_0(x)) = \varphi^{-1}(h_0(x)h(x)) = \varphi^{-1}(f_0(x)1) = \varphi^{-1}(f_0(x)) = f(x)$$

as desired. ∎

Remark 2.4.4. Note both versions of the Tietz Extension Theorem can fail if the set we are trying to extend a continuous function from are not closed. Indeed consider the continuous function $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

for all $x \in (-\infty, 0) \cup (0, \infty)$. Clearly f is continuous on its domain yet it is impossible to extend f to a continuous function on $\mathbb{R}$.

Remark 2.4.5. Note the only requirements used to prove both versions of the Tietz Extension Theorem were that $\mathcal{C}_b(X, \mathbb{R})$ was complete and that a Urysohn's Lemma existed. Thus the same proof may be used to prove the Tietz Extension Theorem for a wider collection of topological spaces.

2.5 Completions

As we have seen complete metric spaces (and in particular Banach spaces) are very nice metric spaces to consider as many of the properties of the real number generalize to these spaces. However, we have also seen metric spaces that are not complete (Example 2.2.9). Thus it is natural to ask, "Given a metric spaces $(\mathcal{X}, d)$, it is possible to view $\mathcal{X}$ inside a complete metric space?" To answer this question, we must first define what we mean by 'view inside'. This leads to the question, "What does it mean for two metric spaces to be the same?"

To begin, we note the following notion of'equality of metric spaces' which is related to topological notions.

Definition 2.5.1. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be a *homeomorphism* if f is invertible and both f and f^{-1} are continuous. Furthermore, its is said that $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are *homeomorphic* if there exists a homeomorphism from $\mathcal{X}$ to $\mathcal{Y}$.

Example 2.5.2. For each $n \in \mathbb{N}$, let $f : \mathbb{K}^n \to \mathbb{K}^n$ be defined by $f(x) = x$. Then f is a homeomorphism from $(\mathbb{K}^n, \|\cdot\|_p)$ to $(\mathbb{K}^n, \|\cdot\|_q)$ for any $p, q \in [1, \infty]$. Indeed, since

$$\|(z_1, \ldots, z_n)\|_\infty \leq \|(z_1, \ldots, z_n)\|_p \leq n^{\frac{1}{p}} \|(z_1, \ldots, z_n)\|_\infty$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$ by Remark 1.3.14, we clearly see that f is a homeomorphism from $(\mathbb{K}^n, \|\cdot\|_\infty)$ to $(\mathbb{K}^n, \|\cdot\|_p)$ for any $p \in [1, \infty]$. Since the composition of homeomorphisms is clearly a homeomorphism, the result follows.

Example 2.5.3. For each $n \in \mathbb{N}$, let $f : \mathbb{R}^{2n} \to \mathbb{C}^n$ be defined by

$$f(x_1, \ldots, x_{2n}) = (x_1 + ix_2, x_3 + ix_4, \ldots, x_{2n-1} + ix_{2n})$$

for all $(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$. Then f is a homeomorphism from $(\mathbb{R}^{2n}, \|\cdot\|_2)$ to $(\mathbb{C}^n, \|\cdot\|_2)$ (and thus from $(\mathbb{R}^{2n}, \|\cdot\|_p)$ to $(\mathbb{C}^n, \|\cdot\|_q)$ for any $p, q \in [1, \infty]$). Indeed as $|a + bi| = \sqrt{a^2 + b^2}$ for all $a, b \in \mathbb{R}$, it is trivial to verify that f is a homeomorphism.

Remark 2.5.4. The notion of homeomorphic metric spaces produces an equivalence relation on metric spaces. Indeed define two metric spaces to be equivalent if and only if there are homeomorphic. It is elementary to verify that this is an equivalence relation on the collection of metric spaces. Consequently homeomorphic is a good notion for when two metric spaces are the same. Indeed note that if $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are homeomorphic and $f : \mathcal{X} \to \mathcal{Y}$ is a homeomorphism, then f provides a bijection between convergent sequences; that is, if $(x_n)_{n\geq1}$ is a sequence in $\mathcal{X}$, then $(x_n)_{n\geq1}$

converges to x in $\mathcal{X}$ if and only if $(f(x_n))_{n\geq 1}$ converges to $f(x)$ in $\mathcal{Y}$. Using Theorem 1.5.7 this implies a subset F of $\mathcal{X}$ is closed if and only if $f(F)$ is closed in $\mathcal{Y}$. Consequently a subset U of $\mathcal{X}$ is open if and only if $f(U)$ is open in $\mathcal{Y}$. Hence both $\mathcal{X}$ and $\mathcal{Y}$ have the same topologies. Thus, for all intents and purposes in this course, $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are the 'same topological spaces'. In particular, Example 2.5.2 implies $(\mathbb{K}^n, \|\cdot\|_p)$ for $p \in [1,\infty]$ are all the same metric space and Example 2.5.3 implies $(\mathbb{R}^{2n}, \|\cdot\|_2)$ to $(\mathbb{C}^n, \|\cdot\|_2)$ can be viewed as the same metric space.

Remark 2.5.5. Furthermore, the notion of a homeomorphism can be used to show something interesting. Given two metric d_1 and d_2 on a non-empty set X, we know that if (X, d_1) and (X, d_2) have the same topology, then they have the same convergent sequences by Proposition 1.4.3. Conversely, if (X, d_1) and (X, d_2) have the same convergent sequences, then the identity map from (X, d_1) to (X, d_2) is a homeomorphism so a set U is open in (X, d_1) if and only if U is open in (X, d_2). Hence convergent sequences completely determine the topology of a metric space!.

Remark 2.5.6. However, there is an issue with using homeomorphic as the notion for two metric spaces to be 'equal'. Indeed we can also show the identity map is a homeomorphism from $(\mathbb{R}, |\cdot|)$ to $(\mathbb{R}, d)$ where $d : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ is defined by $d(x, y) = |e^{-x} - e^{-y}|$ as in Example 1.1.4. Indeed Example 2.1.10 showed that $(\mathbb{R}, |\cdot|)$ and $(\mathbb{R}, d)$ have the same convergent sequences. However, Example 2.1.10 also showed that $(\mathbb{R}, |\cdot|)$ and $(\mathbb{R}, d)$ do not have the same Cauchy sequences. Thus is homeomorphism truly the right notion for metric spaces to be the same?

As the above demonstrates, the notion of homeomorphic metric spaces is weaker than what we desire as although the topological properties are preserved, the metric structures are not. Thus we make the following definitions.

Definition 2.5.7. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces. A function $\varphi : \mathcal{X} \to \mathcal{Y}$ is said to be an *isometry* if $d_{\mathcal{Y}}(\varphi(x_1), \varphi(x_2)) = d_{\mathcal{X}}(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{X}$.

If in addition to being an isometry φ is a bijection, it is said that φ is an *isomorphism*. Finally, its is said that $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are *isomorphic* if there exists an isomorphism from $\mathcal{X}$ to $\mathcal{Y}$.

Remark 2.5.8. Notice that if φ is an isometry, then $\varphi(x_1) = \varphi(x_2)$ if and only if

$$0 = d_{\mathcal{Y}}(\varphi(x_1), \varphi(x_2)) = d_{\mathcal{X}}(x_1, x_2)$$

if and only if $x_1 = x_2$. Hence every isometry is automatically injective. Furthermore, since isometries preserve distances, it is clear that every isometry is continuous.

Example 2.5.9. Let $f : \mathbb{K} \to \mathbb{K}$ where $\mathbb{K}$ is equipped with the absolute value metric. It is elementary to see that f is an isometry (of metric spaces) if and only if there exists an $a \in \mathbb{K}$ and a $u \in \mathbb{K}$ such that $|u| = 1$ and $f(x) = ux + a$ for all $x \in \mathbb{K}$. Indeed, such a function is easily seen to be an isometry. Conversely, given an isometry f, let $a = f(0)$ and $u = f(1) - f(0)$. It is then not difficult to verify that $f(x) = ux + a$ for all $x \in \mathbb{K}$. Note all such isometries are isomorphisms.

Example 2.5.10. Using the results from Chapter 6, it will be possible to show that a function $f : \mathbb{K}^n \to \mathbb{K}^n$ is an isometry with respect to the 2-norms if and only if there exists an $\vec{a} \in \mathbb{K}^n$ and a $n \times n$ matrix U with entries in $\mathbb{K}$ such that $U^*U = I_n$ (where U^* is the conjugate transpose of U and I_n is the $n \times n$ identity matrix) and $f(\vec{x}) = U\vec{x} + \vec{a}$ for all $\vec{x} \in \mathbb{K}^n$.

Example 2.5.11. Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x, 0)$. Clearly f is an isometry that is not an isomorphism.

Example 2.5.12. The function $f : \mathbb{R}^{2n} \to \mathbb{C}^n$ in Example 2.5.3 is an isomorphism from $(\mathbb{R}^{2n}, \|\cdot\|_2)$ to $(\mathbb{C}^n, \|\cdot\|_2)$. However, for any $n \in \mathbb{N}$, f is not an isomorphism from $(\mathbb{R}^{2n}, \|\cdot\|_p)$ to $(\mathbb{C}^n, \|\cdot\|_p)$ for any $p \in [1, \infty)$ since

$$\|f(1, 1, \ldots, 1)\|_p = \left(\sum_{k=1}^{n} |1 + i|^p\right)^{\frac{1}{p}} = n^{\frac{1}{p}}\sqrt{2}$$

whereas $\|(1, \ldots, 1)\|_p = (2n)^{\frac{1}{p}}$. Similarly $\|f(1, \ldots, 1)\|_\infty = \sqrt{2}$ whereas $\|(1, \ldots, 1)\|_\infty = 1$ so f is not an isomorphism from $(\mathbb{R}^{2n}, \|\cdot\|_\infty)$ to $(\mathbb{C}^n, \|\cdot\|_\infty)$.

Remark 2.5.13. The notion of isomorphic metric spaces produces an equivalence relation on metric spaces. Indeed define two metric spaces to be equivalent if and only if there are isomorphic. It is elementary to verify that this is an equivalence relation on the collection of metric spaces. Note this is a superior notion of equivalence than homeomorphic as it preserves the metric structures in addition to the topology!

In fact, the following demonstrates that every metric space can be viewed as a subset of continuous real-valued functions!

Theorem 2.5.14. *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a metric space. Then $(\mathcal{X}, d_{\mathcal{X}})$ is isomorphic to a subset of $(\mathcal{C}_b(\mathcal{X}, \mathbb{R}), \|\cdot\|_\infty)$.*

Proof. Fix a point $a \in \mathcal{X}$. For each $z \in \mathcal{X}$, define a function $f_z : \mathcal{X} \to \mathbb{R}$ by

$$f_z(x) = d(x, z) - d(x, a)$$

for all $x \in \mathcal{X}$. We claim that $f_z \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$. To see this, notice for all $x \in \mathcal{X}$ that

$$|f_z(x)| = |d(x, z) - d(x, a)| \leq d(z, a)$$

by the reverse triangle inequality. Hence f_z is bounded by $d(z,a)$. Furthermore, to see that f_z is continuous, we notice that the functions $x \mapsto d(x,z)$ and $x \mapsto d(x,a)$ are continuous by Example 1.6.5. Hence $f_z \in \mathcal{C}_b(\mathcal{X}, \mathbb{R})$.

Define the map $\varphi : \mathcal{X} \to \mathcal{C}_b(\mathcal{X}, \mathbb{R})$ by

$$\varphi(z) = f_z.$$

We claim that φ is an isomorphism. To see this, notice for all $z_1, z_2 \in \mathcal{X}$ and $x \in \mathcal{X}$ that

$$\begin{aligned} |f_{z_1}(x) - f_{z_2}(x)| &= |(d(x,z_1) - d(x,a)) - (d(x,z_2) - d(x,a))| \\ &= |d(x,z_1) - d(x,z_2)| \leq d(z_1,z_2) \end{aligned}$$

by the triangle inequality. Hence $\|\varphi(z_1) - \varphi(z_2)\|_\infty \leq d(z_1,z_2)$ for all $z_1, z_2 \in \mathcal{X}$. However, since

$$|f_{z_1}(z_2) - f_{z_2}(z_2)| = |(d(z_2,z_1) - d(z_2,a)) - (d(z_2,z_2) - d(z_2,a))| = d(z_2,z_1)$$

we obtain that $\|\varphi(z_1) - \varphi(z_2)\|_\infty = d(z_1,z_2)$. Hence φ is an isometry as desired. ∎

Returning to our original goal of determining whether we can view every metric spaces inside a complete metric space, we are in a position to define what we mean by 'inside a complete metric space'. Of course, we would also like to take the complete metric space to be as small as possible. This returns us to the idea of a closure (Definition 1.5.20).

Definition 2.5.15. Let $(\mathcal{X}, d_\mathcal{X})$ be a metric space. A *completion* of $(\mathcal{X}, d_\mathcal{X})$ is a complete metric space $(\mathcal{Y}, d_\mathcal{Y})$ such that there exists an isometry $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\overline{\varphi(\mathcal{X})} = \mathcal{Y}$.

Example 2.5.16. Consider the metric space $(\mathbb{Q}, d)$ where $d(x,y) = |x-y|$ for all $x, y \in \mathbb{Q}$. Clearly $(\mathbb{R}, |\cdot|)$ is a completion of $(\mathbb{Q}, d)$.

In fact, we have already established everything we require to show that a metric space has a completion.

Corollary 2.5.17. *Every metric space has a completion.*

Proof. Let $(\mathcal{X}, d)$ be a metric space. By Theorem 2.5.14, there exists a subset $A \subseteq (\mathcal{C}_b(\mathcal{X}, \mathbb{R}), \|\cdot\|_\infty)$ such that $\mathcal{X}$ is isomorphic to A. As $(\mathcal{C}_b(\mathcal{X}, \mathbb{R}), \|\cdot\|_\infty)$ is complete by Theorem 2.2.5, $\overline{A}$ is complete by Theorem 2.1.12. Hence $\overline{A}$ is a completion of $\mathcal{X}$ by definition. ∎

Of course, it would be nice if each metric space only had one completion. The following demonstrates this is the case.

Proposition 2.5.18. *Any two completions of a metric space are isomorphic.*

Proof. Let $(\mathcal{X}, d_\mathcal{X})$ be a metric space. Suppose that $(\mathcal{Y}, d_\mathcal{Y})$ and $(\mathcal{Z}, d_\mathcal{Z})$ are completions of $(\mathcal{X}, d_\mathcal{X})$. Therefore there exists isometries $\varphi_\mathcal{Y} : \mathcal{X} \to \mathcal{Y}$ and $\varphi_\mathcal{Z} : \mathcal{X} \to \mathcal{Z}$ such that $\overline{\varphi_\mathcal{Y}(\mathcal{X})} = \mathcal{Y}$ and $\overline{\varphi_\mathcal{Z}(\mathcal{X})} = \mathcal{Z}$. Our goal is to extend the identity map from $\mathcal{X} \subseteq \mathcal{Y}$ to $\mathcal{X} \subseteq \mathcal{Z}$ to obtain an isometry from $\mathcal{Y}$ to $\mathcal{Z}$. To do this, we will make use of the fact that $\mathcal{Y}$ and $\mathcal{Z}$ are complete and thus have convergent Cauchy sequences.

To define an isometry $\varphi : \mathcal{Y} \to \mathcal{Z}$, let $y \in \mathcal{Y}$ be arbitrary. Hence, as $\mathcal{Y}$ is the closure of $\mathcal{X}$ there exists a sequence $(x_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $y = \lim_{n\to\infty} \varphi_\mathcal{Y}(x_n)$. However, as $(\varphi_\mathcal{Y}(x_n))_{n\geq 1}$ converges in $(\mathcal{Y}, d_\mathcal{Y})$, $(\varphi_\mathcal{Y}(x_n))_{n\geq 1}$ is Cauchy in $(\mathcal{Y}, d_\mathcal{Y})$. Therefore, $(x_n)_{n\geq 1}$ is Cauchy in $(\mathcal{X}, d_\mathcal{X})$ as $\varphi_\mathcal{Y}$ is an isometry. Hence $(\varphi_\mathcal{Z}(x_n))_{n\geq 1}$ also must be Cauchy as $\varphi_\mathcal{Z}$ is an isometry. Since $(\mathcal{Z}, d_\mathcal{Z})$ is complete, $(\varphi_\mathcal{Z}(x_n))_{n\geq 1}$ converges in $(\mathcal{Z}, d_\mathcal{Z})$. Let $z_y = \lim_{n\to\infty} \varphi_\mathcal{Z}(x_n)$. We would like to define $\varphi : \mathcal{Y} \to \mathcal{Z}$ such that $f(y) = z_y$.

There is one technical issue with this definition that we should get out of the way; that is, we desire to show that if $(x'_n)_{n\geq 1}$ is another sequence of elements of $\mathcal{X}$ such that $y = \lim_{n\to\infty} \varphi_\mathcal{Y}(x'_n)$, then $z_y = \lim_{n\to\infty} \varphi_\mathcal{Z}(x'_n)$. This will demonstrate that the sequence of elements of $\mathcal{X}$ we choose converging to $y \in \mathcal{Y}$ does not affect the limit in $(\mathcal{Z}, d_\mathcal{Z})$. To see this, notice by the triangle inequality and properties of limits that

$$\begin{aligned}
\lim_{n\to\infty} d_\mathcal{Z}(\varphi_\mathcal{Z}(x'_n), \varphi_\mathcal{Z}(x_n)) &= \lim_{n\to\infty} d_\mathcal{X}(x'_n, x_n) \\
&= \lim_{n\to\infty} d_\mathcal{Y}(\varphi_\mathcal{Y}(x'_n), \varphi_\mathcal{Y}(x_n)) \\
&= d_\mathcal{Y}(y, y) = 0.
\end{aligned}$$

Hence as $z_y = \lim_{n\to\infty} \varphi_\mathcal{Z}(x_n)$, the above easily implies $z_y = \lim_{n\to\infty} \varphi_\mathcal{Z}(x'_n)$. Hence the claim is complete.

Hence we may define $\varphi : \mathcal{Y} \to \mathcal{Z}$ as follows: for each $y \in \mathcal{Y}$ choose a sequence $(x_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $y = \lim_{n\to\infty} \varphi_\mathcal{Y}(x_n)$ and define $\varphi(y) = \lim_{n\to\infty} \varphi_\mathcal{Z}(x_n)$. We claim that φ is an isometry. To see this, let $y, y' \in \mathcal{Y}$ be arbitrary. Choose sequence $(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $y = \lim_{n\to\infty} \varphi_\mathcal{Y}(x_n)$ and $y' = \lim_{n\to\infty} \varphi_\mathcal{Y}(x'_n)$. Then, by the triangle inequality and properties of limits,

$$\begin{aligned}
d_\mathcal{Z}(\varphi(y), \varphi(y')) &= \lim_{n\to\infty} d_\mathcal{Z}(\varphi_\mathcal{Z}(x_n), \varphi_\mathcal{Z}(x'_n)) \\
&= \lim_{n\to\infty} d_\mathcal{X}(x_n, x'_n) \\
&= \lim_{n\to\infty} d_\mathcal{Y}(\varphi_\mathcal{Y}(x_n), \varphi_\mathcal{Y}(x'_n)) \\
&= d_\mathcal{Y}(y, y').
\end{aligned}$$

Hence φ is an isometry (and therefore injective).

To see that φ is surjective (and thus a bijection) let $z \in \mathcal{Z}$ be arbitrary. Note as $\mathcal{Z}$ is the completion of $\varphi_\mathcal{Z}(\mathcal{X})$, there exists a sequence $(x_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $z = \lim_{n\to\infty} \varphi_{\mathcal{Z}}(x_n)$. By similar arguments to those above, $y = \lim_{n\to\infty} \varphi_{\mathcal{Y}}(x_n)$ exists and thus $\varphi(y) = z$. Hence, as $z \in \mathcal{Z}$ was arbitrary, φ is surjective. Hence $\mathcal{Y}$ and $\mathcal{Z}$ are isomorphic. ∎

The above demonstrates everything we could possibly want to know about completions for metric spaces. However, if we are dealing with normed linear spaces, we would like our maps to preserve the vector space structures. In particular, we would like our maps to be linear in order to obtain the appropriate notion of equality. Thus we make the following definitions.

Definition 2.5.19. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces. A function $\varphi : \mathcal{X} \to \mathcal{Y}$ is said to be an *isometry* if φ is linear and $\|\varphi(\vec{x})\|_{\mathcal{Y}} = \|\vec{x}\|_{\mathcal{X}}$ for all $\vec{x} \in \mathcal{X}$.

If in addition to being an isometry φ is a bijection, it is said that φ is an *isomorphism*. Finally, its is said that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are *isomorphic* if there exists a isomorphism from $\mathcal{X}$ to $\mathcal{Y}$.

Note $\|\varphi(\vec{x})\|_{\mathcal{Y}} = \|\vec{x}\|_{\mathcal{X}}$ for all $\vec{x} \in \mathcal{X}$ along with the fact that φ is linear implies

$$\|\varphi(\vec{x}_1) - \varphi(\vec{x}_2)\|_{\mathcal{Y}} = \|\vec{x}_1 - \vec{x}_2\|_{\mathcal{X}}.$$

In particular, isometries for normed linear spaces are isometries for metric spaces.

Example 2.5.20. Note Example 2.5.3 shows that $(\mathbb{R}^{2n}, \|\cdot\|_2)$ and $(\mathbb{C}^n, \|\cdot\|_2)$ were isomorphic metric spaces. If we view $\mathbb{C}^n$ as a vector space over $\mathbb{R}$, then the same proof shows that $(\mathbb{R}^{2n}, \|\cdot\|_2)$ and $(\mathbb{C}^n, \|\cdot\|_2)$ are isomorphic normed linear spaces. However, if we view $\mathbb{C}^n$ as a vector space over $\mathbb{C}$, then $(\mathbb{R}^{2n}, \|\cdot\|_2)$ and $(\mathbb{C}^n, \|\cdot\|_2)$ are not isomorphic. Thus we will always view $\mathbb{C}^n$ as a vector space over $\mathbb{C}$ (as otherwise we should just consider $\mathbb{R}^{2n}$).

Of course, when dealing with normed linear spaces, we would like our completions to behave well with respect to the vector space structures. Thus we provide an alternate and improved definition for the completion of a normed linear space.

Definition 2.5.21. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed linear space. A *completion* of $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ such that there exists an isometry $\varphi : \mathcal{X} \to \mathcal{Y}$ such that $\overline{\varphi(\mathcal{X})} = \mathcal{Y}$.

Of course Corollary 2.5.17 demonstrates that every normed linear space has a completion as a metric space whereas Proposition 2.5.18 shows that there is only one possible completion for each metric space. As a normed linear space completion is a metric space completion, the completion in Corollary 2.5.17 is the only candidate for a normed linear space completion. However, it is not clear whether the function $\vec{z} \mapsto f_{\vec{z}}$ (where $f_{\vec{z}}(\vec{x}) = \|\vec{z} - \vec{x}\| - \|\vec{x} - \vec{a}\|$

for all $\vec{x} \in \mathcal{X}$ and $\vec{a} \in \mathcal{X}$ is fixed) is linear. Thus it is unclear that every normed linear space has a normed linear space completion.

It turns out that every normed linear space has a completion as a normed linear space. There are two methods we could take to proving this. The first is to take the metric space completion of a normed linear space $(\mathcal{X}, \|\cdot\|)$ and define a vector space structure on the completion via the vector space structure on $\mathcal{X}$. The difficulty then comes in definition the norm and verifying the definition does produce a norm.

We will proceed with an alternative description of the completion for normed linear spaces. This description uses equivalence of Cauchy sequences and it of use in showing that structures more specific than normed linear spaces have closures with appropriate properties (see Theorem 6.2.2).

Theorem 2.5.22. *Every normed linear space has a completion.*

Proof. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a normed linear space. Let V denote the set of all Cauchy sequences in $\mathcal{X}$. Note that V is non-empty as every constant sequence is Cauchy. In fact, the constant sequences will give us the embedding of $\mathcal{X}$ into its completion. Furthermore, since given Cauchy sequences $(\vec{x}_n)_{n\geq 1}$ and $(\vec{y}_n)_{n\geq 1}$ and $\alpha \in \mathbb{K}$, the sequences

$$(\vec{x}_n + \vec{y}_n)_{n\geq 1} \qquad \text{and} \qquad (\alpha\vec{x}_n)_{n\geq 1}$$

are Cauchy by the properties of the norm, V is a vector space over $\mathbb{K}$. However, V is not the normed linear space we want. To construct a normed linear space, we require a quotient.

Let

$$W = \left\{(\vec{x}_n)_{n\geq 1} \in V \mid \lim_{n\to\infty} \vec{x}_n = \vec{0}\right\}.$$

Clearly W is a subspace of V. Recall an equivalence relation $\sim$ may be placed on V via $\vec{v}_1 \sim \vec{v}_2$ if and only if $\vec{v}_1 - \vec{v}_2 \in W$. Furthermore, recall if $[\vec{v}]$ denotes the equivalence class of $\vec{v}$ and V/W is the set of all equivalence classes, then $\mathcal{V}/W$ is a vector space with operations $[\vec{v}_1] + [\vec{v}_2] = [\vec{v}_1 + \vec{v}_2]$ and $\alpha[\vec{v}] = [\alpha\vec{v}]$. In particular, two elements $(\vec{x}_n)_{n\geq 1}, (\vec{y}_n)_{n\geq 1} \in V$ produce the same element in V/W if and only if

$$\lim_{n\to\infty} \|\vec{x}_n - \vec{y}_n\|_{\mathcal{X}} = 0$$

Define $\|\cdot\| : V/W \to [0,\infty)$ by

$$\|[(\vec{x}_n)_{n\geq 1}]\| = \limsup_{n\to\infty} \|\vec{x}_n\|_{\mathcal{X}}$$

and note that since $(\vec{x}_n)_{n\geq 1}$ is Cauchy and thus bounded by Lemma 2.1.3, $\|\cdot\|$ does indeed map into $[0,\infty)$. However, since we are dealing with equivalence classes, we must check that $\|\cdot\|$ is well-defined. To see this, notice if $[(\vec{x}_n)_{n\geq 1}] = [(\vec{y}_n)_{n\geq 1}]$ then

$$\lim_{n\to\infty} \|\vec{x}_n - \vec{y}_n\|_{\mathcal{X}} = 0.$$

so

$$\limsup_{n\to\infty} \|\vec{x}_n\|_{\mathcal{X}} = \limsup_{n\to\infty} \|\vec{y}_n\|_{\mathcal{X}}$$

by the reverse triangle inequality. Hence $\|\cdot\|$ is well-defined. To see that $\|\cdot\|$ is indeed a norm, note that $\|[(\vec{x}_n)_{n\geq 1}]\| = 0$ if and only if $\limsup_{n\to\infty} \|\vec{x}_n\|_{\mathcal{X}} = 0$ if and only if $(\vec{x}_n)_{n\geq 1} \in W$ if and only if $[(\vec{x}_n)_{n\geq 1}] = \vec{0}_{V/W}$. As the other properties from Definition 1.1.14 are trivial to verify, $(V/W, \|\cdot\|)$ is a normed linear space.

We will postpone the proof that $(V/W, \|\cdot\|)$ is complete momentarily in order to demonstrate some facts in relation to $\mathcal{X}$. Define $\varphi : \mathcal{X} \to V/W$ by $\varphi(\vec{x}) = [(\vec{x})_{n\geq 1}]$; that is, map each element of $\mathcal{X}$ to a constant sequence. Clearly φ is well-defined, linear, and an isometry. We claim that $\varphi(\mathcal{X})$ is dense in V/W.

To see that $\varphi(\mathcal{X})$ is dense in V/W, let $[(\vec{x}_n)_{n\geq 1}] \in V/W$ be arbitrary and let $\epsilon > 0$ be arbitrary. Since $(\vec{x}_n)_{n\geq 1}$ is Cauchy in $\mathcal{X}$, there exists an $N \in \mathbb{N}$ such that $\|\vec{x}_n - \vec{x}_m\|_{\mathcal{X}} < \epsilon$ for all $n, m \geq N$. Hence

$$\|\varphi(\vec{x}_N) - [(\vec{x}_n)_{n\geq 1}]\| \leq \epsilon$$

by the definition of $\|\cdot\|$. Therefore, as $\epsilon > 0$ was arbitrary, $[(\vec{x}_n)_{n\geq 1}]$ is in the closure of $\varphi(\mathcal{X})$. Therefore, as $[(\vec{x}_n)_{n\geq 1}] \in V/W$ was arbitrary, $\varphi(\mathcal{X})$ is dense in V/W.

To see that $(V/W, \|\cdot\|)$ is complete, let $(\vec{z}_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $(V/W, \|\cdot\|)$. Since $\varphi(\mathcal{X})$ is dense in V/W, for each $n \in \mathbb{N}$ there exists an $\vec{x}_n \in \mathcal{X}$ such that

$$\|\varphi(\vec{x}_n) - \vec{z}_n\| < \frac{1}{n}.$$

We claim that $(\vec{x}_n)_{n\geq 1}$ is a Cauchy sequence of elements of $\mathcal{X}$ and thus is an element of V. To see this, notice for all $n, m \in \mathbb{N}$ that

$$\begin{aligned} \|\vec{x}_n - \vec{x}_m\|_{\mathcal{X}} &= \|\varphi(\vec{x}_n) - \varphi(\vec{x}_m)\| \\ &\leq \|\varphi(\vec{x}_n) - \vec{z}_n\| + \|\vec{z}_n - \vec{z}_m\| + \|\vec{z}_m - \varphi(\vec{x}_m)\| \\ &\leq \frac{1}{n} + \frac{1}{m} + \|\vec{z}_n - \vec{z}_m\|. \end{aligned}$$

Therefore, as $(\vec{z}_n)_{n\geq 1}$ is Cauchy, it is elementary to verify the above inequality implies $(\vec{x}_n)_{n\geq 1}$ is Cauchy. Finally, to see that $(\vec{z}_n)_{n\geq 1}$ converges to $\vec{z} = [(\vec{x}_n)_{n\geq 1}]$, we notice that

$$\lim_{n\to\infty} \|\varphi(\vec{x}_n) - \vec{z}\| = 0$$

as $(\vec{x}_n)_{n\geq 1}$ is Cauchy. Hence as

$$\|\vec{z}_n - \vec{z}\| \leq \|\vec{z}_n - \varphi(\vec{x}_n)\| + \|\varphi(\vec{x}_n) - \vec{z}\| \leq \frac{1}{n} + \|\varphi(\vec{x}_n) - \vec{z}\|,$$

we obtain that $(\vec{z}_n)_{n\geq 1}$ converges to $\vec{z} = [(\vec{x}_n)_{n\geq 1}]$. Therefore, as $(\vec{z}_n)_{n\geq 1}$ was an arbitrary Cauchy sequence, V/W is complete thereby completing the proof. ∎

We saw in Example 2.2.9 that the p-norm on the continuous functions is not complete. Thus a natural question to ask is, "What is the completion of $\mathcal{C}[0,1]$ with respect to the p-norm?" This question can only be answered with knowledge from MATH 4012.

Chapter 3

Banach Space Theorems

We have already seen several properties that make Banach spaces excellent analytic objects to work with. The goal of this chapter is to develop several important theorems in Banach space theory that have wide reaching applications.

3.1 Banach Contractive Mapping Theorem

Our first goal is to prove a theorem that says certain maps on a space have a 'fixed point'; that is, a map $f : X \to X$ has a fixed point if there is an element $a \in X$ such that $f(a) = a$. One corollary of this theorem is that we can find solutions to several types of equations using fixed points.

There are many types of fixed point theorems in analysis. The one specific one we will be investigating involves the following type of map.

Definition 3.1.1. Let $(\mathcal{X}, d)$ be a metric space. A map $f : \mathcal{X} \to \mathcal{X}$ is said to be a *contraction* if there exists a $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in \mathcal{X}$.

It is elementary to see that every contraction is a continuous function. However, not all continuous functions are contractions.

Example 3.1.2. The functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 2$ and $g(x) = x^2$ for all $x \in \mathbb{R}$ are not contractions. However, the function $h : [0, 1] \to \mathbb{R}$ defined by $h(x) = \cos(x)$ is a contraction. Indeed, to see that h is a contraction, we note by the Mean Value Theorem that for all $x, y \in \mathbb{R}$ with $x < y$ that there exists a $c \in (x, y)$ such that

$$|h(x) - h(y)| = |\cos(x) - \cos(y)| \leq |\sin(c)||x - y|.$$

Thus, as $\sin(c) \neq \pm 1$ for all $x \in [0, 1]$, we obtain that h is a contraction.

Example 3.1.3. Let $\mathcal{X}$ be a normed linear space and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{X})$. Then T is a contraction if and only if $\|T\| < 1$.

The reason contractive maps are so important is the following fixed point theorem for complete metric spaces.

Theorem 3.1.4 (Banach Contractive Mapping Theorem). *Let $(\mathcal{X}, d)$ be a complete metric space and let $f : \mathcal{X} \to \mathcal{X}$ be a contraction. Then there exists a unique point $x_0 \in \mathcal{X}$ such that $f(x_0) = x_0$ (that is, f has a unique fixed point).*

Proof. Let $(\mathcal{X}, d)$ be a complete metric space and let $f : \mathcal{X} \to \mathcal{X}$ be a contraction. Therefore there exists a $k \in [0, 1)$ such that

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in \mathcal{X}$. First we will show the existence of an $x_0 \in X$ such that $f(x_0) = x_0$.

Choose any point $x_1 \in \mathcal{X}$. For each $n \in \mathbb{N}$, recursively define $x_n \in \mathcal{X}$ via $x_{n+1} = f(x_n)$. We claim that $(x_n)_{n\geq 1}$ is Cauchy. To see this, let $\epsilon > 0$ be arbitrary. Notice for all $q \in \mathbb{N}$ with $q \geq 2$ that

$$d(x_{q+1}, x_q) = d(f(x_q), f(x_{q-1})) \leq kd(x_q, x_{q-1}) \leq \ldots \leq k^{q-1}d(x_2, x_1).$$

Choose $N \in \mathbb{N}$ such that $\frac{k^{n-1}}{1-k}d(x_2, x_1) < \epsilon$ for all $n \geq N$. Therefore, if $m, n \in \mathbb{N}$ are such that $m \geq n \geq N$, then

$$\begin{aligned} d(x_m, x_n) &\leq \sum_{q=n}^{m-1} d(x_{q+1}, x_q) \quad \text{by the triangle inequality} \\ &\leq \sum_{q=n}^{m-1} k^{q-1}d(x_2, x_1) \\ &\leq \sum_{q=n}^{\infty} k^{q-1}d(x_2, x_1) \\ &= \frac{k^{n-1}}{1-k}d(x_2, x_1) < \epsilon \end{aligned}$$

as $0 \leq k < 1$. Hence $(x_n)_{n\geq 1}$ is Cauchy as desired.

Since $\mathcal{X}$ is complete, $(x_n)_{n\geq 1}$ converges in $\mathcal{X}$. Let $x_0 = \lim_{n\to\infty} x_n$. We claim that $f(x_0) = x_0$. To see this, notice since f is continuous that

$$f(x_0) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$$

as desired.

Finally, to show that x_0 is the unique fixed point, suppose there exists a point $y \in \mathcal{X}$ such that $f(y) = y$. Since

$$d(x_0, y) = d(f(x_0), f(y)) \leq kd(x_0, y),$$

and since $0 \leq k < 1$, we see that $d(x_0, y) = 0$ and thus $y = x_0$ as desired. ∎

Remark 3.1.5. Before moving on, we note that the Banach Contractive Mapping Theorem (Theorem 3.1.4) fails if one replaces the assumption 'f is a contraction' with the weaker notion that '$d(f(x), f(y)) < d(x, y)$ for all $x, y \in \mathcal{X}$ with $x \neq y$'. Indeed consider $\mathcal{X} = [1, \infty)$, which is a complete metric space with respect to the absolute value metric as $\mathcal{X}$ is a closed subset of the complete metric space $(\mathbb{R}, |\cdot|)$.

Define $f : \mathcal{X} \to \mathcal{X}$ by

$$f(x) = x + \frac{1}{x}$$

for all $x \in \mathcal{X}$. To see that f is well-defined (that is, $f(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$), notice that $f(x) > x$ for all $x \in \mathcal{X}$. Hence as $[x, \infty) \subseteq \mathcal{X}$ for all $x \in \mathcal{X}$, f is well-defined.

To see that $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathcal{X}$, notice that f is differentiable on $\mathcal{X}$ with

$$f'(x) = 1 - \frac{1}{x^2}.$$

Therefore, if $x, y \in \mathcal{X}$ are such that $x \neq y$, then the Mean Value Theorem implies there exists a $c \in (1, \infty)$ such that

$$|f(x) - f(y)| \leq |f'(x)||x - y| = \left(1 - \frac{1}{c^2}\right)|x - y| < |x - y|.$$

Hence f has the first desired property.

To see that f does not have a fixed point, suppose otherwise that there exists an $x_0 \in \mathcal{X}$ such that $f(x_0) = x_0$. Hence

$$x_0 = f(x_0) = x_0 + \frac{1}{x_0}$$

so $\frac{1}{x_0} = 0$, which is an obvious contradiction. Hence f does not have a fixed point.

Although it may not seem like it, but the Banach Contractive Mapping Theorem (Theorem 3.1.4) is quite powerful. To emphasize this power, we note we can prove the following result which is assumed in pretty much every differential equations course without proof.

Theorem 3.1.6 (Picard's Theorem). *Let $K \in \mathcal{C}([a, b] \times [c, d], \mathbb{R})$ and suppose there exists an $M \in \mathbb{R}$ such that*

$$|K(x, y_1) - K(x, y_2)| \leq M|y_1 - y_2|$$

for all $x \in [a, b]$ and $y_1, y_2 \in [c, d]$. Then for any $x_0 \in (a, b)$ and $y_0 \in (c, d)$ there exists a unique function f on an open interval I containing x_0 such that $f(x_0) = y_0$ and $f'(x) = K(x, f(x))$ for all $x \in I$.

Proof. First, since

$$|K(x, y_1) - K(x, y_2)| \leq M|y_1 - y_2|$$

for all $x \in [a, b]$ and $y_1, y_2 \in [c, d]$, we see that there exists a constant $k_1 = M(d - c) > 0$ such that

$$|K(x, y) - K(x, c)| \leq k_1$$

for all $y \in [c, d]$ and all $x \in [a, b]$. Furthermore, as $x \mapsto K(x, a)$ is continuous, the Extreme Value Theorem implies there exists a constant k_2 such that $|K(x, c)| \leq k_2$ for all $x \in [a, b]$. Hence

$$|K(x, y)| \leq k_1 + k_2$$

for all $(x, y) \in [a, b] \times [c, d]$. Let $k_0 = k_1 + k_2 > 0$ so that $|K(x, y)| \leq k_0$ for all $(x, y) \in [a, b] \times [c, d]$ (note the existence of k_0 is far simplified with the multivariate Extreme Value Theorem; see Chapter 4).

Choose an $\epsilon > 0$ so that $[y_0 - \epsilon, y_0 + \epsilon] \subseteq [c, d]$ and choose a $\delta > 0$ so that

$$0 < \delta < \min \left\{ \frac{\epsilon}{k_0}, \frac{1}{M}, |x_0 - a|, |x_0 - b| \right\}.$$

Let $I = [x_0 - \delta, x_0 + \delta] \subseteq [a, b]$ and let $J = [y_0 - \epsilon, y_0 + \epsilon] \subseteq [c, d]$. Since $\mathcal{C}(I, J)$ is easily seen to be a closed subset of $(\mathcal{C}_b(I, \mathbb{R}), \|\cdot\|_\infty)$, we obtain that $(\mathcal{C}(I, J), d_\infty)$ is a complete metric space.

Define $\Gamma : \mathcal{C}(I, J) \to \mathcal{C}(I, J)$ as follows: given an $f \in \mathcal{C}(I, J)$, $\Gamma(f) : I \to J$ is the function defined by

$$\Gamma(f)(x) = y_0 + \int_{x_0}^{x} K(t, f(t))\, dt$$

for all $x \in I$. We claim that Γ is a well-defined function; that is $\Gamma(f)(x)$ is well-defined for all $x \in I$ and $f \in \mathcal{C}(I, J)$, that $\Gamma(f)(x) \in J$ for all $x \in I$ and $f \in \mathcal{C}(I, J)$, and that $\Gamma(f)$ is continuous.

To see that $\Gamma(f)(x)$ is well-defined for all $x \in I$ and $f \in \mathcal{C}(I, J)$, notice that $K(t, f(t))$ makes sense for all $t \in [x_0, x] \cup [x, x_0]$ as f is defined on I and $[x_0, x] \cup [x, x_0] \subseteq I$. Furthermore, since $K \in \mathcal{C}([a, b] \times [c, d], \mathbb{R})$, $t \mapsto K(t, f(t))$ is continuous be a composition of continuous functions. Therefore, since continuous functions are Riemann integrable, we have that $\Gamma(f)(x)$ makes sense for all $x \in I$ and $f \in \mathcal{C}(I, J)$.

To see that $\Gamma(f)(x) \in J$ for all $x \in I$ and $f \in \mathcal{C}(I, J)$, notice that

$$\begin{aligned} |\Gamma(f)(x) - y_0| &= \left| \int_{x_0}^{x} K(t, f(t))\, dt \right| \\ &\leq \left| \int_{x_0}^{x} |K(t, f(t))|\, dt \right| \\ &\leq \left| \int_{x_0}^{x} k_0\, dt \right| \\ &\leq k_0 |x - x_0| \leq k_0 \delta < \epsilon. \end{aligned}$$

Hence $\Gamma(f)(x) \in [y_0 - \epsilon, y_0 + \epsilon] = J$ for all $x \in I$ and $f \in \mathcal{C}(I, J)$.

Finally, to see that $\Gamma(f)$ is a continuous function, notice for all $x_1, x_2 \in I$ with $x_1 < x_2$ that

$$\begin{aligned}
|\Gamma(f)(x_2) - \Gamma(f)(x_1)| &\leq \left|\int_{x_1}^{x_2} K(t, f(t))\, dt\right| \\
&\leq \int_{x_1}^{x_2} |K(t, f(t))|\, dt \\
&\leq \int_{x_1}^{x_2} k_0\, dt = k_0(x_2 - x_1).
\end{aligned}$$

From this it is elementary to see that $\Gamma(f)$ is a continuous function. Hence Γ is a well-defined map.

We claim that Γ is a contractive map. To see this, notice for all $f, g \in \mathcal{C}(I, J)$ that for all $x \in I$ we have

$$\begin{aligned}
|\Gamma(f)(x) - \Gamma(g)(x)| &= \left|\int_{x_0}^{x} K(t, f(t)) - K(t, g(t))\, dt\right| \\
&\leq \left|\int_{x_0}^{x} |K(t, f(t)) - K(t, g(t))|\, dt\right| \\
&\leq \left|\int_{x_0}^{x} M|f(t) - g(t)|\, dt\right| \\
&\leq \left|\int_{x_0}^{x} M\, \|f - g\|_\infty\, dt\right| \\
&\leq M|x - x_0|\, \|f - g\|_\infty \leq M\delta\, \|f - g\|_\infty\, .
\end{aligned}$$

Hence, as $M\delta < 1$ by construction, we obtain that Γ is a contractive map. Thus the Banach Contractive Mapping Theorem (Theorem 3.1.4) implies there exists a unique $f \in \mathcal{C}(I, J)$ such that $\Gamma(f) = f$.

As $\Gamma(f) = f$, we have for all $x \in I$ that

$$f(x) = y_0 + \int_{x_0}^{x} K(t, f(t))\, dt.$$

Clearly this implies $f(x_0) = y_0$. Moreover, by the Fundamental Theorem of Calculus, we have that

$$f'(x) = K(x, f(x))$$

for all $x \in (x_0 - \delta, x_0 + \delta)$. Hence the proof is complete. ∎

3.2 The Baire Category Theorem

In this section, we will prove one of the most surprisingly useful theorems pertaining to complete metric spaces. Although it is not be apparent from the statement of the theorem its uses, we will see in the subsequent sections some of its applications.

The theorem we are trying to prove (Theorem 3.2.7) characterizes how specific subsets of a complete metric space behave. The types of sets involved are outlined in the following definition.

Definition 3.2.1. Let $(\mathcal{X}, d)$ be a metric space. A subset $A \subseteq \mathcal{X}$ is said to be

- *nowhere dense* if $\mathrm{int}(\overline{A}) = \emptyset$.
- *first category in* $(\mathcal{X}, d)$ if $A = \bigcup_{n=1}^{\infty} A_n$ where each $A_n \subseteq \mathcal{X}$ is nowhere dense.
- *second category in* $(\mathcal{X}, d)$ if A is not first category.
- *residual* if A^c is first category

Example 3.2.2. Consider $\mathcal{X} = \mathbb{R}$. Clearly for each $x \in \mathbb{R}$ the set $\{x\}$ is nowhere dense. Furthermore, from this it is clear that $\mathbb{Q}$ is of first category in $\mathbb{R}$ and their complements are residual in $\mathbb{R}$. One question that Theorem 3.2.7 will answer is whether $\mathbb{R}$ is of first or of second category.

Example 3.2.3. The Cantor set is nowhere dense. Indeed the Cantor set is closed and has no interior by Corollary 1.5.13. Hence the Cantor set is also of first category.

Remark 3.2.4. More often than not, given a metric space $(\mathcal{X}, d)$, we are interested in whether $\mathcal{X}$ is of first or second category in itself. Consequently, we see that

$$\mathcal{X} = \bigcup_{n=1}^{\infty} A_n \quad \Rightarrow \quad \mathcal{X} = \bigcup_{n=1}^{\infty} \overline{A_n}.$$

Therefore, as the closure of a nowhere dense set is clearly nowhere dense, $\mathcal{X}$ is of first category if and only if $\mathcal{X}$ is a countable union of closed nowhere dense sets. Consequently, the following theorem is useful as it describes an alternate characterization of closed, nowhere dense subsets.

Lemma 3.2.5. *Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. Then A is closed and nowhere dense if and only if A^c is open and dense in $\mathcal{X}$.*

Proof. Suppose A is closed and nowhere dense. Hence $\emptyset = \mathrm{int}(\overline{A}) = \mathrm{int}(A)$ and A^c is open. To see that A^c is dense, let $x \in \mathcal{X}$ be arbitrary. Since $\mathrm{int}(A) = \emptyset$, for each $\epsilon > 0$ we must have that $B(x, \epsilon) \cap A^c \neq \emptyset$. Hence, as $x \in \mathcal{X}$ was arbitrary, A^c is dense in $\mathcal{X}$.

Conversely, suppose A^c is open and dense. Clearly this implies A is closed. To see that A is nowhere dense (that is $\mathrm{int}(\overline{A}) = \mathrm{int}(A)$ is empty) let $a \in A$ be arbitrary. Since A^c is dense in $\mathcal{X}$, for all $\epsilon > 0$ there exists an $x \in A^c$ such that $x \in B(a, \epsilon)$. Hence for all $\epsilon > 0$, $B(a, \epsilon) \not\subseteq A$. Hence $a \notin \mathrm{int}(A)$. Therefore, as $a \in A$ was arbitrary, $\mathrm{int}(A) = \emptyset$ so A is nowhere dense. ∎

Corollary 3.2.6. *Let $(\mathcal{X}, d)$ be a metric space. Then $\mathcal{X}$ is of first category in itself if and only if there exists a sequence $(U_n)_{n\geq 1}$ of open dense subsets of $\mathcal{X}$ with $\bigcap_{n=1}^{\infty} U_n = \emptyset$.*

Proof. The result follows directly from Lemma 3.2.5 and Remark 3.2.4. ∎

Using this corollary, the following important theorem implies every complete metric space is of second category.

Theorem 3.2.7 (Baire's Category Theorem). *Let $(\mathcal{X}, d)$ be a complete metric space. Suppose $(U_n)_{n\geq 1}$ is a sequence of open dense subsets of $\mathcal{X}$. Then $\bigcap_{n=1}^{\infty} U_n$ is dense in $\mathcal{X}$. Hence $\mathcal{X}$ is of second category in itself.*

Proof. To see that $\bigcap_{n=1}^{\infty} U_n$ is dense in $\mathcal{X}$, let $x \in \mathcal{X}$ and $\epsilon > 0$ be arbitrary. We must show that there exists an element of $\bigcap_{n=1}^{\infty} U_n$ within ϵ of x. To do this, it is first useful to note that if $y \in \mathcal{X}$ and $r > 0$ then for any $0 < r' < r$ we have that

$$B[y, r'] \subseteq B(y, r).$$

Let $r_1 = \frac{1}{2}\epsilon$. Since U_1 is dense in $\mathcal{X}$, there exists an element $a_1 \in U_1$ such that $d(a_1, x) < r_1$. Since U_1 is open, by using the above comment there exists an $0 < r_2 < \frac{1}{4}\epsilon$ such that $B[a_1, r_2] \subseteq U_1$ (i.e. choose an open ball around a_1 contained in U_1 and then decrease the radius of the ball).

Since U_2 is dense in $\mathcal{X}$, there exists an element $a_2 \in U_2$ such that $d(a_2, a_1) < r_2$. Hence $a_2 \in B(a_1, r_2)$ so $a_2 \in U_2 \cap B(a_1, r_2)$. Hence, since $U_2 \cap B(a_1, r_2)$ is open, there exists an $0 < r_2 < \frac{1}{2^3}\epsilon$ such that $B[a_2, r_3] \subseteq U_2 \cap B(a_1, r_2)$.

By recursion, for each $n \in \mathbb{N}$ there exists an $a_n \in U_n \cap B(a_{n-1}, r_n)$ and an $0 < r_{n+1} < \frac{1}{2^{n+1}}\epsilon$ such that $d(a_n, a_{n-1}) < r_n$ and $B[a_n, r_{n+1}] \subseteq U_n \cap B(a_{n-1}, r_n)$.

For each $n \in \mathbb{N}$, let $F_n = B[a_n, r_{n+1}]$. Clearly $(F_n)_{n\geq 1}$ is a sequence of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ and $\lim_{n\to\infty} \mathrm{diam}(F_n) = 0$ (as $\mathrm{diam}(F_n) \leq 2r_{n+1}$). Hence Cantor's Theorem (Theorem 2.3.4) implies that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Let $y \in \bigcap_{n=1}^{\infty} F_n$. We claim that $y \in \bigcap_{n=1}^{\infty} U_n$ and $d(x, y) < \epsilon$. To see this, notice that $F_n \subseteq U_n$ for all $n \in \mathbb{N}$. Hence as $y \in F_n$ for all $n \in \mathbb{N}$, $y \in U_n$ for all $n \in \mathbb{N}$ so $y \in \bigcap_{n=1}^{\infty} U_n$. To see that $d(x, y) < \epsilon$, we note that $y \in F_1 = B[a_1, r_2]$ so $d(y, a_1) \leq r_2$. Hence

$$d(x, y) \leq d(x, a_1) + d(a_1, y) \leq r_1 + r_2 < \epsilon$$

by the triangle inequality. Hence the result follows. ∎

There are numerous uses of the Baire Category Theorem. We conclude this section by demonstrating a use related to the structure of $\mathbb{R}$ whereas further uses will be demonstrated in subsequent sections.

Our first use of the Baire Category Theorem will be to analyze certain subsets of metric spaces.

Definition 3.2.8. Let $(\mathcal{X}, d)$ be a metric space. A subset $A \subseteq \mathcal{X}$ is said to be *G_δ* if there exists a collection of open sets $\{U_n\}_{n=1}^\infty$ such that $A = \bigcap_{n=1}^\infty U_n$.

Similarly, a subset $B \subseteq \mathcal{X}$ is said to be *F_σ* if there exists a collection of closed sets $\{F_n\}_{n=1}^\infty$ such that $A = \bigcup_{n=1}^\infty F_n$.

Remark 3.2.9. It is not difficult to see using De Morgan's Laws that A is G_δ if and only if A^c is F_σ.

Example 3.2.10. Every closed subset of a metric space is G_δ. To see this, suppose F be a closed subset of a metric space $(\mathcal{X}, d)$. If $F = \emptyset$ then, as $\emptyset$ is open and as $\bigcap_{n=1}^\infty \emptyset = \emptyset$, we obtain that F is G_δ.

Otherwise, suppose F is not empty. For each $n \in \mathbb{N}$, let

$$U_n = \bigcup_{x \in F} B\left(x, \frac{1}{n}\right).$$

Clearly each U_n is an open subset such that $F \subseteq U_n$. Hence

$$F \subseteq \bigcap_{n=1}^\infty U_n.$$

For the other inclusion, suppose $x \in F^c$. Therefore $x \notin \overline{F} = F$ as F is closed. Hence Lemma 1.6.12 implies that $d(x, F) > 0$. Choose $n \in \mathbb{N}$ such that

$$d(x, F) \geq \frac{1}{n} > 0.$$

Hence $d(x, y) \geq \frac{1}{n}$ for all $y \in \mathcal{F}$. Thus, by the definition of U_n, $x \notin U_n$. Whence $x \notin \bigcap_{n=1}^\infty U_n$. Hence

$$F = \bigcap_{n=1}^\infty U_n$$

so F is G_δ.

For another example, we prove the following.

Proposition 3.2.11. *The rationals are not a G_δ subset of* $(\mathbb{R}, |\cdot|)$.

Proof. Suppose to the contrary that $\mathbb{Q}$ is G_δ. Hence there exists a collection of open sets $\{U_n\}_{n=1}^\infty$ such that $\mathbb{Q} = \bigcap_{n=1}^\infty U_n$. Therefore $\mathbb{Q} \subseteq U_n$ for all n so each U_n is dense in $\mathbb{R}$. Hence each U_n^c is closed and nowhere dense by a result from class.

Notice that

$$\mathbb{R} \setminus \mathbb{Q} = \bigcup_{n=1}^\infty U_n^c$$

so the irrational numbers are a union of closed nowhere dense sets. Moreover, since $\mathbb{Q}$ is countable, we may write $\mathbb{Q} = \{r_n \mid n \in \mathbb{N}\}$. Thus

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$$

so $\mathbb{Q}$ is a countable union of closed nowhere dense sets. Thus

$$\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \left(\bigcup_{n=1}^{\infty} U_n^c \right) \cup \left(\bigcup_{n=1}^{\infty} \{r_n\} \right),$$

we obtain that $\mathbb{R}$ is a countable union of nowhere dense sets and thus $\mathbb{R}$ is of first category. However, as $\mathbb{R}$ is complete, the Baire Category Theorem implies that $\mathbb{R}$ is not of first category thereby providing a contradiction. Hence $\mathbb{Q}$ is not a G_δ set. ∎

Using Proposition 3.2.11, we can demonstrate that certain sets cannot be the discontinuities of a real-valued function.

Theorem 3.2.12. *There does not exists a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at each point in $\mathbb{Q}$ yet discontinuous at each point in $\mathbb{R} \setminus \mathbb{Q}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$. By Theorem 1.6.15 the set of discontinuities of f are F_σ. Thus the points where f is continuous must be a G_δ set. As $\mathbb{Q}$ is not G_δ by Proposition 3.2.11, f cannot be continuous at each point in $\mathbb{Q}$ yet discontinuous at each point in $\mathbb{R} \setminus \mathbb{Q}$. ∎

3.3 Open Mapping Theorem

Another use of the Baire Category Theorem (Theorem 3.2.7) is to study bounded linear maps between Banach spaces. In particular, since bounded linear maps are continuous, the inverse images of open sets are open. The goal of this section is to prove that surjective bounded linear maps map open sets to open sets. This enables us to prove that the inverses of bijective bounded linear maps are bounded and characterize continuous linear maps using their graphs.

To begin, we require the following odd looking result that says if an open ball is in the closure of the image of a bounded linear map of an open ball, then we can expand the later open ball to obtain strict containment.

Lemma 3.3.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space, let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a normed linear space, and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. If $B_{\mathcal{Y}}(\vec{0}, 1) \subseteq \overline{T(B_{\mathcal{X}}(\vec{0}, m))}$ for some $m > 0$, then $B_{\mathcal{Y}}(\vec{0}, 1) \subseteq T(B_{\mathcal{X}}(\vec{0}, 2m))$*

Proof. Let $m > 0$ be such that $B_{\mathcal{Y}}(\vec{0}, 1) \subseteq \overline{T(B_{\mathcal{X}}(\vec{0}, m))}$. Notice for all $\alpha \in \mathbb{R}$ that (where for a set A, $\alpha A = \{\alpha a \mid a \in A\}$)

$$\begin{aligned} B_{\mathcal{Y}}(\vec{0}, \alpha) = \alpha B_{\mathcal{Y}}(\vec{0}, 1) &\subseteq \alpha \overline{T(B_{\mathcal{X}}(\vec{0}, m))} \\ &= \overline{\alpha T(B_{\mathcal{X}}(\vec{0}, m))} \\ &= \overline{T(\alpha B_{\mathcal{X}}(\vec{0}, m))} = \overline{T(B_{\mathcal{X}}(\vec{0}, \alpha m))} \end{aligned}$$

by linearity and continuity of T, and by properties of the norm.

To see that $B_{\mathcal{Y}}(\vec{0}, 1) \subseteq T(B_{\mathcal{X}}(\vec{0}, 2m))$, let $\vec{y} \in B_{\mathcal{Y}}(\vec{0}, 1)$ be arbitrary. Since $\vec{y} \in \overline{T(B_{\mathcal{X}}(\vec{0}, m))}$ there exists an $\vec{x}_1 \in B_{\mathcal{X}}(\vec{0}, m)$ such that

$$\|\vec{y} - T(\vec{x}_1)\|_{\mathcal{Y}} < \frac{1}{2}.$$

Let $\vec{y}_1 = \vec{y} - T(\vec{x}_1) \in \mathcal{Y}$. Then $\vec{y}_1 \in B_{\mathcal{Y}}(\vec{0}, \frac{1}{2}) \subseteq \overline{T(B_{\mathcal{X}}(\vec{0}, \frac{1}{2}m))}$. Hence there exists an $\vec{x}_2 \in B_{\mathcal{X}}(\vec{0}, \frac{1}{2}m)$ such that

$$\|\vec{y}_1 - T(\vec{x}_2)\|_{\mathcal{Y}} < \frac{1}{2^2}.$$

Repeating this process ad nauseum, we obtain a sequence of vectors $(\vec{y}_n)_{n\geq 1}$ in $\mathcal{Y}$ and a sequence of vectors $(\vec{x}_n)_{n\geq 1}$ in $\mathcal{X}$ such that $\vec{y}_n = \vec{y}_{n-1} - T(\vec{x}_n)$, $\vec{y}_n \in B_{\mathcal{Y}}(\vec{0}, \frac{1}{2^n})$, $\vec{x}_{n+1} \in B_{\mathcal{X}}(\vec{0}, \frac{1}{2^n}m)$, and

$$\|\vec{y}_n - T(\vec{x}_{n+1})\|_{\mathcal{Y}} < \frac{1}{2^n}$$

for all $n \in \mathbb{N}$.

Since $\mathcal{X}$ is a Banach space and since

$$\sum_{n=1}^{\infty} \|\vec{x}_n\|_{\mathcal{X}} < \sum_{n=1}^{\infty} \frac{1}{2^n} m = 2m < \infty,$$

we obtain by Theorem 2.3.6 that $\vec{x} = \sum_{n=1}^{\infty} \vec{x}_n$ exists and is an element of $B_{\mathcal{X}}(\vec{0}, 2m)$. To see that $T(\vec{x}) = \vec{y}$ thereby completing the proof, notice since

T is continuous that

$$\begin{aligned}
\|\vec{y} - T(\vec{x})\|_{\mathcal{Y}} &= \lim_{n\to\infty} \left\| \vec{y} - T\left(\sum_{k=1}^{n} \vec{x}_k\right)\right\| \\
&= \lim_{n\to\infty} \left\| \vec{y} - \sum_{k=1}^{n} T(\vec{x}_k)\right\| \\
&= \lim_{n\to\infty} \left\| \vec{y}_1 - \sum_{k=2}^{n} T(\vec{x}_k)\right\| \\
&= \lim_{n\to\infty} \left\| \vec{y}_2 - \sum_{k=3}^{n} T(\vec{x}_k)\right\| \\
&\vdots \\
&= \lim_{n\to\infty} \|\vec{y}_n\| \\
&\leq \limsup_{n\to\infty} \frac{1}{2^n} = 0.
\end{aligned}$$

Hence $T(\vec{x}) = \vec{y}$. Therefore, since $\vec{y} \in B_{\mathcal{Y}}(\vec{0}, 1)$ was arbitrary, $B_{\mathcal{Y}}(\vec{0}, 1) \subseteq T(B_{\mathcal{X}}(\vec{0}, 2m))$. ∎

Combining Lemma 3.3.1 together with the Baire Category Theorem (Theorem 3.2.7), we obtain the following result.

Theorem 3.3.2 (Open Mapping Theorem). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces. If $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is surjective and $U \subseteq \mathcal{X}$ is open, then $T(U)$ is open in $\mathcal{Y}$ (that is, T maps open sets to open sets).*

Proof. Let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be surjective. First we will demonstrate that there exists an $r > 0$ such that $T(B_{\mathcal{X}}(\vec{0}, r))$ is a neighbourhood of $\vec{0}$ in $\mathcal{Y}$.

To begin, for each $m \in \mathbb{N}$ consider the set $F_m = \overline{T(B_{\mathcal{X}}(\vec{0}, m))} \subseteq \mathcal{Y}$. Clearly each F_m is a closed subset of $\mathcal{Y}$. Moreover, since T is surjective,

$$\mathcal{Y} = \bigcup_{m=1}^{\infty} F_m.$$

Therefore, since $\mathcal{Y}$ is complete, the Baire Category Theorem (Theorem 3.2.7) implies that $\mathcal{Y}$ is of second category and thus there must exists an $m_0 \in \mathbb{N}$ such that F_{m_0} is not nowhere dense. Hence $\mathrm{int}(F_{m_0}) \neq \emptyset$. Therefore there exists an $\vec{y}_0 \in F_{m_0}$ and a $\delta > 0$ such that $B_{\mathcal{Y}}(\vec{y}_0, \delta) \subseteq F_{m_0} = \overline{T(B_{\mathcal{X}}(\vec{0}, m))}$.

Since

$$\begin{aligned} B_{\mathcal{Y}}(\vec{0}, \delta) &\subseteq \{\vec{y} - \vec{y_0} \mid \vec{y} \in B_{\mathcal{Y}}(\vec{y_0}, \delta)\} \\ &\subseteq \{\vec{y_1} - \vec{y_2} \mid \vec{y_1}, \vec{y_2} \in F_{m_0}\} \\ &= \{\vec{y_1} + \vec{y_2} \mid \vec{y_1}, \vec{y_2} \in \overline{T(B_{\mathcal{X}}(\vec{0}, m))}\} \\ &\qquad \text{as } T \text{ is linear, } -B_{\mathcal{X}}(\vec{0}, m) = B_{\mathcal{X}}(\vec{0}, m) \\ &\subseteq \overline{T(B_{\mathcal{X}}(\vec{0}, 2m))} \\ &\qquad \text{by continuity, linearity, and the triangle inequality,} \end{aligned}$$

we obtain by Lemma 3.3.1 that $B_{\mathcal{Y}}(\vec{0}, \delta) \subseteq T(B_{\mathcal{X}}(\vec{0}, 4m))$.

To complete the result, let U be an arbitrary open subset of $\mathcal{X}$. To see that $T(U)$ is open in $\mathcal{Y}$, let $\vec{y} \in T(U)$ be arbitrary. Thus there exists a $\vec{x} \in \mathcal{X}$ such that $T(\vec{x}) = \vec{y}$. Since U is open, there exists an $\epsilon > 0$ such that $B_{\mathcal{X}}(\vec{x}, \epsilon) \subseteq U$. However since

$$B_{\mathcal{Y}}\left(\vec{0}, \frac{\epsilon\delta}{4m}\right) = \frac{\epsilon}{4m} B_{\mathcal{Y}}(\vec{0}, \delta) \subseteq \frac{\epsilon}{4m} T(B_{\mathcal{X}}(\vec{0}, 4m)) = T(B_{\mathcal{X}}(\vec{0}, \epsilon))$$

we have that

$$\begin{aligned} B_{\mathcal{Y}}\left(\vec{y}, \frac{\epsilon\delta}{4m}\right) &= \left\{\vec{y} + \vec{z} \,\middle|\, \vec{z} \in B_{\mathcal{Y}}\left(\vec{0}, \frac{\epsilon\delta}{4m}\right)\right\} \\ &\subseteq \left\{T(\vec{x}) + \vec{z} \,|\, \vec{z} \in T(B_{\mathcal{X}}(\vec{0}, \epsilon))\right\} \\ &= \left\{T(\vec{x}) + T(\vec{w}) \,|\, \vec{w} \in B_{\mathcal{X}}(\vec{0}, \epsilon)\right\} \\ &= \left\{T(\vec{x} + \vec{w}) \,|\, \vec{w} \in B_{\mathcal{X}}(\vec{0}, \epsilon)\right\} \\ &= T(B_{\mathcal{X}}(\vec{x}, \epsilon)) \end{aligned}$$

by the linearity of T. Hence $T(U)$ contains an open neighbourhood around $\vec{y}$. Therefore, since $\vec{y} \in T(U)$ was arbitrary, $T(U)$ is open in $\mathcal{Y}$. Hence since U was an arbitrary open subset of $\mathcal{X}$, the result follows. ∎

The Open Mapping Theorem has several applications.

Theorem 3.3.3 (The Inverse Mapping Theorem). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be a bijection. Then $T^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$.*

Proof. Let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be a bijection. Therefore $T^{-1} : \mathcal{Y} \to \mathcal{X}$ exists. Since T is linear, clearly T^{-1} is linear. To see that T^{-1} is bounded (i.e. continuous via Theorem 1.7.11), let $U \subseteq \mathcal{X}$ be open. Then

$$(T^{-1})^{-1}(U) = T(U)$$

is open in $\mathcal{Y}$ by the Open Mapping Theorem (Theorem 3.3.2). Hence T is continuous by Theorem 1.6.7. ∎

Using the Inverse Mapping Theorem (Theorem 3.3.3), we obtain the following property relating different norms on vector spaces.

Corollary 3.3.4. *Let $\mathcal{X}$ be a vector space over $\mathbb{K}$ that is complete with respect to each of two norms $\|\cdot\|_a$ and $\|\cdot\|_b$. If there exists a constant $c_1 \in \mathbb{R}$ such that*

$$\|\vec{x}\|_a \leq c_1 \|\vec{x}\|_b$$

for all $\vec{x} \in \mathcal{X}$, then there exists a constant $c_2 \in \mathbb{R}$ such that

$$\|\vec{x}\|_b \leq c_2 \|\vec{x}\|_a$$

for all $\vec{x} \in \mathcal{X}$.

Proof. Define $T : (\mathcal{X}, \|\cdot\|_b) \to (\mathcal{X}, \|\cdot\|_a)$ by $T(\vec{x}) = \vec{x}$. Clearly T is a linear map. Moreover, since

$$\|\vec{x}\|_a \leq c_1 \|\vec{x}\|_b$$

for all $\vec{x} \in \mathcal{X}$, we see that T is a bounded linear map from $(\mathcal{X}, \|\cdot\|_b)$ to $(\mathcal{X}, \|\cdot\|_a)$. Hence, by the Inverse Mapping Theorem, T^{-1} is a bounded linear map from $(\mathcal{X}, \|\cdot\|_a)$ to $(\mathcal{X}, \|\cdot\|_b)$. Since $T^{-1}(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathcal{X}$, we obtain that

$$\|\vec{x}\|_b = \left\|T^{-1}(\vec{x})\right\|_b \leq \left\|T^{-1}\right\| \|\vec{x}\|_a .$$

Thus letting $c_2 = \|T^{-1}\|$ completes the proof. ∎

To characterize bounded linear maps using their graphs, we require the following. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces. Define a norm $\|\cdot\|_1 : \mathcal{X} \times \mathcal{Y} \to [0, \infty)$ by

$$\|(\vec{x}, \vec{y})\|_1 = \|\vec{x}\|_{\mathcal{X}} + \|\vec{y}\|_{\mathcal{Y}} .$$

Clearly $\|\cdot\|_1$ is a norm and $\mathcal{X} \times \mathcal{Y}$ together with $\|\cdot\|_1$ is denoted $\mathcal{X} \oplus_1 \mathcal{Y}$. Furthermore, it is elementary using the arguments of Section 2.2 to show that if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then $\mathcal{X} \oplus_1 \mathcal{Y}$ is a Banach space.

Theorem 3.3.5 (The Closed Graph Theorem). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be Banach spaces and let $T : \mathcal{X} \to \mathcal{Y}$ be linear. The graph*

$$\mathcal{G}(T) = \{(\vec{x}, T(\vec{x})) \mid \vec{x} \in \mathcal{X}\}$$

is closed in $\mathcal{X} \oplus_1 \mathcal{Y}$ if and only if T is continuous.

Proof. To see that $\mathcal{G}(T)$ is closed when T is continuous, suppose T is continuous and let $((\vec{x}_n, T(\vec{x}_n)))_{n\geq 1}$ be an arbitrary sequence of elements of $\mathcal{G}(T)$ that converges to some element $(\vec{x}, \vec{y}) \in \mathcal{X} \oplus_1 \mathcal{Y}$. Clearly this implies $(\vec{x}_n)_{n\geq 1}$ converges to $\vec{x}$ in $\mathcal{X}$ and $(T(\vec{x}_n))_{n\geq 1}$ converges to $\vec{y} \in \mathcal{Y}$. Since T is continuous, $(\vec{x}_n)_{n\geq 1}$ converging to $\vec{x}$ in $\mathcal{X}$ implies that $(T(\vec{x}_n))_{n\geq 1}$ converges to $T(\vec{x})$. Therefore, due to uniqueness of limits, we must have that $\vec{y} = T(\vec{x})$. Hence $(\vec{x}, \vec{y}) \in \mathcal{G}(T)$ so $\mathcal{G}(T)$ is closed by Theorem 1.5.7.

Conversely, suppose $\mathcal{G}(T)$ is closed in $\mathcal{X} \oplus_1 \mathcal{Y}$. Therefore, since $\mathcal{G}(T)$ is a vector subspace of $\mathcal{X} \oplus_1 \mathcal{Y}$ as T is linear, and since $\mathcal{X} \oplus_1 \mathcal{Y}$ is a Banach space, $\mathcal{G}(T)$ is also a Banach space by Theorem 2.1.12.

Define $S : \mathcal{X} \to \mathcal{G}(T)$ by

$$S(\vec{x}) = (\vec{x}, T(\vec{x}))$$

for all $\vec{x} \in \mathcal{X}$. Clearly S is a linear map that is injective (by the first coordinate) and surjective. Hence S is invertible with $S^{-1} : \mathcal{G}(T) \to \mathcal{X}$ defined by

$$S^{-1}((\vec{x}, T(\vec{x}))) = \vec{x}.$$

Notice for all $(\vec{x}, T(\vec{x})) \in \mathcal{G}(T)$ that

$$\left\| S^{-1}((\vec{x}, T(\vec{x}))) \right\|_{\mathcal{X}} = \|\vec{x}\|_{\mathcal{X}} \leq \|\vec{x}\|_{\mathcal{X}} + \|T(\vec{x})\|_{\mathcal{Y}} = \|(\vec{x}, T(\vec{x}))\|_1 .$$

Therefore S^{-1} is bounded. Hence, as $\mathcal{X}$ and $\mathcal{G}(T)$ are Banach spaces, the Inverse Mapping Theorem (Theorem 3.3.3) implies that S is bounded. Therefore, since

$$\|T(\vec{x})\|_{\mathcal{Y}} \leq \|T(\vec{x})\|_{\mathcal{Y}} + \|\vec{x}\|_{\mathcal{X}} = \|S(\vec{x})\|_1 \leq \|S\| \, \|\vec{x}\|_{\mathcal{X}} ,$$

we see that T is bounded as desired. Hence T is continuous as desired. ∎

3.4 Principle of Uniform Boundedness

For our final major Banach space theorem of this chapter, we will use the Baire Category Theorem (Theorem 3.2.7) to deduce certain pointwise bounded sets are uniformly bounded. These two Uniform Boundness Principles are quite useful.

We begin with the following Uniform Boundness Principles for continuous functions on complete metric spaces.

Theorem 3.4.1 (Uniform Boundedness Principle). *Let $(\mathcal{X}, d_{\mathcal{X}})$ be a complete metric space, let $(\mathcal{Y}, d_{\mathcal{Y}})$ be a metric space, let $y \in \mathcal{Y}$ be a fixed element, and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X}, \mathcal{Y})$ be a non-empty set of functions such that for each $x \in \mathcal{X}$*

$$M_x := \sup_{f \in \mathcal{F}} d_{\mathcal{Y}}(f(x), y) < \infty.$$

Then there exists a non-empty open subset U of X and a constant $M > 0$ such that

$$d_{\mathcal{Y}}(f(x), y) \leq M$$

for all $f \in \mathcal{F}$ and $x \in U$.

Proof. For each $n \in \mathbb{N}$, let

$$F_n = \left\{ x \in \mathcal{X} \,\middle|\, \sup_{f \in \mathcal{F}} d_{\mathcal{Y}}(f(x), y) \leq n \right\}.$$

Clearly each F_n is a closed set as each element of $\mathcal{F}$ is continuous and the distance function is continuous. Furthermore, if $x \in \mathcal{X}$ then $x \in F_n$ for all $n \geq M_x$. Hence

$$\mathcal{X} = \bigcup_{n=1}^{\infty} F_n.$$

Therefore, since $\mathcal{X}$ is second countable by the Baire Category Theorem (Theorem 3.2.7), there exists an $n_0 \in \mathbb{N}$ such that F_{n_0} is not nowhere dense in $\mathcal{X}$. Therefore $\emptyset \neq \mathrm{int}(\overline{F_{n_0}}) = \mathrm{int}(F_{n_0})$ so there exists an open subset U of $\mathcal{X}$ with $U \subseteq F_{n_0}$. Hence for all $x \in U$ we have $d_{\mathcal{Y}}(f(x), y) \leq n_0$ for all $f \in \mathcal{F}$ as desired. ∎

Note the above theorem is most useful when $\mathcal{Y}$ is a normed linear space and $y = \vec{0}$. In this case, the assumption becomes

$$M_x := \sup_{f \in \mathcal{F}} \|f(x)\|_{\mathcal{Y}} < \infty$$

and the conclusion becomes

$$\|f(x)\|_{\mathcal{Y}} \leq M$$

for all $f \in \mathcal{F}$ and $x \in U$.

Building on the above theorem, we obtain the following Uniform Boundedness Principle for bounded linear maps between Banach spaces

Theorem 3.4.2 (Uniform Boundedness Principle - Banach space version). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ a normed linear space, and let $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X}, \mathcal{Y})$ be non-empty. Suppose for each $\vec{x} \in \mathcal{X}$ that*

$$\sup\{\|T(\vec{x})\|_{\mathcal{Y}} \mid T \in \mathcal{F}\} < \infty.$$

Then

$$\sup\{\|T\| \mid T \in \mathcal{F}\} < \infty.$$

Proof. For each $T \in \mathcal{F}$, consider the function $f_T : \mathcal{X} \to \mathbb{R}$ defined by

$$f_T(\vec{x}) = \|T(\vec{x})\|_{\mathcal{Y}}$$

for all $\vec{x} \in \mathcal{X}$. Since T and the norm are continuous functions on $\mathcal{X}$, it is elementary to see that $f_T \in \mathcal{C}(\mathcal{X}, \mathbb{R})$ for all $T \in \mathcal{F}$.

Let

$$\mathcal{F}_0 = \{f_T \mid T \in \mathcal{F}\} \subseteq \mathcal{C}(\mathcal{X}, \mathbb{R}).$$

Since $\mathcal{X}$ is complete and since

$$\sup_{f \in \mathcal{F}_0} |f(\vec{x})| < \infty$$

for all $\vec{x} \in \mathcal{X}$, Theorem 3.4.1 implies that there exists an $M > 0$ and a non-empty open subset U of $\mathcal{X}$ such that

$$\|T(\vec{x})\| = |f_T(\vec{x})| \leq M$$

for all $\vec{x} \in U$ and $T \in \mathcal{F}$.

Since U is a non-empty open set of $\mathcal{X}$, there exists a vector $\vec{x}_0 \in U$ and an $\epsilon > 0$ so that $B_{\mathcal{X}}(\vec{x}_0, \epsilon) \subseteq U$. To obtain the conclusion, let $T \in \mathcal{F}$ be arbitrary. Notice if $\vec{x} \in B_{\mathcal{X}}(\vec{0}, \epsilon)$, then

$$\|T(\vec{x})\|_{\mathcal{Y}} \leq \|T(\vec{x} + \vec{x}_0)\|_{\mathcal{Y}} + \|-T(\vec{x}_0)\|_{\mathcal{Y}} \leq M + \|T(\vec{x}_0)\|_{\mathcal{Y}}.$$

as $\vec{x} + \vec{x}_0 \in B_{\mathcal{X}}(\vec{x}_0, \epsilon)$. Therefore, if $\vec{z} \in B_{\mathcal{X}}(\vec{0}, 1)$, then

$$\|T(\vec{z})\|_{\mathcal{Y}} = \frac{1}{\epsilon} \|T(\epsilon \vec{z})\|_{\mathcal{Y}} \leq \frac{1}{\epsilon} \left(M + \|T(\vec{x}_0)\|_{\mathcal{Y}}\right)$$

as $\epsilon \vec{z} \in B_{\mathcal{X}}(\vec{0}, \epsilon)$. Hence

$$\|T\| \leq \frac{1}{\epsilon} \left(M + \|T(\vec{x}_0)\|_{\mathcal{Y}}\right).$$

Therefore, as $T \in \mathcal{F}$ was arbitrary and as $\sup_{T \in \mathcal{F}} \|T(\vec{x}_0)\|_{\mathcal{Y}} < \infty$, the proof is complete. ∎

The Uniform Boundedness Principle (Theorem 3.4.2) is particularly useful to show the pointwise limit of bounded linear maps products a bounded linear map.

Theorem 3.4.3 (The Banach-Steinhaus Theorem). *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space, let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a normed linear space, and let $(T_n)_{n\geq 1}$ be a sequence of elements of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that for all $\vec{x} \in \mathcal{X}$*

$$\lim_{n\to\infty} T_n(\vec{x})$$

exists in $\mathcal{Y}$. Then $\sup_{n\geq 1} \|T_n\| < \infty$ and the map $T : \mathcal{X} \to \mathcal{Y}$ defined by $T(\vec{x}) = \lim_{n\to\infty} T_n(\vec{x})$ is an element of $\mathcal{B}(\mathcal{X}, \mathcal{Y})$.

Proof. Since for each $\vec{x} \in \mathcal{X}$ the limit $\lim_{n\to\infty} T_n(\vec{x})$ exists, the sequence $(T(\vec{x}_n))_{n\geq 1}$ is bounded. Therefore, by the Principle of Uniform Boundedness (Theorem 3.4.2), we obtain that $\sup_{n\geq 1} \|T_n\| < \infty$.

Define $T : \mathcal{X} \to \mathcal{Y}$ by $T(\vec{x}) = \lim_{n\to\infty} T_n(\vec{x})$. Clearly if $\vec{x}_1, \vec{x}_2 \in \mathcal{X}$ and $\alpha \in \mathbb{K}$ then

$$T(\vec{x}_1 + \alpha\vec{x}_2) = \lim_{n\to\infty} T_n(\vec{x}_1 + \alpha\vec{x}_2) = \lim_{n\to\infty} T_n(\vec{x}_1) + \alpha T_n(\vec{x}_2) = T(\vec{x}_1) + \alpha T(\vec{x}_2)$$

so T is linear. To see that T is bounded, we note for all $\vec{x} \in \mathcal{X}$ that

$$\|T(\vec{x})\| = \lim_{n\to\infty} \|T_n(\vec{x})\| \leq \limsup_{n\to\infty} \|T_n| \, \|\vec{x}\| \leq \left(\sup_{n\geq 1} \|T_n\|\right) \|\vec{x}\| \, .$$

Therefore, as $\sup_{n\geq 1} \|T_n\| < \infty$, T is bounded. ∎

Of course, there are many other uses of the Uniform Boundedness Principle (Theorem 3.4.2) and the Banach-Steinhaus Theorem (Theorem 3.4.3). For example, one can use the Uniform Boundedness Principle (Theorem 3.4.2) to prove that there exists a continuous function whose Fourier series does not converge pointwise. In addition, there are many more uses in functional analysis. However, that is another course.

Chapter 4

Compact Metric Spaces

In the previous chapter we saw several major theorems that will aid in the comprehension of Banach spaces. However, there is one tool that we are still missing: the notion of a compact set. Compact sets are some form of analogue of the notion of closed and bounded subsets of $\mathbb{R}$ in regards to the existence of convergent sequences. In particular, in previous analysis courses, most results for continuous functions hold for continuous functions on a closed bounded interval. The reason these results hold is that closed bounded intervals are compact. For example, we will see it is really the notion of compactness that allows for the Extreme Value Theorem. Thus compact metric spaces are some of the nicest metric spaces we can consider!

Thus the goal of this chapter is to develop the notion of compactness in metric spaces. In particular, several equivalent notions are demonstrated. It should be pointed out that some of these notions only are equivalent only in the metric space setting and not the general topological setting. However, using compactness we will be able to develop several properties that distinguish finite and infinite dimensional Banach spaces and demonstrate that all n-dimensional normed limit spaces are complete and really $\mathbb{K}^n$ in disguise.

4.1 Compact Sets

To study compact sets, we first must define what a compact set it. The notion of a compact set is based on trying to understand specific properties of a set A based on collections of open sets that cover A. As such we define the following.

Definition 4.1.1. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. A collection $\{U_\alpha\}_{\alpha \in I}$ is said to be an *open cover* of A if each U_α is an open subset of $\mathcal{X}$ and $A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

Example 4.1.2. It is not difficult to see that for any set A and $\epsilon > 0$,

$\{B(a, \epsilon) \mid a \in A\}$ is an open cover of A. Similarly, if $U_n = \left(\frac{1}{n}, 1\right)$ for all $n \in \mathbb{N}$, then $\{U_n\}_{n=1}^\infty$ is an open cover of $(0, 1)$.

To study a set A via the open coverings of A, it would be incredibly useful to be able to reduce the number of open sets contained in a covering. In particular, being able to reduce to a finite number of sets would be optimal as things are easy to compute exactly if one only needs to deal with a finite number of elements. Consequently, our notion of a compact set is as follows.

Definition 4.1.3. Let $(\mathcal{X}, d)$ be a metric space. A subset $K \subseteq \mathcal{X}$ is said to be *compact* if whenever $\{U_\alpha\}_{\alpha \in I}$ is an open cover of K there exists an $n \in \mathbb{N}$ and $\alpha_1, \ldots \alpha_n \in I$ such that $K \subseteq \bigcup_{k=1}^n U_{\alpha_k}$; that is, K is compact if every open cover has a finite *subcover*.

Remark 4.1.4. If $(\mathcal{X}, d)$ is a metric space and $A \subseteq \mathcal{X}$, then it is easy to see that A is a compact subset of $(\mathcal{X}, d)$ if and only if A is a compact subset of $(A, d|_A)$. Indeed this follows directly from Definition 4.1.3 and Proposition 1.3.15, which gives the structure of the open subset of $(A, d|_A)$. Consequently, compactness is truly a property of the topology on a set and not the metric space the set is living in.

Example 4.1.5. Technically the empty set is compact as every open cover has a subcover consisting of one element.

Example 4.1.6. The subset $(0, 1)$ of $\mathbb{R}$ is not compact. Indeed we previously saw that if $U_n = \left(\frac{1}{n}, 1\right)$ for all $n \in \mathbb{N}$, then $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of $(0, 1)$. However, clearly $\{U_n\}_{n=1}^\infty$ does not have a finite subcover as if $n_1, \ldots, n_m \in \mathbb{N}$ then $\bigcup_{k=1}^m U_{n_k} = \left(\frac{1}{\max\{n_1, \ldots, n_m\}}, 1\right) \neq (0, 1)$.

Example 4.1.7. The real numbers are not a compact set. Indeed if $U_n = (-n, n)$ for each $n \in \mathbb{N}$, then $\{U_n\}_{n=1}^\infty$ is an open cover of $\mathbb{R}$ that does not have a finite subcover.

The problem with the previous two examples are illustrated with the following result.

Theorem 4.1.8. *Let $(\mathcal{X}, d)$ be a metric space and let $K \subseteq \mathcal{X}$ be compact. Then K is closed and bounded.*

Proof. Let $K \subseteq \mathcal{X}$ be compact. To see that K is closed, let $x \in K^c$ be arbitrary. For each $n \in \mathbb{N}$ consider the closed set $F_n = B\left[x, \frac{1}{n}\right]$ and the open set $U_n = F_n^c$. Since

$$\bigcup_{n=1}^\infty U_n = \left(\bigcap_{n=1}^\infty F_n\right)^c = \{x\}^c = \mathcal{X} \setminus \{x\},$$

we obtain that $K \subseteq \bigcup_{n=1}^\infty U_n$ as $x \in K^c$. Hence $\{U_n\}_{n=1}^\infty$ is an open cover of K. Therefore, since K is compact, there exists $n_1, \ldots, n_q \in \mathbb{N}$ such that

$K \subseteq \bigcup_{k=1}^{q} U_{n_k}$. If $N = \max\{n_1, \ldots, n_q\}$, we clearly obtain that $K \subseteq U_N$ since $U_n \subseteq U_m$ whenever $n \leq m$. Hence $B\left(x, \frac{1}{N}\right) \subseteq F_N \subseteq K^c$. Therefore, since $x \in K^c$ was arbitrary, K^c is open and thus K is closed as desired.

To see that K is bounded, fix some $x \in \mathcal{X}$. For each $n \in \mathbb{N}$, consider the open set $U_n = B(x, n)$. Since for all $y \in \mathcal{X}$ there exists an $m \in \mathbb{N}$ such that $d(x, y) < m$, we see that $\bigcup_{n=1}^{\infty} U_n = \mathcal{X}$. Hence $\{U_n\}_{n=1}^{\infty}$ is an open cover of K. Therefore, since K is compact, there exists $n_1, \ldots, n_q \in \mathbb{N}$ such that $K \subseteq \bigcup_{k=1}^{q} U_{n_k}$. If $N = \max\{n_1, \ldots, n_q\}$, we clearly obtain that $K \subseteq B(x, N)$ and thus K is bounded. ∎

In light of Theorem 4.1.8, we note the following.

Corollary 4.1.9. *Let $(\mathcal{X}, d)$ be a compact metric space and let $A \subseteq \mathcal{X}$. Then A is compact if and only if A is closed.*

Proof. If A is compact, then A is closed by Theorem 4.1.8.

Conversely, suppose that A is closed. To see that A is compact, let $\{U_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of A. Clearly $\{A^c\} \cup \{U_\alpha\}_{\alpha \in I}$ is an open cover of $\mathcal{X}$ and thus must have a finite subcover as $\mathcal{X}$ is compact. Clearly this finite subcover once A^c is removed must be a finite subset of $\{U_\alpha\}_{\alpha \in I}$ that covers A. Hence A is compact. ∎

Unfortunately, compact sets are more than just closed and bounded sets in metric spaces.

Example 4.1.10. Let X be an infinite set and let d be the discrete metric on X. Then X is a closed and bounded subset of (X, d). However, X is not a compact subset of (X, d) since $\{\{x\} \mid x \in X\}$ is an open cover of X with no finite subcovers.

Of course though, at the moment the only example of a compact set we have provided is the empty set, which is not very illuminating. To obtain a plethora of examples of compact sets, we turn to $\mathbb{K}^n$. In particular, the following theorem state that the converse of Theorem 4.1.8 holds for $\mathbb{K}^n$. This may lead those in previous analysis courses to define compact sets to be closed and bounded sets. We will demonstrate why closed bounded sets are not the correct notion to study in metric spaces later.

Theorem 4.1.11 (The Heine-Borel Theorem). *Let $K \subseteq \mathbb{K}^n$. Then K is compact in $(\mathbb{K}^n, \|\cdot\|_\infty)$ if and only if K is closed and bounded.*

Proof. First, if K is compact then K is closed and bounded by Theorem 4.1.8.

For the other direction, let K be closed and bounded. Suppose to the contrary that K is not compact. Hence there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of K that has no finite subcover. We desire to obtain a contradiction to this fact.

Since K is bounded, there exists an $M \in \mathbb{R}$ such that

$$K \subseteq [-M, M] \times \cdots \times [-M, M]$$

when $\mathbb{K} = \mathbb{R}$, and

$$K \subseteq \{(a_1 + b_1 i, \ldots, a_n + b_n i) \mid a_i, b_j \subseteq [-M, M]\}$$

when $\mathbb{K} = \mathbb{C}$. We will proceed with the proof where $\mathbb{K} = \mathbb{R}$ as the case where $\mathbb{K} = \mathbb{C}$ follows by the same arguments using $2n$ in place of n.

Divide $[-M, M]^n$ into 2^n closed balls with side-lengths M. To be specific, for all $q_1, \ldots, q_n \in \{0, 1\}$ let

$$J_{q_1,\ldots,q_n} = [-M + Mq_1, Mq_1] \times \cdots \times [-M + Mq_n, Mq_n].$$

Clearly each $J_{q_1,\ldots,q_n}$ is closed and the union of all possible $J_{q_1,\ldots,q_n}$s contains K. Therefore, since $\{U_\alpha\}_{\alpha\in I}$ does not have a finite subcover of K, there must exist one of these $J_{q_1,\ldots,q_n}$s such that $\{U_\alpha\}_{\alpha\in I}$ does not have a finite subcover of $K \cap J_{q_1,\ldots,q_n}$ (as there are a finite number of $J_{q_1,\ldots,q_n}$s). Denote this $J_{q_1,\ldots,q_n}$ by B_1 and notice $\operatorname{diam}(B_1) = M$.

Suppose for each $k \in \mathbb{N}$ we have constructed closed balls $B_1, \ldots, B_k$ such that $B_{j+1} \subseteq B_j$, $\operatorname{diam}(B_j) = \frac{1}{2^j}M$, and $\{U_\alpha\}_{\alpha\in I}$ does not have a finite subcover of $B_j \cap K$ for all $j \in \{1, \ldots, k-1\}$. By repeating the above process on B_k, there exists a closed ball $B_{k+1} \subseteq B_k$ such that $\operatorname{diam}(B_{k+1}) = \frac{1}{2^{k+1}}M$ and such that $\{U_\alpha\}_{\alpha\in I}$ does not have a finite subcover of $B_{k+1} \cap K$. Thus, by repeating this process ad infinitum, we obtain a collection $\{B_k\}_{k=1}^\infty$ of closed balls of $(\mathbb{K}^n, \|\cdot\|_\infty)$ such that $B_{k+1} \subseteq B_k$, $\operatorname{diam}(B_k) = \frac{1}{2^k}M$, and $\{U_\alpha\}_{\alpha\in I}$ does not have a finite subcover of $V_k \cap K$ for all $k \in \mathbb{N}$ (and thus $B_k \cap K \neq \emptyset$ for all $k \in \mathbb{N}$).

Notice each $B_k \cap K$ is closed as K is closed and that $\operatorname{diam}(B_k \cap K) \leq \operatorname{diam}(B_k) = \frac{1}{2^k}M$. Therefore, since $\mathbb{K}^n$ is complete, Cantor's Theorem (Theorem 2.3.4) implies that

$$Y = \bigcap_{k=1}^\infty (B_k \cap K) \neq \emptyset.$$

We claim that Y has exactly one element. Indeed if $x, y \in Y$ then $x, y \in B_k$ for all $k \in \mathbb{N}$ so $d(x, y) \leq \operatorname{diam}(B_k) = \frac{1}{2^k}M$ for all $k \in \mathbb{N}$ which implies $d(x, y) = 0$, or, equivalently, $x = y$. Hence Y contains exactly one point, say z.

By construction $z \in K$. Therefore, as $\{U_\alpha\}_{\alpha\in I}$ is an open cover of K, there exists an $\alpha_0 \in I$ such that $z \in U_{\alpha_0}$. Thus, since U_{α_0} is open, there exists an $\epsilon > 0$ such that $B(z, \epsilon) \subseteq U_{\alpha_0}$. Since $\operatorname{diam}(B_k) = \frac{1}{2^k}M$ for all $k \in \mathbb{N}$, there exists a $k_0 \in \mathbb{N}$ such that $\operatorname{diam}(B_{k_0}) < \epsilon$. Therefore, as $z \in B_{k_0}$ we obtain for all $x \in B_{k_0}$ that $d(z, x) < \epsilon$ so $x \in B(z, \epsilon) \subseteq U_{\alpha_0}$ for all $x \in B_{k_0}$. This implies $B_{k_0} \cap K \subseteq B_{k_0} \subseteq B(z, \epsilon) \subseteq U_{\alpha_0}$ which contradicts the fact that $\{U_\alpha\}_{\alpha\in I}$ did not have a finite subcover of $B_{k_0} \cap K$. As we have obtained a contradiction, it must be the case that K is compact. ∎

Remark 4.1.12. Of course, since for all $p \in [1, \infty)$ the Banach spaces $(\mathbb{K}^n, \|\cdot\|_\infty)$ and $(\mathbb{K}^n, \|\cdot\|_p)$ have the same open sets (and thus the same open covers of sets) by Remark 1.3.14, a set K is compact in $(\mathbb{K}^n, \|\cdot\|_\infty)$ if and only if K is compact in $(\mathbb{K}^n, \|\cdot\|_p)$. Furthermore, as $(\mathbb{K}^n, \|\cdot\|_\infty)$ and $(\mathbb{K}^n, \|\cdot\|_p)$ have the same closed sets and the same bounded sets (by the same computation as in Remark 1.3.14), the Heine-Borel Theorem (Theorem 4.1.11) also holds for $(\mathbb{K}^n, \|\cdot\|_p)$ for all $p \in [1, \infty)$.

The above demonstrates that the notion of a compact set is the same as that of closed bounded sets in $(\mathbb{K}^n, \|\cdot\|_p)$. Thus why compact sets are nicer than closed bounded sets in metric spaces? The answer is the following result that says the notion of compactness is preserved under continuous maps.

Theorem 4.1.13. *Let $(\mathcal{X}, d_\mathcal{X})$ and $(\mathcal{Y}, d_\mathcal{Y})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be continuous. If K is a non-empty compact subset of $\mathcal{X}$, then $f(K)$ is a compact subset of $\mathcal{Y}$.*

Proof. To see that $f(K)$ is compact, let $\{U_\alpha\}_{\alpha \in I}$ be an arbitrary open cover of $f(K)$ in $\mathcal{Y}$. Therefore $\{f^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of K in $\mathcal{X}$. Hence, as K is compact, there exists $\alpha_1, \ldots, \alpha_n \in I$ such that $K \subseteq \bigcup_{k=1}^n f^{-1}(U_{\alpha_k})$. Therefore $f(K) \subseteq \bigcup_{k=1}^n U_{\alpha_k}$ so $\{U_\alpha\}_{\alpha \in I}$ has a finite subcover of $f(K)$. Therefore, as $\{U_\alpha\}_{\alpha \in I}$ was arbitrary, $f(K)$ is compact. ∎

We note the following example which demonstrate the image under a continuous function of a closed bounded set need not be closed nor bounded for arbitrary metric spaces.

Example 4.1.14. Consider the metric space $(\mathbb{Z}, d)$ where $d : \mathbb{Z} \times \mathbb{Z} \to [0, \infty)$ is defined by

$$d(n, m) = \frac{|n - m|}{1 + |n - m|}$$

for all $n, m \in \mathbb{Z}$. Clearly d is well-defined and a metric by the same arguments as used in Example 1.1.11.

Notice that $d(n, m) \in [0, 1)$ for all $n, m \in \mathbb{Z}$. Hence $\mathbb{Z}$ is a closed bounded set $(\mathbb{Z}, d)$. Furthermore, for each $n \in \mathbb{Z}$,

$$\inf\{d(n, m) \mid m \in \mathbb{Z} \setminus \{n\}\} = \min\{d(n, n+1), d(n, n-1)\} > 0$$

as the function $x \mapsto \frac{x}{1+x}$ is increasing on $[0, \infty)$. Therefore, for each $n \in \mathbb{N}$ there exists an $r_n > 0$ such that $B(n, r_n) = \{n\}$. Thus the topology on $(\mathbb{Z}, d)$ is the discrete topology so every function from $\mathbb{Z}$ to a metric space must be continuous.

Define $f : \mathbb{Z} \to \mathbb{R}$ by

$$f(n) = \begin{cases} n + 1 & \text{if } n \geq 0 \\ -\frac{1}{n} & \text{if } n < 0 \end{cases}.$$

Thus f is continuous. However, as

$$f(\mathbb{Z}) = \mathbb{N} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

we clearly see that $f(\mathbb{Z})$ is neither closed nor bounded in $\mathbb{R}$.

Thus the above gives one reason why compact sets are far nicer than closed bounded sets. In later sections we will see various characterizations and uses of compact sets. For now, we note that since closed intervals in $\mathbb{R}$ are compact by the Heine-Borel Theorem (Theorem 4.1.11), Theorem 4.1.13 is actually a generalization of the Extreme Value Theorem.

Theorem 4.1.15 (The Extreme Value Theorem). *Let $(\mathcal{X}, d)$ be a metric space, let $f : \mathcal{X} \to \mathbb{R}$ be continuous, and let $K \subseteq \mathcal{X}$ be non-empty and compact. Then there exists points $x_1, x_2 \in K$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in K$.*

Proof. Since f is continuous and K is compact, Theorem 4.1.13 implies that $f(K)$ is a compact subset of $\mathbb{R}$. Hence $f(K)$ is closed and bounded by Theorem 4.1.8. Since $f(K)$ is non-empty and bounded, $\sup(f(K))$ and $\inf(f(K))$ are finite and we can construct sequences of elements of $f(K)$ converging to $\sup(f(K))$ and $\inf(f(K))$ respectively. Since $f(K)$ is also closed, this implies $\sup(f(K)), \inf(f(K)) \in f(K)$. Hence there exists $x_1, x_2 \in K$ such that $f(x_1) = \inf(f(K))$ and $f(x_2) = \sup(f(K))$ so $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in K$ as desired. ∎

4.2 Finite Dimensional Normed Linear Spaces

In this section, we will see one important use of the notion of compact sets. In particular, using compactness we will be able to complete our study finite dimensional normed linear spaces. As there is a plethora of finite dimensional normed linear spaces (e.g. take your favourite normed linear space (e.g. $\mathcal{C}[0, 1]$) and take a span of a finite number of elements), we desire to understand properties of such spaces. In particular, as we have seen that $(\mathbb{K}^n, \|\cdot\|_p)$ are complete and characterized compact sets as the closed and bounded sets. We desire a similar analogue for any finite dimensional normed linear space.

To proceed, we note that given a finite dimensional normed linear space $\mathcal{X}$ there are many maps between $\mathcal{X}$ and $(\mathbb{K}^n, \|\cdot\|_p)$. Our hope would be to use one of these maps to obtain that $\mathcal{X}$ is complete. However, we clearly may not be able to produce an isomorphism (i.e. there are many norms on $\mathbb{K}^n$) and the notion of homeomorphisms of metric spaces is not enough to deduce completeness by Example 2.1.10.

However, for normed linear spaces, we have special maps: namely the bounded linear maps. Using the notion of dimension, it is easy to construct an invertible linear map that is bounded when the domain is $\mathbb{K}^n$ equipped with the ∞-norm.

Lemma 4.2.1. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be an n-dimensional normed linear space over $\mathbb{K}$. If $\mathbb{K}^n$ is equipped with the ∞-norm, then there exists a bijective element $T \in \mathcal{B}(\mathbb{K}^n, \mathcal{X})$.*

Proof. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for $\mathcal{X}$. Define $T : \mathbb{K}^n \to \mathcal{X}$ by

$$T((z_1, \ldots, z_n)) = z_1\vec{v}_1 + \cdots + z_n\vec{v}_n.$$

for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$. Clearly T is linear and bijective by construction. Furthermore for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$

$$\begin{aligned} \|T((z_1, \ldots, z_n))\|_{\mathcal{X}} &= \|z_1\vec{v}_1 + \cdots + z_n\vec{v}_n\|_{\mathcal{X}} \\ &\leq \sum_{k=1}^{n} |z_k| \, \|\vec{v}_k\|_{\mathcal{X}} \\ &\leq \left(\sum_{k=1}^{n} \|\vec{v}_k\|_{\mathcal{X}} \right) \|(z_1, \ldots, z_n)\|_{\infty} . \end{aligned}$$

Hence T is bounded with $\|T\| \leq \sum_{k=1}^{n} \|\vec{v}_k\|_{\mathcal{X}}$. ∎

We would like to conclude that the linear map T in Lemma 4.2.1 has a bounded inverse. However, the Inverse Mapping Theorem (Theorem 3.3.3) does not do this for us since we do not a priori know that $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a Banach space (i.e. complete). However, compactness comes to the rescue to give us the following.

Lemma 4.2.2. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be an n-dimensional normed linear space over $\mathbb{K}$. If $\mathbb{K}^n$ is equipped with the ∞-norm, then there exists a bijective linear map $T : \mathbb{K}^n \to \mathcal{X}$ and two numbers $0 < k_1 \leq k_2 < \infty$ such that*

$$k_1 \|\vec{z}\|_{\infty} \leq \|T(\vec{z})\|_{\mathcal{X}} \leq k_2 \|\vec{z}\|_{\infty}$$

for all $\vec{z} \in \mathbb{K}^n$.

Proof. By Lemma 4.2.1 there exists a bijective element $T \in \mathcal{B}(\mathbb{K}^n, \mathcal{X})$. Hence

$$\|T(\vec{z})\|_{\mathcal{X}} \leq \|T\| \, \|\vec{z}\|_{\infty}$$

thereby proving one inequality.

To see the other inequality, let

$$S_1 = \{\vec{z} \in \mathbb{K}^n \mid \|\vec{z}\|_{\infty} = 1\}.$$

Clearly S_1 is a closed bounded subset of $\mathbb{K}^n$ and therefore is compact by the Heine-Borel Theorem (Theorem 4.1.11). Hence $T(S_1)$ is a compact subset of $\mathcal{X}$ by Theorem 4.1.13. Define $f : T(S_1) \to \mathbb{R}$ by

$$f(\vec{x}) = \|\vec{x}\|_{\mathcal{X}}$$

for all $\vec{x} \in T(S_1)$. Since f is continuous and since $T(S_1)$ is compact, by the Extreme Value Theorem (Theorem 4.1.15) there exists a $\vec{x}_0 \in T(S_1)$ such that

$$\alpha = f(\vec{x}_0) \leq f(\vec{x})$$

for all $\vec{x} \in T(S_1)$. Since $\vec{x}_0 \in T(S_1)$ and since T is a bijection, $\vec{x}_0 \neq \vec{0}$ so $\alpha > 0$.

We claim that

$$\alpha \|\vec{z}\|_\infty \leq \|T(\vec{z})\|_{\mathcal{X}}$$

for all $\vec{z} \in \mathbb{K}^n$. Clearly the inequality holds when $\vec{z} = 0$. Otherwise if $\vec{z} \neq 0$ then $\frac{1}{\|\vec{z}\|_\infty}\vec{z} \in S_1$ so

$$\|T(\vec{z})\|_{\mathcal{X}} = \|\vec{z}\|_\infty \left\| T\left(\frac{1}{\|\vec{z}\|_\infty}\vec{z}\right)\right\|_{\mathcal{X}} \geq \alpha \|\vec{z}\|_\infty .$$

Thus the result follows. ∎

The conclusions of Lemma 4.2.2 means that the norms on $\mathbb{K}^n$ and $\mathcal{X}$ behave in a very similar way. For example, clearly $(\vec{z}_n)_{n\geq 1}$ converges in $\mathbb{K}^n$ if and only if $(T(\vec{z}_n))_{n\geq 1}$ converges in $\mathcal{X}$. We will see more applications of these inequalities in Corollary 4.2.5, but for now we encapsulate this idea in the following definition.

Definition 4.2.3. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed linear spaces over $\mathbb{K}$. It is said that $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ are *equivalent* if there exists a bijective linear map $T : \mathcal{X} \to \mathcal{Y}$ and two numbers $0 < k_1 \leq k_2 < \infty$ such that

$$k_1 \|\vec{x}\|_{\mathcal{X}} \leq \|T(\vec{x})\|_{\mathcal{Y}} \leq k_2 \|\vec{x}\|_{\mathcal{X}}$$

for all $\vec{x} \in \mathcal{X}$.

It is clear that the notion of equivalent norms is an equivalence relation on normed linear spaces. In particular, by using Lemma 4.2.2 twice, the following is trivial.

Corollary 4.2.4. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be n-dimensional normed linear spaces over $\mathbb{K}$. Then $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\mathcal{Y}}$ are equivalent.*

The fact that every norm on every finite dimensional normed linear space is equivalent to $\|\cdot\|_\infty$ on $\mathbb{K}^n$ produces the following results properties of finite dimensional normed linear spaces.

Corollary 4.2.5. *Let $(\mathcal{X}, \|\cdot\|)$ be a finite dimensional normed linear space. Then*

(1) $\mathcal{X}$ is complete.

(2) Every vector subspace of $\mathcal{X}$ is closed.

(3) A subset $K \subseteq \mathcal{X}$ is compact if and only if K is closed and bounded.

(4) Every linear map from $\mathcal{X}$ to another normed linear space is bounded.

Proof. Let $(\mathcal{X}, \|\cdot\|)$ be a finite dimensional normed linear space. If $\mathbb{K}^n$ is equipped with the ∞-norm, then by Lemma 4.2.2 there exists a bijective linear map $T : \mathbb{K}^n \to \mathcal{X}$ and two numbers $0 < k_1 \leq k_2 < \infty$ such that

$$k_1 \|\vec{z}\|_\infty \leq \|T(\vec{z})\|_{\mathcal{X}} \leq k_2 \|\vec{z}\|_\infty$$

for all $\vec{z} \in \mathbb{K}^n$.

To see that $\mathcal{X}$ is complete, let $(\vec{x}_n)_{n\geq 1}$ be an arbitrary Cauchy sequence. The above implies $(T^{-1}(\vec{x}_n))_{n\geq 1}$ is a Cauchy sequence and thus converges to some element $\vec{z} \in \mathbb{K}^n$. Since T is a continuous, we see that $(\vec{x}_n)_{n\geq 1}$ converges to $T(\vec{z})$ in $\mathcal{X}$. Therefore, as $(\vec{x}_n)_{n\geq 1}$ was arbitrary, $\mathcal{X}$ is complete.

To see (2), let $\mathcal{Y}$ be a vector subspace of $\mathcal{X}$. Thus $\mathcal{Y}$ is a finite dimensional normed linear space and thus complete by part (1). Hence Theorem 2.1.12 implies $\mathcal{Y}$ is closed.

To see (3), note if K is compact in $\mathcal{X}$ then K is closed and bounded by Theorem 4.1.8. Conversely, suppose K is closed and bounded in $\mathcal{X}$. Hence $T^{-1}(K)$ is closed in $\mathbb{K}^n$. Therefore, since the above inequalities imply $T^{-1}(K)$ is bounded as K is bounded, $T^{-1}(K)$ is compact by the Heine-Borel Theorem. Therefore, by Theorem 4.1.13, $K = T(T^{-1}(K))$ is compact.

To see (4), let n be the dimension of $\mathcal{X}$, let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be another normed linear space, and let $S : \mathcal{X} \to \mathcal{Y}$ be a linear map. If $\mathbb{K}^n$ is equipped with the ∞-norm, by Corollary 4.2.4 there exists a bounded linear map $T : \mathcal{X} \to \mathbb{K}^n$ such that $T^{-1} : \mathbb{K}^n \to \mathcal{X}$ is bounded. As

$$S = (S \circ T^{-1}) \circ T,$$

if it can be demonstrated that $S \circ T^{-1}$ is bounded, then S is a composition of continuous functions and thus will be continuous.

Let $R = S \circ T^{-1} : \mathbb{K}^n \to \mathcal{Y}$ and for each $k \in \{1, \ldots, n\}$, let

$$\vec{y}_k = R((0, \ldots, 0, 1, 0, \ldots, 0))$$

where the 1 occurs in the k^{th} position. Thus for all $(z_1, \ldots, z_n) \in \mathbb{K}^n$

$$\begin{aligned} \|R((z_1, \ldots, z_n))\|_{\mathcal{Y}} &= \|z_1\vec{y}_1 + \cdots + z_n\vec{y}_n\|_{\mathcal{Y}} \\ &\leq \sum_{k=1}^n |z_k| \|\vec{y}_k\|_{\mathcal{Y}} \\ &\leq \left(\sum_{k=1}^n \|\vec{y}_k\|_{\mathcal{Y}}\right) \|(z_1, \ldots, z_n)\|_\infty . \end{aligned}$$

Hence R is bounded with $\|R\| \leq \sum_{k=1}^n \|\vec{y}_k\|_{\mathcal{Y}}$ as desired. ∎

It is not difficult to find infinite dimensional counterexamples to each property in Corollary 4.2.5.

Example 4.2.6. Consider the subset c_{00} of $\ell_\infty(\mathbb{N})$ defined by

$$c_{00} = \{(a_n)_{n\geq 1} \mid a_n = 0 \text{ for all but finitely many } n\}.$$

Clearly c_{00} is a subspace of $\ell_\infty(\mathbb{N})$. However, we claim that c_{00} is not closed. To see this, for each $m, n \in \mathbb{N}$, define

$$a_{m,n} = \begin{cases} \frac{1}{n} & \text{if } n \leq m \\ 0 & \text{otherwise} \end{cases}.$$

Hence for each $m \in \mathbb{N}$, $\vec{z}_m = (a_{m,n})_{n\geq 1} \in c_{00}$. However, $\lim_{m\to\infty} \vec{z}_m = \left(\frac{1}{n}\right)_{n\geq 1}$ in $\ell_\infty(\mathbb{N})$ and $\left(\frac{1}{n}\right)_{n\geq 1} \notin c_{00}$. Hence c_{00} is not closed and thus not complete by Theorem 2.1.12. Hence conclusions (1) and (2) of Corollary 4.2.5 can fail for infinite dimensional normed linear spaces.

Example 4.2.7. Consider the set

$$B = B[\vec{0}, 1] \subseteq \ell_\infty(\mathbb{N}, \mathbb{R}).$$

Clearly B is a closed bounded subset of $\ell_\infty(\mathbb{N})$. However, B is not compact. To see this, let

$$X = \{(a_n)_{n\geq 1} \mid a_n \in \{-1, 0, 1\} \text{ for all } n\}$$

and consider the set

$$\mathcal{U} = \{B(\vec{x}, 1) \mid \vec{x} \in X\}.$$

Clearly $\mathcal{U}$ is an open cover of B. However, $\mathcal{U}$ has no finite subcover of B as for $\vec{x}_1, \vec{x}_2 \in X$, $\vec{x}_1 \in B(\vec{x}_2, 1)$ if and only if $\vec{x}_1 = \vec{x}_2$. Therefore, as X is not finite (specifically uncountable), $\mathcal{U}$ has no finite subcover of B. Hence conclusion (3) of Corollary 4.2.5 can fail for infinite dimensional normed linear spaces. Note we will generalize this example in Theorem 4.4.5.

Example 4.2.8. Let $(\mathcal{X}, \|\cdot\|)$ be an infinite dimensional normed linear space over $\mathbb{R}$ with basis $\{\vec{x}_n\}_{n=1}^\infty$. By scaling if necessary, we may assume that $\|\vec{x}_n\| = 1$ for all $n \in \mathbb{N}$ (i.e. scaling each element in a linearly independent spanning set preserves linear independence and spanning). Define a linear map $T : \mathcal{X} \to \mathbb{R}$ by defining $T(\vec{x}_n) = n$ and by extending the definition of T by linearity. As $|T(\vec{x}_n)| \geq n$ and $\|\vec{x}_n\| = 1$, we see that T is unbounded. Hence conclusion (4) of Corollary 4.2.5 can fail for infinite dimensional normed linear spaces.

Using Corollary 4.2.5 together with the Baire Category Theorem (Theorem 3.2.7), we can characterize the cardinality of any basis of a infinite dimensional Banach space. In particular, any infinite dimensional normed linear space with a countable basis cannot possibly be a Banach space.

Theorem 4.2.9. *Every vector space basis of an infinite dimensional Banach space is uncountable.*

Proof. Suppose $(\mathcal{X}, \|\cdot\|)$ is an infinite dimensional Banach space with a countable basis $\{\vec{x}_n\}_{n=1}^\infty$. For each $n \in \mathbb{N}$, let

$$F_n = \mathrm{span}(\{\vec{x}_1, \ldots, \vec{x}_n\}).$$

Clearly each F_n is a finite dimensional vector space and thus is closed by Corollary 4.2.5.

We claim that $\mathrm{int}(F_n) = \emptyset$ for each $n \in \mathbb{N}$. Indeed, if $\mathrm{int}(F_n) \neq \emptyset$, then there exists an element $\vec{x} \in F_n$ and an $\epsilon > 0$ such that $B(\vec{x}, \epsilon) \subseteq F_n$. However, since F_n is a subspace and closed under translation and scaling, this implies $B(\vec{0}, \epsilon) \subseteq F_n$ by translation and $B(\vec{0}, r) \subseteq F_n$ for all $r > 0$ by scaling. As the later implies $F_n = \mathcal{X}$, we would obtain $\mathcal{X}$ is finite dimensional contradicting the fact that $\mathcal{X}$ is infinite dimensional. Thus $\mathrm{int}(F_n) = \emptyset$ for each $n \in \mathbb{N}$.

The above shows each F_n is nowhere dense. Since $\{\vec{x}_n\}_{n=1}^\infty$ is a basis for $\mathcal{X}$ and

$$\mathcal{X} = \bigcup_{n=1}^\infty F_n,$$

X is a countable union of nowhere dense sets. Hence $\mathcal{X}$ is of first category which contradicts the Baire Category Theorem (Theorem 3.2.7) as $\mathcal{X}$ is a Banach space. As we have a contradiction, the proof is complete. ∎

To conclude our discussions on the differences between finite and infinite dimensional normed linear spaces, we note that every finite dimensional normed linear spaces is a Banach space by Corollary 4.2.5 and all norms on a finite dimensional Banach space are equivalent by Corollary 4.2.4. Of course this is not the case for an infinite dimensional Banach space.

Proposition 4.2.10. *Let $(\mathcal{X}, \|\cdot\|)$ be an infinite dimensional Banach space. Then there exists another norm $\|\cdot\|_0 : \mathcal{X} \to [0, \infty)$ such that $(\mathcal{X}, \|\cdot\|_0)$ is a Banach space, yet $\|\cdot\|$ and $\|\cdot\|_0$ are not equivalent.*

Proof. By Example 4.2.8 there exists a linear map $f : \mathcal{X} \to \mathbb{K}$ and a vector $\vec{y} \in \mathcal{X}$ such that $f(\vec{y}) = 1$ and f is not bounded. Define $S : \mathcal{X} \to \mathcal{X}$ by

$$S(\vec{x}) = \vec{x} - 2f(\vec{x})\vec{y}$$

for all $\vec{x} \in \mathcal{X}$. Clearly S is well-defined and linear as $f : \mathcal{X} \to \mathbb{K}$ is linear.

We claim that S^2 is the identity map on $\mathcal{X}$. To see this, notice for all $\vec{x} \in \mathcal{X}$ that

$$\begin{aligned} S^2(\vec{x}) &= S(\vec{x} - 2f(\vec{x})\vec{y}) \\ &= (\vec{x} - 2f(\vec{x})\vec{y}) - 2f(\vec{x})(\vec{y} - 2f(\vec{y})\vec{y}) \\ &= (\vec{x} - 2f(\vec{x})\vec{y}) - 2f(\vec{x})(-\vec{y}) = \vec{x}. \end{aligned}$$

Therefore, as $\vec{x} \in \mathcal{X}$ was arbitrary, S^2 is the identity map on $\mathcal{X}$.

Define $\|\cdot\|_0 : \mathcal{X} \to [0, \infty)$ by

$$\|\vec{x}\|_0 = \|S(\vec{x})\|_0$$

for all $\vec{x} \in \mathcal{X}$. We claim that $\|\cdot\|_0$ is norm on $\mathcal{X}$. To see this, first notice since $\|\cdot\| : \mathcal{X} \to [0, \infty)$ and $S : \mathcal{X} \to \mathcal{X}$ that $\|\cdot\|_0 : \mathcal{X} \to [0, \infty)$. Next, notice for all $\vec{x} \in \mathcal{X}$ that $\|\vec{x}\|_0 = 0$ if and only if $\|S(\vec{x})\| = 0$ if and only if $S(\vec{x}) = \vec{0}$. Clearly if $\vec{x} = \vec{0}$ then $S(\vec{x}) = \vec{0}$ and thus $\left\|\vec{0}\right\|_0 = 0$. Otherwise, if $\|\vec{x}\|_0 = 0$ then $S(\vec{x}) = \vec{0}$ so $\vec{x} = S(S(\vec{x})) = S(\vec{0}) = \vec{0}$ as S is linear. Hence $\|\cdot\|_0$ satisfies the first property of being a norm.

To see that $\|\cdot\|_0$ satisfies scaling, let $\alpha \in \mathbb{K}$ and let $\vec{x} \in \mathcal{X}$ be arbitrary. Then

$$\|\alpha\vec{x}\|_0 = \|S(\alpha\vec{x})\| = \|\alpha S(\vec{x})\| = |\alpha|\,\|S(\vec{x})\| = |\alpha|\,\|\vec{x}\|_0 .$$

Hence, as $\alpha \in \mathbb{K}$ and $\vec{x} \in \mathcal{X}$ were arbitrary, $\|\cdot\|_0$ satisfies the scaling property of a norm.

Finally, to see that $\|\cdot\|_0$ satisfies the triangle inequality, notice for all $\vec{x}_1, \vec{x}_2 \in \mathcal{X}$ that

$$\begin{aligned}\|\vec{x}_1 + \vec{x}_2\|_0 &= \|S(\vec{x}_1 + \vec{x}_2)\| \\ &= \|S(\vec{x}_1) + S(\vec{x}_2)\| \\ &\leq \|S(\vec{x}_1)\| + \|S(\vec{x}_2)\| \\ &= \|\vec{x}_1\|_0 + \|\vec{x}_2\|_0 .\end{aligned}$$

Hence $\|\cdot\|_0$ satisfies the triangle inequality and thus is a norm on $\mathcal{X}$.

We claim that $(\mathcal{X}, \|\cdot\|_0)$ is complete. To see this, let $(\vec{x}_n)_{n\geq 1}$ be an arbitrary Cauchy in $(\mathcal{X}, \|\cdot\|_0)$. We claim that $(S(\vec{x}_n))_{n\geq 1}$ is Cauchy in $(\mathcal{X}, \|\cdot\|)$. To see this, let $\epsilon > 0$. Since $(\vec{x}_n)_{n\geq 1}$ is Cauchy in $(\mathcal{X}, \|\cdot\|_0)$, there exists an $N \in \mathbb{N}$ such that $\|\vec{x}_n - \vec{x}_m\|_0 < \epsilon$ for all $n, m \geq N$. Hence for all $n, m \geq N$ we have that

$$\|S(\vec{x}_n) - S(\vec{x}_m)\| = \|S(\vec{x}_n - \vec{x}_m)\| = \|\vec{x}_n - \vec{x}_m\|_0 < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $(S(\vec{x}_n))_{n\geq 1}$ is Cauchy in $(\mathcal{X}, \|\cdot\|)$.

Since $(\mathcal{X}, \|\cdot\|)$ is complete, $(S(\vec{x}_n))_{n\geq 1}$ is converges in $(\mathcal{X}, \|\cdot\|)$. Hence there exists a vector $\vec{z} \in \mathcal{X}$ such that $\lim_{n\to\infty} \|S(\vec{x}_n) - \vec{z}\| = 0$. We claim that $(\vec{x}_n)_{n\geq 1}$ converges in $(\mathcal{X}, \|\cdot\|_0)$ to $S(\vec{z})$. To see this, notice that

$$\|\vec{x}_n - S(\vec{z})\|_0 = \|S(\vec{x}_n - S(\vec{z}))\| = \left\|S(\vec{x}_n) - S^2(\vec{z})\right\| = \|S(\vec{x}_n) - \vec{z}\|$$

for all $n \in \mathbb{N}$. Therefore, since $\lim_{n\to\infty} \|S(\vec{x}_n) - \vec{z}\| = 0$ we obtain that $\lim_{n\to\infty} \|\vec{x}_n - S(\vec{z})\| = 0$. Hence $(\vec{x}_n)_{n\geq 1}$ converges in $(\mathcal{X}, \|\cdot\|_0)$ to $S(\vec{z})$. Therefore, as $(\vec{x}_n)_{n\geq 1}$ was arbitrary, $(\mathcal{X}, \|\cdot\|_0)$ is complete.

Finally, we claim that $\|\cdot\|$ and $\|\cdot\|_0$ are not equivalent; that is, there does not exist $c_1, c_2 > 0$ such that

$$\|\vec{x}\| \leq c_1 \|\vec{x}\|_0 \qquad \text{and} \qquad \|\vec{x}\|_0 \leq c_2 \|\vec{x}\|$$

for all $\vec{x} \in \mathcal{X}$. To see this, it suffices to prove only one of these inequalities by Corollary 3.3.4. We will show that there does not exist a constant $c_2 \in \mathbb{R}$ such that $\|\vec{x}\|_0 \leq c_2 \|\vec{x}\|$ for all $\vec{x} \in \mathcal{X}$.

To see this, suppose to the contrary that there exists a constant $C \in \mathbb{R}$ such that

$$\|S(\vec{x})\| = \|\vec{x}\|_0 \leq C \|\vec{x}\|$$

for all $\vec{x} \in \mathcal{X}$. However, if $\{\vec{x}_n\}_{n=1}^\infty$ are the vectors from Example 4.2.8, we see that

$$\begin{aligned} C = C\|\vec{x}_n\| &\geq \|S(\vec{x}_n)\| \\ &= \|\vec{x}_n - T(\vec{x_n})\vec{y}\| \\ &= \|\vec{x}_n - n\vec{y}\| \\ &\geq n\|\vec{y}\| - \|\vec{x}_n\| \\ &\geq n - 1 \end{aligned}$$

for all $n \in \mathbb{N}$. As this is clearly a contradiction, the proof is complete. ∎

4.3 The Finite Intersection Property

In this section, we will begin to produce other characterization of compact sets. Although some of these characterizations we will eventually study do not hold in general topological spaces, the one in this section does. For our first alternate characterization of compactness, our goal is to exchange unions and open sets with intersections and closed sets.

Definition 4.3.1. Let $(\mathcal{X}, d)$ be a metric space. A collection $\{F_\alpha\}_{\alpha \in I}$ is said to have the *finite intersection property* if $\bigcap_{k=1}^n F_{\alpha_k} \neq \emptyset$ for every finite subset $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$.

As the complement of a union of open sets is an intersection of closed sets, the following result is not surprising.

Theorem 4.3.2. *Let $(\mathcal{X}, d)$ be a metric space. The following are equivalent:*

1. *$\mathcal{X}$ is compact.*

2. *Whenever $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed subsets of $\mathcal{X}$ with the finite intersection property, $\bigcap_{\alpha \in I} F_\alpha \neq \emptyset$.*

Proof. Suppose $\mathcal{X}$ is compact and let $\{F_\alpha\}_{\alpha\in I}$ be a collection of closed subsets of $\mathcal{X}$ with the finite intersection property. To see that $\bigcap_{\alpha\in I} F_\alpha \neq \emptyset$ suppose to the contrary that $\bigcap_{\alpha\in I} F_\alpha = \emptyset$. For each $\alpha \in I$, let $U_\alpha = F_\alpha^c$, which are open subsets of $\mathcal{X}$. Since

$$\bigcup_{\alpha\in I} U_\alpha = \bigcup_{\alpha\in I} F_\alpha^c = \left(\bigcap_{\alpha\in I} F_\alpha\right)^c = \emptyset^c = \mathcal{X},$$

we see that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of $\mathcal{X}$. However, for any $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in I$, we see that

$$\bigcup_{m=1}^n U_{\alpha_m} = \bigcup_{m=1}^n F_{\alpha_m}^c = \left(\bigcap_{m=1}^n F_{\alpha_m}\right)^c$$

and, as $\{F_\alpha\}_{\alpha\in I}$ has the finite intersection property,

$$\emptyset \neq \bigcap_{m=1}^n F_{\alpha_m} \quad \text{so} \quad \mathcal{X} \nsubseteq \left(\bigcap_{m=1}^n F_{\alpha_m}\right)^c.$$

Therefore $\{U_\alpha\}_{\alpha\in I}$ is an open subcover of $\mathcal{X}$ without any finite subcovers which contradicts the fact that $\mathcal{X}$ is compact. Hence it must have be the case that $\bigcap_{\alpha\in I} F_\alpha \neq \emptyset$.

For the other direction, suppose any collection of closed subsets of $\mathcal{X}$ with the finite intersection property has non-trivial intersection. To see that $\mathcal{X}$ is compact, suppose to the contrary that there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of $\mathcal{X}$ without any finite subcovers. For each $\alpha \in I$ let $F_\alpha = U_\alpha^c$ which is a closed subset of $\mathcal{X}$. If $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in I$, then by assumption

$$\bigcup_{k=1}^n U_{\alpha_k} \subsetneq \mathcal{X} \quad \text{so} \quad \bigcap_{k=1}^n F_{\alpha_k} = \left(\bigcup_{k=1}^n U_{\alpha_k}\right)^c \neq \emptyset.$$

Hence $\{F_\alpha\}_{\alpha\in I}$ are closed subsets of $\mathcal{X}$ with the finite intersection property. Thus, by assumption,

$$\bigcap_{\alpha\in I} F_\alpha \neq \emptyset \quad \text{so} \quad \bigcup_{\alpha\in I} U_\alpha = \left(\bigcap_{\alpha\in I} F_\alpha\right)^c \neq \mathcal{X}.$$

As this contradicts the fact that $\{U_\alpha\}_{\alpha\in I}$ is an open cover of $\mathcal{X}$, it must be the case that $\mathcal{X}$ is compact. ∎

Of course, the reason why Theorem 4.3.2 is useful is that it is much nicer to deal with closed sets when discussing notions of convergence.

4.4 Sequential Compactness

In this section, we will begin a discussion of another notion of compactness for metric spaces. In particular, the following is often how compactness may be defined (incorrectly) in a first course in analysis.

Definition 4.4.1. A subset A of a metric space $(\mathcal{X}, d)$ is said to be *sequentially compact* if every sequence of elements of A has a subsequence that converges to an element of A.

Remark 4.4.2. Again, it is elementary to see that A is a sequentially compact subset of $(\mathcal{X}, d)$ if and only if A is a sequentially compact subset of $(A, d|_A)$. Thus sequential compactness is a property of the set and the metric, not the encompassing space the set lives in.

For examples of sequentially compact sets, we need look no farther that compact sets.

Theorem 4.4.3. *Every compact metric space is sequentially compact.*

Proof. Let $(\mathcal{X}, d)$ be a compact metric space. To see that $\mathcal{X}$ is sequentially compact, let $(x_n)_{n\geq 1}$ be an arbitrary sequence of elements of $\mathcal{X}$. For each $n \in \mathbb{N}$, let

$$F_n = \overline{\{x_k \mid k \geq n\}}.$$

Therefore $\{F_n\}_{n=1}^\infty$ is a collection of closed subsets of $\mathcal{X}$ which has the finite intersection property since if $n_1, \ldots, n_q \in \mathbb{N}$, then

$$\bigcap_{k=1}^q F_{n_q} = F_{\max\{n_1,\ldots,n_q\}}.$$

Hence, by Theorem 4.3.2, $\bigcap_{n=1}^\infty F_n \neq \emptyset$ as $\mathcal{X}$ is compact.

Let $x \in \bigcap_{n=1}^\infty F_n$ be arbitrary. We claim there exists a subsequence of $(x_n)_{n\geq 1}$ that converges to x. To see this, notice since $x \in F_1$ that there exists an $n_1 \in \mathbb{N}$ such that $d(x, x_{n_1}) \leq 1$ by Corollary 1.5.23. Hence since $x \in F_{n_1+1}$, there exists an $n_2 > n_1$ such that $d(x, x_{n_2}) \leq \frac{1}{2}$ by Corollary 1.5.23. By repeating this process ad infinitum, there exists an increasing sequence $(n_k)_{k\geq 1}$ of natural numbers such that $d(x, x_{n_k}) \leq \frac{1}{k}$. Hence $(x_{n_k})_{k\geq 1}$ is a subsequence of $(x_n)_{n\geq 1}$ that converges to x. Therefore, as $(x_n)_{n\geq 1}$ was an arbitrary sequence in $\mathcal{X}$, $\mathcal{X}$ is sequentially compact. ∎

Using the fact that compact sets are sequentially compact, we have an immediate use of sequential compactness as it can be used to extend Example 4.2.7 and show that many closed balls in infinite dimensional Banach spaces are not compact. To do this, we will follow the idea of Example 4.2.7 and show that we can find unit vectors that are far from other vectors.

Lemma 4.4.4. *Let $(\mathcal{X}, \|\cdot\|)$ be an infinite dimensional Banach space and let $\mathcal{Y}$ be a finite dimensional subspace of $\mathcal{X}$. There exists an element $\vec{x} \in \mathcal{X}$ such that $\|\vec{x}\| = 1$ yet $\|\vec{x} - \vec{y}\| \geq \frac{1}{2}$ for all $\vec{y} \in \mathcal{Y}$.*

Proof. Since $\mathcal{X}$ is infinite dimensional and $\mathcal{Y}$ is finite dimensional, there exists a vector $\vec{x}_0 \in \mathcal{X} \setminus \mathcal{Y}$. Since $\mathcal{Y}$ is closed by Corollary 4.2.5 and since $\vec{x}_0 \notin \mathcal{Y}$,

$$R = \mathrm{dist}(\vec{x}_0, \mathcal{Y}) > 0$$

by Lemma 1.6.12. By the definition of the distance function, there exists a $\vec{y}_0 \in \mathcal{Y}$ such that $\|\vec{x}_0 - \vec{y}_0\| \leq 2R$. Therefore if

$$\vec{x} = \frac{1}{\|\vec{x}_0 - \vec{y}_0\|}(\vec{x}_0 - \vec{y}_0)$$

then $\|\vec{x}\| = 1$ and for all $\vec{y} \in \mathcal{Y}$,

$$\begin{aligned}
\|\vec{x} - \vec{y}\| &= \left\| \frac{1}{\|\vec{x}_0 - \vec{y}_0\|}\vec{x}_0 - \frac{1}{\|\vec{x}_0 - \vec{y}_0\|}\vec{y}_0 - \vec{y} \right\| \\
&= \frac{1}{\|\vec{x}_0 - \vec{y}_0\|} \left\|\vec{x}_0 - (\vec{y}_0 + \|\vec{x}_0 - \vec{y}_0\| \, \vec{y})\right\| \\
&\geq \frac{1}{\|\vec{x}_0 - \vec{y}_0\|} R \quad \text{as } \mathcal{Y} \text{ is a subspace} \\
&\geq \frac{1}{2R} R = \frac{1}{2}
\end{aligned}$$

as desired. ∎

Theorem 4.4.5. *Every non-trivial closed ball in every infinite dimensional Banach space is not compact.*

Proof. Let $(\mathcal{X}, \|\cdot\|)$ be an infinite dimensional Banach space. To prove the result, it suffices to show that the ball $B = B[\vec{0}, 1]$ is not compact as both balls, open subsets of $\mathcal{X}$, and open covers are invariant under translation and non-trivial scaling.

To see that B is not compact, first note we may choose an $\vec{x}_1 \in B$ such that $\|\vec{x}_1\| = 1$ as $\mathcal{X}$ is not zero dimensional. Let $F_1 = \mathrm{span}(\{\vec{x}_1\})$. Since F_1 is one-dimensional, Lemma 4.4.4 implies there exists an $\vec{x}_2 \in B$ such that $\|\vec{x}_2\| = 1$ and $\|\vec{x}_2 - \vec{x}_1\| \geq \frac{1}{2}$.

Let $F_2 = \mathrm{span}(\{\vec{x}_1, \vec{x}_2\})$. Since F_2 is two-dimensional, Lemma 4.4.4 implies there exists a $\vec{x}_3 \in B$ such that $\|\vec{x}_3\| = 1$ and $\|\vec{x}_3 - \vec{x}_k\| \geq \frac{1}{2}$ for each $k \in \{1, 2\}$. Hence, by repeating this process ad infinitum, there exists a sequence $(\vec{x}_n)_{n\geq 1}$ of elements of B such that $\|\vec{x}_n\| = 1$ for all $n \in \mathbb{N}$ and $\|\vec{x}_n - \vec{x}_m\| \geq \frac{1}{2}$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Clearly the sequence $(\vec{x}_n)_{n\geq 1}$ does not have any convergent subsequences since it does not have any Cauchy subsequences. Hence B cannot be sequentially compact so B is not compact by Theorem 4.4.3. ∎

Note the proof of Theorem 4.4.5 also shows that if $\mathcal{X}$ is a Banach space and

$$S = \{\vec{x} \in \mathcal{X} \mid \|\vec{x}\| = 1\}$$

(i.e. the unit sphere in $\mathcal{X}$), then S is not compact. Hence a Banach space is infinite dimensional if and only the unit sphere (or any non-trivial closed unit ball) is not compact.

4.5 Totally Bounded Sets

As Theorem 4.4.3 tells us compact sets are sequentially compact and we have seen one application of sequential compactness, it is natural to ask whether the converse holds. To answer this question, we must examine a notion related to sequential compactness. This turns out to be the correct notion of 'boundedness' to correct the result 'compact if and only if closed and bounded' as we will see in a subsequent section.

In order to develop this correct notion of 'boundedness', we note that we can always cover a metric space with open balls of a certain radius. Consequently, if a metric space (X, d) is compact there must be a finite cover of (X, d) using open balls of a specific radii. This causes us to define the following two terms.

Definition 4.5.1. Let $(\mathcal{X}, d)$ be a metric space, let $A \subseteq \mathcal{X}$ be non-empty, and let $\epsilon > 0$. A subset $\{x_\alpha \mid \alpha \in I\}$ is said to be an *ϵ-net* of A in $\mathcal{X}$ if $x_\alpha \in A$ for all $\alpha \in I$ and $A \subseteq \bigcup_{\alpha \in I} B(x_\alpha, \epsilon)$; that is, for all $a \in A$ there exists an $\alpha \in I$ such that $d(a, x_\alpha) < \epsilon$.

For example, if $A \subseteq \mathcal{X}$, clearly A is an ϵ-net of A for every $\epsilon > 0$. Similarly $\{\frac{k}{n}\}_{k=1}^n$ is an ϵ-net of $[0, 1]$ for every $\epsilon > \frac{1}{2n}$. However, we prefer the later example as it only requires a finite subset of the set under consideration. In particular, given a set we are interested in finding a finite ϵ-net for ever $\epsilon > 0$.

Definition 4.5.2. A subset A of a metric space $(\mathcal{X}, d)$ is said to be *totally bounded* if A has a finite ϵ-net for all $\epsilon > 0$; that is, for each $\epsilon > 0$ there exists $\{a_1, \ldots, a_n\} \subseteq \mathcal{A}$ such that $A \subseteq \bigcup_{k=1}^n B(a_k, \epsilon)$.

Remark 4.5.3. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$. As any ϵ-net of A in $(\mathcal{X}, d)$ is automatically an ϵ-net of A in $(A, d|_A)$ and vice versa, we see that A is totally bounded in $(\mathcal{X}, d)$ if and only if A is totally bounded in $(A, d|_A)$. Hence the notion of total boundedness is a notion of the set and the metric, not the space where the set resides.

Example 4.5.4. It is not difficult to see that $[0, 1]$ is totally bounded. Indeed for every $\epsilon > 0$, choose $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ and consider the $\frac{1}{n}$-net $\{\frac{k}{n}\}_{k=1}^n$, which is clearly an ϵ-net. Similarly $(0, 1)$ is totally bounded.

For more examples of totally bounded sets, we note the following.

Proposition 4.5.5. *Every sequentially compact metric space is totally bounded. Thus compact sets are totally bounded by Theorem 4.4.3.*

Proof. Let $(\mathcal{X}, d)$ be a sequentially compact metric space. To see that $\mathcal{X}$ is totally bounded, suppose to the contrary that there exists an $\epsilon > 0$ such that $\mathcal{X}$ does not have a finite ϵ-net.

Let $x_1 \in \mathcal{X}$ be arbitrary. Since $\{x_1\}$ is not an ϵ-net, there exists an $x_2 \in \mathcal{X} \setminus B(x_1, \epsilon)$. Since $\{x_1, x_2\}$ is not an ϵ-net, there exists an $x_3 \in \mathcal{X} \setminus (B(x_1, \epsilon) \cup B(x_2, \epsilon))$; that is, $d(x_3, x_j) \geq \epsilon$ for all $j \in \{1, 2\}$. By repeating this process ad infinitum, there exists a sequence $(x_n)_{n\geq 1}$ such that $d(x_n, x_m) \geq \epsilon$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Clearly the sequence $(x_n)_{n\geq 1}$ does not have any convergent subsequences since it does not have any Cauchy subsequences. Hence $\mathcal{X}$ cannot be sequentially compact, which is a contradiction. Hence sequentially compact metric spaces are totally bounded. ∎

Remark 4.5.6. It may be very tempting to claim that every totally bounded metric space $(\mathcal{X}, d)$ is automatically compact as if one has an open cover of $(\mathcal{X}, d)$ then we would hope that there is an ϵ-net of X where each ball is contained in a single element of the open cover. However, this argument clearly has the flaw in that how do we know we can cover $(\mathcal{X}, d)$ with balls of size at most ϵ so that that each ϵ-ball is contained in a single element of the given open cover?

To see that totally bounded metric spaces need not be compact consider $\mathcal{X} = (0, 1)$ with the absolute value metric. Then $(0, 1)$ is totally bounded. Indeed for every $\epsilon > 0$, choose $n \in \mathbb{N}$ with $\frac{1}{n} < \epsilon$ and consider the set $\{\frac{k}{n}\}_{k=1}^{n}$, which is clearly an ϵ-net of $(0, 1)$. Hence $(0, 1)$ is totally bounded but not compact by the Heine-Borel Theorem (Theorem 4.1.11) as $(0, 1)$ is not closed. Moreover, a similar argument can be used to show that any bounded subset of $\mathbb{K}^n$ is totally bounded.

Remark 4.5.7. In theory, checking a metric space $(\mathcal{X}, d)$ is totally bounded is easier than it is to check $(\mathcal{X}, d)$ is compact using the definitions of open covers. Indeed it quite difficult to describe all open covers of a metric space and determine whether each open cover has a finite subcover. However, checking a metric space has a finite ϵ-net for every $\epsilon > 0$ is often not too difficult as one need to simply find a correct set of points in the metric space for a given $\epsilon > 0$.

Our goal is to connect the notions of totally boundedness and compactness in metric spaces. To do so, we begin by developing the properties of totally bounded metric spaces. In particular, it is not surprising that total boundedness is a strengthening of the notion of boundedness.

Proposition 4.5.8. *Every totally bounded subset of a metric space is bounded.*

Proof. Let $(\mathcal{X}, d)$ be a metric space and let $A \subseteq \mathcal{X}$ be a totally bounded subset of $\mathcal{X}$. Since A is totally bounded, there exists a finite 1-net for A. Hence there exists an $n \in \mathbb{N}$ and elements $\{a_1, \ldots, a_n\} \subseteq A$ such that for all $a \in A$ there exists a $k \in \{1, \ldots, n\}$ such that $d(a, a_k) < 1$.

Let
$$M = \max\{1 + d(a_k, a_1) \mid k \in \{1, \ldots, n\}\}.$$
We claim that $d(a, a_1) \leq M$ for all $a \in A$ which implies A is bounded by definition. To see this, let $a \in A$ be arbitrary. Then there exists a $k \in \{1, \ldots, n\}$ such that $d(a, a_k) < 1$. As
$$d(a, a_1) \leq d(a, a_k) + d(a_k, a_1) < 1 + d(a_k, a_1) \leq M,$$
the result is complete. ∎

Proposition 4.5.9. *Let $(\mathcal{X}, d)$ be a metric space and suppose $A_1 \subseteq A_2 \subseteq \mathcal{X}$. If A_2 is totally bounded, then A_1 is totally bounded*

Proof. The caveat of this result is that the elements of each ϵ-net for A_1 must come from A_1 and, a priori, they only come from A_2.

To see that A_1 is totally bounded, let $\epsilon > 0$ be arbitrary. Since A_2 is totally bounded, there exists a finite $\frac{\epsilon}{2}$-net for A_2, say $\{x_1, \ldots, x_n\} \subseteq A_2$. Hence
$$A_1 \subseteq \bigcup_{k=1}^{n} B\left(x_k, \frac{\epsilon}{2}\right).$$
Let $I \subseteq \{1, \ldots, n\}$ consist of all indices k such that $A_1 \cap B\left(x_k, \frac{\epsilon}{2}\right) \neq \emptyset$. For each $k \in I$, choose $a_k \in A_1 \cap B\left(x_k, \frac{\epsilon}{2}\right)$.

We claim that $\{a_k\}_{k \in I}$ is an ϵ-net for A_1. To see this, note the claim is trivial if $A_1 = \emptyset$. Otherwise, let $a \in A_1$ be arbitrary. Therefore there exists a $k_0 \in \{1, \ldots, n\}$ such that $a \in B\left(x_{k_0}, \frac{\epsilon}{2}\right)$. Hence $k_0 \in I$ and
$$d(a, a_{k_0}) \leq d(a, x_{k_0}) + d(a_{k_0}, x_{k_0}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
as $a, a_{k_0} \in B\left(x_{k_0}, \frac{\epsilon}{2}\right)$, Therefore, as $a \in A_1$ was arbitrary,
$$\{a_k\}_{k \in I}$$
is an ϵ-net for A_1 by definition. Hence as $\epsilon > 0$ was arbitrary, A_1 is totally bounded. ∎

Proposition 4.5.10. *Let $(\mathcal{X}, d)$ be a metric space. If $A \subseteq \mathcal{X}$ is totally bounded, then $\overline{A}$ is totally bounded.*

Proof. Let $\epsilon > 0$ be arbitrary. Since A is totally bounded, there exists a finite $\frac{\epsilon}{2}$-net for A, say $\{a_1, \ldots, a_n\} \subseteq A$. Hence $a_1, \ldots, a_n \in \overline{A}$ and we claim that $\{a_1, \ldots, a_n\}$ is an ϵ-net for $\overline{A}$. To see this, let $x \in \overline{A}$ be arbitrary. By

Lemma 1.5.23 there exists an $a \in A$ such that $d(x, a) < \frac{\epsilon}{2}$. As $\{a_1, \ldots, a_n\}$ is an $\frac{\epsilon}{2}$-net for A, there exists a $k \in \{1, \ldots, n\}$ such that $d(a, a_k) < \frac{\epsilon}{2}$. Hence $d(x, a_k) < \epsilon$ by the triangle inequality. Therefore, as $x \in \overline{A}$ was arbitrary, $\{a_1, \ldots, a_n\}$ is an ϵ-net for $\overline{A}$. Since $\epsilon > 0$ was arbitrary, $\overline{A}$ is totally bounded by definition. ∎

Proposition 4.5.11. *Every totally bounded metric space is separable.*

Proof. Let $(\mathcal{X}, d)$ be a totally bounded metric space. To see that $\mathcal{X}$ is separable, we must find a countable dense subset of $\mathcal{X}$. Since $\mathcal{X}$ is totally bounded, for each $n \in \mathbb{N}$ there exists a finite subset A_n of $\mathcal{X}$ that is an $\frac{1}{n}$-net. We claim that $A = \bigcup_{n=1}^{\infty} A_n$ is a countable dense subset. The fact that A is countable follows as the countable union of countable sets is countable and the fact that A is dense follows as if $x \in \mathcal{X}$ and $\epsilon > 0$ then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ so x is within ϵ of an element of A_n as A_n was an $\frac{1}{n}$-net. Hence $\mathcal{X}$ is separable. ∎

4.6 The Borel-Lebesgue Theorem

As sequentially compact sets are totally bounded by Proposition 4.5.5, we will use the properties of totally bounded sets to show that sequentially compact sets are compact. In order to prove sequentially compact sets are compact, we require two results. The first demonstrates that the Extreme Value Theorem holds for sequentially compact sets.

Lemma 4.6.1 (Extreme Value Theorem - Sequential Compactness). *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be continuous. If $\mathcal{X}$ is sequentially compact, then $f(\mathcal{X})$ is sequentially compact in $\mathcal{Y}$. Hence, if $\mathcal{Y} = \mathbb{R}$, there exists $x_1, x_2 \in \mathcal{X}$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in \mathcal{X}$.*

Proof. To see that $f(\mathcal{X})$ is sequentially compact, let $(y_n)_{n\geq 1}$ be an arbitrary sequence of elements of $f(\mathcal{X})$. Hence there exists a sequence $(x_n)_{n\geq 1}$ such that $y_n = f(x_n)$ for all $n \in \mathbb{N}$. Since $\mathcal{X}$ is sequentially compact, there exists a subsequence $(x_{k_n})_{n\geq 1}$ that converges to some element $x \in \mathcal{X}$. As f is continuous, $\lim_{n\to\infty} y_{k_n} = \lim_{n\to\infty} f(x_{k_n}) = f(x)$. Hence $(y_{k_n})_{n\geq 1}$ is a convergent subsequence of $(y_n)_{n\geq 1}$. Therefore, as $(y_n)_{n\geq 1}$ was arbitrary, $f(\mathcal{X})$ is sequentially compact by definition.

To see the later claim, suppose $\mathcal{Y} = \mathbb{R}$. Since $f(\mathcal{X})$ is sequentially compact, $f(\mathcal{X})$ is totally bounded by Proposition 4.5.5 and thus bounded by Proposition 4.5.8. Hence $\inf(f(\mathcal{X}))$ and $\sup(f(\mathcal{X}))$ are finite. Since $f(\mathcal{X})$ is sequentially compact, the limits of any convergent sequences with elements in $f(\mathcal{X})$ must be elements of $f(\mathcal{X})$. As we may construct sequences of elements of $f(\mathcal{X})$ converging to $\inf(f(\mathcal{X}))$ and $\sup(f(\mathcal{X}))$ and respectively, we obtain that $\sup(f(\mathcal{X})), \inf(f(\mathcal{X})) \in f(\mathcal{X})$. Hence there exists $x_1, x_2 \in \mathcal{X}$ such that

$f(x_1) = \inf(f(\mathcal{X}))$ and $f(x_2) = \sup(f(\mathcal{X}))$ so $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in \mathcal{X}$ as desired. ∎

Our next lemma enables us to choose 'large' open balls inside any open cover. In particular, once we demonstrate the notions of compact and sequentially compact sets are the same, we may apply the following lemma to any open cover.

Lemma 4.6.2. *Let $(\mathcal{X}, d)$ be a sequentially compact metric space. If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $\mathcal{X}$, then there exists an $\delta_0 > 0$ (called the* Lebesgue number *for $\{U_\alpha\}_{\alpha \in I}$) such that for any $0 < \delta < \delta_0$ and any $x \in \mathcal{X}$ there exists an $\alpha_x \in I$ such that $B(x, \delta) \subseteq U_{\alpha_x}$.*

Proof. To begin, note $(\mathcal{X}, d)$ is totally bounded by Proposition 4.5.5 and thus bounded by Proposition 4.5.8. Hence there exists an $x_0 \in \mathcal{X}$ and an $R > 0$ such that $B(x_0, R) = \mathcal{X}$ by Remark 1.7.2. Hence for any $x \in X$, $B(x, 2R) = \mathcal{X}$ by the triangle inequality.

Fix an open cover $\{U_\alpha\}_{\alpha \in I}$ of $\mathcal{X}$ and consider a function $\varphi : \mathcal{X} \to \mathbb{R}$ defined by

$$\varphi(x) = \sup\{r \in \mathbb{R} \mid r \leq 2R, B(x, r) \subseteq U_\alpha \text{ for some } \alpha \in I\}$$

for all $x \in \mathcal{X}$. To see that φ is well-defined, we claim that $\varphi(x) > 0$ for all $x \in \mathcal{X}$. To see this, notice if $x \in \mathcal{X}$ then $x \in \bigcup_{\alpha \in I} U_\alpha$. Hence there exists an $\alpha_x \in I$ such that $x \in U_{\alpha_x}$. Since U_{α_x} is open, there exists an $r > 0$ such that $B(x, r) \subseteq U_{\alpha_x}$ and thus $\varphi(x) \geq r > 0$.

We claim that φ is continuous. To see this, let $x, y \in \mathcal{X}$ be arbitrary. By definition of φ, if $r < \varphi(x)$ then there exists an $\alpha \in I$ such that $B(x, r) \subseteq U_\alpha$. Thus if $r < \varphi(x)$ and $r - d(x, y) > 0$ then $B(y, r - d(x, y)) \subseteq U_\alpha$ by the triangle inequality so $\varphi(y) \geq r - d(x, y)$. Otherwise, if $r - d(x, y) \leq 0$ then clearly $\varphi(y) \geq 0 \geq r - d(x, y)$. In either case, $\varphi(y) \geq r - d(x, y)$ for all $r < \varphi(x)$ so $\varphi(y) \geq \varphi(x) - d(x, y)$. By replacing the roles of x and y, we see that

$$|\varphi(x) - \varphi(y)| \leq d(x, y).$$

Therefore, as $x, y \in \mathcal{X}$ were arbitrary, φ is clearly continuous.

Since $(\mathcal{X}, d)$ is sequentially compact, Lemma 4.6.1 implies there exists an $x_0 \in \mathcal{X}$ such that $\varphi(x_0) \leq \varphi(x)$ for all $x \in \mathcal{X}$. Hence if $\delta_0 = \varphi(x_0)$, then $\delta_0 > 0$. Furthermore, for all $0 < \delta < \delta_0$ and $x \in \mathcal{X}$ we see that $\delta < \varphi(x)$ so by the definition of φ there exists an $\alpha_x \in I$ with $B(x, \delta) \subseteq U_{\alpha_x}$ as desired. ∎

With the completion of our construction of the Lebesgue number, we can finally prove the equivalence of compactness and sequential compactness in metric spaces.

Theorem 4.6.3 (Borel-Lebesgue Theorem). *A metric space is compact if and only if it is sequentially compact.*

Proof. As compact metric spaces are sequentially compact by Theorem 4.4.3, one direction is complete.

For the other direction, suppose $(\mathcal{X}, d)$ is a sequentially compact metric space. To see that $\mathcal{X}$ is compact, let $\{U_\alpha\}_{\alpha\in I}$ be an arbitrary open cover of $\mathcal{X}$. Therefore, by Lemma 4.6.2 there exists an $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ and any $x \in \mathcal{X}$ there exists an $\alpha_x \in I$ such that $B(x, \delta) \subseteq U_{\alpha_x}$.

Since $\mathcal{X}$ is sequentially compact, $\mathcal{X}$ is totally bounded by Proposition 4.5.5. Hence there exists a finite $\frac{\delta_0}{2}$-net for $\mathcal{X}$, say $\{x_1, \ldots, x_n\}$. Hence

$$\mathcal{X} = \bigcup_{k=1}^{n} B\left(x_k, \frac{\delta_0}{2}\right)$$

By the above paragraph there exists $\alpha_1, \ldots, \alpha_n \in I$ such that $B\left(x_k, \frac{\delta_0}{2}\right) \subseteq U_{\alpha_k}$ for all $k \in \{1, \ldots, n\}$. Hence

$$X = \bigcup_{k=1}^{n} U_{\alpha_k}$$

so $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ is a finite subcover of $\mathcal{X}$. Therefore, as $\{U_\alpha\}_{\alpha\in I}$ was arbitrary, $\mathcal{X}$ is compact by definition. ■

Before moving onto applications of the Borel-Lebesgue Theorem (Theorem 4.6.3), we note that the notions of compact and sequentially compact sets need not agree for general topological spaces. In particular, compactness always implies sequential compactness (the proof of Theorem 4.4.3 pretty much works once the distance functions are changed with open sets) but the converse need not hold. To correct the converse, one needs to generalize the notion of sequences, which we will avoid.

Onto properties of compact sets. Note all but completeness are trivial.

Corollary 4.6.4. *Every compact metric space is complete, totally bounded, and separable.*

Proof. Let $(\mathcal{X}, d)$ be a compact metric space. Since $\mathcal{X}$ is sequentially compact by Theorem 4.6.3, $\mathcal{X}$ is totally bounded by Proposition 4.5.5. Thus $\mathcal{X}$ is separable by Proposition 4.5.11.

To see that $\mathcal{X}$ is complete, let $(x_n)_{n\geq 1}$ be an arbitrary Cauchy sequence in $\mathcal{X}$. Since $\mathcal{X}$ is sequentially compact, $(x_n)_{n\geq 1}$ has a convergent subsequence and thus converges by Lemma 2.1.7. Therefore, as $(x_n)_{n\geq 1}$ was arbitrary, $\mathcal{X}$ is complete. ■

4.7 Compactness and Completeness

In previous sections, we have seen the notions of compactness and sequentially compactness are equal and for finite dimensional spaces, these are equivalent to the notion of closed and bounded sets. Clearly completeness is a generalization of closed sets by Theorem 2.1.12 and total boundedness is a generalization of boundedness by Proposition 4.5.5. Since compact/sequentially compact sets are complete and totally bounded by Proposition 2.5.18, it is natural to ask where the converse is true. As it so happens, it is.

Theorem 4.7.1. *Let $(\mathcal{X}, d)$ be a metric space. Then the following are equivalent:*

1. *$\mathcal{X}$ is compact.*
2. *$\mathcal{X}$ is complete and totally bounded.*

Proof. Since compact metric spaces are complete and totally bounded by Proposition 4.6.4, one direction is complete.

For the other direction, suppose $(\mathcal{X}, d)$ is complete and totally bounded. To show that $(\mathcal{X}, d)$ is compact, we will demonstrate that $(\mathcal{X}, d)$ is sequentially compact and apply the Borel-Lebesgue Theorem (Theorem 4.6.3).

To see that $(\mathcal{X}, d)$ is sequentially compact, let $(x_n)_{n\geq 1}$ be an arbitrary sequence of elements of $\mathcal{X}$. Since $(\mathcal{X}, d)$ is totally bounded, $F_1 = \{x_n\}_{n=1}^\infty$ is totally bounded by Proposition 4.5.9. Hence F_1 has a finite 1-net. This implies there exists an $n_1 \in \mathbb{N}$ such that $x_{n_1} \in F_1$ and

$$I_1 = \{n \in \mathbb{N} \mid n > n_1 \text{ and } x_n \in B(x_{n_1}, 1)\}$$

is infinite. Let $F_2 = \{x_n\}_{n\in I_1}$. Since $(\mathcal{X}, d)$ is totally bounded, F_2 is totally bounded by Proposition 4.5.9 and thus F_2 has finite $\frac{1}{2}$-net. As I_1 is infinite, there exists a $n_2 \in \mathbb{N}$ such that $n_2 \in I_1$ (so $n_2 > n_1$) such that $x_{n_2} \in F_2$ and

$$I_2 = \left\{n \in \mathbb{N} \,\middle|\, n > n_2 \text{ and } x_n \in B\left(x_{n_2}, \frac{1}{2}\right)\right\}$$

is infinite. Let $F_3 = \{x_n\}_{n\in I_2}$. By repeating this process ad infinitum, there exists infinite subsets I_n of I_1 and an increasing sequence $(n_k)_{k\geq 1}$ of natural number such that $x_{n_k} \in F_m$ for all $k \geq m$ and $x_{n_k} \in B(x_{n_m}, \frac{1}{m})$ for all $k > m$. Hence, as $\mathrm{diam}(B(x_{n_m}, \frac{1}{m})) \leq \frac{2}{m}$, $(x_{n_k})_{k\geq 1}$ is a Cauchy subsequence of $(x_n)_{n\geq 1}$. Since $(\mathcal{X}, d)$ is complete, $(x_{n_k})_{k\geq 1}$ is a convergent subsequence of $(x_n)_{n\geq 1}$. Therefore, as $(x_n)_{n\geq 1}$ was arbitrary, $(\mathcal{X}, d)$ is sequentially compact as desired. ∎

As it is easy to verify sets are complete and totally bounded than it is to verify compactness or sequential compactness directly, Theorem 4.7.1 is an excellent tool for verify metric spaces are compact.

Example 4.7.2. Let

$$K = \left\{(x_n)_{n\geq 1} \in \ell_2(\mathbb{N}, \mathbb{R}) \,\middle|\, \sum_{n=1}^\infty n^2|x_n|^2 \leq 1\right\}.$$

We claim that K is a compact subspace of $(\ell_2(\mathbb{N},\mathbb{R}), \|\cdot\|_2)$. To see this, it suffices by Theorem 4.7.1 to show that K is complete and totally bounded with respect to $\|\cdot\|_2$. To see that K is complete, we note since $(\ell_2(\mathbb{N},\mathbb{R}), \|\cdot\|_2)$ is complete, it suffices to prove that K is a closed subset of $(\ell_2(\mathbb{N},\mathbb{R}), \|\cdot\|_2)$.

To see that K is closed in $(\ell_2(\mathbb{N},\mathbb{R}), \|\cdot\|_2)$, let $(\vec{v}_k)_{k\geq 1}$ be an arbitrary sequence of elements of K that converges to some $\vec{x} \in \ell_2(\mathbb{N},\mathbb{R})$. For each $k \in \mathbb{N}$ write

$$\vec{v}_k = (x_{k,n})_{n\geq 1} \qquad \text{and} \qquad \vec{x} = (x_n)_{n\geq 1}.$$

Since

$$\lim_{k\to\infty} \|\vec{v}_k - \vec{x}\|_2 = 0 \qquad \text{and} \qquad |x_{k,n} - x_n| \leq \|\vec{v}_k - \vec{x}\|_2 \text{ for all } k, n \in \mathbb{N},$$

we obtain that $\lim_{k\to\infty} |x_{k,n} - x_n| = 0$ for all $n \in \mathbb{N}$. Furthermore, $\vec{v}_k \in K$ for all $k \in \mathbb{N}$, we obtain by the definition of K that

$$\sum_{n=1}^{\infty} n^2 |x_{k,n}|^2 \leq 1$$

for all $k \in \mathbb{N}$. Hence for all $N \in \mathbb{N}$ we see that

$$\sum_{n=1}^{N} n^2|x_n|^2 = \lim_{k\to\infty} \sum_{n=1}^{N} n^2|x_{k,n}|^2 \leq \limsup_{k\to\infty} \sum_{n=1}^{\infty} n^2 |x_{k,n}|^2 \leq 1.$$

Therefore, since the above holds for all $N \in \mathbb{N}$, we obtain that $\sum_{n=1}^{\infty} n^2|x_n|^2 \leq 1$ and thus $\vec{x} \in K$. Thus, as $(\vec{v}_k)_{k\geq 1}$ was arbitrary, we obtain that K is closed.

To see that K is totally bounded, let $\epsilon > 0$ be arbitrary. Without loss of generality, we may assume that $\epsilon < 1$. To see that K has an ϵ-net, first notice if $\vec{x} = (x_n)_{n\geq 1} \in K$ then $\sum_{n=1}^{\infty} n^2|x_n|^2 \leq 1$ so $n^2|x_n|^2 \leq 1$ for all $n \in \mathbb{N}$ and thus $|x_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. To begin to use this, we note since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ there exists and $N \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \frac{\epsilon^2}{2}.$$

Hence, if $\vec{x} = (x_n)_{n\geq 1} \in K$ then the above shows that

$$\sum_{n=N+1}^{\infty} |x_n|^2 < \frac{\epsilon^2}{2}.$$

Consider the set

$$K_0 = \left\{ (x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; \sum_{n=1}^{N} n^2|x_n|^2 \leq 1 \right\} \subseteq \mathbb{R}^N.$$

By the same arguments used above, K_0 is a closed subset of $\mathbb{R}^N$. Furthermore, if $(x_1, \ldots, x_N) \in K_0$, then $|x_n| \leq \frac{1}{n}$ for all $n \geq N$ and thus

$$\|(x_1, \ldots, x_N)\|_2 \leq \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{\frac{1}{2}}.$$

Therefore K_0 is bounded in $(\mathbb{R}^2, \|\cdot\|_2)$ and hence compact by the Heine-Borel Theorem. Thus K_0 is totally bounded.

Let $\vec{v}_1, \ldots, \vec{v}_m$ be a finite $\frac{\epsilon}{\sqrt{2}}$-net for K_0. Clearly each $\vec{v}_k$ defines an element of K by extending the N-tuple to a sequence by letting every term in the sequence with index greater than N be zero. We claim that $\vec{v}_1, \ldots, \vec{v}_m$ then forms an ϵ-net of K. To see this, let $\vec{x} = (x_n)_{n\geq 1} \in K$ be arbitrary. Then

$$(x_1, \ldots, x_N) \in K_0 \text{ by construction} \qquad \text{and} \qquad \sum_{n=N+1}^{\infty} |x_n|^2 < \frac{\epsilon^2}{2}.$$

Since $\vec{v}_1, \ldots, \vec{v}_m$ is a finite $\frac{\epsilon}{\sqrt{2}}$-net for K_0, there exists a $k \in \{1, \ldots, m\}$ such that

$$\|\vec{v}_k - (x_1, \ldots, x_N)\|_2 < \frac{\epsilon}{\sqrt{2}}.$$

Hence

$$\begin{aligned}\|\vec{v}_k - \vec{x}\|_2^2 &= \|\vec{v}_k - (x_1, \ldots, x_N)\|_2^2 + \sum_{n=N+1}^{\infty} |x_n|^2 \\ &< \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} = \epsilon^2.\end{aligned}$$

Therefore $\|\vec{v}_k - \vec{x}\|_2 < \epsilon$. Hence, as $\vec{x} \in K$ was arbitrary, $\vec{v}_1, \ldots, \vec{v}_m$ is an ϵ-net of K. Therefore, since $\epsilon > 0$ was arbitrary, K is totally bounded. Consequently, K is compact as desired.

4.8 Compactness and Continuous Functions

Using the above, we can see that compact sets are very well-behaved with respect to continuous functions. For example if $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are metric spaces with $\mathcal{X}$ compact, then we can see that

$$\mathcal{C}_b(\mathcal{X}, \mathcal{Y}) = \mathcal{C}(\mathcal{X}, \mathcal{Y}).$$

Indeed if $f \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$, then $f(\mathcal{X})$ is a compact subsets of $\mathcal{Y}$ by Theorem 4.1.13, thus totally bounded (by Theorem 4.7.1), and thus bounded (by Proposition 4.5.8). In particular, if $\mathcal{X}$ is compact and $\mathcal{Y}$ is a complete normed linear space, $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ is a Banach space.

Furthermore, there is a stronger notion of continuity that will hold when consider continuous functions on compact sets.

Definition 4.8.1. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$. It is said that f is *uniformly continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_1, x_2 \in \mathcal{X}$ are such that if $d_{\mathcal{X}}(x_1, x_2) < \delta$ then $d_{\mathcal{Y}}(f(x_1), f(x_2)) < \epsilon$.

That is, there is one δ to rule them all!

Remark 4.8.2. Again, as with continuity and convergence of sequences, either or both of the $< \delta$ and $< \epsilon$ in Definition 4.8.1 may be replaced with $\leq \delta$ and $\leq \epsilon$ respectively. Additionally, note the main use of uniform continuity is that for each $\epsilon > 0$ one may find a δ that works for ANY elements of $\mathcal{X}$.

Example 4.8.3. Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$. It is said that f is *Lipschitz* if there exists a $k \geq 0$ (called a Lipschitz constant) such that

$$d_{\mathcal{Y}}(f(x_1), f(x_2)) \leq k d_{\mathcal{X}}(x_1, x_2)$$

for all $x_1, x_2 \in \mathcal{X}$. It is not difficult to see that Lipschitz functions are uniformly continuous (indeed, given $\epsilon > 0$, take $\delta = \frac{\epsilon}{k+1}$). Hence bounded linear maps between normed linear spaces are uniformly continuous. Similarly, if $f : [a, b] \to \mathbb{R}$ is differentiable such that $|f'(x)| \leq k$ for all $x \in [a, b]$, then the Mean Value Theorem implies that

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2|$$

for all $x_1, x_2 \in [a, b]$ so f is Lipschitz and thus uniformly continuous.

To obtain some examples of continuous functions that are not uniformly continuous, first note the follow.

Remark 4.8.4. Note that if $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ are metric spaces and $f : \mathcal{X} \to \mathcal{Y}$ is not uniformly continuous, then there exists a $\epsilon_0 > 0$ and sequences $(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $d_{\mathcal{X}}(x_n, x'_n) < \frac{1}{n}$ yet $d_{\mathcal{Y}}(f(x_n), f(x'_n)) \geq \epsilon_0$ for all $n \in \mathbb{N}$. This observation is useful in demonstrating functions are not uniformly continuous.

Example 4.8.5. The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous. To see this for each $n \in \mathbb{N}$ let $x_n = \frac{1}{n}$ and $y_n = \frac{2}{n}$. Then $|x_n - y_n| < \frac{1}{n-1}$ yet

$$|f(x_n) - f(y_n)| = \left|\frac{1}{\frac{1}{n}} - \frac{1}{\frac{2}{n}}\right| = \left|n - \frac{n}{2}\right| = \frac{n}{2} \geq 1.$$

Hence f is not uniformly continuous on $(0, 1)$.

Although we could spend time and develop the theory of uniformly continuous functions on $\mathbb{R}$, we desire the following demonstrating continuous functions on compact sets are automatically uniformly continuous. Hence continuous functions on compact sets are especially nice functions.

Theorem 4.8.6. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $f : \mathcal{X} \to \mathcal{Y}$ be continuous. If $\mathcal{X}$ is compact, then f is uniformly continuous.*

Proof. Suppose to the contrary that f is not uniformly continuous. Hence there exists a $\epsilon_0 > 0$ and sequences $(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ of elements of $\mathcal{X}$ such that $d_{\mathcal{X}}(x_n, x'_n) < \frac{1}{n}$ yet $d_{\mathcal{Y}}(f(x_n), f(x'_n)) \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Since $\mathcal{X}$ is compact, $\mathcal{X}$ is sequentially compact by Theorem 4.6.3. Therefore there exists a subsequence $(x_{k_n})_{n\geq 1}$ of $(x_n)_{n\geq 1}$ that converges to some element $z \in \mathcal{X}$. Consider the subsequence $(x'_{k_n})_{n\geq 1}$ of $(x'_n)_{n\geq 1}$. Notice for all $n \in \mathbb{N}$ that

$$\begin{aligned} d_{\mathcal{X}}(x'_{k_n}, z) &\leq d_{\mathcal{X}}(x'_{k_n}, x_{k_n}) + d_{\mathcal{X}}(x_{k_n}, z) \\ &\leq \frac{1}{k_n} + d_{\mathcal{X}}(x_{k_n}, z) \\ &\leq \frac{1}{n} + d_{\mathcal{X}}(x_{k_n}, z). \end{aligned}$$

Therefore, since $\lim_{n\to\infty} d_{\mathcal{X}}(x_{k_n}, z) = 0$, $(x'_{k_n})_{n\geq 1}$ converges to z in $\mathcal{X}$.

Since f is continuous, there exists $N_1, N_2 \in \mathbb{N}$ such that $d_{\mathcal{Y}}(f(z), f(x_{k_n})) < \frac{\epsilon_0}{2}$ for all $n \geq N_1$ and $d_{\mathcal{Y}}(f(z), f(x'_{k_n})) < \frac{\epsilon_0}{2}$ for all $n \geq N_2$. Therefore, if $n = \max\{N_1, N_2\}$, we obtain that

$$d_{\mathcal{Y}}(f(x_{k_n}), f(x'_{k_n})) \leq d_{\mathcal{Y}}(f(x_{k_n}), f(z)) + d_{\mathcal{Y}}(f(z), f(x'_{k_n})) < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0$$

which is a contradiction. Hence it must have been the case that f is uniformly continuous. ∎

Example 4.8.7. There are (uniformly) continuous functions on compact metric spaces that are not Lipschitz. Indeed consider the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = \sqrt{x}$ for all $x \in [0, 1]$. Then f is clearly continuous and thus uniformly continuous by Theorem 4.8.6 as $[0, 1]$ is compact. However, if $x, y \in [0, 1]$ and $y = 0$, then

$$|f(x) - f(y)| = |f(x)| = \sqrt{x} \qquad \text{whereas} \qquad |x - y| = |x| = x.$$

Since

$$\lim_{x\to 0} \frac{|f(x) - f(y)|}{|x - y|} = \lim_{x\to 0} \frac{\sqrt{x}}{x} = \infty,$$

it is impossible that f is Lipschitz.

4.9 Equicontinuity

Now that we have characterized the compact metric spaces as those that are complete and totally bounded in Theorem 4.7.1, we turn our attention back to continuous function spaces. Indeed, recall from Theorem 2.5.14 that every metric space is isomorphic to a subset of a continuous function space. Thus by studying compactness in continuous function spaces, we are studying compactness for all metric spaces!

Thus the remaining goal of the remainder of the chapter is to derive simple conditions to determine when subsets of $\mathcal{C}(\mathcal{X}, \mathbb{R})$ are compact when $\mathcal{X}$ is a compact metric space. Thus to simplify notation, throughout this section for a compact metric space $(\mathcal{X}, d)$, we will use $\mathcal{C}(\mathcal{X})$ to denote $\mathcal{C}(\mathcal{X}, \mathbb{R})$ equipped with the sup-norm.

A characterization of compact subsets of $\mathcal{C}(\mathcal{X})$ is particularly useful in deriving properties of functions from other functions. For example, suppose we have a compact set of functions Φ with a specific property. Then, if we construct a sequence of functions from Φ in a specific way, we know by sequential compactness that this sequence then has a subsequence that converges to an element of Φ and thus must have the same properties.

Of course, we will want to study closed subsets of $\mathcal{C}(\mathcal{X})$ because Theorem 2.1.12 implies the closed set is complete and Theorem 4.7.1 implies being complete is necessary for being compact. As often one desires only to describes a collection of functions that happen not to be closed, we define the following.

Definition 4.9.1. Let $(\mathcal{X}, d)$ be a metric space. A subset $A \subseteq \mathcal{X}$ is said to be *relatively compact* if $\overline{A}$ is compact.

Remark 4.9.2. Notice that if $(\mathcal{X}, d)$ is a complete metric space and $A \subseteq \mathcal{X}$, then A is relatively compact if and only if A is totally bounded by Theorem 2.1.12, Theorem 4.7.1, and Proposition 4.5.9.

Thus, if we want to study relatively compact subsets of $\mathcal{C}(\mathcal{X})$, we need only study which collections of functions are totally bounded. Of course, verifying totally boundedness from definition is easier than verifying compactness from definition, but it still is not simple. Thus we desire to find simpler conditions to verify a collection of functions is totally bounded.

Of course, if a collection of functions is totally bounded with respect to the sup metric, every function will be close to another function from a finite collection. Knowing how each element of this finite collection is continuous at a point then yields information about how the entire collection is continuous at a point. This leads us to the following notion of a collection of functions being 'equally continuous'.

Definition 4.9.3. Let $(\mathcal{X}, d)$ be a compact metric space, let $x_0 \in \mathcal{X}$, and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$. It is said that $\mathcal{F}$ is *equicontinuous at* x_0 if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in \mathcal{X}$ and $d(x, x_0) < \delta$ then $|f(x) - f(x_0)| < \epsilon$ for all $f \in \mathcal{F}$.

If $\mathcal{F}$ is equicontinuous at every point in $\mathcal{X}$, it is said that $\mathcal{F}$ is *equicontinuous*.

Again, as with continuity, uniform continuity, and convergence of sequences, either or both of the $< \delta$ and $< \epsilon$ in Definition 4.9.3 may be replaced with $\leq \delta$ and $\leq \epsilon$ respectively.

Example 4.9.4. For each $n \in \mathbb{N}$ let $f_n : [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for all $x \in [-1, 1]$. The collection $\mathcal{F} = \{f_n\}_{n=1}^\infty$ is equicontinuous at 0. Indeed if $\epsilon > 0$ let $\delta = \min\{\epsilon, 1\} > 0$. Then if $|x| < \delta$ then

$$|f_n(x)| = |x^n| \leq \delta^n \leq \epsilon.$$

Hence $\mathcal{F}$ is equicontinuous at 0. However, $\mathcal{F}$ is not equicontinuous at 1. To see this, notice for all $\delta > 0$,

$$\lim_{n\to\infty} |f_n(1) - f_n(1-\delta)| = \lim_{n\to\infty} |1 - (1-\delta)^n| = 1$$

so no δ can work in Definition 4.9.3 for $\epsilon = \frac{1}{2}$.

Example 4.9.5. Let $\mathcal{F}$ be a collection of functions on $[0, 1]$ with Lipschitz functions with Lipschitz constant at most 1 (see Example 4.8.3). It is not difficult to check that $\mathcal{F}$ is equicontinuous. Hence, if $\mathcal{F}$ consists of all differentiable functions f on $[0, 1]$ with $|f'(x)| \leq 1$ for all $x \in [0, 1]$, then $\mathcal{F}$ is equicontinuous.

To emphasize the idea that equicontinuity should stem from total boundedness, we note the following lemma.

Lemma 4.9.6. *Let $(\mathcal{X}, d_\mathcal{X})$ be a metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$ be totally bounded. Then $\mathcal{F}$ is equicontinuous.*

Proof. To see that $\mathcal{F}$ is equicontinuous, let $\epsilon > 0$ and $x_0 \in \mathcal{X}$ be arbitrary. Since $\mathcal{F}$ is totally bounded, there exists a finite $\frac{\epsilon}{3}$-net for $\mathcal{F}$. Hence there exists an $n \in \mathbb{N}$ and $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that

$$\mathcal{F} \subseteq \bigcup_{k=1}^n B\left(f_k, \frac{\epsilon}{3}\right).$$

Since f_k is continuous at x_0 for all $k \in \{1, \ldots, n\}$, there exists a $\delta_k > 0$ such that

$$d_Y(f_k(x), f_k(x_0)) < \frac{\epsilon}{3}$$

for all $x \in \mathcal{X}$ such that $d_\mathcal{X}(x, x_0) < \delta_k$. Let

$$\delta = \min\{\delta_k \mid k \in \{1, \ldots, n\}\} > 0.$$

We claim that δ works for ϵ in Definition 4.9.3 to show that $\mathcal{F}$ is equicontinuous at x_0. To see this, let $f \in \mathcal{F}$ be arbitrary. Hence, as $\mathcal{F} \subseteq \bigcup_{k=1}^n B\left(f_k, \frac{\epsilon}{3}\right)$, there exists a $k \in \{1, \ldots, n\}$ such that

$$\|f - f_k\|_\infty < \frac{\epsilon}{3}.$$

Thus $|f(x) - f_k(x)| < \frac{\epsilon}{3}$ for all $x \in \mathcal{X}$. Therefore, for all $x \in \mathcal{X}$ such that $d_{\mathcal{X}}(x, x_0) < \delta$, we obtain from above that

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(x_0)| + |f_k(x_0) - f(x_0)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore, as $f \in \mathcal{F}$, $\epsilon > 0$, and $x_0 \in \mathcal{X}$ were arbitrary, $\mathcal{F}$ is equicontinuous as desired. ∎

Unfortunately, equicontinuity does not immediately imply total boundedness.

Example 4.9.7. For each $a \in \mathbb{R}$, let $f_a : [0,1] \to \mathbb{R}$ be defined by $f(x) = x+a$ for all $x \in [0,1]$. Clearly $\{f_a\}_{a\in\mathbb{R}}$ is equicontinuous as

$$|f_a(x) - f_a(y)| = |f_0(x) - f_0(y)|$$

for all $x, y \in [0,1]$. However $\{f_a\}_{a\in\mathbb{R}}$ cannot be totally bounded in $\mathcal{C}[0,1]$ since $\|f_a\|_\infty = a + 1$ for all $a \geq 0$ so $\{f_a\}_{a\in\mathbb{R}}$ is not bounded with respect to $\|\cdot\|_\infty$ and thus cannot be totally bounded by Proposition 4.5.8.

Thus the problem is that equicontinuity does not yield any information about a collection of functions behaving like a bounded collection of functions. Of course we could just ask that the collection of functions is bounded with respect to the sup norm. However, there is also a much simpler notion of boundedness we can ask for.

Before we discuss the correct notion of boundedness to add to equicontinuity, we note that equicontinuity is a nice property as it passes to closures of sets; something we expect as we are studying relative compactness in $\mathcal{C}(\mathcal{X})$.

Proposition 4.9.8. *Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$ be equicontinuous. Then $\overline{\mathcal{F}}$ is equicontinuous.*

Proof. To see that $\overline{\mathcal{F}}$ is equicontinuous, fix an arbitrary element $x_0 \in \mathcal{X}$ and let $\epsilon > 0$ be arbitrary. Since $\mathcal{F}$ is equicontinuous, there exists a $\delta > 0$ such that if $x \in \mathcal{X}$ and $d_{\mathcal{X}}(x, x_0) < \delta$ then $|f(x) - f(x_0)| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$.

To see that δ works with respect to ϵ for $\overline{\mathcal{F}}$ in Definition 4.9.3, let $g \in \overline{\mathcal{F}}$ be arbitrary. By Corollary 1.5.23 there exists an $f \in \mathcal{F}$ such that $\|g - f\|_\infty < \frac{\epsilon}{3}$. Therefore, if $x \in \mathcal{X}$ and $d_{\mathcal{X}}(x, x_0) < \delta$, then

$$\begin{aligned}|g(x) - g(x_0)| &\leq |g(x) - f(x)| + |f(x) - f(x_0)| + |f(x_0) - g(x_0)| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\end{aligned}$$

Therefore, as $g \in \overline{\mathcal{F}}$, $\epsilon > 0$, and $x_0 \in \mathcal{X}$ were arbitrary, we obtain that $\overline{\mathcal{F}}$ is equicontinuous. ∎

As uniform continuity is preferable to continuity as the same δ works for every point in the space, we also desire a strengthening of equicontinuous collections of functions in precisely the same way.

Definition 4.9.9. Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$. It is said that $\mathcal{F}$ is *uniformly equicontinuous* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_1, x_2 \in \mathcal{X}$ and $d(x_1, x_2) < \delta$ then $|f(x_1) - f(x_2)| < \epsilon$ for all $f \in \mathcal{F}$.

As continuous functions on compact sets are automatically uniformly continuous, it is unsurprising that equicontinuous functions on compact sets are automatically uniformly equicontinuous. Of course, the benefit of this theorem is that one obtains the stronger notion of uniformly equicontinuous by just verifying the equicontinuous condition pointwise.

Theorem 4.9.10. *Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$ be equicontinuous. Then $\mathcal{F}$ is uniformly equicontinuous.*

Proof. Our proof will be quite similar to Theorem 4.8.6. Suppose to the contrary that $\mathcal{F}$ is not uniformly equicontinuous. Hence there exists a $\epsilon_0 > 0$, sequences $(x_n)_{n\geq 1}$ and $(x'_n)_{n\geq 1}$ of elements of $\mathcal{X}$, and elements $f_n \in \mathcal{F}$ such that $d(x_n, x'_n) < \frac{1}{n}$ yet $|f_n(x_n) - f_n(x'_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$.

Since $\mathcal{X}$ is compact, $\mathcal{X}$ is sequentially compact by Theorem 4.6.3. Therefore there exists a subsequence $(x_{k_n})_{n\geq 1}$ of $(x_n)_{n\geq 1}$ that converges to some element $z \in \mathcal{X}$. Consider the subsequence $(x'_{k_n})_{n\geq 1}$ of $(x'_n)_{n\geq 1}$. Notice for all $n \in \mathbb{N}$ that

$$d(x'_{k_n}, z) \leq d(x'_{k_n}, x_{k_n}) + d(x_{k_n}, z) \leq \frac{1}{k_n} + d(x_{k_n}, z) \leq \frac{1}{n} + d(x_{k_n}, z).$$

Therefore, since $\lim_{n\to\infty} d(x_{k_n}, z) = 0$, $(x'_{k_n})_{n\geq 1}$ converges to z in $\mathcal{X}$.

Since $\mathcal{F}$ is equicontinuous, there exists a $\delta > 0$ such that if $x \in \mathcal{X}$ and $d(x, z) < \delta$ then $|f(x) - f(z)| < \frac{\epsilon_0}{2}$ for all $f \in \mathcal{F}$. Since both $(x_{k_n})_{n\geq 1}$ and $(x'_{k_n})_{n\geq 1}$ converge to z, there exists an $N \in \mathbb{N}$ such that $d(x_{k_N}, z) < \delta$ and $d(x'_{k_N}, z) < \delta$. Hence

$$\begin{aligned} |f_{k_N}(x_{k_N}) - f_{k_N}(x'_{k_N})| &\leq |f_{k_N}(x_{k_N}) - f_{k_N}(z)| + |f_{k_N}(z) - f_{k_N}(x'_{k_N})| \\ &< \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0 \end{aligned}$$

which is a contradiction. Hence $\mathcal{F}$ is uniformly equicontinuous. ∎

4.10 The Arzelà-Ascoli Theorem

In this section, we will obtain a characterization of compact subsets of $\mathcal{C}(\mathcal{X})$. Of course we desire to obtain conditions that are as easy to check as possible. This was the advantage of proving Theorem 4.9.10 as we obtain uniform equicontinuous from just equicontinuous. As Theorem 4.7.1 demonstrated that compactness was the same as completeness and total boundedness, and as equicontinuous has a 'completeness' feel to it, we desire a notion of boundedness. Of course, the simplest notion of boundedness to check is the following.

Definition 4.10.1. Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$. It is said that $\mathcal{F}$ is *pointwise bounded* if

$$\sup_{f \in \mathcal{F}} |f(x)| < \infty$$

for all $x \in \mathcal{X}$.

However, there no immediate connection between pointwise boundedness and boundedness (and hence total boundedness) of collections of functions.

Example 4.10.2. For each $n \in \mathbb{N}$, let $f_n \in \mathcal{C}[0, 1]$ be defined by

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in \left[0, \frac{1}{n}\right] \\ n^2\left(\frac{2}{n} - x\right) & \text{if } x \in \left[\frac{1}{n}, \frac{2}{n}\right] \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in [0, 1]$. We claim the collection $\mathcal{F} = \{f_n\}_{n=1}^\infty$ is pointwise bounded. To see this, notice $f_n(0) = 0$ for all $n \in \mathbb{N}$ so $\mathcal{F}$ is bounded at 0. Otherwise, if $x \in (0, 1]$, choose $N \in \mathbb{N}$ such that $\frac{2}{N} < x$. Then it is easy to see that $\mathcal{F}$ is bounded by

$$\max\{f_1(x), f_2(x), \ldots, f_{N-1}(x), f_N(x) = 0\}.$$

Hence $\mathcal{F}$ is pointwise bounded. However, $\mathcal{F}$ is not bounded with respect to the sup norm as $|f_n\left(\frac{1}{n}\right)| = n$ for all $n \in \mathbb{N}$ so $\|f_n\|_\infty \geq n$ for all $n \in \mathbb{N}$.

Of course (because we are discussing this), the reason Example 4.10.2 gives a pointwise bounded collection of functions that is not bounded is because the functions fluctuate too much. Thus we have equicontinuity to the rescue.

Proposition 4.10.3. *Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$. If $\mathcal{F}$ is equicontinuous and pointwise bounded, then $\mathcal{F}$ is bounded.*

Proof. Let $\epsilon = 1$. Since $\mathcal{F}$ is equicontinuous, $\mathcal{F}$ is uniformly equicontinuous by Theorem 4.9.10. Hence there exists a $\delta > 0$ such that if $x_1, x_2 \in \mathcal{X}$ and $d(x_1, x_2) < \delta$ then $|f(x_1) - f(x_2)| \leq \epsilon = 1$ for all $f \in \mathcal{F}$.

Since $\mathcal{X}$ is compact, $\mathcal{X}$ is totally bounded by Theorem 4.7.1. Hence $\mathcal{X}$ has a finite δ-net, say $x'_1, \ldots, x'_n$. Note for each $k \in \{1, \ldots, n\}$, we obtain that

$$M_k = \sup_{f \in \mathcal{F}} |f(x'_k)| < \infty$$

as $\mathcal{F}$ is pointwise bounded. Let $M = 1 + \max\{M_1, \ldots, M_n\}$.

We claim that $\|f\|_\infty \leq M$ for all $f \in \mathcal{F}$. To see this, let $f \in \mathcal{F}$ be arbitrary. If $x \in \mathcal{X}$, then as $\{x'_1, \ldots, x'_n\}$ is a δ-net for $\mathcal{X}$, there exists a $k_0 \in \{1, \ldots, n\}$ such that $d(x, x'_{k_0}) < \delta$. Hence $|f(x) - f(x'_{k_0})| \leq 1$ so

$$|f(x)| \leq 1 + |f(x'_{k_0})| \leq 1 + M_{k_0} \leq M.$$

Hence, as this holds for all $x \in \mathcal{X}$, $\|f\|_\infty \leq M$. Hence, as f was arbitrary, $\mathcal{F}$ is bounded. ∎

Using Remark 4.9.2 and the concepts developed above, we can finally prove a characterization of compact sets of functions.

Theorem 4.10.4 (The Arzelà-Ascoli Theorem). *Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$. The following are equivalent:*

1. *$\mathcal{F}$ is relatively compact.*
2. *$\mathcal{F}$ is equicontinuous and pointwise bounded.*

Proof. To begin, suppose $\mathcal{F}$ is relatively compact. Hence $\overline{\mathcal{F}}$ is compact and thus complete and totally bounded by Theorem 4.7.1. Thus $\overline{\mathcal{F}}$ is bounded with respect to $\|\cdot\|_\infty$ by Proposition 4.5.8 so $\mathcal{F}$ is bounded with respect to $\|\cdot\|_\infty$ and hence pointwise bounded. To see that $\mathcal{F}$ is equicontinuous, we note that since $\overline{F}$ is totally bounded, $\mathcal{F}$ is totally bounded by Proposition 4.5.9. Hence $\mathcal{F}$ is equicontinuous by Lemma 4.9.6. Hence the first direction of the proof is complete.

For the other direction, suppose $\mathcal{F}$ is equicontinuous and pointwise bounded. By Remark 4.9.2, it suffices to prove that $\mathcal{F}$ is totally bounded. Thus let $\epsilon > 0$ be arbitrary. Our goal is to divide up $\mathcal{X}$ and the range of $\mathcal{F}$ into suitably small pieces, take one function that maps each piece of $\mathcal{X}$ into a chosen piece of the range of $\mathcal{F}$, and show this is an ϵ-net for $\mathcal{F}$.

Since $\mathcal{F}$ is equicontinuous, $\mathcal{F}$ is uniformly equicontinuous by Theorem 4.9.10. Hence there exists a $\delta > 0$ such that if $x_1, x_2 \in \mathcal{X}$ and $d(x_1, x_2) < \delta$ then $|f(x_1) - f(x_2)| < \frac{\epsilon}{3}$ for all $f \in \mathcal{F}$. Since $\mathcal{X}$ is compact, $\mathcal{X}$ is totally bounded. Hence there exists an $n \in \mathbb{N}$ and $x'_1, \ldots, x'_n \in \mathcal{X}$ such that $\{x'_1, \ldots, x'_n\}$ is a δ-net for $\mathcal{X}$.

Since $\mathcal{F}$ is equicontinuous and pointwise bounded, $\mathcal{F}$ is bounded by Proposition 4.10.3. Hence there exists an $M > 0$ such that $\|f\|_\infty \leq M$ for all $f \in \mathcal{F}$. Choose numbers

$$-M = m_1 < m_2 < \cdots < m_{q+1} = M$$

such that $|m_{k+1} - m_k| < \frac{\epsilon}{3}$ for all $k \in \{1, \ldots, q\}$.

For each n-tuple $(k_1, \ldots, k_n) \in \{1, \ldots, q\}^n$, let

$$\mathcal{F}_{(k_1,\ldots,k_n)} = \{f \in \mathcal{F} \mid f(x'_j) \in [m_{k_j}, m_{k_j+1}] \text{ for all } j \in \{1, \ldots, n\}\}.$$

Clearly

$$\mathcal{F} = \bigcup_{(k_1,\ldots,k_n)\in\{1,\ldots,q-1\}^n} \mathcal{F}_{(k_1,\ldots,k_n)}$$

by construction.

For each $(k_1, \ldots, k_n) \in \{1, \ldots, q\}^n$ for which $\mathcal{F}_{(k_1,\ldots,k_n)} \neq \emptyset$, choose an $f_{(k_1,\ldots,k_n)} \in \mathcal{F}_{(k_1,\ldots,k_n)}$. We claim the collection of all $f_{(k_1,\ldots,k_n)}$ (which is a finite set) is an ϵ-net for $\mathcal{F}$. To see this, let $f \in \mathcal{F}$ be arbitrary. Hence $f \in \mathcal{F}_{(k_1,\ldots,k_n)}$ for some $(k_1, \ldots, k_n) \in \{1, \ldots, q\}^n$. To see that

$$\left\| f - f_{(k_1,\ldots,k_n)} \right\|_\infty \leq \epsilon,$$

let $x \in \mathcal{X}$ be arbitrary. Hence, as $\{x'_1, \ldots, x'_n\}$ is a δ-net for $\mathcal{X}$, there exists a $j \in \{1, \ldots, n\}$ such that $d(x, x'_j) < \delta$. Therefore, by the selection of δ,

$$|f(x) - f(x'_j)| < \frac{\epsilon}{3} \qquad \text{and} \qquad |f_{(k_1,\ldots,k_n)}(x) - f_{(k_1,\ldots,k_n)}(x'_j)| < \frac{\epsilon}{3}.$$

However, as $f \in \mathcal{F}_{(k_1,\ldots,k_n)}$, the fact that $|m_{k+1} - m_k| < \frac{\epsilon}{3}$ for all $k \in \{1, \ldots, q\}$ implies that

$$|f(x'_j) - f_{(k_1,\ldots,k_n)}(x'_j)| < \frac{\epsilon}{3}.$$

Hence the triangle inequality implies

$$|f(x) - f_{(k_1,\ldots,k_n)}(x)| < \epsilon.$$

Therefore, as $x \in \mathcal{X}$ was arbitrary, $\left\| f - f_{(k_1,\ldots,k_n)} \right\|_\infty \leq \epsilon$. Therefore, as $f \in \mathcal{F}$ was arbitrary, we have proven the existence of an ϵ-net for $\mathcal{F}$. Hence, as $\epsilon > 0$ was arbitrary, $\mathcal{F}$ is totally bounded as desired. ∎

To complete our discussion of the Arzelà-Ascoli Theorem (Theorem 4.10.4) we note it is a powerful tool to verify sets of functions are relatively compact. Indeed verifying a set of functions is pointwise bounded is generally trivial and verify a collection of functions is equicontinuous is no more difficult then verifying a single function is continuous. Consequently, if one desires to verify a collection of function is actually compact, one need only verify the collection is relatively compact and closed. One example of this is as follows.

Example 4.10.5. Let

$$K = \left\{ f \in \mathcal{C}[0,1] \,\middle|\, |f(x) - f(y)| \leq \sqrt{|x-y|}\, \forall x, y \in [0,1] \text{ and } f(0) = 0 \right\}.$$

Then K is a compact subset of $\mathcal{C}[0,1]$. To see this via the Arzelà-Ascoli Theorem (Theorem 4.10.4), we need to show that K is equicontinuous, pointwise bounded, and closed in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$. To prove that K is a compact subset of $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, we will show that K is closed and relatively compact as this implies $K = \overline{K}$ is compact.

To see that K is a closed subset of $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, let $(f_n)_{n\geq 1}$ be an arbitrary sequence in K that converges to some $f \in \mathcal{C}[0,1]$ with respect to $\|\cdot\|_\infty$. By the definition of the infinity norm, we see that $(f_n)_{n\geq 1}$ converges pointwise to f. Therefore, since

$$|f_n(x) - f_n(y)| \leq \sqrt{|x-y|} \text{ for all } x, y \in [0,1] \qquad \text{and} \qquad f_n(0) = 0$$

for all $n \in \mathbb{N}$ due to the defining properties of K, we obtain that

$$|f(x) - f(y)| \leq \sqrt{|x-y|} \text{ for all } x, y \in [0,1] \qquad \text{and} \qquad f(0) = 0$$

so $f \in K$ by definition. Therefore, since $(f_n)_{n\geq 1}$ was arbitrary, K is closed in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$.

To see that K is relatively compact in $(\mathcal{C}[0,1], \|\cdot\|_\infty)$, it suffices to show by the Arzelà-Ascoli Theorem (Theorem 4.10.4) that K is equicontinuous and pointwise bounded. To see that K is pointwise bounded, notice for all $f \in K$ and $x \in [0,1]$ that

$$|f(x)| = |f(x) - f(0)| \leq \sqrt{x-0} = \sqrt{x}.$$

Consequently, K is clearly pointwise bounded. To see that K is equicontinuous, let $\epsilon > 0$ and $x \in [0,1]$ be arbitrary. Since the function $g : [0,1] \to \mathbb{R}$ defined by $g(y) = \sqrt{|x-y|}$ is continuous and vanishes at x, there exists a $\delta > 0$ such that if $y \in [0,1]$ and $|x-y| < \delta$ then $g(y) < \epsilon$. Hence for all $f \in K$ and $y \in [0,1]$ such that $|x-y| < \delta$, we obtain that

$$|f(x) - f(y)| \leq \sqrt{|x-y|} < \epsilon.$$

Therefore, since $f \in K$, $\epsilon > 0$, and $x \in [0,1]$ were arbitrary, K is equicontinuous as desired.

Chapter 5

Dense Subsets of Continuous Functions

The notion of relative compactness raises the question about how one goes about taking the closure of a set of functions with respect to the sup metric. In particular, as an element x is in the closure of a set if and only if there is a sequence from the set converging to x, and as a sequence of functions converges with respect to the sup norm if and only if it converges uniformly, we are asking when one function can be uniformly approximated by other functions. This is often useful as there may be a nice collection of functions one understands that approximate all other functions. Hence one may use this nice collection to understand all functions! Thus the goal of this chapter is to develop the theory of dense subsets of $\mathcal{C}(\mathcal{X})$.

5.1 Weierstrass Approximation Theorem

As has been demonstrated in previous analysis courses, every infinitely differentiable function on $\mathbb{R}$ can be approximated 'well' in a little neighbourhood by its Taylor polynomial. Thus it is natural to ask, "How well can we approximate continuous functions using polynomials?"

In this section, we will demonstrate the Weierstrass Approximation Theorem (Theorem 5.1.3) which states every real-valued continuous function on a finite closed interval may be uniformly approximated by a polynomial. To prove the Weierstrass Approximation Theorem (Theorem 5.1.3) we need two ingredients in addition to our previously developed technology on $\mathcal{C}(\mathcal{X})$ with a delicate proof. The first ingredient says we can study any particular finite closed interval we choose.

Lemma 5.1.1. *Consider the linear map $T : \mathcal{C}[a, b] \to \mathcal{C}[0, 1]$ defined by*

$$T(f)(x) = f(a + (b - a)x)$$

for all $x \in [0,1]$ and $f \in \mathcal{C}[a,b]$. Then T is an isometric isomorphism such that $T(p)$ is a polynomial if and only if p is a polynomial.

Proof. Clearly $T(f)$ is well-defined and a continuous function on $[0,1]$ for all $f \in \mathcal{C}[a,b]$. It is elementary to see that T is linear and that $\|T(f)\|_\infty = \|f\|_\infty$ for all $f \in \mathcal{C}[a,b]$. Therefore, as $T^{-1} : \mathcal{C}[0,1] \to \mathcal{C}[a,b]$, defined by

$$T^{-1}(f)(x) = f\left(\frac{x-a}{b-a}\right)$$

for all $x \in [a,b]$ and $f \in \mathcal{C}[0,1]$, exists, we see that T is an isometric isomorphism. In addition, it is clear that if p is a polynomial then $T(p)$ is polynomial and $T^{-1}(p)$ is a polynomial. Hence the result follows. ∎

Our second ingredient is a technical result for a function we will encounter and is proved using elementary calculus.

Lemma 5.1.2. *If $x \in [-1,1]$ and $n \in \mathbb{N}$, then*

$$(1-x^2)^n \geq 1 - nx^2.$$

Proof. Clearly it suffices to consider $x \in [0,1]$ as $(1-(-x)^2)^n = (1-x^2)^n$ and $1 - n(-x)^2 = 1 - nx^2$ for all $x \in [-1,1]$.

Consider the functions $f, g : [0,1] \to \mathbb{R}$ defined by

$$f(x) = (1-x^2)^n \qquad \text{and} \qquad g(x) = 1 - nx^2$$

for all $x \in [0,1]$. Clearly $f(0) = 1 = g(0)$. Furthermore, f and g are differentiable with

$$f'(x) = n(1-x^2)(-2x) \qquad \text{and} \qquad g'(x) = -2nx.$$

As $-2nx \leq 0$ and $0 \leq 1 - x^2 \leq 1$ for all $x \in [0,1]$, we see that $f'(x) \geq g'(x)$ for all $x \in [0,1]$. Hence it follows that $f(x) \geq g(x)$ for all $x \in [0,1]$ as desired. ∎

The above is the little preparation we need to prove the main theorem of this section.

Theorem 5.1.3 (Weierstrass Approximation Theorem). *Let $a, b \in \mathbb{R}$ be such that $a < b$. The set of polynomials is dense in $(\mathcal{C}[a,b], \|\cdot\|_\infty)$; that is, for each $f \in \mathcal{C}[a,b]$ and $\epsilon > 0$ there exists a polynomial p such that $|f(x) - p(x)| < \epsilon$ for all $x \in [a,b]$.*

Proof. By Lemma 5.1.1 we may assume without loss of generality that $a = 0$ and $b = 1$.

Let $g \in \mathcal{C}[0,1]$ be arbitrary. Define the function $f : [0,1] \to \mathbb{R}$ by

$$f(x) = g(x) - (g(0) + (g(1) - g(0))x)$$

for all $x \in [0,1]$. Clearly $f \in \mathcal{C}[0,1]$ and $f(0) = f(1) = 0$. We will demonstrate there exists a sequence $(p_n)_{n\geq 1}$ of polynomials such that

$$\lim_{n\to\infty} \|f - p_n\|_\infty = 0.$$

This will complete the proof as $r_n(x) = p_n(x) + (g(0) + (g(1) - g(0))x)$ are polynomials such that $\lim_{n\to\infty} \|g - r_n\|_\infty = 0$.

To see that f is a uniform limit of polynomials on $[0,1]$, let $\epsilon > 0$ be arbitrary. First note that as $f \in \mathcal{C}[0,1]$ and $f(0) = 0 = f(1)$, we can extend f to be a continuous function on $\mathbb{R}$ by defining $f(x) = 0$ for all $x \in (-\infty, 0) \cup (1, \infty)$. Since f is then continuous on $[-2,2]$, f is uniformly continuous on $[-2,2]$ by Theorem 4.8.6 so there exists a $0 < \delta < 1$ such that if $x \in [-1,1]$ and $|t| < \delta$ then

$$|f(x+t) - f(x)| < \frac{1}{2}\epsilon.$$

Notice for each $n \in \mathbb{N}$ that

$$\int_{-1}^{1} (1-x^2)^n \, dx > 0$$

as $(1-x^2)^n > 0$ for all $x \in (-1,1)$. Hence for each $n \in \mathbb{N}$ there exists a $c_n > 0$ such that

$$c_n \int_{-1}^{1} (1-x^2)^n \, dx = 1.$$

Therefore, by Lemma 5.1.2,

$$\begin{aligned}
\frac{1}{c_n} &= \int_{-1}^{1} (1-x^2)^n \, dx \\
&= 2\int_{0}^{1} (1-x^2)^n \, dx \\
&\geq 2\int_{0}^{\frac{1}{\sqrt{n}}} (1-x^2)^n \, dx \\
&\geq 2\int_{0}^{\frac{1}{\sqrt{n}}} 1 - nx^2 \, dx \\
&= 2\left(x - \frac{n}{3}x^3\right)\Big|_{x=0}^{\frac{1}{\sqrt{n}}} \\
&= \frac{4}{3\sqrt{n}} \geq \frac{1}{\sqrt{n}}.
\end{aligned}$$

Hence $0 < c_n \leq \sqrt{n}$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ define $q_n : \mathbb{R} \to \mathbb{R}$ by

$$q_n(x) = c_n(1-x^2)^n$$

for all $x \in \mathbb{R}$. Thus $q_n(x) \geq 0$ for all $x \in [-1, 1]$ and

$$\int_{-1}^{1} q_n(x)\, dx = 1$$

by the definition of c_n. Notice by the definition of q_n that if $x \in [-1, -\delta] \cup [\delta, 1]$, then

$$q_n(x) = c_n(1 - x^2)^n \leq c_n(1 - \delta^2)^n \leq \sqrt{n}(1 - \delta^2)^n.$$

For each $n \in \mathbb{N}$, define the function $f * q_n : [0, 1] \to \mathbb{R}$ by

$$(f * q_n)(x) = \int_{-1}^{1} f(x + t) q_n(t)\, dt.$$

Due to the translation invariance of the Riemann (Lebesgue) integral, for all $x \in [0, 1]$ we see using the substitution $u = x + t$ that

$$\begin{aligned}
(f * q_n)(x) &= \int_{-1}^{1} f(x + t) q_n(t)\, dt \\
&= \int_{-x}^{1-x} f(x + t) q_n(t)\, dt \qquad f \text{ is } 0 \text{ on } [0, 1]^c \\
&= \int_{0}^{1} f(u) q_n(u - x)\, du.
\end{aligned}$$

Thus, as $q_n(u - x)$ is a polynomial in x with coefficients being continuous functions in u, $f(u)q_n(u-x)$ is a polynomial with coefficients being continuous functions in u. Hence integrating $f(u)q_n(u-x)$ with respect to u is performed by integrating the coefficients of the polynomial in x with respect to u thereby resulting in a polynomial in x. Hence $f * q_n$ is a polynomial on $[0, 1]$.

Finally, we claim that $\lim_{n\to\infty} \|(f * q_n) - f\|_\infty = 0$. To see this, note for each $x \in [0,1]$ that

$$\begin{aligned}
&|(f * q_n)(x) - f(x)| \\
&= \left| \int_{-1}^{1} f(x+t)q_n(t)\,dt - f(x) \right| \\
&= \left| \int_{-1}^{1} f(x+t)q_n(t)\,dt - f(x)\int_{-1}^{1} q_n(t)\,dt \right| \quad \text{as } \int_{-1}^{1} q_n(x)\,dx = 1 \\
&= \left| \int_{-1}^{1} (f(x+t) - f(x))q_n(t)\,dt \right| \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)|q_n(t)\,dt \quad \text{as } q_n(x) \geq 0 \text{ on } [-1,1] \\
&= \int_{[-1,-\delta]\cup[\delta,1]} |f(x+t) - f(x)|q_n(t)\,dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|q_n(t)\,dt \\
&\leq \int_{[-1,-\delta]\cup[\delta,1]} 2\left\|f\right\|_\infty \sqrt{n}(1-\delta^2)^n\,dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|q_n(t)\,dt \\
&= 4\sqrt{n}\left\|f\right\|_\infty (1-\delta^2)^n(1-\delta) + \int_{-\delta}^{\delta} |f(x+t) - f(x)|q_n(t)\,dt \\
&\leq 4\sqrt{n}\left\|f\right\|_\infty (1-\delta^2)^n(1-\delta) + \int_{-\delta}^{\delta} \frac{\epsilon}{2} q_n(t)\,dt \quad \text{by uniform continuity} \\
&\leq 4\sqrt{n}\left\|f\right\|_\infty (1-\delta^2)^n(1-\delta) + \frac{\epsilon}{2}\int_{-1}^{1} q_n(t)\,dt \\
&= 4\sqrt{n}\left\|f\right\|_\infty (1-\delta^2)^n(1-\delta) + \frac{\epsilon}{2}.
\end{aligned}$$

Therefore, as $0 < 1 - \delta^2 < 1$ so

$$\lim_{n\to\infty} 4\sqrt{n}\left\|f\right\|_\infty (1-\delta^2)^n(1-\delta) = 0,$$

we see that for sufficiently large n that $\left\|(f * q_n) - f\right\|_\infty < \epsilon$. Hence, as $\epsilon > 0$ was arbitrary, the result follows. ∎

In the proof of Theorem 5.1.3, functions with similar properties to the q_n and to $f * q_n$ are used in Fourier analysis. In particular, the q_n are nice as they are positive functions which integrate to 1 with all of their mass being closer and closer to zero. Such collections of functions are known as summability kernels, the most famous of which is Fejér's kernel. The functions $f * q_n$ then serve as an 'averaging' of f and we showed they tend to f uniformly.

5.2 Stone-Weierstrass Theorem, Lattice Form

Although the Weierstrass Approximation Theorem (Theorem 5.1.3) is powerful, it is limited as for an arbitrary compact metric space we need not have a notion of polynomials. In this section, we will develop one of two theorems which will produce dense subsets of $\mathcal{C}(\mathcal{X})$. The theorem of this section (Theorem 5.2.10) will be motivated by a poset structure on $\mathcal{C}(\mathcal{X})$.

Given two functions $f, g \in \mathcal{C}(\mathcal{X})$ for some compact metric space $(\mathcal{X}, d)$, is it easy to define a poset structure on $\mathcal{C}(\mathcal{X})$ by defining $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathcal{X}$. However this poset structure is something stronger in that maximums and minimums occur.

Definition 5.2.1. Let $(\mathcal{X}, d)$ be a compact metric space and let $f, g \in \mathcal{C}(\mathcal{X})$. The functions $f \vee g, f \wedge g : \mathcal{X} \to \mathbb{R}$ defined by

$$(f \vee g)(x) = \max(\{f(x), g(x)\}) = \frac{1}{2}f(x) + \frac{1}{2}g(x) - \frac{1}{2}|f(x) - g(x)|$$
$$(f \wedge g)(x) = \min(\{f(x), g(x)\}) = -((-f) \vee (-g))(x)$$

for all $x \in \mathcal{X}$ are continuous functions (as they are a combination of compositions, sums, and scalar multiples of continuous functions) called the *maximum* and *minimum functions* respectively.

It is elementary to see that $f \vee g$ is the smallest function in $\mathcal{C}(\mathcal{X})$ that is larger than both f and g and $f \wedge g$ is the largest function in $\mathcal{C}(\mathcal{X})$ that is smaller than both f and g. In particular, we will be interested in the following subspaces of $\mathcal{C}(\mathcal{X})$.

Definition 5.2.2. Let $(\mathcal{X}, d)$ be a compact metric space. A vector subsspace $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$ is said to be a *lattice* if $f \vee g \in \mathcal{F}$ whenever $f, g \in \mathcal{F}$.

Example 5.2.3. It is elementary to see that $\mathcal{C}(\mathcal{X})$ is lattice.

For $a, b \in \mathbb{R}$ with $a < b$, consider the subspace $\mathcal{F}$ of $\mathcal{C}[a, b]$ consisting of all piecewise linear functions; that is;

$$\mathcal{F} = \left\{ f : [a, b] \to \mathbb{R} \,\middle|\, \substack{f \in \mathcal{C}[a,b] \text{ and there exists a partition } \{t_k\}_{k=0}^n \\ \text{such that } f \text{ is linear on } [t_{k-1}, t_k] \text{ for all } k.} \right\}.$$

It is not difficult to check that $\mathcal{F}$ is lattice in $\mathcal{C}[a, b]$ (i.e. the max of two linear functions is linear and the union of two partitions is a partition). In fact, it is not difficult to check that $\mathcal{F}$ is the smallest lattice in $\mathcal{C}[a, b]$ that contains the functions $f(x) = x$ and $g(x) = 1$ for all $x \in [a, b]$.

Remark 5.2.4. The reason we require a lattice of $\mathcal{C}(\mathcal{X})$ to be a vector subspace is that we know $\mathcal{C}(\mathcal{X})$ is a vector space so it would be difficult to approximate a general function from a set that is not a vector subspace of $\mathcal{C}(\mathcal{X})$. This is not too much of a restriction since we may always take the span of a given a subset of $\mathcal{C}(\mathcal{X})$.

Remark 5.2.5. As it is not difficult to see that $(-f) \vee (-g) = -(f \wedge g)$, we have that any lattice in $\mathcal{C}(\mathcal{X})$ is closed under taking the maximum and minimum of the functions it contains (as lattices are subspaces and thus closed under scalar multiplication).

Of course, not every lattice can be dense in $\mathcal{C}(\mathcal{X})$ as the constant functions are clearly a lattice. The additional property required of our lattices is as follow.

Definition 5.2.6. Let $(\mathcal{X}, d_{\mathcal{X}})$ be a compact metric space and let $(\mathcal{Y}, d_{\mathcal{Y}})$ be a metric space. A collection of continuous functions $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X}, \mathcal{Y})$ is said to *separates points* if for all $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ there exists a function $f \in \mathcal{F}$ such that $f(x_1) \neq f(x_2)$.

Of course it is not difficult to find examples of sets of continuous functions that separate points.

Example 5.2.7. For $a, b \in \mathbb{R}$ with $a < b$, clearly the piecewise linear functions on $\mathcal{C}[a, b]$ separate points as the function $f(x) = x$ is (piecewise) linear and clearly separates points.

Example 5.2.8. If $(\mathcal{X}, d)$ is a compact metric space, then $\mathcal{C}(\mathcal{X})$ separates points. Indeed if $x_1, x_2 \in \mathcal{X}$ are such that $x_1 \neq x_2$, then the function $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = d(x, x_1)$ for all $x \in \mathcal{X}$ is continuous, $f(x_1) = 0$, and $f(x_2) = d(x_1, x_2) > 0 = f(x_1)$.

Using the above example, we may obtain the following.

Proposition 5.2.9. *If $(\mathcal{X}, d)$ be a compact metric space and $\mathcal{F}$ be a dense subset of $\mathcal{C}(\mathcal{X})$, then $\mathcal{F}$ separates points.*

Proof. Let $\mathcal{F}$ be a dense subset of $\mathcal{C}(\mathcal{X})$. To see that $\mathcal{F}$ separates points, let $x_1, x_2 \in \mathcal{X}$ such that $x_1 \neq x_2$ be arbitrary. Define the function $f : \mathcal{X} \to \mathbb{R}$ by $f(x) = d(x, x_1)$ for all $x \in \mathcal{X}$. Clearly f is continuous, $f(x_1) = 0$ and $f(x_2) = d(x_1, x_2) > 0$.

Let $\epsilon = \frac{1}{3}d(x_1, x_2) > 0$. Since $\mathcal{F}$ is dense in $\mathcal{C}(\mathcal{X})$, there exists a $g \in \mathcal{F}$ such that $\|f - g\|_\infty < \epsilon$. Hence

$$\begin{aligned} 3\epsilon = d(x_2, x_1) = |f(x_1) - f(x_2)| \\ &\leq |f(x_1) - g(x_1)| + |g(x_1) - g(x_2)| + |g(x_2) - f(x_2)| \\ &\leq 2\epsilon + |g(x_1) - g(x_2)| \end{aligned}$$

Hence $|g(x_1) - g(x_2)| \geq \epsilon > 0$ so $g(x_1) \neq g(x_2)$. Hence, as $x_1, x_2 \in \mathcal{X}$ were arbitrary, we obtain that $\mathcal{F}$ separates points. ∎

In fact, the following theorem demonstrates (after adding in the constant functions) that 'separating points' is the only obstacle for a lattice to be dense in $\mathcal{C}(\mathcal{X})$.

Theorem 5.2.10 (Stone-Weierstrass Theorem - Lattice Version). *Let $(\mathcal{X}, d)$ be compact metric space and let $\mathcal{F} \subseteq \mathcal{C}(\mathcal{X})$ be such that*

1. *$1 \in \mathcal{F}$ (the constant function that is one everywhere),*

2. *$\mathcal{F}$ separates points, and*

3. *$\mathcal{F}$ is a lattice.*

Then $\mathcal{F}$ is dense in $\mathcal{C}(\mathcal{X})$.

Proof. First we claim that for all $x_1, x_2 \in \mathcal{X}$ with $x_1 \neq x_2$ and for all $\alpha, \beta \in \mathbb{R}$ there exists a function $h \in \mathcal{F}$ such that

$$h(x_1) = \alpha \qquad \text{and} \qquad h(x_2) = \beta.$$

To see this, let $x_1, x_2 \in \mathcal{X}$ be arbitrary points such that $x_1 \neq x_2$. Since $\mathcal{F}$ separates points, there exists a function $g \in \mathcal{F}$ such that $g(x_1) \neq g(x_2)$. Hence if we define $h : \mathcal{X} \to \mathbb{R}$ by

$$h(x) = \alpha + \frac{\beta - \alpha}{g(x_2) - g(x_1)}(g(x) - g(x_1)),$$

then clearly $h \in \mathcal{F}$ as $\mathcal{F}$ is a vector subspace and $1 \in \mathcal{F}$, $h(x_1) = \alpha$, and $h(x_2) = \beta$ as desired. We will use these functions to build-up our approximates.

To prove that $\mathcal{F}$ is dense in $(\mathcal{C}(\mathcal{X}), \|\cdot\|_\infty)$, let $f \in \mathcal{C}(\mathcal{X})$ be arbitrary and let $\epsilon > 0$ be arbitrary. To begin, we will demonstrate that for each $z \in \mathcal{X}$ there exists a function $h_z \in \mathcal{F}$ such that $h_z(z) = f(z)$ and $h_z(x) < f(x) + \epsilon$ for all $x \in \mathcal{X}$ (and thus $h_z - f < \epsilon$, which is close to what we want).

To the function $h_z \in \mathcal{F}$ exists, fix $z \in \mathcal{X}$. By the above paragraph for each $y \in \mathcal{X}$ there exists a $h_{z,y} \in \mathcal{F}$ such that $h_{z,y}(z) = f(z)$ and $h_{z,y}(y) = f(y)$. Since the function $h_{z,y} - f$ is continuous and $h_{z,y}(y) - f(y) = 0$, there exists a open set U_y containing y such that $h_{z,y}(x) - f(x) < \epsilon$ for all $x \in U_y$. However, since $\{U_y\}_{y \in \mathcal{X}}$ is an open cover of $\mathcal{X}$ and as $\mathcal{X}$ is compact, there exists an $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in \mathcal{X}$ such that $\mathcal{X} \subseteq \bigcup_{k=1}^n U_{y_k}$. Let

$$h_z = h_{z,y_1} \wedge h_{z,y_2} \wedge \cdots \wedge h_{z,y_n},$$

which is an element of $\mathcal{F}$ as $\mathcal{F}$ is a lattice. In addition, as $h_{z,y_k}(z) = f(z)$ for all $k \in \{1, \ldots, n\}$, we clearly see that $h_z(z) = f(z)$. Moreover if $x \in \mathcal{X}$ is an arbitrary element, then there exists a $k_0 \in \{1, \ldots, n\}$ such that $x \in U_{y_{k_0}}$ and thus

$$h_z(x) \leq h_{z,y_{k_0}}(x) < f(x) + \epsilon.$$

Hence, as $x \in \mathcal{X}$ was arbitrary, h_z has the desired properties.

We may now use the $h_z \in \mathcal{F}$ along with a similar technique to obtain an $h \in \mathcal{F}$ such that $\|f - h\|_\infty \leq \epsilon$. To see this, notice for each $z \in \mathcal{X}$ that $h_z - f$ is continuous and $h_z(z) - f(z) = 0$ so thee exists an open set V_z containing z such that $h_z(x) - f(x) > -\epsilon$ for all $x \in V_z$. However, since $\{V_z\}_{z \in \mathcal{X}}$ is an open cover of $\mathcal{X}$ and as $\mathcal{X}$ is compact, there exists an $m \in \mathbb{N}$ and $z_1, \ldots, z_m \in \mathcal{X}$ such that $\mathcal{X} \subseteq \bigcup_{k=1}^m V_{z_k}$. Let

$$h = h_{z_1} \vee h_{z_2} \vee \cdots \vee h_{z_m}.$$

which is an element of $\mathcal{F}$ as $\mathcal{F}$ is a lattice. Furthermore, as $h_{z_k}(x) < f(x) + \epsilon$ for all $x \in \mathcal{X}$ and for all $k \in \{1, \ldots, m\}$, we see that $h(x) < f(x) + \epsilon$ for all $x \in \mathcal{X}$ by the definition of the maximum. Moreover, if $x \in \mathcal{X}$ is arbitrary then there exists a $k_0 \in \{1, \ldots, m\}$ such that $x \in V_{z_{k_0}}$ and thus

$$h(x) \geq h_{z_k}(x) > f(x) - \epsilon.$$

Therefore, as $x \in \mathcal{X}$ was arbitrary, we have that

$$f(x) - \epsilon < h(x) < f(x) + \epsilon$$

for all $x \in \mathcal{X}$. Hence $\|h - f\|_\infty \leq \epsilon$. Therefore, as $\epsilon > 0$ and $f \in \mathcal{C}(\mathcal{X})$ were arbitrary, the result follows. ∎

Using Theorem 5.2.10, Example 5.2.3, and Example 5.2.7 imply that the piecewise linear functions on $\mathcal{C}[a, b]$ are dense in $\mathcal{C}[a, b]$. Of course, one could verify the density of piecewise linear functions in $\mathcal{C}[a, b]$ directly using uniform continuity. Indeed given $f \in \mathcal{C}[a, b]$ and an $\epsilon > 0$, choose the δ from uniform continuity. Then choose a partition with intervals of length at most δ and define a piecewise linear function g that takes the values that f does at each end of each interval in the partition. Uniform continuity and piecewise linearity will then implies that $\|f - g\|_\infty \leq 2\epsilon$.

5.3 Stone-Weierstrass Theorem, Subalgebra Form

Of course constructing a vector subspace that is a lattice in $\mathcal{C}(\mathcal{X})$ may not be an easy task as making a vector subspace be closed under maximum and minimum is highly non-trivial. In this section, we will discuss another version of the Stone-Weierstrass Theorem (Theorem 5.3.5) that is vastly easier to verify in general. Furthermore, note the lattice version of the Stone-Weierstrass Theorem (Theorem 5.2.10) cannot possible extend to complex-valued functions as there is no natural ordering on $\mathbb{C}$. However, using our new version of the Stone-Weierstrass Theorem (Theorem 5.3.5) we will be able to develop a version of the Stone-Weierstrass Theorem (Theorem 5.3.7) for complex-valued functions.

To replace the lattice structure for dense subsets, we will consider the following structure.

Definition 5.3.1. Let $(\mathcal{X}, d)$ be a compact metric space. A vector subspace $\mathcal{A} \subseteq \mathcal{C}(X, \mathbb{K})$ is said to be a *subalgebra of* $\mathcal{C}(X, \mathbb{K})$ if whenever $f, g \in \mathcal{A}$ it is the case that $fg \in \mathcal{A}$.

Example 5.3.2. Clearly $\mathcal{C}(X, \mathbb{K})$ is a subalgebra of $\mathcal{C}(X, \mathbb{K})$ and any ideal of $\mathcal{C}(X, \mathbb{K})$ is a subalgebra of $\mathcal{C}(X, \mathbb{K})$. Furthermore, it is clear that the polynomials are a subalgebra of $C[a, b]$.

Example 5.3.3. Let
$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}.$$
The *trigonometric polynomials* on $\mathbb{T}$ is the subset of $\mathcal{C}(\mathbb{T}, \mathbb{C})$ defined by

$$Trig(\mathbb{T}) = \mathrm{span}_{\mathbb{C}}(\{f_n : \mathbb{T} \to \mathbb{C} \mid n \in \mathbb{Z}, f_n(z) = z^n \text{ for all } z \in \mathbb{T}\}).$$

As $f_n f_m = f_{n+m}$ for all $n, m \in \mathbb{Z}$, clearly $Trig(\mathbb{T})$ is a subalgebra of $\mathcal{C}(\mathbb{T}, \mathbb{C})$.

There are many other ways to view $Trig(\mathbb{T})$. Indeed notice that if $z \in \mathbb{T}$ then $f_n(z) = z^n$ if $n \geq 0$ and $f_n(z) = \overline{z}^{-n}$ if $n < 0$. Therefore, it is easy to see that

$$Trig(\mathbb{T}) = \{p(z, \overline{z}) : \mathbb{T} \to \mathbb{C} \mid p \text{ a polynomial in two variables}\}.$$

Finally, to see why these are called the trigonometric polynomials, recall if $z \in \mathbb{T}$ then $z = e^{i\theta}$ for some $\theta \in [0, 2\pi]$. Therefore, we see that

$$f_n(z) = e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

for all $n \in \mathbb{Z}$. Therefore, as

$$\frac{1}{2}(f_n(z) + f_{-n}(z)) = \cos(n\theta) \qquad \text{and} \qquad \frac{1}{2i}(f_n(z) - f_{-n}(z)) = \sin(n\theta),$$

we see that

$$Trig(\mathbb{T}) = \mathrm{span}\{\cos(n\theta), \sin(n\theta) : [0, 2\pi] \to \mathbb{C} \mid n \in \mathbb{N} \cup \{0\}\}.$$

This is why $Trig(\mathbb{T})$ is called the trigonometric polynomials.

As the closure of a vector subspace is a vector subspace and as we appear to want to show that specific algebras are dense in $\mathcal{C}(X, \mathbb{K})$, which is an algebra, it is not difficult to believe the closure of a subalgebra is a subalgebra.

Lemma 5.3.4. *Let $(\mathcal{X}, d)$ be a compact metric space and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{X}, \mathbb{K})$ be a subalgebra of $\mathcal{C}(\mathcal{X}, \mathbb{K})$. Then $\overline{\mathcal{A}}$ is a subalgebra of $\mathcal{C}(\mathcal{X}, \mathbb{K})$*

Proof. To begin, notice for all functions $f, g \in \mathcal{C}(\mathcal{X}, \mathbb{K})$ that

$$\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$$

due to the definition of the sup-norm.

To see that $\overline{\mathcal{A}}$ is a subalgebra, let $f, g \in \overline{\mathcal{A}}$ be arbitrary. Therefore there exists sequences $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ of functions in $\mathcal{A}$ such that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0 = \lim_{n\to\infty} \|g - g_n\|_\infty$. Clearly for all $\alpha \in \mathbb{K}$ the sequence $(\alpha f_n + g_n)_{n\geq 1}$ consists of elements of $\mathcal{A}$ as $\mathcal{A}$ is a subspace and

$$\lim_{n\to\infty} \|(\alpha f + g) - (\alpha f_n + g_n)\|_\infty \leq \limsup_{n\to\infty} |\alpha| \|f - f_n\|_\infty + \|g - g_n\|_\infty = 0.$$

Therefore $\alpha f + g \in \overline{\mathcal{A}}$ so $\overline{\mathcal{A}}$ is subspace. To see that $\overline{\mathcal{A}}$ is a subalgebra, notice the sequence $(f_n g_n)_{n\geq 1}$ consists of elements of $\mathcal{A}$ as $\mathcal{A}$ is subalgebra. Since $\sup_{n\geq 1} \|f_n\|_\infty < \infty$ as $0 = \lim_{n\to\infty} \|f - f_n\|_\infty$, we obtain that

$$\begin{aligned}\lim_{n\to\infty} \|fg - f_n g_n\|_\infty &\leq \limsup_{n\to\infty} \|fg - f_n g\|_\infty + \|f_n g - f_n g_n\|_\infty \\ &\leq \limsup_{n\to\infty} \|f - f_n\|_\infty \|g\|_\infty + \|f_n\|_\infty \|g - g_n\|_\infty \\ &= 0.\end{aligned}$$

Therefore $fg \in \overline{\mathcal{A}}$ so $\overline{\mathcal{A}}$ is a subalgebra. ∎

Using nothing but Lemma 5.3.4, the Weierstrass Approximation Theorem (Theorem 5.1.3), and the lattice version of the Stone-Weierstrass Theorem (Theorem 5.2.10), we obtain the following Stone-Weierstrass Theorem with next to no difficulty.

Theorem 5.3.5 (Stone-Weierstrass Theorem - Algebra Version). *Let $(\mathcal{X}, d)$ be compact metric space and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{X})$ be a subalgebra such that*

1. $1 \in \mathcal{A}$ *and*
2. $\mathcal{A}$ *separates points.*

Then $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{X})$.

Proof. By Lemma 5.3.4 it is clear that $\overline{\mathcal{A}}$ is a closed subalgebra of $\mathcal{C}(\mathcal{X})$ that contains one and separates points. Our goal is to prove that $\overline{\mathcal{A}} = \mathcal{C}(\mathcal{X})$.

First we claim that if $f \in \overline{\mathcal{A}}$ then $|f| \in \overline{\mathcal{A}}$. To see this, consider the function $a : [-\|f\|_\infty, \|f\|_\infty] \to \mathbb{R}$ defined by $a(x) = |x|$ for all $x \in [-\|f\|_\infty, \|f\|_\infty]$. As $a \in C[-\|f\|_\infty, \|f\|_\infty]$, the Weierstrass Approximation Theorem (Theorem 5.1.3) implies there exists a sequence of polynomials p_n such that $\lim_{n\to\infty} \|p_n - a\|_\infty = 0$ as continuous functions on $[-\|f\|_\infty, \|f\|_\infty]$. Hence, as $f : \mathcal{X} \to [-\|f\|_\infty, \|f\|_\infty]$, we see that

$$\lim_{n\to\infty} \|p_n \circ f - a \circ f\|_\infty = 0$$

as continuous functions on $\mathcal{X}$. Clearly $a \circ f = |f|$. Moreover, notice for any polynomial $p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ where $a_0, a_1, \ldots, a_m \in \mathbb{R}$ that

$$p \circ f = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0 1 \in \overline{\mathcal{A}}$$

as $1, f \in \overline{\mathcal{A}}$ and $\overline{\mathcal{A}}$ is a subalgebra. Hence we see that $p_n \circ f \in \mathcal{A}$ for all $n \in \mathbb{N}$ and hence $|f| \in \overline{\mathcal{A}}$.

Next let $f, g \in \overline{\mathcal{A}}$ be arbitrary. Then $f + g, f - g \in \overline{\mathcal{A}}$ as $\overline{\mathcal{A}}$ is subspace so $|f - g| \in \overline{\mathcal{A}}$ by the above paragraph. Hence, as $\overline{\mathcal{A}}$ is a subspace, we see that

$$f \vee g = \frac{1}{2}(f + g) + \frac{1}{2}|f - g| \in \overline{\mathcal{A}}.$$

Hence $\overline{\mathcal{A}}$ is a lattice that contains one and separates points. Therefore the lattice form of the Stone-Weierstrass Theorem (Theorem 5.2.10 implies that $\overline{\mathcal{A}}$ is dense in $\mathcal{C}(\mathcal{X})$. Therefore, as $\overline{\mathcal{A}}$ is closed, we obtain that $\overline{\mathcal{A}} = \mathcal{C}(\mathcal{X})$ so $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{X})$ as desired. ∎

Example 5.3.6. As it is not difficult to verify that

$$\mathcal{A} = \operatorname{span}\{x^n \mid n \in \{0, 3, 6, 9, \ldots\}\}$$

is a subalgebra of $\mathcal{C}[0,1]$ that separates points and contains 1, the Stone-Weierstrass Theorem (Theorem 5.3.5) implies that $\mathcal{F}$ is dense in $\mathcal{C}[0,1]$.

It is not difficult to develop a version of the Stone-Weierstrass Theorem (Theorem 5.3.5) for complex-valued functions now. To do so, recall that if $f \in \mathcal{C}(\mathcal{X}, \mathbb{C})$, then the function $\overline{f} : \mathcal{X} \to \mathbb{C}$ defined by $\overline{f}(x) = \overline{f(x)}$ (the complex conjugate) is a continuous function being the composition of two continuous functions.

Theorem 5.3.7 (Stone-Weierstrass Theorem - Complex Version). *Let $(\mathcal{X}, d)$ be compact metric space and let $\mathcal{A} \subseteq \mathcal{C}(\mathcal{X}, \mathbb{C})$ be a subalgebra such that*

1. $1 \in \mathcal{A}$,
2. $\mathcal{A}$ *separates points, and*
3. $\overline{f} \in \mathcal{A}$ *whenever* $f \in \mathcal{A}$.

Then $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{C})$.

Proof. Consider the set

$$\mathcal{A}_0 = \{f \in \mathcal{A} \mid f(\mathcal{X}) \subseteq \mathbb{R}\}.$$

Clearly $\mathcal{A}_0$ is a subalgebra of $\mathcal{C}(\mathcal{X}, \mathbb{R})$ that contains the constant function 1. We claim that $\mathcal{A}_0$ separates points. To see this, let $x_1, x_2 \in \mathcal{X}$ be arbitrary points such that $x_1 \neq x_2$. Since $\mathcal{A}$ separates points, there exists an $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$. Hence it must be the case that either $\operatorname{Re}(f)(x_1) \neq \operatorname{Re}(f)(x_2)$ or $\operatorname{Im}(f)(x_1) \neq \operatorname{Im}(f)(x_2)$. Since $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{A}_0$ as $\mathcal{A}$ is a vector subspace closed under complex conjugates, we obtain that $\mathcal{A}_0$ separates points.

By the algebra version of the Stone-Weierstrass Theorem (Theorem 5.3.5), we obtain that $\mathcal{A}_0$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{R})$. To see that $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{C})$, let $f \in \mathcal{C}(\mathcal{X}, \mathbb{C})$ be arbitrary and let $\epsilon > 0$ be arbitrary. Since $\operatorname{Re}(f), \operatorname{Im}(f) \in \mathcal{C}(\mathcal{X}, \mathbb{R})$ and since $\mathcal{A}_0$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{R})$, there exists $g_1, g_2 \in \mathcal{A}_0$ such that

$$\|\operatorname{Re}(f) - g_1\|_\infty < \frac{\epsilon}{2} \qquad \text{and} \qquad \|\operatorname{Im}(f) - g_2\|_\infty < \frac{\epsilon}{2}.$$

As $\mathcal{A}$ is a subalgebra over $\mathbb{C}$, we see that $g_1 + ig_2 \in \mathcal{A}$ and

$$\begin{aligned} \|f - (g_1 + ig_2)\|_\infty &= \|(\mathrm{Re}(f) + i\mathrm{Im}(f)) - (g_1 + ig_2)\|_\infty \\ &\leq \|\mathrm{Re}(f) - g_1\|_\infty + |i| \, \|\mathrm{Im}(f) - g_2\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, as $f \in \mathcal{C}(\mathcal{X}, \mathbb{C})$ and $\epsilon > 0$ were arbitrary, we obtain that $\mathcal{A}$ is dense in $\mathcal{C}(\mathcal{X}, \mathbb{C})$. ∎

Corollary 5.3.8. *The trigonometric polynomials $Trig(\mathbb{T})$ is dense in $\mathcal{C}(\mathbb{T}, \mathbb{C})$.*

Proof. By Example 5.3.3 we know that $Trig(\mathbb{T})$ is a subalgebra of $\mathcal{C}(\mathbb{T}, \mathbb{C})$. Since $z^0 = 1$ for all $z \in \mathbb{T}$, clearly $1 \in Trig(\mathbb{T})$. Furthermore, as every $z \in \mathbb{T}$ can be written as $z = e^{i\theta}$ for some $\theta \in [0, 2\pi]$, we see that

$$\overline{z^n} = \overline{e^{in\theta}} = e^{-in\theta} = z^{-n}$$

for all $n \in \mathbb{Z}$ and $z \in \mathbb{T}$. Hence $Trig(\mathbb{T})$ is closed under complex conjugates. Finally, to see that $Trig(\mathbb{T})$ separates points, we note that the function $f(z) = z$ for all $z \in \mathbb{T}$ is an element of $Trig(\mathbb{T})$ and clearly separate points. Hence, by the Stone-Weierstrass Theorem (Theorem 5.3.7), we obtain that $Trig(\mathbb{T})$ is dense in $\mathcal{C}(\mathbb{T}, \mathbb{C})$. ∎

Chapter 6

Hilbert Spaces

As we have seen, Banach spaces are nice because their norm structure produces many analytical results and their completeness structure furthers these results. In this chapter, we will begin an investigation of a more specific structure that enables a far deeper theory. In particular, we will study Banach spaces whose norm is induced by an inner product. These so called Hilbert spaces have an incredible rich structure. In particular, we will be able to complete describe in very simple terms all Hilbert spaces. Furthermore, we will see that the structure of bounded linear maps between Hilbert spaces is incredibly nice.

6.1 Inner Product Spaces

To begin our study of Hilbert spaces, we recall the relevant facts about inner product spaces.

Definition 6.1.1. Let V be a vector space over $\mathbb{K}$. An *inner product* on V is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ such that

1. $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v} \in V$,
2. $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$,
3. $\langle \vec{x} + \lambda \vec{y}, \vec{v} \rangle = \langle \vec{x}, \vec{v} \rangle + \lambda \langle \vec{y}, \vec{v} \rangle$ for all $\vec{v}, \vec{x}, \vec{y} \in V$ and $\lambda \in \mathbb{K}$ (i.e. $\langle \cdot, \cdot \rangle$) is linear in the first entry), and
4. $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$ for all $\vec{x}, \vec{y} \in V$ (where $\overline{z}$ is the complex conjugate of z).

Remark 6.1.2. Combining properties (3) and (4) in Definition 6.1.1, we obtain for all $\vec{v}, \vec{x}, \vec{y} \in V$ and $\lambda \in \mathbb{K}$ that

$$\begin{aligned}\langle \vec{v}, \vec{x} + \lambda\vec{y}\rangle &= \overline{\langle \vec{x} + \lambda\vec{y}, \vec{v}\rangle} \\ &= \overline{\langle \vec{x}, \vec{v}\rangle + \lambda\langle \vec{y}, \vec{v}\rangle} \\ &= \overline{\langle \vec{x}, \vec{v}\rangle} + \overline{\lambda}\overline{\langle \vec{y}, \vec{v}\rangle} \\ &= \langle \vec{v}, \vec{x}\rangle + \overline{\lambda}\langle \vec{v}, \vec{y}\rangle.\end{aligned}$$

That is, every inner product is *conjugate linear* in the second entry.

Remark 6.1.3. Notice that if $\langle \cdot, \cdot \rangle$ is an inner product on a vector space V, the fact that $\langle \cdot, \cdot \rangle$ is linear in the first entry and conjugate linear in the second entries implies that $\langle \vec{0}, \vec{v}\rangle = \langle \vec{v}, \vec{0}\rangle = 0$ for all $\vec{v} \in V$.

As we are interested in the vector space together with a fixed inner product, we make the following definition.

Definition 6.1.4. An *inner product space* is a pair $(V, \langle \cdot, \cdot \rangle)$ where V is a vector space over $\mathbb{K}$ and $\langle \cdot, \cdot \rangle$ is an inner product on V.

Remark 6.1.5. Again, we will often abuse notation by said that V is an inner product space without specifying $\langle \cdot, \cdot \rangle$.

Example 6.1.6. Let $n \in \mathbb{N}$. Define $\langle \cdot, \cdot \rangle_2 : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ by

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n)\rangle_2 = \sum_{k=1}^{n} z_k \overline{w_k}$$

for all $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{K}^n$. It is elementary to verify that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathbb{K}^n$. We call $\langle \cdot, \cdot \rangle$ the *standard inner product* on $\mathbb{K}^n$.

Example 6.1.7. Let $n \in \mathbb{N}$ and let $\mathcal{M}_n(\mathbb{K})$ denote the set of $n \times n$ matrices with entries in $\mathbb{K}$. Define $\langle \cdot, \cdot \rangle : \mathcal{M}_n(\mathbb{K}) \times \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$ by

$$\langle A, B\rangle = \mathrm{Tr}(AB^*)$$

for all $A, B \in \mathcal{M}_n(\mathbb{K})$ where B^* is the conjugate transpose of B and $\mathrm{Tr} : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$ is the trace. As the trace is linear, it is elementary to verify that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{M}_n(\mathbb{K})$.

Notice if we write $A = [a_{i,j}]$ and $B = [b_{i,j}]$ then

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{i,j}\overline{b_{i,j}}.$$

Therefore, by comparing with Example 6.1.6, it is elementary to see that there is an invertible linear map $\varphi : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}^{n^2}$ such that $\langle \varphi(A), \varphi(B)\rangle_{\mathbb{K}^{n^2}} = \mathrm{Tr}(AB^*)$ for all $A, B \in \mathcal{M}_n(\mathbb{K})$. In particular, $\mathcal{M}_n(\mathbb{K})$ with this inner product is really $\mathbb{K}^{n^2}$ with the standard inner product in disguise.

Example 6.1.8. Let $n \in \mathbb{N}$ and let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathbb{K}^n$. It is then possible to show that there exists a matrix $A = [a_{i,j}] \in \mathcal{M}_n(\mathbb{K})$ such that A is invertible and positive definite, and

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle = \sum_{i,j=1}^{n} a_{i,j} z_j \overline{w_i}$$

for all $(z_1, \ldots, z_n), (w_1, \ldots, w_n) \in \mathbb{K}^n$. We leave the proof as an exercise that will make use of the theory we will develop in this chapter and the fact that

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle = \langle A(z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle_2$$

where $\langle \cdot, \cdot \rangle_2$ is the standard inner product from Example 6.1.6 and where $A(z_1, \ldots, z_n)$ represents the vector obtained by matrix multiplication of A against the column vector with entries $(z_1, \ldots, z_n)$. Note if A is the identity matrix, then the standard inner product is recovered.

Example 6.1.9. Define $\langle \cdot, \cdot \rangle : \mathcal{C}[0,1] \times \mathcal{C}[0,1] \to \mathbb{R}$ by

$$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$$

for all $f, g \in \mathcal{C}[0,1]$. It is elementary to verify that $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{C}[0,1]$.

Example 6.1.10. Define $\langle \cdot, \cdot \rangle : \ell_2(\mathbb{N}) \times \ell_2(\mathbb{N}) \to \mathbb{K}$ by

$$\langle (z_n)_{n\geq 1}, (w_n)_{n \geq 1} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n}$$

for all $(z_n)_{n\geq 1}, (w_n)_{n\geq 1} \in \ell_2(\mathbb{N})$. It is not difficult to see that $\langle \cdot, \cdot \rangle$ will satisfy the conditions in Definition 6.1.1 provided the sum under consideration actually converges in $\mathbb{K}$. Since

$$\sum_{n=1}^{\infty} |z_n \overline{w_n}| \leq \|(z_n)_{n\geq 1}\|_2 \, \|(w_n)_{n\geq 1}\|_2$$

by Hölder's Inequality (Theorem 1.2.9), and since $\mathbb{K}$ is complete (so absolutely summable series converge by Theorem 2.3.6), the sum is finite.

We desire to show that each inner product space has a norm induced by the inner product, which happens to be the 2-norm in (almost) all of the above examples. To do this, we first prove the following very useful inequality.

Theorem 6.1.11 (Cauchy-Schwarz Inequality). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. For all $\vec{x}, \vec{y} \in V$,*

$$|\langle \vec{x}, \vec{y} \rangle| \leq \langle \vec{x}, \vec{x} \rangle^{\frac{1}{2}} \langle \vec{y}, \vec{y} \rangle^{\frac{1}{2}}.$$

Furthermore, the above inequality is an equality if and only if $\{\vec{x}, \vec{y}\}$ is linearly dependent.

Proof. First notice if $\vec{x} = \vec{0}$ or $\vec{y} = \vec{0}$, then the proof is trivial by Remark 6.1.3. Thus we may assume that $\vec{x}, \vec{y} \neq \vec{0}$.

Choose $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ such that

$$\langle \lambda\vec{x}, \vec{y}\rangle = \lambda\langle \vec{x}, \vec{y}\rangle = |\langle \vec{x}, \vec{y}\rangle|,$$

and notice for all $t \in \mathbb{R}$ that

$$\begin{aligned} 0 &\leq \langle \lambda\vec{x} + t\vec{y}, \lambda\vec{x} + t\vec{y}\rangle \\ &= |\lambda|^2\langle \vec{x}, \vec{x}\rangle + t\langle \vec{y}, \lambda\vec{x}\rangle + t\langle \lambda\vec{x}, \vec{y}\rangle + t^2\langle \vec{y}, \vec{y}\rangle \\ &= \langle \vec{x}, \vec{x}\rangle + 2t|\langle \vec{x}, \vec{y}\rangle| + t^2\langle \vec{y}, \vec{y}\rangle. \end{aligned}$$

By substituting

$$t_0 = -\frac{|\langle \vec{x}, \vec{y}\rangle|}{\langle \vec{y}, \vec{y}\rangle}$$

which is well-defined as $\vec{y} \neq 0$, we obtain that

$$0 \leq \langle \vec{x}, \vec{x}\rangle - 2\frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\langle \vec{y}, \vec{y}\rangle} + \frac{|\langle \vec{x}, \vec{y}\rangle|^2}{\langle \vec{y}, \vec{y}\rangle}$$

which clearly implies the inequality.

For the additional claim, notice if $\vec{x} = \alpha\vec{y}$ for some $\alpha \in \mathbb{K}$, then

$$|\langle \vec{x}, \vec{y}\rangle| = |\alpha|\langle \vec{y}, \vec{y}\rangle = \alpha^{\frac{1}{2}}\overline{\alpha}^{\frac{1}{2}}\langle \vec{y}, \vec{y}\rangle^{\frac{1}{2}}\langle \vec{y}, \vec{y}\rangle^{\frac{1}{2}} = \langle \vec{x}, \vec{x}\rangle^{\frac{1}{2}}\langle \vec{y}, \vec{y}\rangle^{\frac{1}{2}}.$$

For the other direction, notice if the Cauchy-Schwarz inequality is an equality then the above proof shows

$$\langle \lambda\vec{x} + t_0\vec{y}, \lambda\vec{x} + t_0\vec{y}\rangle = 0.$$

Hence $\lambda\vec{x} + t_0\vec{y} = \vec{0}$ so $\{\vec{x}, \vec{y}\}$ is linearly dependent (as $\lambda \neq 0$). ∎

Theorem 6.1.12. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Then V is a normed linear space with a norm $\|\cdot\| : V \to [0, \infty)$ defined by*

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v}\rangle}$$

for all $\vec{v} \in V$.

Proof. It is elementary using Definition 6.1.1 to see that $\|\cdot\|$ is well-defined, $\|\vec{v}\| \geq 0$ for all $\vec{v} \in V$, $\|\vec{v}\| = 0$ if and only if $\vec{v} = \vec{0}$, and $\|\alpha\vec{v}\| = |\alpha|\,\|\vec{v}\|$ for all $\vec{v} \in V$ and $\alpha \in \mathbb{K}$. To see that $\|\cdot\|$ satisfies the triangle inequality, notice for all $\vec{x}, \vec{y} \in V$ that

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 &= \langle \vec{x} + \vec{y}, \vec{x} + \vec{y}\rangle \\ &= \langle \vec{x}, \vec{x}\rangle + \langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{x}\rangle + \langle \vec{y}, \vec{y}\rangle \\ &= \|\vec{x}\|^2 + 2\mathrm{Re}(\langle \vec{x}, \vec{y}\rangle) + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2|\langle \vec{x}, \vec{y}\rangle| + \|\vec{y}\|^2 \\ &\leq \|\vec{x}\|^2 + 2\,\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2 \quad \text{by Cauchy-Schwarz (Theorem 6.1.11)} \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2. \end{aligned}$$

Hence the triangle inequality follows. ∎

Remark 6.1.13. For the triangle inequality to be an equality, notice we require equality in the Cauchy-Schwarz inequality which implies $\vec{x}$ and $\vec{y}$ are linearly dependent. Furthermore, we notice we require $\mathrm{Re}(\langle \vec{x}, \vec{y}\rangle) = |\langle \vec{x}, \vec{y}\rangle|$ will then occur only if $\vec{x} = \alpha\vec{y}$ or $\vec{y} = \alpha\vec{x}$ for some $\alpha \in [0, \infty)$. Clearly this later condition implies equality in the triangle inequality.

Remark 6.1.14. By the Cauchy-Schwarz inequality $|\langle \vec{x}, \vec{y}\rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$, we see that the inner product is simultaneously continuous in its entry. Indeed if $\vec{x} = \lim_{n\to\infty} \vec{x}_n$ and $\vec{y} = \lim_{n\to\infty} \vec{y}_n$, then

$$\begin{aligned} \limsup_{n\to\infty} |\langle \vec{x}, \vec{y}\rangle - \langle \vec{x}_n, \vec{y}_n\rangle| &\leq \limsup_{n\to\infty} |\langle \vec{x}, \vec{y}\rangle - \langle \vec{x}, \vec{y}_n\rangle| + |\langle \vec{x}, \vec{y}_n\rangle - \langle \vec{x}_n, \vec{y}_n\rangle| \\ &\leq \limsup_{n\to\infty} \|\vec{x}\| \, \|\vec{y} - \vec{y}_n\| + \|\vec{x} - \vec{x}_n\| \, \|\vec{y}_n\| = 0 \end{aligned}$$

as $\vec{x} = \lim_{n\to\infty} \vec{x}_n$ and $\vec{y} = \lim_{n\to\infty} \vec{y}_n$, with the later implying that $(\vec{y}_n)_{n\geq 1}$ is bounded.

Remark 6.1.15. The proof of Theorem 6.1.12 also enables us to develop a notion of an angle. To motivate this, recall the cosine law for a triangle which states

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

for a triangle with sides a, b, c and angle θ opposite to c. Thinking of a 'triangle' formed by two vectors $\vec{x}, \vec{y}$ and their difference in a real inner product space, the proof of Theorem 6.1.12 demonstrates

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\langle \vec{x}, \vec{y}\rangle.$$

Thus, for a real inner product space, we would like to define the angle θ between $\vec{x}$ and $\vec{y}$ to be such that

$$\cos(\theta) = \frac{\langle \vec{x}, \vec{y}\rangle}{\|\vec{x}\| \, \|\vec{y}\|}$$

which exists by the Cauchy-Schwarz Inequality.

Using the above notion of an angle, we obtain the definition of what it means for two vectors to be perpendicular.

Definition 6.1.16. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Two vectors $\vec{v}, \vec{w} \in V$ are said to be *orthogonal* if $\langle \vec{v}, \vec{w}\rangle = 0$.

Using the properties of the inner product, it is nearly trivial to obtain the following theorems. Thus we omit the proofs.

Theorem 6.1.17 (Pythagorean Theorem). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. If $\{\vec{v}_k\}_{k=1}^n$ is a set of orthogonal vectors, then*

$$\left\| \sum_{k=1}^n \vec{v}_k \right\|^2 = \sum_{k=1}^n \|\vec{v}_k\|^2 .$$

Theorem 6.1.18 (Parallelogram Law). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. If $\vec{x}, \vec{y} \in V$, then*

$$\|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2 = 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 .$$

Remark 6.1.19. It is difficult but possible to show that any norm on any vector space over $\mathbb{K}$ that satisfies the Parallelogram Law actually comes from an inner product.

Theorem 6.1.20 (Polarization Identity). *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. If $\vec{x}, \vec{y} \in V$, then*

- $\langle \vec{x}, \vec{y} \rangle = \frac{1}{4}\|\vec{x} + \vec{y}\|^2 - \frac{1}{4}\|\vec{x} - \vec{y}\|^2$ *if $\mathbb{K} = \mathbb{R}$, and*
- $\langle \vec{x}, \vec{y} \rangle = \frac{1}{4}\sum_{k=1}^4 i^k \left\|\vec{x} + i^k \vec{y}\right\|^2$ *if $\mathbb{K} = \mathbb{C}$.*

6.2 Definition of a Hilbert Space

Now that we have norms induced by inner products, it is natural to ask whether these normed linear spaces are complete (i.e. Banach spaces). As these are a special type of Banach spaces, they are given a special name.

Definition 6.2.1. A *Hilbert space* is a complete inner product space.

Often we will use $\mathcal{H}$ to denote a Hilbert space abusing notation by not mentioning the norm nor inner product.

Already we have seen several examples of Hilbert spaces including $(\mathbb{K}^n, \|\cdot\|_2)$ and $(\ell_2(\mathbb{N}), \|\cdot\|_2)$. However, $(\mathcal{C}[0,1], \|\cdot\|_2)$ is an inner product product space that is not complete by Example 2.2.9. Therefore, it is natural to ask, "Is the completion of $(\mathcal{C}[0,1], \|\cdot\|_2)$ a Hilbert space?"

Theorem 6.2.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\mathcal{H}$ be the normed linear space completion of V from Theorem 2.5.22. There exists an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ such that $\langle \vec{x}, \vec{y} \rangle_{\mathcal{H}} = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x}, \vec{y} \in V$.*

Proof. To prove this result, we could proceed in one of two ways. The first way would be to complete Remark 6.1.19 and show that the norm on the completion of an inner product space then satisfies the the Parallelogram Law. Instead we will use an argument similar to Proposition 2.5.18 to define an inner product on the completion.

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\mathcal{H}$ be the normed linear space completion of V from Theorem 2.5.22. Define $\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{K}$ as follows: Given $\vec{x}, \vec{y} \in \mathcal{H}$, choose sequences $(\vec{x}_n)_{n\geq 1}$ and $(\vec{y}_n)_{n\geq 1}$ such that $\vec{x} = \lim_{n\to\infty} \vec{x}_n$ and $\vec{y} = \lim_{n\to\infty} \vec{y}_n$. We then define

$$\langle \vec{x}, \vec{y} \rangle_{\mathcal{H}} = \lim_{n\to\infty} \langle \vec{x}_n, \vec{y}_n \rangle.$$

To complete the proof, we will first need to demonstrate three things: that the above limit exists, that the definition did not depend on the sequences selected, and that the resulting definition does indeed yield an inner product.

To see that the limit exists, notice for all $n, m \in \mathbb{N}$ that

$$\begin{aligned} |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}_m, \vec{y}_m \rangle| &\leq |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}_n, \vec{y}_m \rangle| + |\langle \vec{x}_n, \vec{y}_m \rangle - \langle \vec{x}_m, \vec{y}_m \rangle| \\ &= |\langle \vec{x}_n, \vec{y}_n - \vec{y}_m \rangle| + |\langle \vec{x}_n - \vec{x}_m, \vec{y}_m \rangle| \\ &\leq \|\vec{x}_n\| \, \|\vec{y}_n - \vec{y}_m\| + \|\vec{x}_n - \vec{x}_m\| \, \|\vec{y}_m\| \end{aligned}$$

with the last inequality coming from the Cauchy-Schwarz inequality. Since $(\vec{x}_n)_{n\geq 1}$ and $(\vec{y}_n)_{n\geq 1}$ converge in $\mathcal{H}$, $(\vec{x}_n)_{n\geq 1}$ and $(\vec{y}_n)_{n\geq 1}$ are bounded and Cauchy. Hence the above inequality demonstrates that $(\langle \vec{x}_n, \vec{y}_n \rangle)_{n\geq 1}$ is Cauchy in $\mathbb{K}$ and thus converges. Hence the limit exists.

Similarly, if $(\vec{x}'_n)_{n\geq 1}$ and $(\vec{y}'_n)_{n\geq 1}$ are such that $\vec{x} = \lim_{n\to\infty} \vec{x}'_n$ and $\vec{y} = \lim_{n\to\infty} \vec{y}'_n$, the above computation shows that

$$\begin{aligned} |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}'_n, \vec{y}'_n \rangle| &\leq |\langle \vec{x}_n, \vec{y}_n \rangle - \langle \vec{x}_n, \vec{y}'_n \rangle| + |\langle \vec{x}_n, \vec{y}'_n \rangle - \langle \vec{x}'_n, \vec{y}'_n \rangle| \\ &= |\langle \vec{x}_n, \vec{y}_n - \vec{y}'_n \rangle| + |\langle \vec{x}_n - \vec{x}'_n, \vec{y}'_n \rangle| \\ &\leq \|\vec{x}_n\| \, \|\vec{y}_n - \vec{y}'_n\| + \|\vec{x}_n - \vec{x}'_n\| \, \|\vec{y}'_n\|. \end{aligned}$$

Hence we see that

$$\lim_{n\to\infty} \langle \vec{x}_n, \vec{y}_n \rangle = \lim_{n\to\infty} \langle \vec{x}'_n, \vec{y}'_n \rangle.$$

Thus the definition of $\langle \vec{x}, \vec{y} \rangle_{\mathcal{H}}$ does not depend on the sequences representing $\vec{x}$ and $\vec{y}$.

To see that $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product on $\mathcal{H}$, first we notice that $\langle \vec{x}, \vec{x} \rangle_{\mathcal{H}} \geq 0$ as the limit of positive real numbers is positive. Furthermore, notice that $\langle \vec{x}, \vec{x} \rangle_{\mathcal{H}} = 0$ if and only if there exists a sequence $(\vec{x}_n)_{n\geq 1}$ such that $\vec{x} = \lim_{n\to\infty} \vec{x}_n$ and $\lim_{n\to\infty} \langle \vec{x}_n, \vec{x}_n \rangle = 0$. As the later is equivalent to $\lim_{n\to\infty} \|\vec{x}_n\| = 0$, we see that $\langle \vec{x}, \vec{x} \rangle_{\mathcal{H}} = 0$ if and only if $\vec{x} = \vec{0}$. Moreover

$$\overline{\langle \vec{x}, \vec{y} \rangle_{\mathcal{H}}} = \lim_{n\to\infty} \overline{\langle \vec{x}_n, \vec{y}_n \rangle} = \lim_{n\to\infty} \langle \vec{y}_n, \vec{x}_n \rangle = \langle \vec{y}, \vec{x} \rangle_{\mathcal{H}}.$$

Finally, we see for all $\alpha \in \mathbb{K}$ and $\vec{x}, \vec{y}, \vec{v} \in \mathcal{H}$ that $(\vec{x})_{n\geq 1}$, $(\vec{y}_n)_{n\geq 1}$, and $(\vec{v}_n)_{n\geq 1}$ are sequences in V that converge to $\vec{x}$, $\vec{y}$, and $\vec{v}$ respectively, then

$$\begin{aligned} \langle \vec{x} + \alpha \vec{y}, \vec{v} \rangle_{\mathcal{H}} &= \lim_{n\to\infty} \langle \vec{x}_n + \alpha \vec{y}_n, \vec{v}_n \rangle \\ &= \lim_{n\to\infty} \langle \vec{x}_n, \vec{v}_n \rangle + \alpha \langle \vec{y}_n, \vec{v}_n \rangle \\ &= \langle \vec{x}, \vec{v} \rangle_{\mathcal{H}} + \alpha \langle \vec{y}, \vec{v} \rangle_{\mathcal{H}}. \end{aligned}$$

Hence $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is an inner product. ∎

Thus the completion of $(\mathcal{C}[0,1], \|\cdot\|_2)$ is a Hilbert space. But this still does not answer the question, "What is the completion of $(\mathcal{C}[0,1], \|\cdot\|_2)$?" Unfortunately, technology from MATH 4012 is required to answer this question.

6.3 Orthogonal Projections

Hilbert spaces are nice as their inner products provide an additional structure for the norm that Banach spaces do not have. Hence Hilbert spaces will automatically have 'nicer geometry' than Banach spaces. In this section, we will begin to exploit this geometry.

Our first geometric result enables us to minimize the distance from a point to the following specific type of sets.

Definition 6.3.1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. A non-empty subset $C \subseteq \mathcal{X}$ is said to be *convex* if $\lambda\vec{a} + (1-\lambda)\vec{b} \in C$ for every $\vec{a}, \vec{b} \in C$ and $\lambda \in [0,1]$ (that is, the line connecting $\vec{a}$ to $\vec{b}$ is contained in C).

Example 6.3.2. Clearly every vector subspace of a normed linear space is a convex subset. Furthermore, every open (closed) ball in a normed linear space is a convex subset. Indeed if $r > 0$ and $\vec{x} \in \mathcal{X}$, let $\lambda \in [0,1]$ and $\vec{a}, \vec{b} \in B(\vec{x}, r)$ be arbitrary. Thus $\|\vec{a} - \vec{x}\| < r$ and $\left\|\vec{b} - \vec{x}\right\| < r$. Therefore

$$\begin{aligned}\left\|\lambda\vec{a} + (1-\lambda)\vec{b} - \vec{x}\right\| &\leq \|\lambda(\vec{a} - \vec{x})\| + \left\|(1-\lambda)(\vec{b} - \vec{x})\right\| \\ &\leq \lambda\|\vec{a} - \vec{x}\| + (1-\lambda)\left\|\vec{b} - \vec{x}\right\| \\ &< \lambda r + (1-\lambda)r = r\end{aligned}$$

and thus $\lambda\vec{a} + (1-\lambda)\vec{b} \in B(\vec{x}, r)$.

Moreover, convex sets have a property we seen previously in this course.

Proposition 6.3.3. *Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space and let C be a convex set. Then C is connected.*

Proof. Let C be an arbitrary convex subset of a Hilbert space $\mathcal{H}$. If C is empty, then C is connected trivially. Hence we may assume without loss of generality that C is non-empty.

To see that C is connected, suppose to the contrary that C is not connected. Hence C is disconnected. Therefore there exists non-empty disjoint open subsets U and V of $\mathcal{H}$ such that $U \cap C \neq \emptyset$, $V \cap C \neq \emptyset$, and $C \subseteq U \cup V$.

Let $\vec{x} \in U \cap C$ and $\vec{y} \in V \cap C$. Define $f : [0,1] \to \mathcal{H}$ by

$$f(\lambda) = \lambda\vec{x} + (1-\lambda)\vec{y}$$

for all $\lambda \in [0,1]$. Since $\vec{x}, \vec{y} \in C$ and C is convex, we have that $f : [0,1] \to C$. Moreover, note that U and V are open subsets of $\mathcal{H}$ such that $\vec{x} \in U \cap f([0,1])$, $\vec{y} \in V \cap f([0,1])$, and $f([0,1]) \subseteq C \subseteq U \cup V$. Hence $f([0,1])$ is disconnected. However, since scalar multiplication, vector addition, and composition of continuous functions are continuous functions, f is continuous. Hence $f([0,1])$ must be a connected subset of C by the Intermediate Value Theorem. As this contradicts the fact that $f([0,1])$ is disconnected, we have a contradiction. Hence C must be connected. ∎

The reason convex sets are so nice in Hilbert spaces is the following.

Theorem 6.3.4. *Let $\mathcal{H}$ be a Hilbert space and let $C \subseteq \mathcal{H}$ be a non-empty, closed, convex subset of $\mathcal{H}$. For each $\vec{x} \in \mathcal{H}$ there exists a unique point $\vec{y} \in C$ that is closest to $\vec{x}$; that is*

$$\|\vec{x} - \vec{z}\| = \operatorname{dist}(\vec{x}, C)$$

if and only if $\vec{z} = \vec{y}$.

Proof. To begin, let $d = \operatorname{dist}(\vec{x}, C)$. We will first demonstrate there exists a point $\vec{y} \in C$ such that $\|\vec{x} - \vec{y}\| = d$. By definition of the distance, for each $n \in \mathbb{N}$ there exists $\vec{y}_n \in C$ such that

$$\|\vec{x} - \vec{y}_n\|^2 < d^2 + \frac{1}{n}.$$

We claim that $(\vec{y}_n)_{n \geq 1}$ is Cauchy in $\mathcal{H}$. To see this, notice by the Parallelogram Law we have for all $n, m \in \mathbb{N}$ that

$$\begin{aligned}
\|\vec{y}_n - \vec{y}_m\|^2 &= \|(\vec{x} - \vec{y}_m) - (\vec{x} - \vec{y}_n)\|^2 \\
&= 2\|\vec{x} - \vec{y}_m\|^2 + 2\|\vec{x} - \vec{y}_n\|^2 - \|(\vec{x} - \vec{y}_m) + (\vec{x} - \vec{y}_n)\|^2 \\
&= 2\|\vec{x} - \vec{y}_m\|^2 + 2\|\vec{x} - \vec{y}_n\|^2 - 4\left\|\vec{x} - \frac{\vec{y}_m + \vec{y}_n}{2}\right\|^2 \\
&\leq 2\left(d^2 + \frac{1}{n}\right) + 2\left(d^2 + \frac{1}{m}\right) - 4d^2 \\
&= \frac{2}{n} + \frac{2}{m}
\end{aligned}$$

(where the third to fourth line follows as $\frac{\vec{y}_m + \vec{y}_n}{2} \in C$ since C was convex). Hence we obtain that $(\vec{y}_n)_{n \geq 1}$ is Cauchy in $\mathcal{H}$. Therefore $\vec{y} = \lim_{n\to\infty} \vec{y}_n$ exists as $\mathcal{H}$ is complete. Since C was closed in $\mathcal{H}$, we obtain that $\vec{y} \in C$. Furthermore, as

$$\|\vec{x} - \vec{y}\| = \lim_{n\to\infty} \|\vec{x} - \vec{y}_n\| \leq d,$$

we obtain that $\|\vec{x} - \vec{y}\| = d$.

To see that $\vec{y}$ is the unique vector with this property, suppose $\vec{z} \in C$ is such that $\|\vec{x} - \vec{z}\| = d$. A similar computation to the one above show that

$$\begin{aligned}\|\vec{y} - \vec{z}\|^2 &= 2\|\vec{x} - \vec{y}\|^2 + 2\|\vec{x} - \vec{z}\|^2 - 4\left\|\vec{x} - \frac{\vec{y} + \vec{z}}{2}\right\|^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0.\end{aligned}$$

Hence $\vec{y} = \vec{z}$ as desired. ∎

Of course it would be nice to be able to determine the vector $\vec{y}$ in Theorem 6.3.4. In general this is a difficult task for arbitrary closed convex subsets of Hilbert spaces. However, closed vector subspaces of a Hilbert space are an abundant collection of examples of closed convex sets for which we can solve this problem!

To begin, we must use the geometry of Hilbert spaces and the following.

Definition 6.3.5. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $S \subseteq V$. The *orthogonal complement* of S in V is the set

$$S^\perp = \{\vec{x} \in V \mid \langle \vec{x}, \vec{z} \rangle = 0 \text{ for all } \vec{z} \in S\}.$$

Example 6.3.6. The orthogonal complement of the x-axis in $\mathbb{R}^2$ with respect to the standard inner product is the y-axis. Similarly, the orthogonal complement of the y-axis in $\mathbb{R}^3$ with respect to the standard inner product is the yz-plane.

Remark 6.3.7. Clearly if $S \subseteq V$, then $S^\perp$ is a closed vector subspace of V. Furthermore $S^\perp = (\mathrm{span}(S))^\perp$ and $S^\perp = (\overline{S})^\perp$. Thus the notion of the orthogonal complement is really a notion for closed vector subspaces of inner product spaces.

Returning to Theorem 6.3.4, we can obtain a description of the closed vector using orthogonal complements.

Theorem 6.3.8. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be a closed subspace of $\mathcal{H}$. Given $\vec{x} \in \mathcal{H}$ and $\vec{y} \in \mathcal{K}$, $\|\vec{x} - \vec{y}\| = \mathrm{dist}(\vec{x}, \mathcal{K})$ and and only if $\vec{x} - \vec{y} \in \mathcal{K}^\perp$.*

Proof. First suppose $\vec{y} \in \mathcal{K}$ is such that $\|\vec{x} - \vec{y}\| = \mathrm{dist}(\vec{x}, \mathcal{K})$. To see that $\vec{x} - \vec{y} \in \mathcal{K}^\perp$, suppose to the contrary that there exists a $\vec{z} \in \mathcal{K}$ such that $\alpha = \langle \vec{x} - \vec{y}, \vec{z} \rangle \neq 0$. Note this implies $\vec{z} \neq \vec{0}$. By scaling $\vec{z}$ if necessary (changing the value of α), we may assume that $\|\vec{z}\| = 1$.

Consider the vector $\vec{v} = \vec{y} + \alpha\vec{z}$ which is an element of $\mathcal{K}$ as $\mathcal{K}$ is a vector subspace. Then

$$\begin{aligned}\|\vec{x} - \vec{v}\|^2 &= \langle \vec{x} - \vec{y} - \alpha\vec{z}, \vec{x} - \vec{y} - \alpha\vec{z} \rangle \\ &= \|\vec{x} - \vec{y}\|^2 - \alpha\langle \vec{z}, \vec{x} - \vec{y} \rangle - \overline{\alpha}\langle \vec{x} - \vec{y}, \vec{z} \rangle + |\alpha|^2 \|\vec{z}\|^2 \\ &= \|\vec{x} - \vec{y}\|^2 - |\alpha|^2 \\ &< \mathrm{dist}(\vec{x}, \mathcal{K})^2,\end{aligned}$$

which is a contradiction as $\vec{v} \in \mathcal{K}$. Hence it must be the case that $\vec{x} - \vec{y} \in \mathcal{K}^\perp$.

Conversely, suppose $\vec{x} - \vec{y} \in \mathcal{K}^\perp$. Clearly $\|\vec{x} - \vec{y}\| \geq \mathrm{dist}(\vec{x}, \mathcal{K})$ whereas for all $\vec{z} \in \mathcal{K}$,

$$\begin{aligned} \|\vec{x} - \vec{z}\|^2 &= \|(\vec{x} - \vec{y}) - (\vec{z} - \vec{y})\|^2 \\ &= \|\vec{x} - \vec{y}\|^2 + \|\vec{z} - \vec{y}\|^2 \geq \|\vec{x} - \vec{y}\|^2 \end{aligned}$$

by the Pythagorean Theorem since $\vec{z} - \vec{y} \in \mathcal{K}$ (as $\mathcal{K}$ is a vector subspace) and $\vec{x} - \vec{y} \in \mathcal{K}^\perp$. Hence $\|\vec{x} - \vec{y}\| = \mathrm{dist}(\vec{x}, \mathcal{K})$. ∎

Using the above, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{K}$, we can decompose $\mathcal{H}$ nicely.

Theorem 6.3.9. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be a closed vector subspace of $\mathcal{H}$. Then $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$; that is, every element $\vec{x} \in \mathcal{H}$ can be written uniquely as a sum of elements from $\mathcal{K}$ and $\mathcal{K}^\perp$. Moreover, for all $\vec{y} \in \mathcal{K}$ and $\vec{z} \in \mathcal{K}^\perp$, $\|\vec{y} + \vec{z}\| \leq \sqrt{\|\vec{y}\|^2 + \|\vec{z}\|^2}$.*

Proof. Let $\vec{x} \in \mathcal{H}$. By Theorems 6.3.4 and 6.3.8, there exists a unique vector $\vec{y} \in \mathcal{K}$ such that $\vec{z} = \vec{x} - \vec{y} \in \mathcal{K}^\perp$. Hence as $\vec{x} = \vec{y} + \vec{z}$, we obtain that $\mathcal{H} = \mathcal{K} + \mathcal{K}^\perp$. Furthermore, the uniqueness follows from the uniqueness of $\vec{y}$. The norm inequality then follows from the Pythagorean Theorem. ∎

Corollary 6.3.10. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{K}$ be a closed vector subspace of $\mathcal{H}$. There is a unique linear map $P : \mathcal{H} \to \mathcal{K} \subseteq \mathcal{H}$ such that $P(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathcal{K}$ and $P(\vec{y}) = \vec{0}$ for all $\vec{y} \in \mathcal{K}^\perp$. The linear map P is called the* orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$. *Furthermore, P is bounded with $\|P\| \leq 1$ (with equality whenever $\mathcal{K} \neq \{\vec{0}\}$), $P^2 = P$, and $\|\vec{x} - P(\vec{x})\| = \mathrm{dist}(\vec{x}, \mathcal{K})$ for all $\vec{x} \in \mathcal{H}$.*

Proof. For each $\vec{x} \in \mathcal{H}$, by Theorem 6.3.9 we may write $\vec{x} = \vec{y} + \vec{z}$ with $\vec{y} \in \mathcal{K}$ (such that $\|\vec{x} - \vec{y}\| = \mathrm{dist}(\vec{x}, \mathcal{K})$) and $\vec{z} \in \mathcal{K}^\perp$, and define $P(\vec{x}) = \vec{y}$. It is elementary to verify that P is a well-defined linear map such that $P(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathcal{K}$ and $P(\vec{y}) = \vec{0}$ for all $\vec{y} \in \mathcal{K}^\perp$. To see that P is bounded, notice

$$\|P(\vec{x})\|^2 = \|\vec{y}\|^2 \leq \|\vec{y}\|^2 + \|\vec{z}\|^2 = \|\vec{x}\|^2$$

so $\|P\| \leq 1$. Clearly $\|P\| = 1$ when $\mathcal{K} \neq \{\vec{0}\}$ and clearly $P^2 = P$. Finally the fact that $\|\vec{x} - P(\vec{x})\| = \mathrm{dist}(\vec{x}, \mathcal{K})$ for all $\vec{x} \in \mathcal{H}$ follows by construction. ∎

It is elementary to verify that if P is the orthogonal projection onto a subspace $\mathcal{K}$ and $I_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}$ is the identity map, then $I_{\mathcal{H}} - P$ is the orthogonal projection on $\mathcal{K}^\perp$.

We will see how useful orthogonal projections are in the following section. For now, we can use the concept of a direct sum in Hilbert spaces to prove the following.

Corollary 6.3.11. *Let $\mathcal{H}$ be a Hilbert space and let $S \subseteq \mathcal{H}$ be non-empty. Then $(S^\perp)^\perp = \overline{\mathrm{span}(S)}$.*

Proof. To begin, let $\vec{x} \in \overline{\mathrm{span}(S)}$ be arbitrary. Thus there exists a sequence $(\vec{x}_n)_{n\geq 1}$ of elements of $\mathrm{span}(S)$ such that $\vec{x} = \lim_{n\to\infty} \vec{x}_n$. Let $\vec{y} \in S^\perp$ be arbitrary. Then

$$\langle \vec{x}, \vec{y}\rangle = \lim_{n\to\infty} \langle \vec{x}_n, \vec{y}\rangle = 0$$

as $\vec{y} \in S^\perp$ and $\vec{x}_n \in \mathrm{span}(S)$ for all n. Therefore, as $\vec{y} \in S^\perp$ was arbitrary, $\vec{x} \in (S^\perp)^\perp$. Thus, as $\vec{x} \in \overline{\mathrm{span}(S)}$ was arbitrary, $\overline{\mathrm{span}(S)} \subseteq (S^\perp)^\perp$.

For the other inclusion, let $\vec{x} \in (S^\perp)^\perp$ be arbitrary. Since $\overline{\mathrm{span}(S)}$ is a closed vector subspace, Theorem 6.3.8 implies there exists a vector $\vec{y} \in \overline{\mathrm{span}(S)}$ such that $\vec{x} - \vec{y} \in \overline{\mathrm{span}(S)}^\perp$. Notice for all $\vec{z} \in \overline{\mathrm{span}(S)}$ that

$$\langle \vec{x} - \vec{y}, \vec{z}\rangle = 0$$

as $\vec{x} - \vec{y} \in \overline{\mathrm{span}(S)}^\perp$. Similarly, if $\vec{z} \in \overline{\mathrm{span}(S)}^\perp$ then $\vec{z} \in S^\perp$ so

$$\langle \vec{x} - \vec{y}, \vec{z}\rangle = \langle \vec{x}, \vec{z}\rangle - \langle \vec{y}, \vec{z}\rangle = 0 - 0 = 0.$$

Therefore, as every vector in $\mathcal{H}$ can be written as the sum of elements from $\overline{\mathrm{span}(S)}$ and $\overline{\mathrm{span}(S)}$, we obtain that $\langle \vec{x} - \vec{y}, \vec{z}\rangle = 0$ for all $\vec{z} \in \mathcal{H}$. Hence by choosing $\vec{z} = \vec{x} - \vec{y}$, we obtain that $\vec{x} = \vec{y} \in \overline{\mathrm{span}(S)}$. Therefore, as $\vec{x} \in (S^\perp)^\perp$ was arbitrary, we obtain that $(S^\perp)^\perp \subseteq \overline{\mathrm{span}(S)}$ as desired. ∎

6.4 Orthonormal Bases

Using the theory of orthogonal projections, we can develop a notion of bases for Hilbert spaces that is far superior to taking a vector space basis. In particular, recall from Theorem 4.2.9 that any vector space basis for an infinite dimensional Banach space must be uncountable. Thus we desire 'nice' bases for Hilbert spaces that to avoid this problem and use the geometry of Hilbert spaces. Thus we begin with the following.

Definition 6.4.1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. An element $\vec{x} \in \mathcal{X}$ is said to be a *unit vector* if $\|\vec{x}\| = 1$.

Definition 6.4.2. Let $\mathcal{H}$ be a Hilbert space. A subset $\{e_\alpha\}_{\alpha\in\Lambda}$ is said to be an *orthonormal set* if each e_α is a unit vector and $\langle e_\alpha, e_\beta\rangle = 0$ $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ (i.e. an orthogonal set of unit vectors).

Remark 6.4.3. It is not difficult to see that every orthonormal set of vectors is automatically linearly independent. Indeed suppose $\{e_\alpha\}_{\alpha\in\Lambda}$ is orthonormal and there exists $\alpha_1, \ldots, \alpha_n \in \Lambda$ and $a_1, \ldots, a_n \in \mathbb{K}$ are such that

$$\sum_{k=1}^{n} a_k e_{\alpha_k} = \vec{0}.$$

For each $j \in \{1, \ldots, n\}$, taking the inner product with e_{α_j} produces

$$0 = \langle \vec{0}, e_{\alpha_j} \rangle = \sum_{k=1}^{n} a_k \langle e_{\alpha_k}, e_{\alpha_j} \rangle = a_j.$$

Hence $a_j = 0$ for all $j \in \{1, \ldots, n\}$ so $\{e_\alpha\}_{\alpha \in \Lambda}$ is linearly independent.

We desire to construct special orthonormal sets. Unfortunately, unlike with finite dimensional theory that students may have seen previously, the notion of spanning orthonormal sets is not the correct notion for infinite dimensional Hilbert spaces.

For the correct notion, given a Hilbert space $\mathcal{H}$, let $\mathcal{E}_{\mathcal{H}}$ denote the set of all orthonormal subsets of $\mathcal{H}$. Notice we may place a partial ordering on $\mathcal{E}_{\mathcal{H}}$ via inclusion. Since the union of any chain of orthonormal sets under this ordering is an upper bound for the chain (and as $\mathcal{E}_{\mathcal{H}} \neq \emptyset$), Zorn's Lemma (Axiom A.5.10) implies there is a maximal element of $\mathcal{E}_{\mathcal{H}}$ under inclusion. These are the objects we are after.

Definition 6.4.4. Let $\mathcal{H}$ be a Hilbert space. An *orthonormal basis* of $\mathcal{H}$ is a maximal orthonormal set.

Example 6.4.5. For $n \in \mathbb{N}$, consider the vectors $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{K}^n$ where for each $j \in \{1, \ldots, n\}$

$$\vec{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$$

where the unique 1 occurs in the j^{th} spot. Clearly $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is orthonormal with respect to the standard inner product. Suppose that $\mathcal{E}$ were not a maximal orthonormal set. Then there would exist a vector $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{E}^\perp$ with $\|\vec{x}\| = 1$. The fact that $\vec{x} \in \mathcal{E}^\perp$ implies

$$0 = \langle \vec{x}, \vec{e}_j \rangle = x_j$$

for all $j \in \{1, \ldots, n\}$. Thus $\vec{x} = \vec{0}$, an obvious contradiction. Hence $\mathcal{E}$ is an orthonormal basis for $\mathbb{K}^n$.

Example 6.4.6. For each $n \in \mathbb{N}$, let $\vec{e}_n \in \ell_2(\mathbb{N})$ be the sequence $\vec{e}_n = (e_{n,k})_{k\geq 1}$ where $e_{n,n} = 1$ and $e_{n,k} = 0$ for all $k \neq n$. By the same arguments as above, $\mathcal{E} = \{\vec{e}_n\}_{n=1}^{\infty}$ is an orthonormal basis for $\ell_2(\mathbb{N})$. However, it is elementary to see that $\mathcal{E}$ does not span $\ell_2(\mathbb{N})$ (indeed the sequence $(\frac{1}{n})_{n\geq 1} \in \ell_2(\mathbb{N})$ is not a finite linear combination of elements of $\mathcal{E}$).

Remark 6.4.7. Using the argument preceding Definition 6.4.4, it is easy to see if $\mathcal{F}$ is an orthonormal subset of a Hilbert space $\mathcal{H}$ then there exists an orthonormal basis $\mathcal{E}$ for $\mathcal{H}$ containing $\mathcal{F}$ (i.e. restrict the Zorn's Lemma argument to orthonormal sets containing $\mathcal{F}$).

In the finite dimensional world, we every have an algorithm for constructing orthonormal bases.

Theorem 6.4.8 (Gram-Schmidt Orthogonalization Process). *Let V be an inner product space and let $L = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a linearly independent subset of V. Then there exists an orthonormal set $O = \{\vec{e}_1, \ldots, \vec{e}_n\}$ such that* $\mathrm{span}(L) = \mathrm{span}(O)$.

Proof. As $\vec{v}_1 \neq \vec{0}$ as L is linearly independent, let $\vec{e}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1$. Then

$$\|\vec{e}_1\| = \left\| \frac{1}{\|\vec{v}_1\|}\vec{v}_1 \right\| = \frac{1}{\|\vec{v}_1\|}\|\vec{v}_1\| = 1.$$

Suppose for some $k \in \{1, \ldots, n-1\}$ we have constructed $\vec{e}_1, \ldots, \vec{e}_k$ such that $\{\vec{e}_1, \ldots, \vec{e}_k\}$ is orthonormal and $\{\vec{e}_1, \ldots, \vec{e}_k, \vec{v}_{k+1}, \ldots, \vec{v}_n\}$ is linearly independent with the same span as L. Let

$$\vec{x}_{k+1} = \vec{v}_{k+1} - \sum_{j=1}^{k} \langle \vec{v}_{k+1}, \vec{e}_j \rangle \vec{e}_j.$$

Since $\{\vec{e}_1, \ldots, \vec{e}_k\}$ is orthonormal, it is easy to see that $\vec{x}_{k+1}$ is orthogonal to $\{\vec{e}_1, \ldots, \vec{e}_k\}$. Furthermore, as $\{\vec{e}_1, \ldots, \vec{e}_k, \vec{v}_{k+1}\}$ is linearly independent, we see that $\vec{x}_k$ is non-zero and $\{\vec{e}_1, \ldots, \vec{e}_k, \vec{x}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ is linearly independent. If we define $\vec{e}_{k+1} = \frac{1}{\|\vec{x}_{k+1}\|}\vec{x}_{k+1}$, we easily obtain that $\{\vec{e}_1, \ldots, \vec{e}_{k+1}\}$ is orthonormal and $\{\vec{e}_1, \ldots, \vec{e}_{k+1}, \vec{v}_{k+2}, \ldots, \vec{v}_n\}$ is linearly independent with the same span as L. The proof is then complete by recursion. ∎

Remark 6.4.9. The proof of the Gram-Schmidt Orthogonalization Process actually makes use of a formula for the orthogonal projection onto a finite subspace. Notice that if $\mathcal{K}$ is a finite dimensional vector subspace of a Hilbert space $\mathcal{H}$, $\mathcal{K}$ is closed by Corollary 4.2.5 and the Gram-Schmidt Orthogonalization Process implies $\mathcal{K}$ has a orthonormal basis which is a vector space basis, say $\{\vec{e}_1, \ldots, \vec{e}_n\}$. If P is the orthogonal projection onto $\mathcal{K}$, we claim that

$$P(\vec{x}) = \sum_{k=1}^{n} \langle \vec{x}, \vec{e}_k \rangle \vec{e}_k$$

for all $\vec{x} \in \mathcal{H}$. Indeed if $\vec{y}$ denotes the right-hand side of the above expression, clearly $\vec{x} - \vec{y}$ is orthogonal to each $\vec{e}_k$ and thus $\vec{x} - \vec{y} \in \mathcal{K}^\perp$. As $P(\vec{x})$ is the unique vector such that $\vec{x} - P(\vec{x}) \in \mathcal{K}^\perp$ by Theorems 6.3.4 and 6.3.8, and by Corollary 6.3.10, we obtain that $\vec{y} = P(\vec{x})$.

Although orthonormal bases for finite dimensional vector subspaces are useful for the above projection formula, as orthonormal bases need not be vector space bases in infinite dimensional Hilbert spaces, we must ask, "How close are orthonormal bases to actual vector spaces bases?" We will see that orthonormal bases are 'bases with respect to analytic conditions'. To begin, we first note the following result for countable orthonormal bases.

Theorem 6.4.10 (Bessel's Inequality, Countable). *Let $\mathcal{H}$ be a Hilbert space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set with Λ countable. For each $\vec{x} \in \mathcal{H}$,*

$$\sum_{\alpha\in\Lambda} |\langle \vec{x}, e_\alpha\rangle|^2 \leq \|\vec{x}\|^2 .$$

Proof. Without loss of generality $\Lambda = \mathbb{N}$ (the proof of the result for finite Λ is contained within). For each $n \in \mathbb{N}$, let $\mathcal{K}_n = \mathrm{span}(\{e_1, \ldots, e_n\})$. Then, if P_n is the orthogonal projection onto $\mathcal{K}_n$, we obtain for all $\vec{x} \in \mathcal{H}$ that

$$\begin{aligned}
\|\vec{x}\|^2 &\geq \|P(\vec{x})\|^2 \\
&= \left\| \sum_{k=1}^{n} \langle \vec{x}, e_k\rangle e_k \right\|^2 \\
&= \sum_{k=1}^{n} |\langle \vec{x}, e_\alpha\rangle|^2
\end{aligned}$$

by the Pythagorean Theorem (Theorem 6.1.17), Corollary 6.3.10 and Remark 6.4.9. Hence the result follows by taking the limit as n tends to infinity. ∎

Using Bessel's Inequality for countable orthonormal sets, we obtain the following important result in the case of uncountable orthonormal bases.

Lemma 6.4.11. *Let $\mathcal{H}$ be a Hilbert space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set. For each $\vec{x} \in \mathcal{H}$, the set $\{\alpha \in \Lambda \mid \langle \vec{x}, e_\alpha\rangle \neq 0\}$ is countable.*

Proof. For each $n \in \mathbb{N}$ let

$$\mathcal{E}_n = \left\{ \alpha \in \Lambda \,\middle|\, |\langle \vec{x}, e_\alpha\rangle| > \frac{1}{n} \right\}.$$

We claim that each $\mathcal{E}_n$ is finite. Indeed suppose to the contrary that $\mathcal{E}_n$ is infinite. Hence there exists a collection $\{\alpha_m\}_{m\in\mathbb{N}} \subseteq \mathcal{E}_n$ such that $\alpha_m \neq \alpha_k$ whenever $k \neq m$. By Theorem 6.4.10 we obtain that

$$\|\vec{x}\|^2 \geq \sum_{m\in\mathbb{N}} |\langle \vec{x}, e_{\alpha_m}\rangle|^2 \geq \sum_{m\in\mathbb{N}} \frac{1}{n^2},$$

which is impossible. Hence each $\mathcal{E}_n$ must be finite.

Since

$$\{\alpha \in \Lambda \mid \langle \vec{x}, e_\alpha\rangle \neq 0\} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n,$$

the set under consideration is a countable union of countable sets and thus is countable. ∎

Using the above, we immediately obtain a version of Bessel's Inequality for uncountable sets. In that which follows, we will be summing over an uncountable set. However, as only countable many terms in the sum are non-zero, by summing over this uncountable set we mean summing over the countable number of non-zero terms and, as all the terms are positive, the sum converges absolutely so the order of summation does not matter. There is a formal way to sum over uncountable sets, but this would take us into a deep realm of topology.

Theorem 6.4.12 (Bessel's Inequality). *Let $\mathcal{H}$ be a Hilbert space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set. For each $\vec{x} \in \mathcal{H}$,*

$$\sum_{\alpha\in\Lambda} |\langle \vec{x}, e_\alpha\rangle|^2 \leq \|\vec{x}\|^2 .$$

Corollary 6.4.13. *Let $\mathcal{H}$ be a Hilbert space and let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set. For each $\vec{x} \in \mathcal{H}$, the sum*

$$\sum_{\alpha\in\Lambda} \langle \vec{x}, e_\alpha\rangle e_\alpha$$

converges.

Proof. By Lemma 6.4.11, only a countable number of coefficients are non-zero in the desired sum are non-zero. Thus, to show the sum converges, we can assume without loss of generality that $\Lambda = \mathbb{N}$.

For each $N \in \mathbb{N}$, consider the partial sum $S_N = \sum_{n=1}^{N} \langle x, e_n\rangle e_n$. To see that $(S_N)_{N\geq 1}$ is Cauchy and thus converges as $\mathcal{H}$ is complete, let $\epsilon > 0$. Since $\sum_{\alpha\in\Lambda} |\langle x, e_\alpha\rangle|^2$ converges by Bessel's Inequality, there exists an $N_0 \in \mathbb{N}$ such that $\sum_{n=N_0}^{\infty} |\langle x, e_n\rangle|^2 < \epsilon^2$. Notice for all $M, N \in \mathbb{N}$ with $M \geq N \geq N_0$ that

$$\begin{aligned}
\|S_M - S_N\|^2 &= \left\| \sum_{n=N+1}^{M} \langle x, e_n\rangle e_n \right\|^2 \\
&= \sum_{n=N+1}^{M} \|\langle x, e_n\rangle e_n\|^2 \qquad \text{by the Pythagorean Theorem} \\
&= \sum_{n=N+1}^{M} |\langle x, e_n\rangle|^2 \\
&\leq \sum_{n=N_0}^{\infty} |\langle x, e_n\rangle|^2 < \epsilon^2.
\end{aligned}$$

Therefore $(S_N)_{N\geq 1}$ is Cauchy and thus converges. Hence the desired sum converges. ∎

Finally, we obtain another characterization of an orthonormal basis that shows orthonormal bases are good analytical bases for Hilbert spaces.

Theorem 6.4.14. *Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. The following are equivalent:*

(1) $\{e_\alpha\}_{\alpha\in\Lambda}$ *is an orthonormal basis for $\mathcal{H}$.*

(2) $\mathrm{span}(\{e_\alpha\}_{\alpha\in\Lambda})$ *is dense in $\mathcal{H}$.*

(3) For all $\vec{x}\in\mathcal{H}$, $\vec{x} = \sum_{\alpha\in\Lambda}\langle \vec{x}, e_\alpha\rangle e_\alpha$.

(4) For all $\vec{x}\in\mathcal{H}$, $\|\vec{x}\|^2 = \sum_{\alpha\in\Lambda}|\langle \vec{x}, e_\alpha\rangle|^2$.

Proof. To see that (1) implies (2), suppose $\{e_\alpha\}_{\alpha\in\Lambda}$ is an orthonormal basis for $\mathcal{H}$. If $\mathrm{span}(\{e_\alpha\}_{\alpha\in\Lambda})$ is not dense in $\mathcal{H}$, then $\mathcal{K} = \overline{\mathrm{span}(\{e_\alpha\}_{\alpha\in\Lambda})}$ is a closed vector subspace of $\mathcal{H}$ that is not equal to $\mathcal{H}$. Hence $\mathcal{K}^\perp \neq \emptyset$ by Theorem 6.3.9 so$\mathcal{K}^\perp$ must contain a vector $\vec{x}$ of length 1. Since $\vec{x}$ is orthogonal to each element of $\mathcal{K}$ and thus each e_α, we obtain that $\{\vec{x}\}\cup\{e_\alpha\}_{\alpha\in\Lambda}$ is an orthonormal set, which is larger than $\{e_\alpha\}_{\alpha\in\Lambda}$. As this contradicts the fact that $\{e_\alpha\}_{\alpha\in\Lambda}$ is a maximal orthonormal set, we have obtained a contradiction. Hence (1) implies (2).

To see that (2) implies (3), let $\vec{x}\in\mathcal{H}$ be arbitrary. By Corollary 6.4.13 the vector $\vec{y} = \sum_{\alpha\in\Lambda}\langle \vec{x}, e_\alpha\rangle e_\alpha$ is an element of $\mathcal{H}$. Hence there exists an increasing sequence of finite subsets Λ_n of Λ such that

$$\vec{y} = \lim_{n\to\infty}\sum_{\alpha\in\Lambda_n}\langle \vec{x}, e_\alpha\rangle e_\alpha.$$

Therefore, by the continuity of the inner product, we obtain that

$$\begin{aligned}\langle \vec{x}-\vec{y}, e_\alpha\rangle &= \lim_{n\to\infty}\left\langle \vec{x} - \sum_{\alpha\in\Lambda_n}\langle \vec{x}, e_\alpha\rangle e_\alpha, e_\beta\right\rangle\\ &= \lim_{n\to\infty}\langle \vec{x}, e_\beta\rangle - \sum_{\alpha\in\Lambda_n}\langle \vec{x}, e_\alpha\rangle\langle e_\alpha, e_\beta\rangle\\ &= 0\end{aligned}$$

for all $\beta\in\Lambda$. Hence $\vec{x}-\vec{y}\in(\mathrm{span}(\{e_\alpha\}_{\alpha\in\Lambda}))^\perp = \mathcal{H}^\perp = \{\vec{0}\}$. Thus $\vec{x}=\vec{y}$ as desired. Therefore, as $\vec{x}\in\mathcal{H}$ was arbitrary, (2) implies (3).

To see that (3) implies (4), let $\vec{x}\in\mathcal{H}$ be arbitrary. Notice there exists an increasing sequence of finite subsets Λ_n of Λ such that

$$\vec{x} = \lim_{n\to\infty}\sum_{\alpha\in\Lambda_n}\langle \vec{x}, e_\alpha\rangle e_\alpha \qquad\text{and}\qquad \sum_{\alpha\in\Lambda}|\langle \vec{x}, e_\alpha\rangle|^2.$$

Thus, by the continuity of the inner product

$$\begin{aligned} \|\vec{x}\|^2 &= \lim_{n\to\infty} \left\langle \sum_{\alpha\in\Lambda_n} \langle \vec{x}, e_\alpha\rangle e_\alpha, \sum_{\alpha\in\Lambda_n} \langle \vec{x}, e_\alpha\rangle e_\alpha \right\rangle \\ &= \lim_{n\to\infty} \sum_{\alpha\in\Lambda_n} |\langle \vec{x}, e_\alpha\rangle|^2 \\ &= \sum_{\alpha\in\Lambda} |\langle \vec{x}, e_\alpha\rangle|^2. \end{aligned}$$

Hence (3) implies (4).

Finally, to see that (4) implies (1), suppose to the contrary that $\{e_\alpha\}_{\alpha\in\Lambda}$ was not an orthonormal basis. Thus there exists a vector $\vec{x} \in \mathcal{H}$ such that $\|\vec{x}\|^2 = 1$ yet $\vec{x}$ is orthogonal to each e_α. However, the formula in (4) then implies $1 = 0$ which is impossible. Hence $\{e_\alpha\}_{\alpha\in\Lambda}$ is an orthonormal basis. ■

Using the same arguments as in Remark 6.4.9, we obtain a version of the orthogonal projection formula for infinite dimensional subspaces.

Corollary 6.4.15. *Let $\mathcal{K}$ be a closed vector subspace of a Hilbert space $\mathcal{H}$. If $\{e_\alpha\}_{\alpha\in\Lambda}$ is an orthonormal basis for $\mathcal{K}$ and P is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{K}$, then for all $\vec{x} \in \mathcal{H}$*

$$P(\vec{x}) = \sum_{\alpha\in\Lambda} \langle \vec{x}, e_\alpha\rangle e_\alpha.$$

6.5 Isomorphisms of Hilbert Spaces

We have seen in the previous section that every Hilbert space has an orthonormal basis and some of the properties of orthonormal bases. One question becomes, "Can we use orthonormal bases to distinguish Hilbert spaces?" The following is our first step.

Proposition 6.5.1. *If $\mathcal{H}$ is a Hilbert space, then any two orthonormal basis for $\mathcal{H}$ have the same cardinality.*

Proof. If $\mathcal{H}$ has a finite orthonormal basis, then $\mathcal{H}$ is finite dimensional. Since each orthonormal basis for a finite dimensional Hilbert space is a vector space basis, the result trivial follows. Hence we will assume $\mathcal{H}$ has only infinite dimensional orthonormal bases.

Let $\{e_\alpha\}_{\alpha\in\mathcal{E}}$ and $\{f_\beta\}_{\beta\in\mathcal{F}}$ be orthonormal bases for $\mathcal{H}$. Recall for each $\alpha \in \mathcal{E}$ the set

$$\mathcal{F}_\alpha = \{\beta \in \mathcal{F} \mid \langle e_\alpha, f_\beta\rangle \neq 0\}$$

is countable by Lemma 6.4.11. Furthermore, by Theorem 6.4.14 applied to $\{e_\alpha\}_{\alpha\in\mathcal{E}}$, for each $\beta \in \mathcal{F}$ there exists an $\alpha \in \mathcal{E}$ such that $\beta \in F_\alpha$. Therefore $\mathcal{F} = \bigcup_{\alpha\in\mathcal{E}} \mathcal{F}_\alpha$ so, as $|\mathcal{F}_\alpha| \leq |\mathbb{N}|$,

$$|\mathcal{F}| \leq |\mathbb{N}||\mathcal{E}| = |\mathcal{E}|$$

by cardinality theory. By replacing the roles of $\mathcal{F}$ and $\mathcal{E}$, we obtain that $|\mathcal{E}| \leq |\mathcal{F}|$ so $|\mathcal{E}| = |\mathcal{F}|$ as desired. ∎

Because of Proposition 6.5.1, we can now make the following definition.

Definition 6.5.2. The *dimension* of a Hilbert space $\mathcal{H}$, denoted $\dim(\mathcal{H})$, is the cardinality of an orthonormal basis for $\mathcal{H}$.

One interesting question is, "Does the dimension of a Hilbert space uniquely determines the Hilbert space?" In order to answer this question, we must ask what it means for two Hilbert spaces to be the same. Modelling Example 6.1.7, we easily see the following is the correct maps to use to determine when two Hilbert spaces are the same.

Definition 6.5.3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. A *unitary operator* from $\mathcal{H}_1$ to $\mathcal{H}_2$ is a surjective map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\langle U(\vec{x}), U(\vec{y})\rangle_{\mathcal{H}_2} = \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}_1}$$

for all $\vec{x}, \vec{y} \in \mathcal{H}_1$.

Definition 6.5.4. Two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are said to be *isomorphic* if there exists a unitary operator from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Remark 6.5.5. Clearly $\|U\vec{x}\|_{\mathcal{H}_2} = \|\vec{x}\|_{\mathcal{H}_1}$ so unitary operators are injective (and thus bijective). Notice since

$$\begin{aligned}\langle U(\vec{x} + \alpha\vec{z}), U(\vec{y})\rangle_{\mathcal{H}_2} &= \langle \vec{x} + \alpha\vec{z}, \vec{y}\rangle_{\mathcal{H}_1} \\ &= \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}_1} + \alpha\langle \vec{z}, \vec{y}\rangle_{\mathcal{H}_1} \\ &= \langle U(\vec{x}), U(\vec{y})\rangle_{\mathcal{H}_1} + \alpha\langle U(\vec{z}), U(\vec{y})\rangle_{\mathcal{H}_2} \\ &= \langle U(\vec{x}) + \alpha U(\vec{z}), U(\vec{y})\rangle_{\mathcal{H}_2}\end{aligned}$$

for all $\vec{x}, \vec{z}, \vec{y} \in \mathcal{H}_1$ and $\alpha \in \mathbb{K}$, the fact that U is surjective (so for each $\vec{v} \in \mathcal{H}_2$ there is a $\vec{y} \in \mathcal{H}_1$ such that $U(\vec{y}) = \vec{v}$) implies that U is linear. Hence $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ by the first part of the proof and the inverse of U is also linear and can also clearly be seen to be a unitary. Thus, as the composition of unitaries is clearly a unitary, isomorphism for Hilbert spaces is an equivalence relation.

The following demonstrates that dimension of a Hilbert space completely determines the Hilbert space.

Theorem 6.5.6. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if and only if $\dim(\mathcal{H}_1) = \dim(\mathcal{H}_2)$.*

Proof. First suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic. Therefore there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis for $\mathcal{H}_1$. By the definition and properties of a unitary (see Remark 6.5.5), we see that $\{U(e_\alpha)\}_{\alpha\in\Lambda}$ is an orthonormal set. Furthermore, Theorem 6.4.14 implies that $\mathrm{span}(\{e_\alpha\}_{\alpha\in\Lambda})$ is dense in $\mathcal{H}_1$. Therefore, as U is a linear map and a homeomorphism, $\mathrm{span}(\{U(e_\alpha)\}_{\alpha\in\Lambda})$ must also be dense in $\mathcal{H}_2$. Hence $\{U(e_\alpha)\}_{\alpha\in\Lambda}$ is an orthonormal basis of $\mathcal{H}_2$. Hence $\dim(\mathcal{H}_1) = |\Lambda| = \dim(\mathcal{H}_2)$ as desired.

For the converse direction, we note that since isomorphism of Hilbert spaces is an equivalence relation that it suffices to prove the following.

Corollary 6.5.7. *Let $\mathcal{H}$ be a Hilbert space and let Λ be a set such that $|\Lambda| = \dim(\mathcal{H})$. Then $\mathcal{H}$ is isomorphic to the Hilbert space*

$$\ell_2(\Lambda,\mathbb{K}) = \left\{ f : \Lambda \to \mathbb{K} \,\middle|\, \begin{smallmatrix} \{\alpha\in\Lambda \,|\, f(\alpha)\neq 0\} \textit{ is countable} \\ \textit{and } \sum_{\alpha\in\Lambda} |f(\alpha)|^2 < \infty \end{smallmatrix} \right\}$$

equipped with the inner product

$$\langle f, g\rangle_{\ell_2(\Lambda,\mathbb{K})} = \sum_{\alpha\in\Lambda} f(\alpha)\overline{g(\alpha)}.$$

Proof. First we must proof that $\ell_2(\Lambda,\mathbb{K})$ together with the inner product described is indeed a Hilbert space. The proof that $\langle f, g\rangle_{\ell_2(\Lambda,\mathbb{K})}$ is a well-defined inner product is as in Example 6.1.10. The proof that $\ell_2(\Lambda,\mathbb{K})$ is a Banach space follows the proof given in Proposition 2.2.2 using the fact that a countable union of countable sets is countable so that given a Cauchy sequence there are only a countable number of entries of Λ that need to be considered when demonstrating convergence. Hence $\ell_2(\Lambda,\mathbb{K})$ is a Hilbert space.

To complete the proof, it suffices to show that $\mathcal{H}$ is isomorphic to such a space. Let $\{e_\alpha\}_{\alpha\in\Lambda}$ be an orthonormal basis of $\mathcal{H}$. Define $U : \mathcal{H} \to \ell_2(\Lambda,\mathbb{K})$ by $U(h)(\alpha) = \langle h, e_\alpha\rangle_{\mathcal{H}}$ for all $\alpha \in \Lambda$ and $h \in \mathcal{H}$. Note if $h \in \mathcal{H}$ then $U(h)$ is indeed an element of $\ell_2(\Lambda,\mathbb{K})$ by Bessel's inequality (Theorem 6.4.12). Hence U is a well-defined linear map that maps the orthonormal basis $\{e_\alpha\}_{\alpha\in\Lambda}$ to the orthonormal basis $\{f_\alpha\}_{\alpha\in\Lambda}$ where

$$f_\alpha(\beta) = \begin{cases} 1 & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases}.$$

Hence Theorem 6.4.14 implies that U is surjective. To see that U is a unitary (and thus injective), notice for all $\vec{x}, \vec{y} \in \mathcal{H}$ that by Theorem 6.4.14 and the fact that the inner product is continuous in each entry, we have

$$\begin{aligned}
\langle U(\vec{x}), U(\vec{y})\rangle_{\ell_2(\Lambda,\mathbb{K})} &= \sum_{\alpha\in\Lambda} \langle \vec{x}, e_\alpha\rangle_{\mathcal{H}} \overline{\langle e_\alpha, \vec{y}\rangle_{\mathcal{H}}} \\
&= \sum_{\alpha\in\Lambda} \langle \langle \vec{x}, e_\alpha\rangle_{\mathcal{H}} e_\alpha, \langle e_\alpha, \vec{y}\rangle_{\mathcal{H}} e_\alpha\rangle_{\mathcal{H}} \\
&= \sum_{\alpha,\beta\in\Lambda} \langle \langle \vec{x}, e_\alpha\rangle_{\mathcal{H}} e_\alpha, \langle e_\beta, \vec{y}\rangle_{\mathcal{H}} e_\beta\rangle_{\mathcal{H}} \\
&= \left\langle \sum_{\alpha\in\Lambda} \langle \vec{x}, e_\alpha\rangle_{\mathcal{H}} e_\alpha, \sum_{\beta\in\Lambda} \langle e_\beta, \vec{y}\rangle_{\mathcal{H}} e_\beta \right\rangle_{\mathcal{H}} \\
&= \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}}
\end{aligned}$$

Hence U is a unitary so $\mathcal{H}$ is isomorphic to $\ell_2(\Lambda, \mathbb{K})$. ∎

This completes the proof of Theorem 6.5.6. ∎

6.6 The Riesz-Representation Theorem

Orthonormal bases are important structures for Hilbert spaces. For example, orthonormal bases are a central part of Fourier Analysis. Instead of heading in that direction, we turn our focus to another important structure of Hilbert spaces; the structure of all continuous linear maps from a Hilbert space to the complex numbers. As such linear maps are also important outside of Hilbert spaces, we make the following definition.

Definition 6.6.1. Let $(\mathcal{X}, \|\cdot\|)$ be a normed linear space. The *dual space* of $\mathcal{X}$ is $\mathcal{X}^* = B(\mathcal{X}, \mathbb{K})$ and an element of $\mathcal{X}^*$ is called a *linear functional*.

Recall that the norm on $\mathcal{X}^*$ is given by

$$\|f\| = \sup\{|f(\vec{x})| \mid \vec{x} \in \mathcal{X}, \|\vec{x}\| \leq 1\}.$$

Furthermore, it is useful to note the following result whose proof is trivial (based on Theorem 1.5.7) and thus is omitted.

Lemma 6.6.2. *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ are normed linear spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then*

$$\ker(T) = \{\vec{x} \in \mathcal{X} \mid T(\vec{x}) = \vec{0}\}$$

is a closed vector subspace of $\mathcal{X}$.

One fascinating thing about a Hilbert space $\mathcal{H}$ is that $\mathcal{H}^*$ is $\mathcal{H}$ in a specific way.

Theorem 6.6.3 (Riesz Representation Theorem). *Let $\mathcal{H}$ be a Hilbert space and let $\varphi \in \mathcal{H}^*$. Then there exists a unique vector $\vec{y} \in \mathcal{H}$ such that*

$$\varphi(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$$

for all $\vec{x} \in \mathcal{H}$. Moreover $\|\varphi\| = \|\vec{y}\|_{\mathcal{H}}$.

Proof. First, let $\vec{y} \in \mathcal{H}$ be arbitrary and define $\varphi_{\vec{y}} : \mathcal{H} \to \mathbb{C}$ by $\varphi_{\vec{y}}(\vec{x}) = \langle \vec{x}, \vec{y} \rangle$ for all $\vec{x} \in \mathcal{H}$. Let us check that $\varphi_{\vec{y}}$ is a linear functional and compute its norm. It is elementary to see that $\varphi_{\vec{y}}$ is a linear map. To see that φ is continuous, note by the Cauchy-Schwarz inequality that

$$|\varphi_{\vec{y}}(\vec{x})| \leq \|\vec{x}\| \, \|\vec{y}\| \, .$$

Hence φ is continuous and $\|\varphi\| \leq \|\vec{y}\|$. For the other inequality, notice said inequality is trivial if $\vec{y} = \vec{0}$. Otherwise let $\vec{z} = \frac{1}{\|\vec{y}\|}\vec{y}$ so that $\vec{z}$ is a unit vector. Since

$$\varphi_{\vec{y}}(\vec{z}) = \left\langle \frac{1}{\|\vec{y}\|}\vec{y}, \vec{y} \right\rangle = \|\vec{y}\| \, ,$$

the other inequality follows.

Now let $\varphi \in \mathcal{H}^*$ be arbitrary. If $\varphi(\vec{x}) = 0$ for all $\vec{x} \in \mathcal{H}$, then clearly $\varphi = \varphi_{\vec{0}}$. Otherwise, suppose φ is not the zero linear functional. Therefore, by Lemma 6.6.2, $\ker(\varphi)$ is a closed vector subspace of $\mathcal{H}$ that does not equal $\mathcal{H}$. Hence there exists a vector $\vec{z} \in \ker(\varphi)^\perp \setminus \{\vec{0}\}$. As $\varphi(\vec{z}) \neq 0$, by scaling if necessary we may assume that $\varphi(\vec{z}) = 1$.

We claim that $\operatorname{span}(\{\vec{z}\}) = \ker(\varphi)^\perp$. To see this, it suffices to show that if $\vec{z}_1 \in \ker(\varphi)^\perp \setminus \{\vec{0}\}$ and $\varphi(\vec{z}_1) = 1$, then $\vec{z} = \vec{z}_1$. Indeed if $\vec{z}_1$ has the desired properties, then $\vec{z} - \vec{z}_1 \in \ker(\varphi)^\perp$ and

$$\varphi(\vec{z} - \vec{z}_1) = 1 - 1 = 0$$

so $\vec{z} - \vec{z}_1 \in \ker(\varphi)$. Hence $\vec{z} - \vec{z}_1 \in \ker(\varphi) \cap \ker(\varphi)^\perp = \{\vec{0}\}$ so $\vec{z} = \vec{z}_1$ as desired. Hence $\operatorname{span}(\{\vec{z}\}) = \ker(\varphi)^\perp$ and thus $\{\vec{z}\}^\perp = \ker(\varphi)$

As $\vec{z} \neq \vec{0}$, let $\vec{y} = \frac{1}{\|\vec{z}\|^2}\vec{z}$. Therefore $\{\vec{y}\}^\perp = \{\vec{z}\}^\perp = \ker(\varphi)$. We claim that $\varphi = \varphi_{\vec{y}}$. To see this, we notice for all $\vec{x} \in \ker(\varphi)$ that $\vec{x} \in \{\vec{y}\}^\perp$ so

$$\langle \vec{x}, \vec{y} \rangle = 0 = \varphi(\vec{x}).$$

Otherwise, if $\vec{x} = \beta\vec{y}$ for some $\beta \in \mathbb{K}$, we see that

$$\begin{aligned}\varphi(\vec{x}) &= \beta\varphi(\vec{y}) \\ &= \beta\frac{1}{\|\vec{z}\|^2}\varphi(\vec{z}) \\ &= \frac{\beta}{\|\vec{z}\|^2} \\ &= \frac{\beta}{\|\vec{z}\|^4}\langle\vec{z},\vec{z}\rangle \\ &= \beta\langle\vec{y},\vec{y}\rangle \\ &= \langle\vec{x},\vec{y}\rangle.\end{aligned}$$

Therefore, as $\mathcal{H} = \ker(\varphi) \oplus \ker(\varphi)^\perp = \ker(\varphi) \oplus \mathrm{span}(\{\vec{y}\})$ by Theorem 6.3.9, it follows that $\varphi = \varphi_{\vec{y}}$ as desired.

Finally, for uniqueness, suppose there exists $\vec{y}_1, \vec{y}_2 \in \mathcal{H}$ such that $\varphi_{\vec{y}_1} = \varphi_{\vec{y}_2}$. Then

$$\langle\vec{x},\vec{y}_1\rangle = \langle\vec{x},\vec{y}_2\rangle \text{ for all } \vec{x} \in \mathcal{H} \quad \Rightarrow \quad \langle\vec{x},\vec{y}_1 - \vec{y}_2\rangle = 0 \text{ for all } \vec{x} \in \mathcal{H}.$$

By selecting $\vec{x} = \vec{y}_1 - \vec{y}_2$, we obtain that $\|\vec{y}_1 - \vec{y}_2\| = 0$ so $\vec{y}_1 = \vec{y}_2$ as desired. ∎

The above description of $\mathcal{H}^*$ is useful in many ways. First we note the following result which allows us to use the elements of $\mathcal{H}^*$ to compute the norm of elements of $\mathcal{H}$. Note the following also has an analogue for Banach spaces which we will not have time to develop.

Lemma 6.6.4. *Let $\mathcal{H}$ be a Hilbert space. If $\vec{x} \in \mathcal{H}$ then*

$$\|\vec{x}\| = \sup\{|f(\vec{x})| \mid f \in \mathcal{H}^*, \|f\| \leq 1\}.$$

Proof. First notice for all $f \in \mathcal{H}^*$ with $\|f\| \leq 1$ that

$$|f(\vec{x})| \leq \|f\| \, \|\vec{x}\| \leq \|\vec{x}\|$$

so we obtain that

$$\|\vec{x}\| \geq \sup\{|f(\vec{x})| \mid f \in \mathcal{H}^*, \|f\| \leq 1\}.$$

For the other inequality, notice if $\vec{x} = \vec{0}$ then the inequality is trivial. If $\vec{x} \neq \vec{0}$, let $\vec{y} = \frac{1}{\|\vec{x}\|}\vec{x} \in \mathcal{H}$ so that $\|\vec{y}\| = 1$. Hence if we define $\varphi_{\vec{y}} : \mathcal{H} \to \mathbb{C}$ by

$$\varphi_{\vec{y}}(\vec{z}) = \langle\vec{z},\vec{y}\rangle$$

for all $\vec{z} \in \mathcal{H}$, then $\|\varphi_{\vec{y}}\| = 1$ by the Riesz Representation Theorem (Theorem 6.6.3). Since

$$|\varphi_{\vec{y}}(\vec{x})| = \left|\left\langle\vec{x}, \frac{1}{\|\vec{x}\|}\vec{x}\right\rangle\right| = \frac{\|\vec{x}\|^2}{\|\vec{x}\|} = \|\vec{x}\|$$

the other inequality follows. ∎

Using Lemma 6.6.4 and the definition of the operator norm, we obtain the following useful description of the operator norm of a linear map between Hilbert spaces.

Lemma 6.6.5. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T : \mathcal{H} \to \mathcal{K}$ be linear. Then*

$$\|T\| = \sup\{|\langle T(\vec{x}), \vec{y}\rangle_{\mathcal{K}}| \mid \vec{x} \in \mathcal{H}, \vec{y} \in \mathcal{K}, \|\vec{x}\|_{\mathcal{H}}, \|\vec{y}\|_{\mathcal{K}} \leq 1\}$$

(with both sides being infinity if T is not bounded).

The above norm description of a bounded linear map is quite useful. For example, using the Riesz Representation Theorem (Theorem 6.6.3), when given a linear map we can construct a nice 'reverse' linear map that plays well with respect to the inner product.

Theorem 6.6.6. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then there exists a unique linear map $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, called the* adjoint *of T, such that*

$$\langle T^*(\vec{y}), \vec{x}\rangle_{\mathcal{H}} = \langle \vec{y}, T(\vec{x})\rangle_{\mathcal{K}}$$

for all $\vec{x} \in \mathcal{H}$ and $\vec{y} \in \mathcal{K}$. Furthermore $\|T^*\| = \|T\|$.

Proof. Fix $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For each $\vec{y} \in \mathcal{K}$, consider the linear map $f_{\vec{y}} : \mathcal{K} \to \mathbb{C}$ defined by

$$f_{\vec{y}}(\vec{x}) = \langle T(\vec{x}), \vec{y}\rangle_{\mathcal{K}}$$

for all $\vec{x} \in \mathcal{H}$. Since

$$|f_{\vec{y}}(\vec{x})| = |\langle T(\vec{x}), \vec{y}\rangle_{\mathcal{K}}| \leq \|T(\vec{x})\|_{\mathcal{K}} \|\vec{y}\|_{\mathcal{K}} \leq \|T\| \|\vec{x}\|_{\mathcal{K}} \|\vec{y}\|_{\mathcal{K}}$$

via the Cauchy-Schwarz inequality, we see that $f_{\vec{y}}$ is a bounded linear map. Therefore, by the Riesz Representation Theorem (Theorem 6.6.3) there exists a unique vector, denoted $T^*_{\vec{y}} \in \mathcal{H}$ such that

$$\langle T(\vec{x}), \vec{y}\rangle_{\mathcal{K}} = f_{\vec{y}}(\vec{x}) = \langle \vec{x}, T^*_{\vec{y}}\rangle_{\mathcal{H}}$$

for all $\vec{x} \in \mathcal{H}$.

We claim that the map $T^* : \mathcal{K} \to \mathcal{H}$ defined by $T^*(\vec{y}) = T^*_{\vec{y}}$ is a bounded linear map. To see linearity, notice for all $\vec{x} \in \mathcal{H}$, $\vec{y}_1, \vec{y}_2 \in \mathcal{K}$, and $\alpha \in \mathbb{K}$ that

$$\begin{aligned} \langle \vec{x}, T^*_{\vec{y}_1+\alpha\vec{y}_2}\rangle_{\mathcal{H}} &= \langle T(\vec{x}), \vec{y}_1 + \alpha\vec{y}_2\rangle_{\mathcal{K}} \\ &= \langle T(\vec{x}), \vec{y}_1\rangle_{\mathcal{K}} + \overline{\alpha}\langle T(\vec{x}), \vec{y}_2\rangle_{\mathcal{K}} \\ &= \langle \vec{x}, T^*_{\vec{y}_1}\rangle_{\mathcal{H}} + \overline{\alpha}\langle \vec{x}, T^*_{\vec{y}_2}\rangle_{\mathcal{H}} \\ &= \langle \vec{x}, T^*_{\vec{y}_1} + \alpha T^*_{\vec{y}_2}\rangle_{\mathcal{H}}. \end{aligned}$$

Therefore, as the above holds for all $\vec{x} \in \mathcal{H}$, we see (for example, by the uniqueness part of the Riesz Representation Theorem (Theorem 6.6.3)) that

$$T^*_{\vec{y}_1 + \alpha\vec{y}_2} = T^*_{\vec{y}_1} + \alpha T^*_{\vec{y}_2}.$$

Therefore, as $\vec{y}_1, \vec{y}_2 \in \mathcal{K}$ and $\alpha \in \mathbb{K}$ were arbitrary, T^* is linear.

To see that T^* is bounded, we notice that

$$\begin{aligned}
&\sup\{|\langle T^*(\vec{y}), \vec{x}\rangle_{\mathcal{H}}| \mid \vec{x} \in \mathcal{H}, \vec{y} \in \mathcal{K}, \|\vec{x}\|_{\mathcal{H}}, \|\vec{y}\|_{\mathcal{K}} \leq 1\} \\
&= \sup\{|\langle \vec{y}, T(\vec{x})\rangle_{\mathcal{K}}| \mid \vec{x} \in \mathcal{H}, \vec{y} \in \mathcal{K}, \|\vec{x}\|_{\mathcal{H}}, \|\vec{y}\|_{\mathcal{K}} \leq 1\} \\
&= \sup\{|\langle T(\vec{x}), \vec{y}\rangle_{\mathcal{K}}| \mid \vec{x} \in \mathcal{H}, \vec{y} \in \mathcal{K}, \|\vec{x}\|_{\mathcal{H}}, \|\vec{y}\|_{\mathcal{K}} \leq 1\} \\
&= \|T\|.
\end{aligned}$$

Thus it follows from Lemma 6.6.5 that T^* is bounded with $\|T^*\| = \|T\|$. Finally, uniqueness of T^* comes from construction and the uniqueness in the Riesz Representation Theorem. ∎

Often it will be the case that we consider $\mathcal{K} = \mathcal{H}$ in Theorem 6.4.14. This means the adjoint becomes an operator on $\mathcal{B}(\mathcal{H}, \mathcal{H})$. Often we will use $\mathcal{B}(\mathcal{H})$ to denote $\mathcal{B}(\mathcal{H}, \mathcal{H})$ for simplicity.

It turns out that many standard operations and linear maps we have seen are related to the adjoint.

Example 6.6.7. Let $A \in \mathcal{M}_n(\mathbb{K})$ and define $L_A : \mathbb{K}^n \to \mathbb{K}^n$ by $L_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{K}^n$ (where we write $\vec{x}$ as a column vector and use matrix multiplication). Then, with respect to the standard inner product on $\mathbb{K}^n$, $(L_A)^* = L_{A^*}$ where A^* is the conjugate transpose of A. To see this, write $A = [a_{i,j}]$. Then for each $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n) \in \mathbb{K}^n$,

$$\begin{aligned}
\langle (L_A)^*\vec{x}, \vec{y}\rangle &= \langle \vec{x}, L_A(\vec{y})\rangle \\
&= \left\langle (x_1, \ldots, x_n), \left(\sum_{j=1}^n a_{1,j}y_j, \ldots, \sum_{j=1}^n a_{n,k}y_j\right)\right\rangle \\
&= \sum_{i,j=1}^n x_i\overline{a_{i,j}y_j} \\
&= \sum_{i,j=1}^n \overline{a_{i,j}}x_i\overline{y_j} \\
&= \left\langle \left(\sum_{i=1}^n \overline{a_{i,1}}x_i, \ldots, \sum_{i=1}^n \overline{a_{i,n}}x_i\right), (y_1, \ldots, y_n)\right\rangle \\
&= \langle L_{A^*}(\vec{x}), \vec{y}\rangle.
\end{aligned}$$

Therefore, as the above holds for all $\vec{y} \in \mathbb{K}^n$, we see (for example, by the uniqueness part of the Riesz Representation Theorem (Theorem 6.6.3)) that $(L_A)^*(\vec{x}) = L_{A^*}(\vec{x})$ for all $\vec{x} \in \mathbb{K}^n$. Hence $(L_A)^* = L_{A^*}$ as claimed.

One very important result in operator theory is the following result.

Theorem 6.6.8. *Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Then $\|T\|^2 = \|T^*T\|$.*

Proof. First, we note for all $\vec{x} \in \mathcal{H}$ that

$$\|T^*(T(\vec{x}))\|_{\mathcal{H}} \leq \|T^*\| \, \|T(\vec{x})\|_{\mathcal{H}} \leq \|T^*\| \, \|T\| \, \|\vec{x}\|_{\mathcal{H}}.$$

Hence $\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2$ as $\|T^*\| = \|T\|$ by Theorem 6.6.6.

To see the other inequality, notice that

$$\begin{aligned}
\|T\|^2 &= \sup\{\|T(\vec{x})\|^2 \mid \vec{x} \in \mathcal{H}, \|\vec{x}\|_{\mathcal{H}} \leq 1\} \\
&= \sup\{\langle T(\vec{x}), T(\vec{x})\rangle_{\mathcal{H}} \mid \vec{x} \in \mathcal{H}, \|\vec{x}\|_{\mathcal{H}} \leq 1\} \\
&= \sup\{\langle T^*T(\vec{x}), \vec{x}\rangle_{\mathcal{H}} \mid \vec{x} \in \mathcal{H}, \|\vec{x}\|_{\mathcal{H}} \leq 1\} \\
&\leq \sup\{\langle T^*T(\vec{x}), \vec{y}\rangle_{\mathcal{H}} \mid \vec{x}, \vec{y} \in \mathcal{H}, \|\vec{x}\|_{\mathcal{H}}, \|\vec{y}\|_{\mathcal{H}} \leq 1\} \\
&= \|T^*T\|
\end{aligned}$$

by Lemma 6.6.5. Hence the proof is complete. ∎

Our next goal is to use the adjoint to describe orthogonal projections. To do this, we first note the following.

Lemma 6.6.9. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $(T^*)^* = T$ and*

$$(\mathrm{Im}(T))^{\perp} = \ker(T^*).$$

Hence $\overline{\mathrm{Im}(T)} = \ker(T^*)^{\perp}$.

Proof. The fact that $(T^*)^*$ follows trivially from the definition of the adjoint and clearly $\overline{\mathrm{Im}(T)} = \ker(T^*)^{\perp}$ will follow from $(\mathrm{Im}(T))^{\perp} = \ker(T^*)$ using Corollary 6.3.11.

To prove that $(\mathrm{Im}(T))^{\perp} = \ker(T^*)$, let $\vec{x} \in \ker(T^*)$ be arbitrary. Let $\vec{y} \in \mathrm{Im}(T)$ be arbitrary. Then there exists a vector $\vec{z} \in \mathcal{H}$ such that $\vec{y} = T(\vec{z})$. Hence

$$\langle \vec{y}, \vec{x}\rangle_{\mathcal{K}} = \langle T(\vec{z}), \vec{x}\rangle_{\mathcal{K}} = \langle \vec{z}, T^*(\vec{x})\rangle_{\mathcal{H}} = \langle \vec{z}, \vec{0}\rangle_{\mathcal{H}} = 0.$$

Therefore, as $\vec{y} \in \mathrm{Im}(T)$ was arbitrary, it follows that $\vec{x} \in (\mathrm{Im}(T))^{\perp}$. Hence, as $\vec{x} \in \ker(T^*)$ was arbitrary, $\ker(T^*) \subseteq (\mathrm{Im}(T))^{\perp}$.

For the other direction, let $\vec{x} \in (\mathrm{Im}(T))^{\perp}$ be arbitrary. Then for all $\vec{y} \in \mathcal{H}$ we see that

$$\langle T^*(\vec{x}), \vec{y}\rangle_{\mathcal{H}} = \langle \vec{x}, T(\vec{y})\rangle_{\mathcal{K}} = 0$$

as $T(\vec{y}) \in \mathrm{Im}(T)$ and $\vec{x} \in (\mathrm{Im}(T))^{\perp}$. Therefore, as $\vec{y} \in \mathcal{H}$ was arbitrary, we see (for example, by the uniqueness part of the Riesz Representation Theorem (Theorem 6.6.3)) that $\vec{x} \in \ker(T^*)$. Hence, as $\vec{x} \in (\mathrm{Im}(T))^{\perp}$ was arbitrary, $\ker(T^*) = (\mathrm{Im}(T))^{\perp}$ as desired. ∎

Using this characterization, we obtain the following.

Proposition 6.6.10. *Let $\mathcal{H}$ be a Hilbert space. An element $P \in \mathcal{B}(\mathcal{H})$ is an orthogonal projection onto a closed vector subspace of $\mathcal{H}$ if and only if $P^2 = P$ and $P^* = P$.*

Proof. Suppose P is the orthogonal projection onto a closed vector subspace $\mathcal{K}$ of $\mathcal{H}$. As we have previously seen that $P^2 = P$, it suffices to show that $P^* = P$. To see this, let $\vec{x}, \vec{y} \in \mathcal{H}$ be arbitrary. By Theorem 6.3.9 we can write $\vec{x} = \vec{x}_P + \vec{x}_0$ and $\vec{y} = \vec{y}_P + \vec{y}_0$ where $\vec{x}_P, \vec{y}_P \in \mathcal{K}$ and $\vec{x}_0, \vec{y}_0 \in \mathcal{K}^\perp$. Therefore we have that

$$P(\vec{x}_P) = \vec{x}_P, \quad P(\vec{y}_P) = \vec{y}_P, \quad P\vec{x}_0 = \vec{0}, \quad \text{and} \quad P\vec{y}_0 = \vec{y}_0.$$

Hence

$$\begin{aligned}
\langle P^*(\vec{x}), \vec{y}\rangle &= \langle \vec{x}, P(\vec{y})\rangle \\
&= \langle \vec{x}_P + \vec{x}_0, P(\vec{y}_P + \vec{y}_0)\rangle \\
&= \langle \vec{x}_P + \vec{x}_0, \vec{y}_P\rangle \\
&= \langle \vec{x}_P, \vec{y}_P\rangle + \langle \vec{x}_0, \vec{y}_P\rangle \\
&= \langle \vec{x}_P, \vec{y}_P\rangle \\
&= \langle \vec{x}_P, \vec{y}_P\rangle + \langle \vec{x}_P, \vec{y}_0\rangle \\
&= \langle \vec{x}_P, \vec{y}_P\rangle + \vec{y}_0\rangle \\
&= \langle P(\vec{x}_P + \vec{x}_0), \vec{y}_P\rangle + \vec{y}_0\rangle \\
&= \langle P(\vec{x}), \vec{y}\rangle.
\end{aligned}$$

Therefore, as the above holds for all $\vec{y} \in \mathcal{H}$, we see (for example, by the uniqueness part of the Riesz Representation Theorem (Theorem 6.6.3)) that $P^*(\vec{x}) = P(\vec{x})$ for all $\vec{x} \in \mathcal{H}$. Hence $P^* = P$ as claimed.

For the other direction, let $P \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be such that $P^2 = P = P^*$. Let $\mathcal{K} = \ker(P)$, which is a closed vector subspace by Lemma 6.6.2. Notice by Lemma 6.6.9 that

$$\mathcal{K}^\perp = \ker(P)^\perp = \overline{\mathrm{Im}(P^*)} = \overline{\mathrm{Im}(P)}.$$

We claim that P is the orthogonal projection onto $\overline{\mathrm{Im}(P)}$. To see this, first we notice that

$$\overline{\mathrm{Im}(P)}^\perp = (\mathcal{K}^\perp)^\perp = \mathcal{K}$$

by Corollary 6.3.11. Therefore, as $P(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathcal{K}$, it suffices to show that P is the identity on $\overline{\mathrm{Im}(P)}$. If $\vec{x} \in \mathrm{Im}(P)$, then $\vec{x} = P(\vec{y})$ for some $\vec{y} \in \mathcal{H}$ and thus

$$P(\vec{x}) = P^2(\vec{y}) = P(\vec{y}) = \vec{x}.$$

Therefore, P is the identity on $\mathrm{Im}(P)$. Hence P is the identity on $\overline{\mathrm{Im}(P)}$ by continuity. Thus the result follows. ∎

Finally, we can re-discuss unitary operators in the context of adjoints. First we note the following for isometries.

Proposition 6.6.11. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following are equivalent:*

1. $V^*V = I_{\mathcal{H}}$.
2. $\|V(\vec{x})\|_{\mathcal{K}} = \|\vec{x}\|_{\mathcal{H}}$ *for all* $\vec{x} \in \mathcal{H}$ *(that is, V is an isometry).*
3. $\langle V(\vec{x}), V(\vec{y})\rangle_{\mathcal{K}} = \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}}$ *for all* $\vec{x}, \vec{y} \in \mathcal{H}$.

Proof. First, to see that (1) implies (2), suppose (1) holds. Then for all $\vec{x} \in \mathcal{H}$

$$\|V(\vec{x})\|_{\mathcal{K}}^2 = \langle V(\vec{x}), V(\vec{x})\rangle_{\mathcal{K}} = \langle V^*V(\vec{x}), \vec{x}\rangle_{\mathcal{H}} = \langle \vec{x}, \vec{x}\rangle_{\mathcal{H}} = \|\vec{x}\|_{\mathcal{H}}^2.$$

Hence (2) holds so (1) implies (2)

Next, to see that (2) implies (3), suppose that (2) holds. By the same proof of the Polarization Identity (Theorem 6.1.20), we see that

$$\begin{aligned}\langle V(\vec{x}), V(\vec{y})\rangle_{\mathcal{K}} &= \frac{1}{4}\|V(\vec{x}) + V(\vec{y})\|^2 - \frac{1}{4}\|V(\vec{x}) - V(\vec{y})\|^2 \\ &= \frac{1}{4}\|V(\vec{x} + \vec{y})\|^2 - \frac{1}{4}\|V(\vec{x} - \vec{y})\|^2 \\ &= \frac{1}{4}\|\vec{x} + \vec{y}\|^2 - \frac{1}{4}\|\vec{x} - \vec{y}\|^2 \\ &= \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}}\end{aligned}$$

if $\mathbb{K} = \mathbb{R}$ and

$$\begin{aligned}\langle V(\vec{x}), V(\vec{y})\rangle_{\mathcal{K}} &= \frac{1}{4}\sum_{k=1}^{4}\left\|V(\vec{x}) + i^k V(\vec{y})\right\|^2 \\ &= \frac{1}{4}\sum_{k=1}^{4}\left\|V(\vec{x} + i^k \vec{y})\right\|^2 \\ &= \frac{1}{4}\sum_{k=1}^{4}\left\|\vec{x} + i^k \vec{y}\right\|^2 \\ &= \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}}\end{aligned}$$

if $\mathbb{K} = \mathbb{C}$. Hence (3) follows so (2) implies (3)

Finally, to see that (3) implies (1), suppose (3) holds. Then for all $\vec{x}, \vec{y} \in \mathcal{H}$

$$\langle I_{\mathcal{H}}(\vec{x}), \vec{y}\rangle_{\mathcal{H}} = \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}} = \langle V(\vec{x}), V(\vec{y})\rangle_{\mathcal{K}} = \langle V^*V(\vec{x}), \vec{y}\rangle_{\mathcal{H}}.$$

Hence it follows that $V^*V = I$ as desired. ∎

Using the above, we obtain the following.

Corollary 6.6.12. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The following are equivalent:*

1. $U^*U = I_{\mathcal{H}}$ *and* $UU^* = I_{\mathcal{K}}$.
2. $\|U(\vec{x})\|_{\mathcal{K}} = \|\vec{x}\|_{\mathcal{H}}$ *for all* $\vec{x} \in \mathcal{H}$ *and* U *is surjective.*
3. $\langle U(\vec{x}), U(\vec{y})\rangle_{\mathcal{K}} = \langle \vec{x}, \vec{y}\rangle_{\mathcal{H}}$ *for all* $\vec{x}, \vec{y} \in \mathcal{H}$ *and* U *is surjective (i.e. U is a unitary).*

Hence, if $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a unitary, then $\|U\| = 1$.

Proof. Clearly (1) implies (2) and (2) implies (3) by Proposition 6.6.11. Suppose (3) holds. Then $U^*U = \mathcal{I}_{\mathcal{H}}$ by Proposition 6.6.11. Since (3) holds, we obtain U is an isometry by Proposition 6.6.11. Hence U is injective and thus invertible as a linear map between vector spaces. Therefore, due to the uniqueness of the inverses, we obtain that $UU^* = I_{\mathcal{K}}$. ∎

Using all of the above (in fact, using substantially less technology), we can prove the following.

Theorem 6.6.13. *Let $A \in \mathcal{M}_n(\mathbb{K})$ and define $L_A : \mathbb{K}^n \to \mathbb{K}^n$ by $L_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{K}^n$ (where we write $\vec{x}$ as a column vector and use matrix multiplication). Then*

$$\|L_A\| = \max\left\{\sqrt{\lambda} \mid \lambda \text{ an eigenvalue for } A^*A\right\}.$$

Proof. First, consider the case $A = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and let $M = \max\{|d_1|, |d_2|, \ldots, |d_n|\}$. To see that $\|L_A\| = M$, first notice for all $k \in \{1, \ldots, n\}$ that if $\vec{e}_k$ is the vector in $\mathbb{C}^n$ with a 1 in the k^{th} entry and 0s elsewhere, then $\|\vec{e}_k\|_2 = 1$ and

$$\|L_A(\vec{e}_k)\|_2 = \|d_k\vec{e}_k\|_2 = |d_k|.$$

Hence $\|L_A\| \geq M$.

To see the reverse inequality, notice for all $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$ such that $\|\vec{x}\|_2 = \sqrt{\sum_{k=1}^n |x_k|^2} \leq 1$ that

$$\begin{aligned}\|L_A(\vec{x})\|_2 &= \|(d_1x_1, d_2x_2, \ldots, d_nx_n)\|_2 \\ &= \sqrt{\sum_{k=1}^n |d_kx_k|^2} \\ &= \sqrt{\sum_{k=1}^n |d_k|^2|x_k|^2} \\ &\leq \sqrt{\sum_{k=1}^n M^2|x_k|^2} \\ &= M\sqrt{\sum_{k=1}^n |x_k|^2} = M.\end{aligned}$$

Hence $\|L_A\| \leq M$ so $\|L_A\| = M$ as desired.

Next,l et $A \in \mathcal{M}_n(\mathbb{C})$ be arbitrary and let $U \in \mathcal{M}_n(\mathbb{C})$ be an arbitrary unitary matrix. Then $L_{U^*AU} = L_{U^*}L_AL_U = L_U^*L_AL_U$ and L_U is a unitary operator. Hence

$$\|L_{U^*AU}\| = \|L_U^*L_AL_U\| \leq \|L_U^*\|\,\|L_A\|\,\|L_U\| = \|L_A\|$$

as unitary operators have norm 1. Moreover, since $L_{U^*AU} = L_U^*L_AL_U$ implies

$$L_A = L_UL_{U^*AU}L_{U^*}$$

as $(L_U^*)^{-1} = L_U$ and $(L_U)^{-1} = L_{U^*}$, we also have that

$$\|L_A\| = \|L_UL_{U^*AU}L_U^*\| \leq \|L_U\|\,\|L_{U^*AU}\|\,\|L_U^*\| = \|L_A\|$$

Hence $\|L_{U^*AU}\| = \|L_A\|$ as desired.

Since A^*A is a self-adjoint matrix and positive semi-definite, the Spectral Theorem for Self-Adjoint Matrices implies there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ and a diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ such that $A^*A = U^*DU$ and $\lambda_1, \ldots, \lambda_n \in [0, \infty)$ are the eigenvalues of A^*A. Hence we have that

$$\begin{aligned}\|L_A\| &= \|L_A^*L_A\|^{\frac{1}{2}} \\ &= \|L_{A^*A}\|^{\frac{1}{2}} \\ &= \|L_{U^*DU}\|^{\frac{1}{2}} \\ &= \|L_D\|^{\frac{1}{2}} \\ &= \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}^{\frac{1}{2}} \\ &= \max\left\{\sqrt{\lambda} \mid \lambda \text{ an eigenvalue for } A^*A\right\}\end{aligned}$$

as desired. ∎

Appendix A

Basic Set Theory

All mathematics must contain some notation in order for one to adequately describe the objects of study. As such, we begin the notation for the most basic structures in this course.

A.1 Sets

One of the most natural mathematical objects is the following:

Heuristic Definition. A *set* is a collection of distinct objects.

The following table list several sets, the symbol used to represent the set, and a set notational way to describe the set.

Set	**Symbol**	**Set Notation**
natural numbers	$\mathbb{N}$	$\{1, 2, 3, 4, \ldots\}$
integers	$\mathbb{Z}$	$\{0, 1, -1, 2, -2, 3, -3, \ldots\}$
rational numbers	$\mathbb{Q}$	$\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
real numbers	$\mathbb{R}$	$\{$real numbers$\}$
complex numbers	$\mathbb{C}$	$\{a + bi \mid a, b \in \mathbb{R}\}$

Notice two different types of notation are used in the above table to describe sets: namely $\{$objects$\}$ and $\{$objects $\mid$ conditions on the objects$\}$. Furthermore, the symbol $\emptyset$ will denote the *empty set*; that is, the set with no elements.

Given a set X and an object x, we say that x is an *element* of X, denoted $x \in X$, when x is one of the objects that make up X. Furthermore, we will use $x \notin X$ when x is not an element of X. For example, $\sqrt{2} \in \mathbb{R}$ yet $\sqrt{2} \notin \mathbb{Q}$ and $0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$. Furthermore, given two sets X and Y, we say that Y is a *subset* of X, denoted $Y \subseteq X$, if each element of Y is an element of X; that is, if $a \in Y$ then $a \in X$. For example $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. Furthermore, note the empty set is a subset of all sets, and if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$.

One important question that has not been addressed is, "What exactly is a set?" This questions must be asked as we have not provided a rigorous definition of a set. This leads to some interesting questions, such as, "Does the collection of all sets form a set?"

Let us suppose that there is a set of all sets; that is

$$Z = \{X \mid X \text{ is a set}\}$$

makes sense. Note Z has the interesting property that $Z \in Z$. Furthermore, if Z exists, then

$$Y = \{X \mid X \text{ is a set and } X \notin X\}$$

would be a valid subset of Z. However, we clearly have two disjoint cases: either $Y \in Y$ or $Y \notin Y$ (that is, either Y is an element of Y or Y is not an element of Y).

If $Y \in Y$, then the definition of Y implies $Y \notin Y$ which is a contradiction since we cannot have both $Y \in Y$ and $Y \notin Y$. Thus, if $Y \in Y$ is false, then it must be the case that $Y \notin Y$.

However, $Y \notin Y$ implies by the definition of Y that $Y \in Y$. Again this is a contradiction since we cannot have both $Y \notin Y$ and $Y \in Y$. This argument is known as Russell's Paradox and demonstrates that there cannot be a set of all sets.

The above paradox illustrates the necessity of a rigorous definition of a set. However, said definition takes us in a different direction than desired in this course. That being said, a rigorous definition of a set would provide us with the ability to take subsets of a given set and would permit the following operations on sets.

Definition A.1.1. Let X be a set. The *power set of* X, denote $\mathcal{P}(X)$, is

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

Note $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$.

Definition A.1.2. Let $\{X_\alpha\}_{\alpha\in I}$ denote a collection of subsets of a set X indexed by a set I.

- The *union* of $\{X_\alpha\}_{\alpha\in I}$, denoted $\bigcup_{\alpha\in I} X_\alpha$, is the set

$$\bigcup_{\alpha\in I} X_\alpha = \{a \mid a \in X_\alpha \text{ for at least one } \alpha \in I\}.$$

- The *intersection* of $\{X_\alpha\}_{\alpha\in I}$, denoted $\bigcap_{\alpha\in I} X_\alpha$, is the set

$$\bigcap_{\alpha\in I} X_\alpha = \{a \mid a \in X_\alpha \text{ for all } \alpha \in I\}.$$

Definition A.1.3. Given two sets X and Y, the *set difference* of X and Y, denoted $X \setminus Y$, is the set

$$X \setminus Y = \{a \mid a \in X \text{ and } a \notin Y\}.$$

In this course, we will often have a set X and will be considering subsets of X. Consequently, given a subset Y of X, the set difference $X \setminus Y$ will be called the *complement* of Y (in X) and will be denoted Y^c for convenience.

To conclude this section, we note the following set inequalities that will be used surprisingly often in this course.

Theorem A.1.4 (De Morgan's Laws). *Let X and I be non-empty sets and for each $\alpha \in I$ let X_α be a subset of X. Then*

$$X \setminus \left(\bigcup_{\alpha \in I} X_\alpha\right) = \bigcap_{\alpha \in I} (X \setminus X_\alpha) \qquad \text{and} \qquad X \setminus \left(\bigcap_{\alpha \in I} X_\alpha\right) = \bigcup_{\alpha \in I} (X \setminus X_\alpha).$$

Proof. Notice that

$$\begin{aligned} x \in \left(\bigcup_{i \in I} EXi\right)^c &\iff x \notin \bigcup_{i \in I} X_i \\ &\iff x \notin X_i \text{ for all } i \in I \\ &\iff x \in X_i^c \text{ for all } i \in I \\ &\iff x \in \bigcap_{i \in I} X_i^c \end{aligned}$$

which completes the proof since we have shown that $x \in (\bigcup_{i \in I} X_i)^c$ if and only if $x \in \bigcap_{i \in I} X_i^c$ (which implies the sets are the same).

We can play a similar game to prove that

$$\left(\bigcap_{i \in I} X_i\right)^c = \bigcup_{i \in I} X_i^c.$$

Alternatively, we can use the first result to prove the second. To do this, we must first show that if $E \subseteq X$ and $F = E^c$, then $F^c = E$. Indeed notice $x \in F^c$ if and only if $x \notin F$ if and only if $x \notin E^c$ if and only if $x \in E$. Hence $F^c = E$.

To prove this new equality using the old, for each $i \in I$ let $F_i = X_i^c$. By applying the first equation using the F_i's instead of the X_i's, we obtain that

$$\left(\bigcup_{i \in I} F_i\right)^c = \bigcap_{i \in I} F_i^c.$$

Since $F_i = X_i^c$ so $F_i^c = X_i$ by the above proof, we have that

$$\left(\bigcup_{i \in I} X_i^c\right)^c = \bigcap_{i \in I} X_i.$$

Hence

$$\bigcup_{i\in I} X_i^c = \left(\bigcap_{i\in I} X_i\right)^c$$

by taking the complement of both sides and using the proof in the above paragraph. ∎

A.2 Functions

In any analysis course, functions will play a fundamental role. The most useful and accurate method for defining functions is to use the following operation on sets (which is also valid by the actual definition of what a set is).

Definition A.2.1. Given two non-empty sets X and Y, the *Cartesian product* of X and Y, denoted $X \times Y$, is the set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Definition A.2.2. Given two non-empty sets X and Y, a *function* f from X to Y, denoted $f : X \to Y$, is a subset S of $X \times Y$ such that for each $x \in X$ there is an unique element denoted $f(x) \in Y$ such that $(x, f(x)) \in S$ (that is, a function is defined by its graph).

Example A.2.3. Given two non-empty sets X and Y, there is a natural way to view

$$X \times Y = \{f : \{1, 2\} \to X \cup Y \mid f(1) \in X, f(2) \in Y\}.$$

Indeed, a function $f : \{1, 2\} \to X \cup Y$ is uniquely determined by the values $f(1)$ and $f(2)$. Consequently, an $f : \{1, 2\} \to X \cup Y$ as defined in the above set can be viewed as the pair $(f(1), f(2))$. Conversely a pair $(x, y) \in X \times Y$ can be represented by the function $f : \{1, 2\} \to X \cup Y$ defined by $f(1) = x$ and $f(2) = y$.

The above example can be extended from a pair of sets to a finite number of sets. Let $X_1, \ldots, X_n$ be non-empty sets. We define the *product* of these sets to be

$$X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_j \in X_j \text{ for all } j \in \{1, \ldots, n\}\}.$$

If $X = X_1 = \cdots = X_n$, we will write X^n for $X_1 \times \cdots \times X_n$.

Notice we can view $X_1 \times \cdots \times X_n$ as a set of functions in a similar manner to Example A.2.3. Indeed

$$X_1 \times \cdots \times X_n = \left\{ f : \{1, \ldots, n\} \to \bigcup_{k=1}^{n} X_k \,\middle|\, f(j) \in X_j \ \forall\, j \in \{1, \ldots, n\} \right\}.$$

But what happens if we want to take a product of an infinite number of sets?

Given a non-empty set I and a collection of non-empty sets $\{X_\alpha\}_{\alpha\in I}$, we define the product

$$\prod_{\alpha\in I} X_\alpha = \left\{ f : I \to \bigcup_{\alpha\in I} X_\alpha \,\middle|\, f(i) \in X_i \text{ for all } i \in I \right\}.$$

However, we must ask, "Is the above set non-empty?" That is, how do we know there is always such a function? The answer is, because we add an axiom to make it so.

Axiom A.2.4 (The Axiom of Choice). *Given a non-empty set I and a collection of non-empty sets $\{X_\alpha \mid \alpha \in I\}$, the product $\prod_{\alpha\in I} X_\alpha$ is non-empty. Any function $f \in \prod_{\alpha\in I} X_\alpha$ is called a choice function.*

One may ask, "Why Mr. Anderson? Why? Why do we include the Axiom of Choice?" The short answer is, of course, "Because I choose to."

It turns out that the Axiom of Choice is independent from the axioms of (Zermelo–Fraenkel) set theory. This means that if one starts with the standard axioms of set theory, one can neither prove nor disprove the Axiom of Choice. Thus we have the option on whether to include or exclude the Axiom of Choice from our theory. We will allow the use of the Axiom of Choice (and almost surely you have made use of it in a previous analysis course and didn't even know it!).

A.3 Bijections

As we will be using functions throughout the remainder of the course, we will need some notation and definitions.

Given a function $f : X \to Y$ and $A \subseteq X$, we define

$$f(A) = \{f(x) \mid x \in A\} \subseteq Y.$$

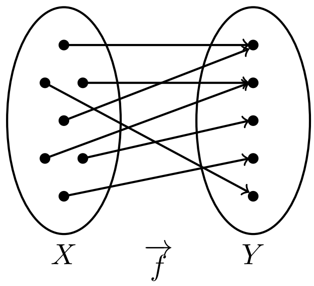
Definition A.3.1. Given a function $f : X \to Y$, the *range* of f is $f(X)$.

Using the notion of the range, we can define an important property we may desire our functions to have.

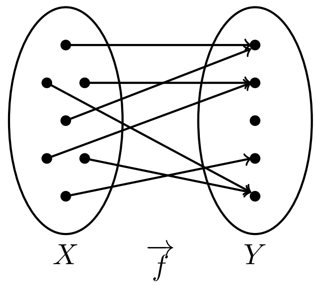
Definition A.3.2. A function $f : X \to Y$ is said to be *surjective* (or *onto*) if $f(X) = Y$; that is, for each $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

To illustrate when a function is surjective or not, consider the following diagrams.

f is surjective



f is not surjective



Example A.3.3. Consider the function $f : [0, 1] \to [0, 2]$ defined by $f(x) = x^2$. Notice f is not surjective since $f(x) \neq 2$ for all $x \in [0, 1]$. However, the function $g : [0, 1] \to [0, 1]$ defined by $g(x) = x^2$ is surjective. Consequently, the target set (known as the *co-domain*) matters.

One useful tool when dealing with functions is to be able to describe all points in the initial space that map into a predetermined set. Thus we make the following definition.

Definition A.3.4. Given a function $f : X \to Y$ and a $B \subseteq Y$, the *preimage* of B under f is the set

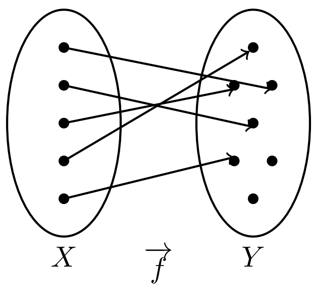
$$f^{-1}(B) = \{x \in X \mid f(x) \in B\} \subseteq X.$$

Note the notation used for the preimage does not assume the existence of an inverse of f (see Theorem A.3.8). Using preimages, we can define an important property we may desire our functions to have.

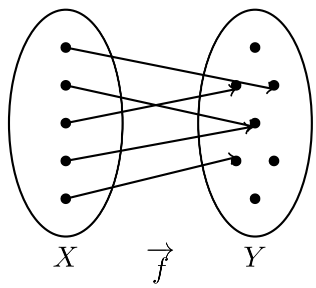
Definition A.3.5. A function $f : X \to Y$ is said to be *injective* (or *one-to-one*) if for all $y \in Y$, the preimage $f^{-1}(\{y\})$ has at most one element; that is, if $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$, then $x_1 = x_2$.

To illustrate when a function is injective or not, consider the following diagrams.

f is injective



f is not injective



Example A.3.6. Consider the function $f : [-1, 1] \to [0, 1]$ defined by $f(x) = x^2$. Notice f is not injective since $f(-1) = f(1)$. However, the function $g : [0, 1] \to [0, 1]$ defined by $g(x) = x^2$ is injective. Consequently, the initial set (known as the *domain*) matters.

We desire to combine the notions of injective and surjective.

Definition A.3.7. A function $f : X \to Y$ is said to be a *bijection* if f is injective and surjective.

Using the above examples, we have seen several functions that are not bijective. Furthermore, we have seen that $f : [0, 1] \to [0, 1]$ defined by $f(x) = x^2$ is bijective. One way to observe that f is injective is to consider the function $g : [0, 1] \to [0, 1]$ defined by $g(x) = \sqrt{x}$. Notice that f and g 'undo' what the other function does. In fact, this is true of all bijections.

Theorem A.3.8. *A function $f : X \to Y$ is a bijection if and only if there exists a function $g : Y \to X$ such that*

- *$g(f(x)) = x$ for all $x \in X$, and*
- *$f(g(y)) = y$ for all $y \in Y$.*

Furthermore, if f is a bijection, there is exactly one function $g : Y \to X$ that satisfies these properties, which is called the inverse of f and is denoted by $f^{-1} : Y \to X$. Notice this implies f^{-1} is also a bijection with $(f^{-1})^{-1} = f$.

Proof. Suppose that f is a bijection. Since f is surjective, for each $y \in Y$ there exists an $z_y \in X$ such that $f(z_y) = y$. Furthermore, note z_y is the unique element of X that f maps to y since f is injective.

Define $g : Y \to X$ by $g(y) = z_y$. Clearly g is a well-defined function.

To see that g satisfies the two properties, first let $x \in X$ be arbitrary. Then $y = f(x) \in Y$. However, since $f(z_y) = y = f(x)$, it must be the case that $z_y = x$ as f is injective. Therefore

$$g(f(x)) = g(y) = z_y = x$$

as desired. For the second property, let $y \in Y$ be arbitrary. Then

$$f(g(y)) = f(z_y) = y$$

by the definition of z_y. Hence g satisfies the desired properties.

Conversely, suppose $g : Y \to X$ satisfies the two properties. To see that f is injective, suppose $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$. Then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2$$

as desired. To see that f is surjective, let $y \in Y$ be arbitrary. Then $g(y) \in X$ so

$$y = f(g(y)) \in f(X).$$

Since $y \in Y$ is arbitrary, we have $Y \subseteq f(X)$. Hence $f(X) = Y$ so f is surjective. Therefore, as f is both injective and surjective, f is bijective by definition.

Finally, suppose f is bijective and $g : Y \to X$ satisfies the above properties. Suppose $h : Y \to X$ is another function such that $h(f(x)) = x$ for all $x \in X$, and $f(h(y)) = y$ for all $y \in Y$. Then for all $y \in Y$,

$$h(y) = g(f(h(y))) = g(y)$$

(where we have used $g(f(x_1)) = x_1$ when $x_1 = h(y)$ and $f(h(y)) = y$). Therefore $g = h$ as desired. ∎

Remark A.3.9. If $f : X \to Y$ is injective, consider the function $g : X \to f(X)$ defined by $g(x) = f(x)$ for all $x \in X$. Clearly g is injective since f is, and, by construction, g is surjective. Hence g is bijective and thus has an inverse $g^{-1} : f(X) \to X$. The function g^{-1} is called the *inverse of f on its image*.

A.4 Equivalence Relations

Using the same idea as we used for defining functions (i.e. as subsets of a Cartesian product), we can define another useful notion in mathematics.

Definition A.4.1. Given two non-empty sets X and Y, a *relation* is a subset of the product $X \times Y$. Given a relation R, we write xRy if $(x, y) \in R$.

Given a non-empty set X, by a relation on X we will mean a relation on $X \times X$.

Using a specific type of relation, we can generalize the notion of equality.

Definition A.4.2. Let X be a set. A relation $\sim$ on the elements of X is said to be an *equivalence relation* if:

(1) (reflexive) $x \sim x$ for all $x \in X$,

(2) (symmetric) if $x \sim y$, then $y \sim x$ for all $x, y \in X$, and

(3) (transitive) if $x \sim y$ and $y \sim z$, then $x \sim z$ for all $x, y, z \in X$.

Given an $x \in X$, the set $\{y \in X \mid y \sim x\}$ is called the *equivalence class* of x and is denoted $[x]$.

Notice that $[x] \cap [y] \neq \emptyset$ if and only if $x \sim y$. Thus by taking an index set consisting of one element from each equivalence class, the set X can be written as the disjoint union of its equivalence classes.

Example A.4.3. Let V be a vector space and let W be a subspace of V. It is elementary to check that if we define $\vec{x} \sim \vec{y}$ if and only if $\vec{x} - \vec{y} \in W$, then $\sim$ is an equivalence relation on V. Note that the equivalence classes of V then become a vector space, denoted V/W, with the operations $[\vec{x}] + [\vec{y}] = [\vec{x} + \vec{y}]$ and $\alpha[\vec{x}] = [\alpha\vec{x}]$. Note the necessity of checking that these operations are well-defined; that is, for addition to make sense, one must show that if $\vec{x}_1 \sim \vec{x}_2$ and $\vec{y}_1 \sim \vec{y}_2$ then $\vec{x}_1 + \vec{y}_1 \sim \vec{x}_2 + \vec{y}_2$.

A.5 Zorn's Lemma

In this section, we will briefly review Zorn's Lemma. We begin with the basics.

Definition A.5.1. Let X be a set. A relation $\preceq$ on the elements of X is called a *partial ordering* if:

(1) (reflexivity) $a \preceq a$ for all $a \in X$,

(2) (antisymmetry) if $a \preceq b$ and $b \preceq a$, then $a = b$ for all $a, b \in X$, and

(3) (transitivity) if $a, b, c \in X$ are such that $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Clearly $\leq$ is a partial ordering on $\mathbb{R}$. Here is another example:

Example A.5.2. Given a set X, the relation $\preceq$ on $\mathcal{P}(X)$ defined by

$$Z \preceq Y \qquad \text{if and only if} \qquad Z \subseteq Y$$

is a partial ordering on $\mathcal{P}(X)$.

The partial ordering in the previous example is not as nice as our ordering on $\mathbb{R}$. To see this, consider the sets $Z = \{1\}$ and $Y = \{2\}$. Then $Z \npreceq Y$ and $Y \npreceq Z$; that is, we cannot use the partial ordering to compare X and Z. However, if $x, y \in \mathbb{R}$, then either $x \leq y$ or $y \leq x$. Consequently, a partial ordering is nicer if it has the following property:

Definition A.5.3. Let X be a set. A partial ordering $\preceq$ on X is called a *total ordering* if for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$ (or both).

Of course, we desire to equip a set with a partial ordering. Thus we give the following name to such an object.

Definition A.5.4. A *partially ordered set* (or *poset*) is a pair $(X, \preceq)$ where X is a non-empty set and $\preceq$ is a partial ordering on X.

Our main focus is a 'result' about totally ordered subsets of partially ordered sets:

Definition A.5.5. Let $(X, \preceq)$ be a partially ordered set. A non-empty subset $Y \subseteq X$ is said to be a *chain* if Y is totally ordered with respect to $\preceq$; that is, if $a, b \in Y$, then either $a \preceq b$ or $b \preceq a$.

Clearly any non-empty subset of a totally ordered set is a chain. Here is a less obvious example.

Example A.5.6. Recall that the power set $\mathcal{P}(\mathbb{R})$ of $\mathbb{R}$ has a partial ordering $\preceq$ where

$$A \preceq B \quad \Longleftrightarrow \quad A \subseteq B.$$

If $Y = \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathbb{R})$ are such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then Y is a chain.

Like with the real numbers, upper bounds play an important role with respect to chains.

Definition A.5.7. Let $(X, \preceq)$ be a partially ordered set. A non-empty subset $Y \subseteq X$ is said to be a *bounded above* if there exists a $z \in X$ such that $y \leq z$ for all $y \in Y$. Such an element z is said to be an *upper bound* for Y.

Example A.5.8. Recall from Example A.5.6 that if $Y = \{A_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathbb{R})$ are such that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$, then Y is a chain with respect to the partial ordering defined by inclusion. If

$$A = \bigcup_{n=1}^{\infty} A_n$$

then clearly $A \in \mathcal{P}(\mathbb{R})$ and $A_n \subseteq A$ for all $n \in \mathbb{N}$. Hence A is an upper bound for Y.

Recall there are optimal upper bounds of subsets of $\mathbb{R}$ called least upper bounds which need not be in the subset. We desire a slightly different object when it comes to partially ordered sets as the lack of a total ordering means there may not be a unique 'optimal' upper bound.

Definition A.5.9. Let X be a non-empty set and let $\preceq$ be a partial ordering on X. An element $x \in X$ is said to be *maximal* if there does not exist a $y \in X \setminus \{x\}$ such that $x \preceq y$; that is, there is no element of X that is larger than x with respect to $\preceq$.

Notice that $\mathbb{R}$ together with its usual ordering $\leq$ does not have a maximal element (by, for example, the Archimedean Property). However, many partially ordered sets do have maximal elements. For example $([0, 1], \leq)$ has 1 as a maximal element (although $((0, 1), \leq)$ does not).

For an example involving a partial ordering that is not a total ordering, suppose $X = \{x, y, z, w\}$ and $\preceq$ is defined such that $a \preceq a$ for all $a \in X$, $a \preceq b$ for all $a \in \{x, y\}$ and $b \in \{z, w\}$, and $a \not\preceq b$ for all other pairs $(a, b) \in X \times X$.

It is not difficult to see that z and w are maximal elements and x and y are not maximal elements. Thus it is possible, when dealing with a partial ordering that is not a total ordering, to have multiple maximal elements.

The result we require for the next subsection may now be stated using the above notions.

Axiom A.5.10 (Zorn's Lemma). *Let $(X, \preceq)$ be a non-empty partially ordered set. If every chain in X has an upper bound, then X has a maximal element.*

We will not prove Zorn's Lemma. To do so, we would need to use the Axiom of Choice. In fact, Zorn's Lemma and the Axiom of Choice are logically equivalent; that is, assuming the axioms of (Zermelo-Fraenkel) set theory, one may use the Axiom of Choice to prove Zorn's Lemma, and one may use Zorn's Lemma to prove the Axiom of Choice.

As a simple example of the use of Zorn's Lemma, we present the following.

Example A.5.11. Let V be a (non-zero) vector space. We claim that V has a basis; that is, a linearly independent spanning set. To see this, let $\mathcal{L}$ denote the collection of all linearly independent subsets of V (which is clearly non-empty) and define a partial ordering on $\mathcal{L}$ by $A \preceq B$ if and only if $A \subseteq B$ (clearly this is a partial ordering on $\mathcal{L}$).

To invoke Zorn's Lemma (Axiom A.5.10), we need to demonstrate that every chain in $\mathcal{L}$ has an upper bound. Let $\{A_\alpha\}_{\alpha \in I}$ be an arbitrary chain in $\mathcal{L}$ and let

$$A = \bigcup_{\alpha \in I} A_\alpha.$$

We claim that $A \in \mathcal{L}$. To see this, suppose $\vec{v}_1, \ldots, \vec{v}_n \in A$ and $a_1\vec{v}_1 + \cdots a_n\vec{v}_n = 0$ for some scalars a_k. By the definition of A and the fact that $\{A_\alpha\}_{\alpha \in I}$ is a chain, there exists an $i \in I$ such that $\vec{v}_1, \ldots, \vec{v}_n \in A_i$ (that is, each $\vec{v}_k$ is in some A_α and as the A_α are totally ordered, take the largest). Hence, as A_i is a linearly independent set, $a_1\vec{v}_1 + \cdots a_n\vec{v}_n = 0$ implies $a_1 = \cdots = a_n = 0$. Hence $A \in \mathcal{L}$. As A is clearly an upper bound for $\{A_\alpha\}_{\alpha \in I}$, ever chain in $\mathcal{L}$ has an upper bound.

By Zorn's Lemma there exists a maximal element $B \in \mathcal{L}$. We claim that B is a basis for V. To see this, suppose to the contrary that $\mathrm{span}(B) \neq V$. Thus there exists a non-zero vector $\vec{v} \in V \setminus \mathrm{span}(B)$. This implies that $B \cup \{\vec{v}\}$ is linearly independent. However, as $B \preceq B \cup \{\vec{v}\}$ and $B \neq B \cup \{\vec{v}\}$, we have a contradiction to the fact that B is a maximal element in $\mathcal{L}$. Hence it must have been the case that $\mathrm{span}(B) = V$ and thus B is a basis for V.

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