MATH 2001
Real Analysis I

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Preface:

These are the first edition of these lecture notes for MATH 2001 (Real Analysis I). Consequently, there may be several typographical errors, missing exposition on necessary background, or unclear explanations. If you come across any typos, errors, omissions, or unclear explanations, please feel free to contact me so that I may continually improve these notes.

Please note that any text in the colour blue was not covered in class. Some of this text was not covered in class as its difficult level is not suitable for this course (most of this is in Chapter 1). Some of this text was not covered in class as it was suitable for assignment work and appears there. The remainder of the text I would love to cover if there was sufficient time, but there will not be. Thus I have selected the most essential material and examples to present in class. Students taking this course are only responsible for the material covered in class and on the assignments, and a student can skip any text in blue and still obtain a deep introduction to real analysis. However, understanding this additional material will deepen a student's comprehension and abilities in real analysis thereby enhancing their success in this and future courses.

Contents

Motivation for this Course

Calculus as we know it was developed in the 17th century to tackle the prominent physics problems at the time. As the study of kinematics (the study of motion) was paramount to describe the motion of celestial bodies, the theory to compute rates of change (i.e. derivatives) and areas (i.e. integrals) was developed. However, eventually scientists encountered issues in their theory which yielded bizarre contradictions in specific cases.

For one such example, consider the problem of adding up the following infinite number of numbers:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots .$$

It is not difficult to use elementary techniques to show this infinite sum makes sense. But what is its value? Let S denote the value of this infinite sum. On the one hand

$$S = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \cdots .$$

Thus S is a sum of strictly positive numbers so we expect that $S > 0$. However, on the other hand

$$\begin{aligned} S &= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \cdots \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) \\ &= \frac{1}{2}S. \end{aligned}$$

This latter equation implies $S = 0$ in direct contradiction to our initial claim that $S > 0$. So what went wrong?

The root of our error comes from the question, "What do we mean by adding up an infinite number of elements?" As we have not made a precise definition of "adding up an infinite number of elements", we cannot find our mathematical mistake. This is where analysis and this course begin.

This course will serve as an introduction to analysis for mathematics majors. In particular, we will rebuild calculus up from base axioms through logic and rigorous proofs. Consequently the apparent contradictions from calculus are resolved and one has a rich theory that can go beyond studying kinematics. In particular, the notions of limits, topology, continuity, derivatives, and integrals will be developed and studied in a rigorous proof-based manner. More importantly, students will learn the proper way to think and prove facts about analysis, which is an essential skill for more advanced courses in analysis and mathematics.

Chapter 1

The Real Numbers

To develop a completely rigorous and logically based mathematical theory of analysis, the first question we must ask is, "What is real? How do you define 'real'? If you're talking about what you can feel, what you can smell, what you can taste and see, then 'real' is simply electrical signals interpreted by your brain." That is,

- How do we write down a rigorous definition of the real numbers?
- What properties do the real numbers have?
- What properties of the real numbers distinguish them from other number systems?

In this chapter, we will answer the above questions. As this is a second-year undergraduate course, the focus in this chapter will be to introduce and develop understanding of the defining properties of the real numbers over studying the advanced set theoretic and logical considerations of this construction.

To begin, it is best to first understand how one constructs a much more simple mathematical object.

1.1 The Natural Numbers

To motivate the complexity of providing a rigorous definition of the real numbers, we begin by showing even defining the natural numbers rigorously in mathematics is not a non-trivial task.

1.1.1 Peano's Axioms

Peano's Axioms are the five properties one must impose on a number system to uniquely identify the natural numbers. They are as follows:

Definition 1.1.1 (Peano's Axioms). The natural numbers, denoted $\mathbb{N}$, are the unique number system satisfying the following five axioms:

(P1) There is a number, denoted 1, such that $1 \in \mathbb{N}$.

(P2) For each number $n \in \mathbb{N}$, there is a number $S(n) \in \mathbb{N}$ called the *successor* of n (i.e. $S(n) = n + 1$).

(P3) The number 1 is not the successor of any number in $\mathbb{N}$.

(P4) If $m, n \in \mathbb{N}$ and $S(n) = S(m)$, then $n = m$.

(P5) (Induction Axiom) If $X \subseteq \mathbb{N}$ is such that

(a) $1 \in X$, and

(b) if $k \in \mathbb{N}$ and $k \in X$, then $S(k) \in X$,

then $X = \mathbb{N}$.

Each of the above five axioms are necessary to uniquely identify the natural numbers as the following examples show. Note (P2) is necessary to discuss (P4) so no example is provided where (P1), (P3), (P4), and (P5) hold but (P2) does not.

Example 1.1.2. The empty set $\emptyset$ does not satisfy (P1) but satisfies (P2), (P3), (P4), and (P5) vacuously.

Example 1.1.3. Consider the set $X = \{1, 2\}$ where we define $S(1) = 2$ and $S(2) = 1$. One may verify that X satisfies (P1), (P2), (P4), and (P5) but does not satisfy (P3) since 1 is the successor of 2.

Example 1.1.4. Consider the set $X = \{1, 2\}$ where we define $S(1) = 2$ and $S(2) = 2$. One may verify that X satisfies (P1), (P2), (P3), and (P5) but does not satisfy (P4) since $S(1) = S(2)$ but $1 \neq 2$.

Example 1.1.5. Consider the set $\mathbb{N}^2 = \{(n, m) \mid n, m \in \mathbb{Z}\}$ where we define $1 = (1, 1)$ and $S(n, m) = (n+1, m+1)$. One may verify that $\mathbb{N}^2$ satisfies (P1), (P2), (P3), and (P4) but does not satisfy (P5) since $X = \{(n, n) \mid n \in \mathbb{N}\}$ has properties (a) and (b) but is not all of $\mathbb{N}^2$.

Remark 1.1.6. It should be pointed out that Peano's Axioms immediately give us our notion of $<$ on $\mathbb{N}$. Indeed we define $n < m$ to mean that there exists a chain $n = k_0, k_1, \ldots, k_\ell = m$ so that $k_j = S(k_{j-1})$ for all j.

Instead of focusing on set theoretic and logical implications of the Peano's Axioms, our focus will be on what tools they provide for us.

1.1.2 The Principle of Mathematical Induction

The Induction Axiom from Definition 1.1.1 leads to the following principle which students should be familiar with from MATH 1200.

Theorem 1.1.7 (The Principle of Mathematical Induction). *For each $k \in \mathbb{N}$, let P_k be a mathematical statement. Suppose*

- *(base case) P_1 is true, and*
- *(inductive step) if $k \in \mathbb{N}$ and P_k is true, then P_{k+1} is true.*

Then P_n is true for all $n \in \mathbb{N}$.

Proof. Let
$$X = \{n \in \mathbb{N} \mid P_n \text{ is true}\}.$$
Our goal is to show that $X = \mathbb{N}$ using the Induction Axiom in Definition 1.1.1. Hence we must show two things:

(a) $1 \in X$, and

(b) if $k \in \mathbb{N}$ and $k \in X$, then $k + 1 \in X$.

By assumption we see that $1 \in X$ as P_1 is true. Thus (a) is true.

To see that (b) is true, suppose that $k \in \mathbb{N}$ and $k \in X$. By the definition of X, we know P_k is true. Thus, by the assumptions in the statement of the theorem, P_{k+1} is true and hence $k + 1 \in X$ by the definition of X. Therefore (b) is true.

Hence the Induction Axiom in Definition 1.1.1 implies $X = \mathbb{N}$. Hence P_n is true for all $n \in \mathbb{N}$. ∎

The Principle of Mathematical Induction is a very convenient method for proving a collection of mathematical statements indexed by the natural numbers are true. The following is one specific example (which is reliant on properties of the real numbers we will discuss shortly).

Theorem 1.1.8 (Binomial Theorem). *For all $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$,*
$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$
[Here $x^0 = 1 = y^0$ by definition.]

Proof. To see this result is true, fix $x, y \in \mathbb{R}$ and for each $n \in \mathbb{N}$ let P_n be the statement that $(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$. To show that P_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction. To do so, we must demonstrate the two conditions in Theorem 1.1.7.

Base Case: To see that P_1 is true, notice that when $n = 1$,

$$(x+y)^1 = x + y = \binom{1}{1}x^1y^0 + \binom{1}{0}x^0y^1.$$

Hence P_1 is true.

Inductive Step: Assume that P_k is true; that is, assume $(x+y)^k = \sum_{j=0}^{k}\binom{k}{j}x^jy^{k-j}$ (this assumption is known as the induction hypothesis). To see that P_{k+1} is true, first notice for all $\ell \in \{1, \ldots, k\}$ that

$$\begin{aligned}
\binom{k}{\ell-1} + \binom{k}{\ell} &= \frac{k!}{(\ell-1)!(k-(\ell-1))!} + \frac{k!}{\ell!(k-\ell)!} \\
&= \frac{k!}{(\ell-1)!(k-\ell)!}\left(\frac{1}{k+1-\ell} + \frac{1}{\ell}\right) \\
&= \frac{k!}{(\ell-1)!(k-\ell)!}\left(\frac{\ell + (k+1-\ell)}{(k+1-\ell)\ell}\right) \\
&= \frac{k!}{(\ell-1)!(k-\ell)!}\left(\frac{k+1}{(k+1-\ell)\ell}\right) \\
&= \frac{(k+1)!}{\ell!((k+1)-\ell)!} = \binom{k+1}{\ell}
\end{aligned}$$

and that $1 = \binom{k}{0} = \binom{k}{k} = \binom{k+1}{0} = \binom{k+1}{k+1}$. Therefore, we have that

$$\begin{aligned}
&(x+y)^{k+1} \\
&= (x+y)(x+y)^k \\
&= (x+y)\sum_{j=0}^{k}\binom{k}{j}x^jy^{k-j} \qquad \text{by the induction hypothesis} \\
&= \left(x\sum_{j=0}^{k}\binom{k}{j}x^jy^{k-j}\right) + \left(y\sum_{j=0}^{k}\binom{k}{j}x^jy^{k-j}\right) \\
&= \left(\binom{k}{k}x^{k+1} + \sum_{j=0}^{k-1}\binom{k}{j}x^{j+1}y^{k-j}\right) + \left(\binom{k}{0}y^{k+1} + \sum_{j=1}^{k}\binom{k}{j}x^jy^{k-j+1}\right) \\
&= \left(x^{k+1} + \sum_{\ell=1}^{k}\binom{k}{\ell-1}x^\ell y^{(k+1)-\ell}\right) + \left(y^{k+1} + \sum_{\ell=1}^{k}\binom{k}{\ell}x^\ell y^{(k+1)-\ell}\right) \\
&= \binom{k+1}{k+1}x^{k+1} + \left(\sum_{\ell=1}^{k}\left(\binom{k}{\ell-1} + \binom{k}{\ell}\right)x^\ell y^{(k+1)-\ell}\right) + \binom{k+1}{0}y^{k+1} \\
&= \binom{k+1}{k+1}x^{k+1}y^0 + \left(\sum_{\ell=1}^{k}\binom{k+1}{\ell}x^\ell y^{(k+1)-\ell}\right) + \binom{k+1}{0}x^0y^{k+1} \\
&= \sum_{\ell=0}^{k+1}\binom{k}{\ell}x^\ell y^{k-\ell}.
\end{aligned}$$

Hence P_{k+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the result follows by the Principle of Mathematical Induction. ■

In addition to the Principle of Mathematical Induction, students saw in MATH 1200 the following improvement which allows for one to assume all the previous statements are true in the inductive step.

Theorem 1.1.9 (The Principle of Strong Induction). *For each $k \in \mathbb{N}$, let P_k be a mathematical statement. Suppose*

1. *P_1 is true, and*
2. *if $k \in \mathbb{N}$ and P_m is true for all $m \leq k$, then P_{k+1} is true.*

Then P_n is true for all $n \in \mathbb{N}$.

Proof. For each $k \in \mathbb{N}$, let Q_k be the mathematical statement "$P_1, P_2, \ldots, P_k$ are all true". Thus, to show that P_n is true for all $n \in \mathbb{N}$, it suffices to show that Q_n is true for all $n \in \mathbb{N}$.

To show that Q_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction. To do so, we must demonstrate the two conditions in Theorem 1.1.7.

Base Case: To see that Q_1 is true, notice that P_1 is true by our assumptions. Hence Q_1 is true.

Inductive Step: Assume that Q_k is true; that is, assume $P_1, P_2, \ldots, P_k$ are all true. To see that Q_{k+1} is true, notice since $P_1, P_2, \ldots, P_k$ are all true, our assumptions imply that P_{k+1} is true. Hence $P_1, P_2, \ldots, P_k, P_{k+1}$ are all true so Q_{k+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the result follows by the Principle of Mathematical Induction. ■

1.1.3 The Well-Ordering Principle

There is another way to think about the Principle of Mathematical Induction which is quite useful in analysis. Instead of thinking about whether we can show P_n is true for all $n \in \mathbb{N}$, we can think about whether there is a first natural number where P_n fails. This leads to the following additional form of the Principle of Mathematical Induction.

Theorem 1.1.10 (The Well-Ordering Principle). *Every non-empty subset of $\mathbb{N}$ has a least element; that is, if $X \subseteq \mathbb{N}$ and $X \neq \emptyset$, then there is an element $m \in X$ such that $m \leq k$ for all $k \in X$.*

Proof. Suppose for the sake of a contradiction that X is a non-empty subset of $\mathbb{N}$ that does not have a least element. Let

$$Y = \mathbb{N} \setminus X = \{n \in \mathbb{N} \mid n \notin X\}.$$

For each $n \in \mathbb{N}$ let P_n be the statement that "$n \in Y$". We will apply the Principle of Strong Induction to show that P_n is true for all $n \in \mathbb{N}$ and thus $Y = \mathbb{N}$. This will complete the proof since $Y = \mathbb{N}$ implies $X = \emptyset$, which contradicts the fact that X is non-empty.

To apply Principle of Strong Induction, we must demonstrate the two necessary assumptions in Theorem 1.1.9.

Base Case: To see that P_1 is true, note since X does not have a least element that we know that $1 \notin X$ or else 1 would be the least element of X. Hence $1 \in Y$ so P_1 is true.

Inductive Step: Assume $k \in \mathbb{N}$ and P_m is true for all $m \leq k$. Hence $\{1, \ldots, k\} \subseteq Y$. Thus each element of $\{1, \ldots, k\}$ is not in X. Therefore $k+1 \notin X$ for otherwise $k+1$ would be the least element of X since none of $1, \ldots, k$ are in X. Hence $k+1 \in Y$ as $k+1 \notin X$ so P_{k+1} is true.

Hence, by Strong Induction, $Y = \mathbb{N}$ thereby completing the proof by earlier discussions. ∎

The Well-Ordering Principle is quite useful in proofs where one wants to know there is a first natural number where something fails. For example, we can prove the following results from MATH 1200 using the Well-Ordering Principle.

Proposition 1.1.11 (The Division Algorithm). *Let $a, b \in \mathbb{N}$. Then there exists unique $q, r \in \mathbb{N} \cup \{0\}$ such that $0 \leq r < b$ and $a = bq + r$.*

Proof. Let $a, b \in \mathbb{N}$ be fixed. First we will show the existence of $q, r \in \mathbb{N} \cup \{0\}$ such that $0 \leq r < b$ and $a = bq + r$. To show this, we divide the discussion into three cases.

Case 1: $a < b$. In this case, let $r = a$ and $q = 0$. Clearly $0 \leq r = a < b$ and $a = r = 0b + r = bq + r$ as desired.

Case 2: $a = b$. In this case, let $r = 0$ and $q = 1$. Clearly $0 \leq r < b$ and $a = b = 1b + 0 = bq + r$ as desired.

Case 3: $a > b$. In this case, let

$$X = \{n \in \mathbb{N} \mid a < nb\}.$$

Since $a \geq 1$ we know that $2a = a + a > a + 0 = a$. Moreover, since $b \geq 1$ we know that $ab \geq a$ and thus

$$(a+1)b = ab + a \geq a + a = 2a > a.$$

Hence $a + 1 \in X$ so $X \neq \emptyset$.

Since $X \neq \emptyset$, the Well Ordering Principle (Theorem 1.1.10) implies there exists a least element $c \in X$. Let

$$q = c - 1 \in \mathbb{N} \cup \{0\}.$$

Since $q \in \mathbb{N} \cup \{0\}$ and $q < c$, $q \notin X$ as c was the least element of X. Therefore $a \geq qb$. Hence if we let $r = a - qb$, then $r \geq 0$ and $a = qb + r$.

To complete the proof in this case, it remains only to show that $r < b$. To see this, suppose to the contrary that $r \geq b$. Hence $a - qb \geq b$ so that

$$a \geq qb + b = (q+1)b = cb.$$

This implies that $c \notin X$. However, this contradicts the fact that c was the least element of X (and thus, in particular, was an element of X). As we have obtained a contradiction it must be the case that $r < b$. Thus this case holds.

As the above three cases cover all possible cases for $a, b \in \mathbb{N}$, it follows that there exist $q, r \in \mathbb{N} \cup \{0\}$ such that $0 \leq r < b$ and $a = bq + r$.

To see that $q, r \in \mathbb{N} \cup \{0\}$ are unique, assume $q', r' \in \mathbb{N} \cup \{0\}$ are such that $0 \leq r' < b$ and $a = bq' + r'$. To see that $q' = q$ and $r' = r$, note that

$$bq + r = a = bq' + r'$$

so that

$$b(q - q') = r' - r.$$

Suppose for the sake of a contradiction that $q \neq q'$. Hence $q - q' \leq -1$ or $q - q' \geq 1$. Therefore, as $b \geq 1$, we have that

$$r' - r = b(q - q') \leq -b \quad \text{or} \quad r' - r = b(q - q') \geq b.$$

However, since $0 \leq r < b$ and $0 \leq r' < b$, we must have that

$$-b < r' - r < b.$$

As this is a contradiction, we have that $q = q'$. Hence $r' - r = b(q - q') = b(0) = 0$ so $r' = r$ as desired. ■

To complete our discussion of the natural numbers, we should note we used the Induction Axiom from Definition 1.1.1 to prove the Principle of Mathematical Induction (Theorem 1.1.7), which we then used to prove the Principle of Strong Induction (Theorem 1.1.9), which we then used to prove the Well-Ordering Principle (Theorem 1.1.10). It turns out that if one replaces the Induction Axiom in Definition 1.1.1 with the Well-Ordering Principle, then one can deduce the Induction Axiom. That is, the Induction Axiom and the Well-Ordering Principle are logically equivalent! The proof of this fact is below.

Theorem 1.1.12. *Let (P1), (P2), (P3), (P4), and (P5) be as in Definition 1.1.1. Consider the mathematical statement.*

(P′5) (Well-Ordering Principle) If $X \subseteq \mathbb{N}$ and $X \neq \emptyset$, then X has a least element.

If (P1), (P2), (P3), (P4), and (P′5) are true, then (P5) is true.

Proof. Assume (P1), (P2), (P3), (P4), and (P′5) are true. Before showing that (P5) is true, we claim that if $n \in \mathbb{N} \setminus \{1\}$, then $n = S(k)$ for some $k \in \mathbb{N}$. To see this, let

$$Y = \{n \in \mathbb{N} \mid n \neq S(k) \text{ for all } k \in \mathbb{N}\}.$$

Note $1 \in Y$ by (P3) so $Y \neq \emptyset$. Hence Y has a least element. This means that 1 is the least element of Y. Hence the definition of $<$ implies that $Y = \{1\}$ as desired.

To see that (P5) is true, let $X \subseteq \mathbb{N}$ be such that

(a) $1 \in X$, and

(b) if $k \in \mathbb{N}$ and $k \in X$, then $S(k) \in X$.

Our goal is to show that $X = \mathbb{N}$.

To see this, let $Y = \mathbb{N} \setminus X$. Thus, to show that $X = \mathbb{N}$, it suffices to show that $Y = \emptyset$.

Suppose to the contrary that $Y \neq \emptyset$. Hence (P′5) implies that Y has a least element. Let m be the least element of Y. Note since $1 \in X$, $1 \notin Y$ and thus $m \neq 1$. Since $m \neq 1$, there exists a $k \in \mathbb{N}$ such that $S(k) = m$. Since m is the least element of Y, $k \notin Y$ so $k \in X$. However (b) implies then implies that $m = S(k) \in X$ so $m \notin Y$. As this contradicts the fact that m is the least element of Y, we have a contradiction. Hence $Y = \emptyset$ thereby completing the proof. ■

1.2 What are the Real Numbers?

With a rigorous formal definition of the natural numbers now complete, we turn our attention to more interesting pursuits. Using the natural numbers and equivalence relations, it is possible to construct the integers (denoted $\mathbb{Z}$) and the rational numbers (denoted $\mathbb{Q}$). For the interested reader, this is done in Appendices B.1 and B.2. However, our interest lies in the real numbers (denoted $\mathbb{R}$). In particular, how does one construct the real numbers, what properties do the real numbers have, and what properties do we need to enforce on a number system to guarantee it is the real numbers? That is, how do we rigorously define the real numbers? Of course, since defining $\mathbb{N}$ was not a simple task, we expect defining $\mathbb{R}$ to be very non-trivial.

1.2.1 Fields

To begin our discussion of what are the real numbers, we note there are some natural operations we want to apply to the real numbers: namely addition, subtraction, multiplication, and division. These operations have specific properties that we shall explore and formalize.

We begin with addition and multiplication. Recall that addition and multiplication are operations on pairs of real numbers; that is, for every $x, y \in \mathbb{R}$ there are numbers, denoted $x + y$ and $x \cdot y$, which are elements of $\mathbb{R}$. Furthermore, there are two properties we require for addition and multiplication to behave well, and one property that says addition and multiplication play together nicely:

(F1) (Commutativity) $x + y = y + x$ and $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$.

(F2) (Associativity) $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$.

(F3) (Distributivity) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ for all $x, y, z \in \mathbb{R}$.

Instead of adding in the operations of subtraction and division, we can realize how they may be derived from addition and multiplication. For example, what does subtracting 3 from 4 mean in terms of addition? Well, it really means add the number -3 to 4. And how are 3 and -3 related? Well, -3 is the unique number x such that $3 + x = 0$. And what is 0 in terms of addition? Well, 0 is the unique number y that when you add y to any number z, you end up with z.

Similarly, what does dividing by 7 mean in terms of multiplication? Well, it really means multiply by $\frac{1}{7}$. And how are 7 and $\frac{1}{7}$ related? Well, $\frac{1}{7}$ is the unique number x such that $7x = 1$. And what is 1 in terms of multiplication? Well, 1 is the unique number y that when you multiply y to any number z, you end up with z.

Using the above, we added the following properties to our list of properties defining $\mathbb{R}$:

(F4) (Existence of Identities) There are numbers $0, 1 \in \mathbb{R}$ with $0 \neq 1$ such that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.

(F5) (Existence of Inverses) For all $x, y \in \mathbb{R}$ with $y \neq 0$, there exists $-x, y^{-1} \in \mathbb{R}$ such that $x + (-x) = 0$ and $y \cdot y^{-1} = 1$.

Using these two properties, one then defines subtraction and division via $x - y = x + (-y)$ and $x \div z = x \cdot z^{-1}$ for all $x, y, z \in \mathbb{R}$ with $z \neq 0$. Furthermore, it is possible to show that all of the numbers listed in (F4) and (F5) are unique; that is, any number with the same properties as one of 0, 1, $-x$, or y^{-1} must be the corresponding number. The following lemma elaborates and shows that numbers behave in an arbitrary field as one would expect.

Lemma 1.2.1. *Assuming (F1), (F2), and (F3), the following hold:*

a) If $0_1, 0_2 \in \mathbb{R}$ are such that $0_1 + x = x$ and $0_2 + x = x$ for all $x \in \mathbb{R}$, then $0_1 = 0_2$.

b) If $1_1, 1_2 \in \mathbb{R}$ are such that $1_1 \cdot x = x$ and $1_2 \cdot x = x$ for all $x \in \mathbb{R}$, then $1_1 = 1_2$.

c) If 0 is as in (F4), if $x \in \mathbb{R}$, and if $y_1, y_2 \in \mathbb{R}$ are such that $x + y_1 = 0$ and $x + y_2 = 0$, then $y_1 = y_2$ (i.e. $y_1 = y_2 = -x$).

d) If 1 is as in (F4), if $x \in \mathbb{R} \setminus \{0\}$, and if $y_1, y_2 \in \mathbb{R}$ are such that $x \cdot y_1 = 1$ and $x \cdot y_2 = 1$, then $y_1 = y_2$ (i.e. $y_1 = y_2 = x^{-1}$).

e) If $x \in \mathbb{R}$, then $0 \cdot x = 0$.

f) If $x \in \mathbb{R}$, then $-x = (-1) \cdot x$.

Proof. a) Let $0_1, 0_2 \in \mathbb{R}$ be such that $0_1 + x = x$ and $0_2 + x = x$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} 0_1 &= 0_2 + 0_1 && \text{i.e. take } x = 0_1 \text{ in } 0_2 + x = x \\ &= 0_1 + 0_2 && \text{by commutivity} \\ &= 0_2 && \text{i.e. take } x = 0_2 \text{ in } 0_1 + x = x \end{aligned}$$

as desired.

b) Let $1_1, 1_2 \in \mathbb{R}$ are such that $1_1 \cdot x = x$ and $1_2 \cdot x = x$ for all $x \in \mathbb{R}$. Then

$$\begin{aligned} 1_1 &= 1_2 \cdot 1_1 && \text{i.e. take } x = 1_1 \text{ in } 1_2 \cdot x = x \\ &= 1_1 \cdot 1_2 && \text{by commutivity} \\ &= 1_2 && \text{i.e. take } x = 1_2 \text{ in } 1_1 \cdot x = x \end{aligned}$$

as desired.

c) Let 0 be as in (F4) and suppose $x \in \mathbb{R}$ and $y_1, y_2 \in \mathbb{R}$ are such that $x + y_1 = 0$ and $x + y_2 = 0$. Then

$$\begin{aligned} y_1 &= 0 + y_1 && \text{by identity properties} \\ &= (x + y_2) + y_1 && \text{by assumption} \\ &= x + (y_2 + y_1) && \text{by associativity} \\ &= x + (y_1 + y_2) && \text{by commutivity} \\ &= (x + y_1) + y_2 && \text{by associativity} \\ &= 0 + y_2 && \text{by assumption} \\ &= y_2 && \text{by identity properties} \end{aligned}$$

as desired.

d) Let 1 be as in (F4) and suppose $x \in \mathbb{R} \setminus \{0\}$ and $y_1, y_2 \in \mathbb{R}$ are such that $x \cdot y_1 = 1$ and $x \cdot y_2 = 1$. Then

$$\begin{aligned} y_1 &= 1 \cdot y_1 && \text{by identity properties} \\ &= (x \cdot y_2) \cdot y_1 && \text{by assumption} \\ &= x \cdot (y_2 \cdot y_1) && \text{by associativity} \\ &= x \cdot (y_1 \cdot y_2) && \text{by commutivity} \\ &= (x \cdot y_1) \cdot y_2 && \text{by associativity} \\ &= 1 \cdot y_2 && \text{by assumption} \\ &= y_2 && \text{by identity properties} \end{aligned}$$

as desired.

e) Let $x \in \mathbb{R}$. To see that $0 \cdot x = 0$, notice that

$$\begin{aligned} 0 \cdot x &= (0 + 0) \cdot x && \text{by identity properties} \\ &= (0 \cdot x) + (0 \cdot x) && \text{by distributivity.} \end{aligned}$$

Hence

$$\begin{aligned} 0 &= (0 \cdot x) + (-(0 \cdot x)) && \text{by inverses} \\ &= ((0 \cdot x) + (0 \cdot x)) + (-(0 \cdot x)) && \text{by the above equation} \\ &= (0 \cdot x) + ((0 \cdot x) + (-(0 \cdot x))) && \text{by associativity} \\ &= (0 \cdot x) + 0 && \text{by inverses} \\ &= 0 \cdot x && \text{by identity properties} \end{aligned}$$

as desired.

f) Finally, let $x \in \mathbb{R}$. To see that $-x = (-1) \cdot x$, we recall by part c) that $-x$ is the unique element $y \in \mathbb{R}$ such that $x + y = 0$. Thus, to complete the proof, it suffices to show that $x + (-1) \cdot x = 0$. Notice that

$$\begin{aligned} x + (-1) \cdot x &= 1 \cdot x + (-1) \cdot x && \text{by identity properties} \\ &= (1 + (-1)) \cdot x && \text{by distributivity} \\ &= 0 \cdot x && \text{by inverses} \\ &= 0 && \text{by part (e)} \end{aligned}$$

as desired. ■

Although we want the real numbers have the above five properties, they are not the only number system that has all five properties. Consequently, we make the following definition.

Definition 1.2.2. A *field* is a set $\mathbb{F}$ together with two operations $+$ and $\cdot$ such that $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$ for all $a, b \in \mathbb{F}$, and $+$ and $\cdot$ satisfy (F1), (F2), (F3), (F4), and (F5) as written above (replacing $\mathbb{R}$ with $\mathbb{F}$ - note Lemma 1.2.1 also automatically holds for $\mathbb{F}$) with $0 \neq 1$.

Thus the simple way to say we want the real numbers to have the above five properties is to say that we want the real numbers to be a field. However, some of our other number systems may or may not be fields.

Example 1.2.3. The natural numbers and integers are not a field. One way to see this is that the number 2 does not have a multiplicative inverse inside the natural numbers nor integers. To see this, suppose $k \in \mathbb{Z}$ were such that $2k = 1$. If $k \leq 0$, then $2x \leq k \neq 1$. Otherwise, if $k > 0$ then $2k = k + k > k \geq 1$ so $2k \neq 1$.

Example 1.2.4. The rational numbers are a field. This is shown in Appendix B.2.

Example 1.2.5. The complex numbers are a field. This should have been shown in MATH 1200 assuming that the real numbers are a field.

Notice if one is given a field $\mathbb{F}$ and a subset E of $\mathbb{F}$ then E is a field with the same operations of $+$ and $\cdot$ provided

- (closed under $+$ and $\cdot$) if $a, b \in E$, then $a + b \in E$ and $a \cdot b \in E$,
- (contains the identities) $0, 1 \in E$,
- (closed under additive inverses) if $a \in E$, then $-a \in E$, and
- (closed under multiplicative inverses) if $a \in E$ and $a \neq 0$ then $a^{-1} \in E$.

Indeed, if these properties hold for E, it is easy to see that (F1), (F2), (F3), (F4), and (F5) hold because of these properties and because these field axioms hold in $\mathbb{F}$. In this case, we call E a *subfield* of $\mathbb{F}$. There are many examples of subfields of $\mathbb{R}$ with the following just being one example (assuming we have made sense of $\sqrt{2}$ and shown $\sqrt{2} \in \mathbb{R}$).

Proposition 1.2.6. *The set*

$$\mathbb{Q}[\sqrt{2}] := \{x + y\sqrt{2} \mid x, y \in \mathbb{Q}\}$$

is a subfield of $\mathbb{R}$ *such that* $\mathbb{Q} \subsetneq \mathbb{Q}[\sqrt{2}] \subsetneq \mathbb{R}$.

Proof. First, we claim that $\mathbb{Q} \subsetneq \mathbb{Q}[\sqrt{2}]$. Clearly $\mathbb{Q} \subseteq \mathbb{Q}[\sqrt{2}]$. To see that the inclusion is strict, we will reprove the fact that $\sqrt{2} \notin \mathbb{Q}$ (which student surely saw in MATH 1200). To see this, suppose to the contrary that there exists $a, b \in \mathbb{Z}$ with $b \neq 0$ such that

$$\sqrt{2} = \frac{a}{b}.$$

By removing any common divisors, we can assume without loss of generality that a and b have no common divisors. Since $b\sqrt{2} = a$, we have that $2b^2 = a^2$.

Therefore 2 divides a^2. Since the square of an odd number is odd, we have that a must be even and thus $a = 2c$ for some $c \in \mathbb{Z}$. Hence

$$2b^2 = a^2 = (2c)^2 = 4c^2$$

so that $b^2 = 2c^2$. Hence 2 divides b^2. Since the square of an odd number is odd, we have that b must be even. However, we have now shown that both a and b are even which contradicts the fact that a and b have no common divisors. Thus $\sqrt{2} \notin \mathbb{Q}$ so $\mathbb{Q} \subsetneq \mathbb{Q}[\sqrt{2}]$.

Assuming we know that $\mathbb{Q} \subseteq \mathbb{R}$ and $\sqrt{2} \in \mathbb{R}$ (see Proposition 1.3.10), we have that $\mathbb{Q}[\sqrt{2}] \subseteq \mathbb{R}$. To see that the inclusion is strict, we will show that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ (the fact that $\sqrt{3} \in \mathbb{R}$ is similar to Proposition 1.3.10). To see that $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$, suppose to the contrary that there exists $x, y \in \mathbb{Q}$ such that

$$x + y\sqrt{2} = \sqrt{3}.$$

By squaring both sides, we obtain that

$$(x^2 + 2y^2) + 2xy\sqrt{2} = 3.$$

If $x \neq 0$ and $y \neq 0$, then we have that

$$\sqrt{2} = \frac{3 - x^2 - 2y^2}{2xy}.$$

However, since $x, y \in \mathbb{Q}$, this implies $\sqrt{2} \in \mathbb{Q}$, which contradicts the fact that $\sqrt{2} \notin \mathbb{Q}$. Thus $x = 0$ or $y = 0$.

By a similar proof to that used to show that $\sqrt{2} \notin \mathbb{Q}$ (i.e. use divisibility by 3 instead of divisibility by 2), we obtain $\sqrt{3} \notin \mathbb{Q}$. If $y = 0$ then $x = \sqrt{3}$ which implies that $\sqrt{3} \in \mathbb{Q}$, which is a contradiction. Thus it must be the case that $x = 0$ so that $y\sqrt{2} = \sqrt{3}$.

Since $y \in \mathbb{Q}$, there exists $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $y = \frac{a}{b}$. By removing any common divisors, we can assume without loss of generality that a and b have no common divisors. Thus

$$a\sqrt{2} = b\sqrt{3}$$

so that $2a^2 = 3b^2$. Since 2 does not divide 3, this implies that 2 divides b^2. Since the square of an odd number is odd, we have that b must be even and thus $b = 2c$ for some $c \in \mathbb{Z}$. Hence

$$2a^2 = 3b^2 = 3(2c)^2 = 12c^2$$

so that $a^2 = 6c^2$. Hence 2 divides a^2. Since the square of an odd number is odd, we have that a must be even. However, we have now shown that both a and b are even which contradicts the fact that a and b have no common divisors. Therefore, we have a contradiction in all possible cases. Hence $\sqrt{3} \notin \mathbb{Q}[\sqrt{2}]$ and thus $\mathbb{Q}[\sqrt{2}] \subsetneq \mathbb{R}$.

Before moving on to the proof that $\mathbb{Q}[\sqrt{2}]$ is a field, we claim that if $x, y \in \mathbb{Q}$ are such that $x + y\sqrt{2} = 0$, then $x = 0$ and $y = 0$. To see thus, notice if $y \neq 0$ then

$$\sqrt{2} = -\frac{x}{y} \in \mathbb{Q},$$

which is a contradiction. Hence $y = 0$. Therefore $x + 0\sqrt{2} = 0$ and thus $x = 0$ as desired.

To prove that $\mathbb{Q}[\sqrt{2}]$ is a field, we note by the comments before this proposition that it suffices to show that

- If $a, b \in \mathbb{Q}[\sqrt{2}]$ then $a + b \in \mathbb{Q}[\sqrt{2}]$ and $a \cdot b \in \mathbb{Q}[\sqrt{2}]$,
- $0, 1 \in \mathbb{Q}[\sqrt{2}]$,
- if $a \in \mathbb{Q}[\sqrt{2}]$ then $-a \in \mathbb{Q}[\sqrt{2}]$, and
- if $a \in \mathbb{Q}[\sqrt{2}]$ and $a \neq 0$, then $a^{-1} \in \mathbb{Q}[\sqrt{2}]$.

Notice for all $x_1, x_2, y_1, y_2 \in \mathbb{Q}$ that

$$(x_1 + y_1\sqrt{2}) + (x_2 + y_2\sqrt{2}) = (x_1 + x_2) + (y_1 + y_2)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$$
$$(x_1 + y_1\sqrt{2}) \cdot (x_2 + y_2\sqrt{2}) = (x_1x_2 + 2y_1y_2) + (x_2y_1 + x_1y_2)\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

Moreover $0 = 0 + 0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$ and $1 = 1 + 0\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$. Clearly if $x, y \in \mathbb{Q}$ then $-(x + y\sqrt{2}) = (-x) + (-y)\sqrt{2} \in \mathbb{Q}[\sqrt{2}]$.

Finally, suppose $x, y \in \mathbb{Q}$ are such that $x + y\sqrt{2} \neq 0$. Thus $x \neq 0$ or $y \neq 0$. Moreover, we claim that $x^2 - 2y^2 \neq 0$. To see this, suppose to the contrary that $x^2 - 2y^2 = 0$. If $y = 0$ then $x^2 = 0$ which implies that $x = 0$ contradicting the fact that $x \neq 0$ or $y \neq 0$. Thus $y \neq 0$ so that

$$2 = \frac{x^2}{y^2}.$$

This implies $\sqrt{2} = \pm\frac{x}{y} \in \mathbb{Q}$ which is a contradiction. Hence $x^2 - 2y^2 \neq 0$.

By the above, we have that

$$\frac{x}{x^2 - 2y^2} + \frac{-y}{x^2 - 2y^2}\sqrt{2} \in \mathbb{Q}[\sqrt{2}].$$

Moreover, since

$$\left(\frac{x}{x^2 - 2y^2} + \frac{-y}{x^2 - 2y^2}\sqrt{2}\right)(x + y\sqrt{2}) = 1$$

we have that

$$\frac{x}{x^2 - 2y^2} + \frac{-y}{x^2 - 2y^2}\sqrt{2} = (x + y\sqrt{2})^{-1} \in \mathbb{Q}[\sqrt{2}]$$

as desired. Hence $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{R}$. ■

Although there are many subfields of $\mathbb{R}$, there are also many fields that look strikingly different from $\mathbb{R}$ that we need to consider if we are to rigorously define $\mathbb{R}$.

Example 1.2.7. Recall from MATH 1200 that if $n \in \mathbb{N}$ then

$$\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$$

denotes the integers modulo n. It is elementary to verify that (F1), (F2), and (F3) hold and (F4) holds with $[0]$ the zero and $[1]$ the one. Moreover, clearly $-[k] = [-k] = [n-k]$ for all $k \in \{0, 1, \ldots, n-1\}$.

However, in general, it need not be true that for all $[k] \in \{[1], \ldots, [n-1]\}$ there exists an $[m] \in \{[0], [1], \ldots, [n-1]\}$ such that $[k][m] = [1]$; that is, $km \equiv 1 \mod n$. However, if n is prime, then the Euclidean algorithm does imply that for all $k \in \{1, \ldots, n-1\}$ there exists integers s and t so that $ks + nt = 1$ and thus $ks \equiv 1 \mod n$. Hence $\mathbb{Z}_p$ for p prime satisfies (F5) and thus is a field.

Notice that all the field properties are algebraic in nature (and can be studied future in the Algebra sequence - MATH 3021, MATH 3022, and MATH 4021). Are there other properties of $\mathbb{R}$ we can include to distinguish $\mathbb{R}$ from other fields?

1.2.2 Partially Ordered Sets

One way we think of the real numbers beyond their algebraic properties is that we think of them as a line of numbers. What this line implicitly means is that we have a nice ordering of the real numbers; that is, given two numbers, we have a notion which tells us which number is bigger. Thus, in order to rigorously define the real numbers and distinguish it from other fields, it is useful for us to examine mathematical notions of orderings.

The base mathematical notion of an ordering is formalized as follows.

Definition 1.2.8. Let X be a set. A relation $\preceq$ on the elements of X is called a *partial ordering* if:

1. (reflexivity) $a \preceq a$ for all $a \in X$,
2. (antisymmetry) if $a \preceq b$ and $b \preceq a$, then $a = b$ for all $a, b \in X$, and
3. (transitivity) if $a, b, c \in X$ are such that $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

Example 1.2.9. Let $\leq$ be our usual notion of "less than or equal to" on $\mathbb{R}$. Then $\leq$ is a partial ordering on $\mathbb{R}$. In fact, $\leq$ defines a partial ordering on any subfield of $\mathbb{R}$ (e.g. $\mathbb{Q}$).

It is useful to note that there are many other orderings that do not look like $\leq$ does on $\mathbb{R}$.

Example 1.2.10. Let X be a non-empty set and consider the power set of X
$$\mathcal{P}(X) := \{Y \mid Y \subseteq X\}.$$
We define a relation $\preceq$ on $\mathcal{P}(X)$ as follows: given $A, B \in \mathcal{P}(X)$,
$$A \preceq B \qquad \text{if and only if} \qquad A \subseteq B.$$
It is not difficult to verify that $\preceq$ is a partial ordering on $\mathcal{P}(\mathbb{R})$.

The partial ordering in Example 1.2.10 is not as nice as our ordering on $\mathbb{R}$. To see this, for example consider $X = \mathbb{Z}$ and the sets $A = \{1\}$ and $B = \{2\}$. Then $A \not\preceq B$ and $B \not\preceq A$; that is, we cannot use the partial ordering to compare A and B. However, if $x, y \in \mathbb{R}$, then either $x \leq y$ or $y \leq x$. The ability to compare any two elements is a property we would like to impose on our partial orderings.

Definition 1.2.11. Let X be a set. A partial ordering $\preceq$ on X is called a *total ordering* if for all $x, y \in X$ we have $x \preceq y$ or $y \preceq x$.

Example 1.2.12. Let $\leq$ be our usual notion of "less than or equal to" on $\mathbb{R}$. Then $\leq$ is a total ordering on $\mathbb{R}$. In fact, $\leq$ defines a total ordering on any subfield of $\mathbb{R}$ (e.g. $\mathbb{Q}$).

Unfortunately for our goal of isolating $\mathbb{R}$ among all fields, it is easy to place a total ordering on $\mathbb{Z}_p$.

Example 1.2.13. Let $\mathbb{Z}_p$ be as in Example 1.2.7 with p prime. For $k, m \in \{0, \ldots, n-1\}$, define $[k] \preceq [m]$ if and only if $k \leq m$. It is easy to verify that $\preceq$ is a total ordering on $\mathbb{Z}_p$.

As $\mathbb{Z}_p$ has a total ordering, are there any other nice properties that $\leq$ on $\mathbb{R}$ has that we would like to impose? Indeed by combining the algebraic notions of a field together with the notion of a total ordering, we can describe properties $\leq$ on $\mathbb{R}$ has that the total ordering on $\mathbb{Z}_p$ does not.

Definition 1.2.14. An *ordered field* is a pair $(\mathbb{F}, \preceq)$ where $\mathbb{F}$ is a field and $\preceq$ is a total ordering on $\mathbb{F}$ such that the following two properties hold:

- (Additive Property) if $x, y, z \in \mathbb{F}$ are such that $x \preceq y$, then $x+z \preceq y+z$.
- (Multiplicative Property) if $x, y \in \mathbb{F}$ are such that $0 \preceq x$ and $0 \preceq y$, then $0 \preceq x \cdot y$.

Example 1.2.15. The pair $(\mathbb{R}, \leq)$ is an ordered field. In addition, if E is any subfield of $\mathbb{R}$ (e.g. $\mathbb{Q}$ or $\mathbb{Q}[\sqrt{2}]$), then $(E, \leq)$ is an ordered field.

It is possible to show that there are no total orderings on $\mathbb{Z}_p$ nor $\mathbb{C}$ which turns them into ordered fields. To show this, we first prove the following property. While we are at it, we might as well prove that our usual operations on inequalities hold.

Lemma 1.2.16. *If* $(\mathbb{F}, \preceq)$ *is an ordered field, then* $0 \preceq 1$*.*

Proof. As $(\mathbb{F}, \preceq)$ is an ordered field, either $0 \preceq 1$ or $1 \preceq 0$.

Suppose for the sake of a contradiction that $1 \preceq 0$. Thus, by the Additive Property of an ordered field,

$$0 = 1 + (-1) \preceq 0 + (-1) = -1.$$

Hence, by the Multiplicative Property of an ordered field,

$$0 \preceq (-1) \cdot (-1) = 1.$$

However, since $1 \preceq 0$ and $0 \preceq 1$, it follows that $0 = 1$ by antisymmetry. As this contradicts property (F4) of a field, we have a contradiction. Hence it must be the case that $0 \preceq 1$. ∎

Lemma 1.2.17. *If* $(\mathbb{F}, \preceq)$ *is an ordered field and* $a \in \mathbb{F} \setminus \{0\}$ *is such that* $0 \preceq a$*, then* $0 \preceq a^{-1}$*.*

Proof. Let $(\mathbb{F}, \preceq)$ be an ordered field and let $a \in \mathbb{F} \setminus \{0\}$ be such that $0 \preceq a$. Since $(\mathbb{F}, \preceq)$ is an ordered field and thus $\preceq$ is a total ordering, either $0 \preceq a^{-1}$ or $a^{-1} \preceq 0$.

Suppose for the sake of a contradiction that $a^{-1} \preceq 0$. By the Additive Property of an ordered field, this implies that

$$0 = a^{-1} + (-a^{-1}) \leq 0 + (-a^{-1}) = -a^{-1}.$$

Since $a > 0$, the Multiplicative Property of an ordered field, this implies that

$$0 \leq (a)(-a^{-1}) = (a)(-1)(a^{-1}) = (a)(a^{-1})(-1) = (1)(-1) = -1.$$

However, the Additive Property of an ordered field then implies that

$$1 = 1 + 0 \leq 1 + (-1) = 0.$$

However, since $0 \leq 1$ by Lemma 1.2.16, we obtain that $0 \leq 1$ and $1 \leq 0$ so $1 = 0$ by anti-symmetry. As this contradicts the fact that $0 \neq 1$ in any field, we have our contradiction. Hence the result follows. ∎

Lemma 1.2.18. *Let* $(\mathbb{F}, \preceq)$ *be an ordered field. Then the following hold:*

a) If $a, b, c \in \mathbb{F}$ *are such that* $0 \preceq a$ *and* $b \preceq c$*, then* $a \cdot b \preceq a \cdot c$*.*

b) If $a, b, c \in \mathbb{F}$ *are such that* $a \preceq 0$ *and* $b \preceq c$*, then* $a \cdot c \preceq a \cdot b$*.*

c) If $a, b \in \mathbb{F} \setminus \{0\}$ *are such that* $0 \preceq a \preceq b$*, then* $0 \preceq b^{-1} \preceq a^{-1}$*.*

Proof. a) Let $a, b, c \in \mathbb{F}$ be such that $0 \preceq a$ and $b \preceq c$. By the Additive Property of an ordered field, we know that

$$0 = b + (-b) \preceq c + (-b).$$

Hence the Multiplicative Property of an ordered field implies that

$$0 \preceq a \cdot (c + (-b)) = (a \cdot c) + (a \cdot (-b)).$$

Note

$$\begin{aligned}
a \cdot (-b) &= a \cdot ((-1) \cdot b) && \text{by Lemma 1.2.1 part f)} \\
&= (a \cdot (-1)) \cdot b && \text{by associativity} \\
&= ((-1) \cdot a) \cdot b && \text{by commutivity} \\
&= (-1) \cdot (a \cdot b) && \text{by associativity} \\
&= -(a \cdot b) && \text{by Lemma 1.2.1 part f).}
\end{aligned}$$

Thus the Additive Property of an ordered field implies that

$$\begin{aligned}
a \cdot b &\preceq ((a \cdot c) + (a \cdot (-b)))) + (a \cdot b) \\
&= (a \cdot c) + ((a \cdot (-b) + (a \cdot b)) && \text{by associativity} \\
&= (a \cdot c) + 0 && \text{by } a \cdot (-b) = -(a \cdot b) \\
&= a \cdot c
\end{aligned}$$

as desired.

b) Let $a, b, c \in \mathbb{F}$ be such that $a \preceq 0$ and $b \preceq c$. By the Additive Property of an ordered field, we know that

$$0 = (-a) + a \preceq (-a) + 0 = -a.$$

Therefore, by part a) we know that $(-a) \cdot b \preceq (-a) \cdot c$. By a similar computation to that used in the proof of part a), we can show that

$$(-a) \cdot b = -(a \cdot b) \qquad \text{and} \qquad (-a) \cdot c = -(a \cdot c).$$

Thus $-(a \cdot b) \preceq -(a \cdot c)$. Thus the Additive Property of an ordered field implies that

$$\begin{aligned}
a \cdot c &= (a \cdot c) + 0 && \text{by identity properties} \\
&= (a \cdot c) + ((a \cdot b) + (-(a \cdot b))) && \text{by inverses} \\
&= ((a \cdot c) + (a \cdot b)) + (-(a \cdot b)) && \text{by associativity} \\
&= (-(a \cdot b)) + ((a \cdot c) + (a \cdot b)) && \text{by commutivity} \\
&\preceq (-(a \cdot c)) + ((a \cdot c) + (a \cdot b)) && \text{by the Additive Property} \\
&= ((-(a \cdot c)) + (a \cdot c)) + (a \cdot b) && \text{by associtivity} \\
&= ((a \cdot c) + (-(a \cdot c))) + (a \cdot b) && \text{by commutivity} \\
&= 0 + (a \cdot b) && \text{by inverses} \\
&= (a \cdot b) + 0 && \text{by commutivity} \\
&= a \cdot b
\end{aligned}$$

as desired.

c) Let $a, b \in \mathbb{F} \setminus \{0\}$ be such that $0 \preceq a \prec b$. Thus $0 \preceq a^{-1}$ and $0 \preceq b^{-1}$ by Lemma 1.2.17. Since $0 \preceq a^{-1}$ and $a \preceq b$, part a) implies that

$$1 = a^{-1} \cdot a \preceq a^{-1} \cdot b.$$

Therefore, since $0 \preceq b^{-1}$ and $1 \preceq a^{-1} \cdot b$, again part a) implies that

$$b^{-1} = b^{-1} \cdot 1 \preceq b^{-1} \cdot (a^{-1} \cdot b).$$

Since

$$\begin{aligned} b^{-1} \cdot (a^{-1} \cdot b) &= b^{-1} \cdot (b \cdot a^{-1}) && \text{by commutivity} \\ &= (b^{-1} \cdot b) \cdot a && \text{by associativity} \\ &= 1 \cdot a && \text{by inverses} \\ &= a && \text{by identity properties,} \end{aligned}$$

the proof is complete. ■

Proposition 1.2.19. *For all prime numbers p, there is no total ordering $\preceq$ on $\mathbb{Z}_p$ such that $(\mathbb{Z}_p, \preceq)$ is an ordered field.*

Proof. Suppose for the sake of a contradiction that there was a total ordering $\preceq$ on $\mathbb{Z}_p$ so that $(\mathbb{Z}_p, \preceq)$ is an ordered field. For each $n \in \mathbb{N}$ let P_n be the statement that $[n-1] \preceq [n]$. To show that P_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction. To do so, we must demonstrate the two conditions in Theorem 1.1.7.

Base Case: To see that P_1 is true, notice that $[0] \preceq [1]$ by Lemma 1.2.16. Hence P_1 is true.

Inductive Step: Assume that P_k is true; that is, assume $[k-1] \prec [k]$. To see that P_{k+1} is true, notice by the Additive Property of an ordered field that

$$[k] = [k-1] + [1] \preceq [k] + [1] = [k+1].$$

Hence P_{k+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the Principle of Mathematical Induction implies that $[n-1] \preceq [n]$ for all $n \in \mathbb{N}$.

By the transitivity of a partial ordering, the fact that $[n-1] \preceq [n]$ for all $n \in \mathbb{N}$ implies that $[1] \preceq [p] = [0]$. However, as $[0] \preceq [1]$, it follows that $[0] = [1]$ by antisymmetry. As $1 \not\equiv 0 \mod p$ for p prime, we have a contradiction. Hence the result follows. ■

Proposition 1.2.20. *There is no total ordering $\preceq$ on $\mathbb{C}$ such that $(\mathbb{C}, \preceq)$ is an ordered field.*

Proof. Suppose for the sake of a contradiction that there was a total ordering $\preceq$ on $\mathbb{C}$ so that $(\mathbb{C}, \preceq)$ is an ordered field. As $\preceq$ is a total ordering, either $0 \preceq i$ or $i \preceq 0$. We will show that both $0 \preceq i$ and $i \preceq 0$ lead to contradictions thereby completing the proof.

Suppose that $0 \preceq i$. Hence the Multiplicative Property of an ordered field implies that

$$0 = 0 \cdot 0 \preceq i \cdot i = -1.$$

Thus the Additive Property of an ordered field implies that

$$1 = 1 + 0 \preceq 1 + (-1) = 0.$$

However, as $0 \preceq 1$ by Lemma 1.2.16, it follows that $0 = 1$ by antisymmetry which is a clear contradiction.

Suppose that $i \preceq 0$. Thus the Additive property of an ordered field implies that

$$0 = i + (-i) \preceq 0 + (-i) = -i.$$

Hence the Multiplicative Property of an ordered field implies that

$$0 = 0 \cdot 0 \preceq (-i) \cdot (-i) = -1.$$

Thus the Additive Property of an ordered field implies that

$$1 = 1 + 0 \preceq 1 + (-1) = 0.$$

However, as $0 \preceq 1$ by Lemma 1.2.16, it follows that $0 = 1$ by antisymmetry which is a clear contradiction. Hence the result follows. ∎

Thus by requiring that the real numbers are an ordered field, we have distinguished the real numbers from $\mathbb{Z}_p$ and from $\mathbb{C}$ by Propositions 1.2.19 and 1.2.20 respectively. However, every subfield of $\mathbb{R}$ (e.g. $\mathbb{Q}$) will automatically be an ordered field. What other properties do we need?

1.2.3 The Least Upper Bound Property

It turns out that there is only one final property we need to distinguish $\mathbb{R}$ from all other fields! To introduce this property, we first define the following.

Definition 1.2.21. Let X be a set, let $\preceq$ be a partial ordering on X, and let $A \subseteq X$. An element $\alpha \in X$ is said to be

- an *upper bound* for A if $a \preceq \alpha$ for all $a \in A$.
- a *lower bound* for A if $\alpha \preceq a$ for all $a \in A$.

Moreover, we say that

- A is *bounded above* if A has an upper bound,

- A is *bounded below* if A has a lower bound, and
- A is *bounded* if A has both an upper and lower bound.

For the sake of examples, we will be focusing on $\mathbb{R}$ together with its usual ordering $\leq$. To begin (and for much future use in this course), it is useful to define the following sets.

Notation 1.2.22. For all $a, b \in \mathbb{R}$ with $a \leq b$, we define

$$\begin{aligned}(a, b) &:= \{x \in \mathbb{R} \mid a < x < b\}\\ [a, b) &:= \{x \in \mathbb{R} \mid a \leq x < b\}\\ (a, b] &:= \{x \in \mathbb{R} \mid a < x \leq b\}\\ [a, b] &:= \{x \in \mathbb{R} \mid a \leq x \leq b\}.\end{aligned}$$

For the first two, we permit ∞ to replace b, and, for the first and third, we permit $-\infty$ to replace a. Each of the above sets is called an *interval* with (a, b) called an *open* interval and $[a, b]$ called a *closed* interval.

Example 1.2.23. Let $A = (0, 1)$. Then 1 is an upper bound of X and 0 is a lower bound of A. Thus A is bounded. Furthermore, note that 5 is also an upper bound of A and -7 is a lower bound of A. In particular, $[1, \infty)$ is the set of all upper bounds for A and $(-\infty, 0]$ is the set of all lower bound for A.

Example 1.2.24. Let $A = [0, 1]$. Then 1 is an upper bound of A and 0 is a lower bound of A. Thus A is bounded. Furthermore, note that 5 is also an upper bound of A and -7 is a lower bound of A. In particular, $[1, \infty)$ is the set of all upper bounds for A and $(-\infty, 0]$ is the set of all lower bound for A.

Example 1.2.25. Let $A = \emptyset$. Then every number in $\mathbb{R}$ is both an upper and lower bound of A vacuously (that is, there are no elements of A to which to check the defining property).

Example 1.2.26. Let $A = \mathbb{N}$. Clearly A is bounded below by 1 and the set of lower bounds for A is precisely $(-\infty, 1]$. However, does $\mathbb{N}$ have an upper bound? Our intuition says no so that $\mathbb{N}$ is not bounded above. However, how do we prove this? What property of the $\mathbb{R}$ gives us that $\mathbb{N}$ is not bounded above?

Example 1.2.27. Let $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$. Clearly A is bounded above by 1 and the set of upper bounds for A is precisely $[1, \infty)$. Moreover, clearly every element of $(-\infty, 0]$ is a lower bound for A. However, are there any other lower bounds of A in $\mathbb{R}$? How do we prove this?

Example 1.2.28. Let $A = \{q \in \mathbb{Q} \mid q^2 \leq 2\}$. It is not difficult to see that 2 is an upper bound for A since if $q \in \mathbb{Q}$ and $q \geq 2$ then $q^2 \geq 4$ and thus $q \notin A$. Moreover, we see that $\sqrt{2}$ is an upper bound for A in $\mathbb{R}$. Is the set of upper bounds of A in $\mathbb{R}$ exactly $[\sqrt{2}, \infty)$?

Example 1.2.28 is quite interesting when it comes to defining $\mathbb{R}$. In particular, instead of trying to add in $\sqrt{2}$ to $\mathbb{Q}$ via the desire to have $\sqrt{2}^2 = 2$, we could instead think of $\sqrt{2}$ as the upper bound of the set A that is smallest in size. We formalize this notion as follows.

Definition 1.2.29. Let X be a set, let $\preceq$ be a partial ordering on X, and let $A \subseteq X$. An element $\alpha \in X$ is said to be the *least upper bound* of A if

- α is an upper bound of A, and
- if β is an upper bound of A, then $\alpha \preceq \beta$.

We write $\text{lub}(A)$ in place of α, provided α exists.

Similarly, an element $\alpha \in X$ is said to be the *greatest lower bound* of A if

- α is a lower bound of A, and
- if β is a lower bound of A, then $\beta \preceq \alpha$.

We write $\text{glb}(A)$ in place of α, provided α exists.

Remark 1.2.30. In the above definition, notice we have used the term 'the least upper bound' instead of 'a least upper bound'. This is because it is elementary to show that a set with a least upper bound has exactly one least upper bound. Indeed if α_1 and α_2 are both least upper bounds of a set A, then $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$ by the two defining properties of a least upper bound, so $\alpha_1 = \alpha_2$.

Example 1.2.31. Let $A = (0, 1)$. Then $0 = \text{glb}(A)$ and $1 = \text{lub}(A)$.

Example 1.2.32. Let $A = [0, 1]$. Then $0 = \text{glb}(A)$ and $1 = \text{lub}(A)$.

Example 1.2.33. Clearly a set that is not bounded above cannot have a least upper bound and a set that is not bounded below cannot have a greatest lower bound. Consequently $\emptyset \subseteq \mathbb{R}$ has no least upper bound nor greatest lower bound.

Remark 1.2.34. Let

$$A = \{q \in \mathbb{Q} \mid q^2 \leq 2\}.$$

Using this set and the notion of least upper bounds, we can illustrates a substantial difference between $\mathbb{Q}$ and $\mathbb{R}$. Notice that $A \subseteq \mathbb{Q}$. However, if we only consider numbers in $\mathbb{Q}$, then A does not have a least upper bound in $\mathbb{Q}$ as if $b \in \mathbb{Q}$ and $\sqrt{2} < b$, there is always an $r \in \mathbb{Q}$ such that $\sqrt{2} < r < b$ (see Proposition 1.3.8). However, inside of $\mathbb{R}$, our hope is that $\text{lub}(A) = \sqrt{2}$.

Based on the above, we can use the following property of $\mathbb{R}$ to distinguish it from its subfields.

Axiom 1.2.35 (The Least Upper Bound Property). *Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound.*

Note the term 'non-empty' must be included because of Example 1.2.33.

1.2.4 Constructing the Real Numbers

Using the above, we can finally define the real numbers!

Definition 1.2.36. The real numbers, denoted $\mathbb{R}$, are the unique ordered field with the Least Upper Bound Property.

There are two parts to Definition 1.2.36: 1) that we can actually construct an ordered fields with the Least Upper Bound Property, and 2) all such constructions give the same number system (which we call $\mathbb{R}$).

To construct the real numbers, one first must construct the rational numbers (see Appendix B.2). Then there are many ways one can construct the real numbers from the rational numbers.

One such way is to use sets of rational numbers similar to the one discussed in Example 1.2.28. This is done in Appendix B.3. Another method uses an equivalence relation on the set of Cauchy sequences (see Chapter 3) of rational numbers. This approach is discussed in Appendix B.4.

Finally, one can show that every ordered field with the Least Upper Bound Property can be identified with the real numbers "up to relabeling" (i.e. for example, instead of calling $\sqrt{2}$ 'the square root of two', perhaps we want to all it 'Frodo'). This is done in Appendix B.5.

1.3 Some Properties of the Real Numbers

Instead of focusing on the unproven (unless you read the appendices) aspects of the previous section, we will focus using the fact that $\mathbb{R}$ is an ordered field with the Least Upper Bound Property to show that $\mathbb{R}$ has other useful properties one would expect.

1.3.1 Comparing Least Upper and Greatest Lower Bounds

To begin, at the moment there is an an asymmetry in our definition of the real number. Indeed we have said that the real numbers had the Least Upper Bound Property, but we have completely ignored the concept of lower bounds. Due to the ordered field structure of $\mathbb{R}$, the following lemma shows that we need not worry.

Lemma 1.3.1. *Let A be a non-empty subset of $\mathbb{R}$ and let*

$$B = -A = \{-a \mid a \in A\}.$$

Then $\alpha \in \mathbb{R}$ is a lower bound of A if and only if $-\alpha$ is a upper bound of B.

Proof. Notice that

$$\begin{aligned}\alpha \text{ is a lower bound for } A &\text{ if and only if } \alpha \leq a \text{ for all } a \in A\\ &\text{ if and only if } -a \leq -\alpha \text{ for all } a \in A\\ &\text{ if and only if } b \leq -\alpha \text{ for all } b \in B\\ &\text{ if and only if } -\alpha \text{ is an upper bound for } B. \end{aligned}$$ ∎

Consequently, the real numbers also have the Greatest Lower Bound Property.

Proposition 1.3.2 (The Greatest Lower Bound Property). *Every non-empty subset of* $\mathbb{R}$ *that is bounded below has a greatest lower bound.*

Proof. Let A be a non-empty subset of $\mathbb{R}$ that is bounded below. Let

$$B = -A = \{-a \mid a \in A\}.$$

Since A is bounded below, there exists an $\alpha \in \mathbb{R}$ such that α is a lower bound for A. Hence Lemma 1.3.1 implies that $-\alpha$ is an upper bound for B. Thus B is bounded above.

Since B is bounded above, the Least Upper Bound Property implies that B has a least upper bound. Let $\beta = \text{lub}(B)$. We claim that $-\beta$ is the greatest lower bound of A.

First, since β is an upper bound for B, Lemma 1.3.1 implies that $-\beta$ is a lower bound of A. To see that $-\beta$ is a lower bound of A, let α be a lower bound of A. Thus $-\alpha$ is an upper bound of B by Lemma 1.3.1. Since β is the least upper bound for B, we obtain that $\beta \leq -\alpha$. Hence $\alpha \leq -\beta$. Therefore, as α was an arbitrary lower bound for A, $-\beta$ is the greatest lower bound of A. Hence the result follows. ∎

In fact, combining Lemma 1.3.1 and the proof of Proposition 1.3.2, we immediately have the following.

Corollary 1.3.3. *Let* A *be a set and let*

$$-A = \{-a \mid a \in A\}.$$

Then A *is bounded below if and only if* $-A$ *is bounded above. Moreover*

$$\text{glb}(A) = -\text{lub}(-A).$$

Similar to how the least upper bound and greatest lower bounds behave well with respect to negation, the following shows they behave well with respect to multiplication.

Lemma 1.3.4. *Let $c \in \mathbb{R}$ be such that $c > 0$. If A is a non-empty subset of $\mathbb{R}$ that is bounded above, then*

$$cA = \{ca \mid a \in A\}$$

is bounded above. Moreover $\mathrm{lub}(cA) = c\,\mathrm{lub}(A)$.

Similarly, if A is a non-empty subset of $\mathbb{R}$ that is bounded below, then cA is bounded below and $\mathrm{glb}(cA) = c\,\mathrm{glb}(A)$.

Proof. We will only prove the result for when A is bounded above as the proof when A is bounded below is similar (or follows from the connection between least upper bounds and greatest lower bounds in Corollary 1.3.3).

Assume A is non-empty and bounded above. Therefore cA is non-empty and $\alpha = \mathrm{lub}(A)$ exists. We claim that $c\alpha$ is an upper bound for cA. To see this, let $b \in cA$ be arbitrary. Hence, by the definition of cA there exists an $a \in A$ such that $b = ca$. Since α is an upper bound for A and since $a \in A$, we have that $a \leq \alpha$. Therefore, since $c > 0$, we have that

$$b = ca \leq c\alpha.$$

Therefore, since $b \in cA$ was arbitrary, $c\alpha$ is an upper bound for cA. Hence cA is bounded above. Thus $\mathrm{lub}(cA)$ exists and

$$\mathrm{lub}(cA) \leq c\alpha = c\,\mathrm{lub}(A).$$

To prove the other inequality, we can interchange the roles of A and cA above. To begin, note that $\frac{1}{c} > 0$ and

$$A = \frac{1}{c}(cA).$$

Therefore, by the above proof using cA in place of A and $\frac{1}{c}$ in place of c, we obtain that

$$\mathrm{lub}(A) = \mathrm{lub}\left(\frac{1}{c}(cA)\right) \leq \frac{1}{c}\mathrm{lub}(cA).$$

Hence, since $c > 0$, we obtain that

$$c\,\mathrm{lub}(A) \leq \mathrm{lub}(cA).$$

Therefore, by combining the two inequalities, we obtain that $\mathrm{lub}(cA) = c\,\mathrm{lub}(A)$ as desired. ∎

Furthermore, the least upper bounds and greatest lower bounds play well with respect to translation.

Lemma 1.3.5. *Let $c \in \mathbb{R}$. If A is a non-empty subset of $\mathbb{R}$ that is bounded above, then*

$$c + A = \{c + a \mid a \in A\}$$

is bounded above. Moreover $\mathrm{lub}(c + A) = c + \mathrm{lub}(A)$.

Similarly, if A is a non-empty subset of $\mathbb{R}$ that is bounded below, then $c + A$ is bounded below and $\mathrm{glb}(c + A) = c + \mathrm{glb}(A)$.

Proof. We will only prove the result for when A is bounded above as the proof when A is bounded below is similar (or follows from the connection between least upper bounds and greatest lower bounds in Corollary 1.3.3).

Assume A is non-empty and bounded above. Therefore $c+A$ is non-empty and $\alpha = \mathrm{lub}(A)$ exists. We claim that $c+\alpha$ is an upper bound for $c+A$. To see this, let $b \in c+A$ be arbitrary. Hence, by the definition of $c+A$ there exists an $a \in A$ such that $b = c+a$. Since α is an upper bound for A and since $a \in A$, we have that $a \leq \alpha$. Therefore we have that

$$b = c + a \leq c + \alpha.$$

Therefore, since $b \in c+A$ was arbitrary, $c+\alpha$ is an upper bound for $c+A$. Hence $c+A$ is bounded above. Thus $\mathrm{lub}(c+A)$ exists and

$$\mathrm{lub}(c+A) \leq c+\alpha = c + \mathrm{lub}(A).$$

To prove the other inequality, we can interchange the roles of A and $c+A$ above. To begin, note that $-c \in \mathbb{R}$ and

$$A = (-c) + (c + A).$$

Therefore, by the above proof using $c+A$ in place of A and $-c$ in place of c, we obtain that

$$\mathrm{lub}(A) = \mathrm{lub}\left((-c)+(c+A)\right) \leq (-c) + \mathrm{lub}(c+A).$$

Hence, we obtain that

$$c + \mathrm{lub}(A) \leq \mathrm{lub}(c+A).$$

Therefore, by combining the two inequalities, we obtain that $\mathrm{lub}(c+A) = c + \mathrm{lub}(A)$ as desired. ∎

1.3.2 Density of $\mathbb{Q}$ in $\mathbb{R}$

Using the Least Upper Bound Property (or Greatest Lower Bound Property), we can answer some nagging questions posed in Examples 1.2.26 and 1.2.27. Indeed we can prove two results (both known as the Archimedean Property as they are logically equivalent) with the first showing that $\mathbb{N}$ is not bounded above in $\mathbb{R}$ and the second which shows that no number $\epsilon > 0$ is a lower bound of $\{\frac{1}{n} \mid n \in \mathbb{N}\}$.

Theorem 1.3.6 (The Archimedean Property - I). *The natural numbers are not bounded above in $\mathbb{R}$.*

Proof. Suppose for the sake of a contradiction that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Thus the Least Upper Bound Property implies that $\mathbb{N}$ must has a least upper bound. Let $\alpha \in \mathbb{R}$ be the least upper bound of $\mathbb{N}$.

Since α is an upper bound of $\mathbb{N}$, we know that $n \leq \alpha$ for all $n \in \mathbb{N}$. Hence $m + 1 \leq \alpha$ for all $m \in \mathbb{N}$ and thus $m \leq \alpha - 1$ for all $m \in \mathbb{N}$. Thus $\alpha - 1$ is an upper bound for $\mathbb{N}$. However, since $\alpha - 1 \leq \alpha$, this contradicts the fact that α is the least upper bound of $\mathbb{N}$. Hence the result follows. ∎

Theorem 1.3.7 (The Archimedean Property - II). *For all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$.*

Proof. Fix $\epsilon > 0$. Thus $\frac{1}{\epsilon} > 0$. Since $\mathbb{N}$ is not bounded above, $\frac{1}{\epsilon}$ is a not an upper bound for $\mathbb{N}$. Thus there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{\epsilon} < N$. Hence $0 < \frac{1}{N} < \epsilon$ as desired. ∎

Using the Archimedean Property, we can show that every open interval contains a rational number; that is, for all $a, b \in \mathbb{R}$ with $a < b$ there exists a $q \in \mathbb{Q}$ such that $q \in (a, b)$. This is known as the density of the rational numbers in the real numbers (a concept that can be visited in Chapter 3).

Proposition 1.3.8 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *For all $a, b \in \mathbb{R}$ with $a < b$, there exists a $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. Fix $a, b \in \mathbb{R}$ with $a < b$. We will divide the proof into two cases.

Case 1: $a > 0$. In this case, let $\epsilon = b - a > 0$. By the Archimedean Property (Theorem 1.3.7) there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$.

Let
$$X = \left\{ m \in \mathbb{N} \,\middle|\, \frac{m}{N} > a \right\}.$$
Recall that the natural numbers are not bounded above in $\mathbb{R}$ by the Archimedean Property (Theorem 1.3.6). Therefore there exists an $n \in \mathbb{N}$ such that $a < n$. Hence $\frac{nN}{N} = n > a$ so $nN \in X$ and thus $X \neq \emptyset$. Therefore the Well-Ordering Principle (Theorem 1.1.10) implies that X has a least element.

Let k be the least element of X. Hence $a < \frac{k}{N}$ and if $m \in \mathbb{N}$ is such that $m < k$, then $\frac{m}{N} \leq a$. In particular, since $a > 0$, we have that $\frac{k-1}{N} \leq a < \frac{k}{N}$ (i.e. we need $a > 0$ in the case $k = 1$).

We claim that $\frac{k}{N} < b$. To see this, note that
$$\begin{aligned} k - 1 &\leq Na \\ &= N(b - \epsilon) \\ &= Nb - N\epsilon \\ &< Nb - 1 \end{aligned}$$
since $1 < N\epsilon$. Therefore $k < Nb$ so $\frac{k}{N} < b$ as desired.

Hence $a < \frac{k}{N} < b$ which completes the proof of this case since $\frac{k-1}{N} \in \mathbb{Q}$.

Case 2: $a \leq 0$. Recall that the natural numbers are not bounded above in $\mathbb{R}$ by the Archimedean Property (Theorem 1.3.6). Therefore there exists an $n \in \mathbb{N}$ such that $-a < n$. Hence $0 < a+n < b+n$. Therefore, by the first case of this proof, there exists a $q \in \mathbb{Q}$ such that $a+n < q < b+n$. Hence $a < q-n < b$. Therefore, since $q, n \in \mathbb{Q}$ implies $q - n \in \mathbb{Q}$, the proof of this case is complete.

Therefore, since the above two cases cover all possible cases, the proof is complete. ■

Of course, the irrational numbers are also 'dense' in the real numbers.

Proposition 1.3.9 (Density of $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$). *For all $a, b \in \mathbb{R}$ with $a < b$, there exists a $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.*

Proof. Fix $a, b \in \mathbb{R}$ with $a < b$. Thus $a - \sqrt{2} < b - \sqrt{2}$. By the density of the rational numbers, there exists a $q \in \mathbb{Q}$ such that $a - \sqrt{2} < q < b - \sqrt{2}$. Hence $a < q + \sqrt{2} < b$.

To complete the proof, it suffices to show that $q + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$. To see this, suppose for the sake of a contradiction that $q + \sqrt{2} \in \mathbb{Q}$. Hence there exists a $q' \in \mathbb{Q}$ such that $q + \sqrt{2} = q'$. Hence $\sqrt{2} = q - q' \in \mathbb{Q}$. Therefore, since $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$, we have a contradiction. Hence $q + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ and $a < q + \sqrt{2} < b$ as desired. ■

1.3.3 Existence of Square Roots

Using the definition of the real numbers, it is possible to show that $\sqrt{2}$ is a real number. In fact, the following proof can be considered are first true 'real analysis' proof of the course! In particular, the ideas and techniques used in this proof are a preview of the type of arguments will be used again and again throughout the course.

Proposition 1.3.10. *There exists a unique $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and $\alpha^2 = 2$ (i.e. α is what we call $\sqrt{2}$).*

Proof. First we will show that such an α exists.

Let

$$A = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

Notice if $x \in \mathbb{R}$ and $x \geq 2$ then $x^2 \geq 4$ and thus $x \notin A$. Hence 2 is an upper bound for A so A is bounded above. Therefore, by the Least Upper Bound Property, A has a least upper bound.

Let α be the least upper bound of A. We expect that α is the droid... I mean number we are looking for. To see this, first note since $1 \in A$ and α is an upper bound of A that $\alpha \geq 1$. We claim that $\alpha^2 = 2$. To see this, we will show that $\alpha^2 < 2$ and $\alpha^2 > 2$ both lead to contradictions.

Suppose for the sake of a contradiction that $\alpha^2 < 2$. We will show that this contradicts the fact that α is an upper bound of A by finding an $x_1 \in \mathbb{R}$ such that $x_1 > \alpha$ yet $x_1^2 < 2$ so that $x_1 \in A$. When reading this proof, it is far more helpful to look at how we define x_1, look at the computation, and then go back and pick things correctly to make the calculations work.

Let

$$\epsilon_1 = \frac{2-\alpha^2}{2\alpha+1}.$$

Note ϵ_1 is well-defined since $\alpha \geq 1$ so that $2\alpha + 1 > 0$. Moreover, since $2 - \alpha^2 > 0$, we see that $\epsilon_1 > 0$. Hence the Archimedean Property (Theorem 1.3.7) implies there exists an $N_1 \in \mathbb{N}$ such that $0 < \frac{1}{N_1} < \epsilon_1$.

Let $x_1 = \alpha + \frac{1}{N_1}$. Clearly $x_1 > \alpha$. Moreover

$$\begin{aligned}
x_1^2 &= \left(\alpha + \frac{1}{N_1}\right)^2 \\
&= \alpha^2 + 2\alpha\frac{1}{N_1} + \frac{1}{N_1^2} \\
&\leq \alpha^2 + 2\alpha\frac{1}{N_1} + \frac{1}{N_1} && \text{since } \frac{1}{N_1^2} \leq \frac{1}{N_1} \\
&= \alpha^2 + (2\alpha+1)\frac{1}{N_1} \\
&< \alpha^2 + (2\alpha+1)\epsilon_1 && \text{since } \frac{1}{N_1} < \epsilon_1 \\
&= \alpha^2 + (2\alpha+1)\left(\frac{2-\alpha^2}{2\alpha+1}\right) \\
&= \alpha^2 + (2-\alpha^2) = 2.
\end{aligned}$$

Hence $x_1 \in A$. Since $x_1 > \alpha$ and $x_1 \in A$, we have a contradiction to the fact that α was an upper bound of A. Hence $\alpha^2 \geq 2$.

Suppose for the sake of a contradiction that $\alpha^2 > 2$. We will show that this contradicts the fact that α is the least upper bound of A by finding an $x_2 \in \mathbb{R}$ such that $x_2 < \alpha$ yet x_2 is an upper bound of A. Again, it is far more helpful to look at how we define x_2, look at the computation, and then go back and pick things correctly to make the calculations work.

Let

$$\epsilon_2 = \frac{\alpha^2 - 2}{2\alpha}.$$

Note ϵ_2 is well-defined since $\alpha \geq 1$ so that $2\alpha > 0$. Moreover, since $\alpha^2 - 2 > 0$, we see that $\epsilon_2 > 0$. Hence the Archimedean Property (Theorem 1.3.7) implies there exists an $N_2 \in \mathbb{N}$ such that $0 < \frac{1}{N_2} < \epsilon_2$.

Let $x_2 = \alpha - \frac{1}{N_2}$. Clearly $x_2 < \alpha$. Moreover, since $\alpha \geq 1$ and $\frac{1}{N_2} \leq 1$, we have that $x_2 \geq 0$.

Notice that

$$\begin{aligned}
x_2^2 &= \left(\alpha - \frac{1}{N_2}\right)^2 \\
&= \alpha^2 - 2\alpha\frac{1}{N_2} + \frac{1}{N_2^2} \\
&> \alpha^2 - 2\alpha\frac{1}{N_2} && \text{since } \frac{1}{N_2^2} > 0 \\
&> \alpha^2 - 2\alpha\epsilon_2 && \text{since } \frac{1}{N_2} < \epsilon_2 \\
&= \alpha^2 - 2\alpha\left(\frac{\alpha^2-2}{2\alpha}\right) \\
&= \alpha^2 - (\alpha^2 - 2) = 2.
\end{aligned}$$

We claim that $x_2^2 > 2$ implies that x_2 is an upper bound of A. To see this, suppose to the contrary that x_2 is not an upper bound of A. Thus there must exists an $x \in A$ such that $x_2 < x$. However, this implies since $x_2 \geq 0$ that $x_2^2 < x^2 < 2$. Since this contradicts the fact that $x_2^2 > 2$, we have our contradiction. Hence x_2 is an upper bound of A.

Since x_2 is an upper bound of A and since $x_2 < \alpha$, α is not the least upper bound of A. Hence we have a contradiction.

Since $\alpha^2 < 2$ and $\alpha^2 > 2$ have both lead to contradictions, we have that $\alpha^2 = 2$ as desired.

Finally, to see that α is the unique element of $\mathbb{R}$ with the desired properties, assume $\beta \in \mathbb{R}$ is such that $\beta > 0$ and $\beta^2 = 2$. To see that $\beta = \alpha$, we will show that $\beta < \alpha$ and $\beta > \alpha$ both lead to contradictions.

Suppose for the sake of a contradiction that $\beta < \alpha$. Then $0 < \beta < \alpha$ and thus $2 = \beta^2 < \alpha^2 = 2$, which is absurd. Hence we have a contradiction in this case.

Similarly, suppose for the sake of a contradiction that $\beta > \alpha$. Then $\beta > \alpha > 0$ and thus $2 = \beta^2 > \alpha^2 = 2$, which is absurd. Hence we have a contradiction in this case.

Since $\beta < \alpha$ and $\beta > \alpha$ have both lead to contradictions, we have that $\beta = \alpha$. Hence α is the unique element of $\mathbb{R}$ such that $\alpha > 0$ and $\alpha^2 = 2$. ∎

Of course, one could repeat the proof of Proposition 1.3.10 to show that $\sqrt{3}, \sqrt{5}, \sqrt{7}$, etc. all are real numbers. Furthermore, one can repeat the same idea using the Binomial Theorem to show that n^{th} roots exist. To show the existence such elements (and more elements) in the real numbers, it is prefer to take the approach that $x \mapsto \sqrt[n]{x}$ is the inverse of $x \mapsto x^n$ although with the theory of Chapter 5 (see Corollary 4.5.5).

1.3.4 The Absolute Value Function

To complete this section, it is useful throughout the course to consider the following function.

Definition 1.3.11. Given $x \in \mathbb{R}$, the *absolute value* of x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

The absolute value has many important properties and uses in analysis. Notice that $|x|$ represents the distance from x to 0. Consequently, we can also see that $|x - y|$ represents the distance from x to y for all $x, y \in \mathbb{R}$. In particular, the absolute value function has many properties one would expect of a 'distance function'.

Lemma 1.3.12. *The absolute value function has the following properties:*

a) $|x| = 0$ *if and only if* $x = 0$.

b) For $x, y \in \mathbb{R}$, *the distance* x *to* y *in* $\mathbb{R}$ *is zero if and only if* $x = y$.

c) $|-x| = |x|$ *for all* $x \in \mathbb{R}$.

d) $|xy| = |x||y|$ *for all* $x, y \in \mathbb{R}$.

e) For all $x, y \in \mathbb{R}$, *the distance from* x *to* y *is equal to the distance from* y *to* x.

Proof. a) This follows immediately from the definition of the absolute value.

b) The distance from x to y is zero if and only if $|x - y| = 0$ if and only if $x - y = 0$ (by part a)) if and only if $x = y$ as desired.

c) Note if $x = 0$ then $|-x| = |-0| = |0| = |x|$. If $x > 0$ then $-x < 0$ so $|-x| = -(-x) = x = |x|$. Finally, if $x < 0$ then $-x > 0$ so $|-x| = -x = |x|$. Hence the result follows.

d) Consider the following four cases.

Case 1: $x, y \geq 0$. In this case $xy \geq 0$ so $|xy| = xy = |x||y|$.

Case 2: $x, y < 0$. In this case $xy > 0$ so that $|xy| = xy = (-1)^2 xy = (-x)(-y) = |x||y|$.

Case 3: $x \geq 0$ and $y < 0$. In this case $xy \leq 0$ so that $|xy| = -xy = x(-y) = |x||y|$.

Case 4: $x < 0$ and $y \geq 0$. In this case $xy \leq 0$ so $|xy| = -xy = (-x)y = |x||y|$.

Thus, as these four cases cover all possible cases, the result is true.

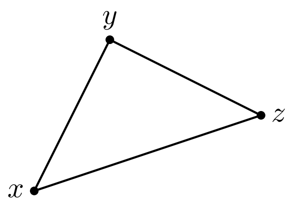
e) Notice for all $x, y \in \mathbb{R}$ that $|y-x| = |(-1)(x-y)| = |-1||x-y| = |x-y|$. Hence the distance from x to y is equal to the distance from y to x. ∎

Finally, there is one more property our ‘distance function’ should have.

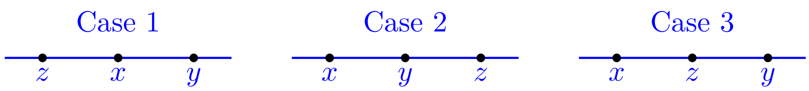
Proposition 1.3.13 (The Triangle Inequality). *Let $x, y, z \in \mathbb{R}$. Then*

$$|x - y| \leq |x - z| + |z - y|.$$

That is, the distance from x to y is no more than the sum of the distance from x to z and the distance from z to y.



Proof. If $x = y$, the result is trivial to verify. Consequently we will assume $x < y$ (if $y < x$, we can relabel y with x and x with y to run the following proof). We have three cases to consider.



Case 1. $z < x$: In this case, notice

$$|x - y| \leq |z - y| = 0 + |z - y| \leq |x - z| + |z - y|$$

as desired.

Case 2. $y < z$: In this case, notice

$$|x - y| \leq |x - z| = |x - z| + 0 \leq |x - z| + |z - y|$$

as desired.

Case 3. $x \leq z \leq y$: In this case, we easily see that

$$|x - y| = |x - z| + |z - y|.$$

Hence, as we have exhausted all cases (up to flipping x and y), the proof is complete. ∎

The Triangle Inequality is an incredibly useful tool in analysis. Furthermore, there are many other forms of the Triangle Inequality. For example, letting $x = a$, $y = -b$, and $z = 0$ in the inequality in Proposition 1.3.13 produces

$$|a + b| \leq |a| + |b| \qquad \text{for all } a, b \in \mathbb{R}.$$

In addition, if we let $x = a$, $y = 0$, and $z = b$, we obtain

$$|a| \leq |a - b| + |b| \qquad \text{so} \qquad |a| - |b| \leq |a - b|,$$

and if we let $x = b$, $y = 0$, and $z = a$, we obtain

$$|b| \leq |a - b| + |a| \qquad \text{so} \qquad -(|a| - |b|) \leq |a - b|.$$

Consequently, we obtain that

$$||a| - |b|| \leq |a - b| \qquad \text{for all } a, b \in \mathbb{R}.$$

All of these inequalities will be considered the Triangle Inequality and will be of incredible use.

Chapter 2

Limits of Sequences

With our understanding of the real numbers, we can move onto studying more mathematical topics in analysis. The whole premise of mathematical analysis is the ability to take limits; that is, the ability to approximation one object with an collection of other objects. In this section, we will study the formal definition of a limit of a sequence of real numbers. Subsequently, we will develop several tools to determine when sequences of real numbers have a limit and several tools to help compute the limit. By understanding the formal definitions, properties, and results in this section, one has all the basic tools necessary to comprehend more advanced topics in analysis.

2.1 The Formal Definition of a Limit of a Sequence

2.1.1 Sequences

Before discussing limits, let us provide a precise mathematical definition of a sequence.

Definition 2.1.1. A *sequence of real numbers* is an ordered list of real numbers indexed by the natural numbers.

Notation 2.1.2. If we have $a_k \in \mathbb{R}$ for all $k \in \mathbb{N}$, we will use $(a_n)_{n\geq 1}$ or $(a_1, a_2, a_3, \ldots)$ to denote a sequences whose first element is a_1, whose second element is a_2, etc.

Example 2.1.3. If $c \in \mathbb{R}$ and $a_n = c$ for all $n \in \mathbb{N}$, then the sequence $(a_n)_{n\geq 1}$ is the *constant sequence* with value c.

Example 2.1.4. For all $n \in \mathbb{N}$, let $a_n = \frac{1}{n}$. Then $(a_n)_{n\geq 1}$ is the sequence $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$.

Example 2.1.5. For all $n \in \mathbb{N}$, let $a_n = (-1)^{n+1}$. Then $(a_n)_{n\geq 1}$ is the sequence $(1, -1, 1, -1, 1, -1, \ldots)$.

Example 2.1.6. Let $a_1 = 1$ and $a_2 = 1$. For $n \in \mathbb{N}$ with $n \geq 3$, let $a_n = a_{n-1} + a_{n-2}$. Then $(a_n)_{n\geq 1}$ is the sequence

$$(1, 1, 2, 3, 5, 8, 13, \ldots).$$

This sequence is known as the *Fibonacci sequence* and is an example of a *recursively defined sequence* (a sequence where subsequent terms are defined using the previous terms under a fixed pattern).

2.1.2 Definition of a Limit

With the above definition of a sequence, we turn to providing a precise definition of a limit of a sequence of real numbers. If we consider the sequence $(\frac{1}{n})_{n\geq 1}$, we intuitively know that as n gets larger and larger, the sequence gets closer and closer to zero. Thus we would want to use this to say that 0 is the limit of $(\frac{1}{n})_{n\geq 1}$. This may lead us to take the following as our definition of a limit:

"A sequence $(a_n)_{n\geq 1}$ has limit L (as n tends to infinity) if as n gets larger and larger, a_n gets closer to L."

However, the fault in the above idea of a limit is that $(\frac{1}{n})_{n\geq 1}$ also gets 'closer and closer' to -1. We prefer 0 over -1 as the limit of $(\frac{1}{n})_{n\geq 1}$ since $\frac{1}{n}$ better and better approximates 0 whereas we intuitively know that $(\frac{1}{n})_{n\geq 1}$ cannot approximate -1. This leads us to the following better idea of what a limit is:

Heuristic Definition. A sequence $(a_n)_{n\geq 1}$ has limit L if the terms of $(a_n)_{n\geq 1}$ are eventually all approximately L.

Using the above as a guideline, we obtain a rigorous, mathematical definition of the limit of a sequence of real numbers.

Definition 2.1.7. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. A number $L \in \mathbb{R}$ is said to be the *limit* of $(a_n)_{n\geq 1}$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ (which depends on ϵ) such that $|a_n - L| < \epsilon$ for all $n \geq N$.

If $(a_n)_{n\geq 1}$ has limit L, we say that $(a_n)_{n\geq 1}$ *converges* to L and write $L = \lim_{n\to\infty} a_n$. Otherwise, if L is not a limit of $(a_n)_{n\geq 1}$ for all $L \in \mathbb{R}$, we say that $(a_n)_{n\geq 1}$ *diverges*.

Example 2.1.8. Consider the constant sequence $(a_n)_{n\geq 1}$ where $a_n = c$ for all $n \in \mathbb{N}$ and some $c \in \mathbb{R}$. Notice for all $\epsilon > 0$, $|a_n - c| = 0 < \epsilon$ for all $n \in \mathbb{N}$. Hence $(a_n)_{n\geq 1}$ converges to c.

Example 2.1.9. To see that $\lim_{n\to\infty} \frac{1}{n} = 0$ using the definition of the limit, let $\epsilon > 0$ be arbitrary. Then by the Archimedean Property (Theorem 1.3.7) there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{N} < \epsilon$. Therefore, for all $n \geq N$

we obtain that $0 < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Hence $\left|\frac{1}{n} - 0\right| < \epsilon$ for all $n \geq N$. Hence $\lim_{n\to\infty} \frac{1}{n} = 0$. Note that $(\frac{1}{n})_{n\geq 1}$ has limit zero, but no term in the sequence is zero.

Remark 2.1.10. By negating Definition 2.1.7, we see that a sequence $(a_n)_{n\geq 1}$ does not converge to L if there is an $\epsilon > 0$ (depending on the L) such that for every $N \in \mathbb{N}$ there is an $n \geq N$ such that $|a_n - L| \geq \epsilon$.

Hence a sequence $(a_n)_{n\geq 1}$ diverges if for all $L \in \mathbb{R}$ there is an $\epsilon > 0$ (depending on the L) such that for every $N \in \mathbb{N}$ there is an $n \geq N$ such that $|a_n - L| \geq \epsilon$.

Example 2.1.11. Using Remark 2.1.10, we can show that $((-1)^{n+1})_{n\geq 1}$ does not converge. Indeed let $L \in \mathbb{R}$ be arbitrary and let $\epsilon = \frac{1}{2}$. Suppose there exists an $N \in \mathbb{N}$ such that $|(-1)^{n+1} - L| < \epsilon$ for all $n \geq N$. Since there exists an odd number n greater than N, we obtain that $|1 - L| < \epsilon$. Therefore, since $\epsilon = \frac{1}{2}$, we obtain that $\frac{1}{2} < L < \frac{3}{2}$. Similarly, since there exists an even number n greater than N, we obtain that $|-1-L| < \epsilon$. Therefore, since $\epsilon = \frac{1}{2}$, we obtain that $-\frac{3}{2} < L < -\frac{1}{2}$. Hence $L < -\frac{1}{2}$ and $L > \frac{1}{2}$, which is absurd. Hence we have a contradiction so L is not the limit of $((-1)^{n+1})_{n\geq 1}$. Therefore, since $L \in \mathbb{R}$ was arbitrary, $((-1)^{n+1})_{n\geq 1}$ does not converge.

2.1.3 Uniqueness of the Limit

Notice in the definition of 'the' limit of a sequence, we used 'the' instead of 'a'; that is, how do we know that there is at most one limit to a sequence? The following justifies the use of the word 'the' and demonstrates one important proof technique when dealing with limits.

Proposition 2.1.12. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. If L and K are limits of $(a_n)_{n\geq 1}$, then $L = K$.*

Proof. We will provide two different (but basically the same) proofs of this fact.

For the first, we will provide a direct proof. Assume L and K are limits of $(a_n)_{n\geq 1}$. To see that $L = K$, let $\epsilon > 0$. Since L is a limit of $(a_n)_{n\geq 1}$, we know by the definition of a limit that there exists an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|a_n - L| < \epsilon$. Similarly, since K is a limit of $(a_n)_{n\geq 1}$, we know by the definition of a limit that there exists an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|a_n - K| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. By the above paragraph, we have that $|a_N - L| < \epsilon$ and $|a_N - K| < \epsilon$. Hence by the Triangle Inequality

$$|L - K| \leq |L - a_N| + |a_N - K| < \epsilon + \epsilon = 2\epsilon.$$

Therefore, we have obtained that $|L - K| < 2\epsilon$ for all $\epsilon > 0$.

We claim that this implies $|L - K| = 0$ thereby completing the proof. To see this, we claim that if $x \in \mathbb{R}$ and $0 \leq x < 2\epsilon$ for all $\epsilon > 0$, then $x = 0$. To see this, suppose to the contrary that $x \in \mathbb{R}$, $0 \leq x < 2\epsilon$ for all $\epsilon > 0$, but $x \neq 0$. Thus $x > 0$. Therefore if we take $\epsilon = \frac{1}{2}x$, then $\epsilon > 0$ and $2\epsilon = x$. Since this contradicts the fact that $x < 2\epsilon$ for all $\epsilon > 0$, we have our contradiction. Hence $|L - K| = 0$ so $L = K$ as desired.

For the second, we will provide a proof by contradiction. Suppose for the sake of a contradiction that $L \neq K$. Let $\epsilon = \frac{|L-K|}{2}$. Since $L \neq K$, we know that $\epsilon > 0$.

Since L is a limit of $(a_n)_{n\geq 1}$, we know by the definition of a limit that there exists an $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|a_n - L| < \epsilon$. Similarly, since K is a limit of $(a_n)_{n\geq 1}$, we know by the definition of a limit that there exists an $N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then $|a_n - K| < \epsilon$.

Let $N = \max\{N_1, N_2\}$. By the above paragraph, we have that $|a_N - L| < \epsilon$ and $|a_N - K| < \epsilon$. Hence by the Triangle Inequality

$$|L - K| \leq |L - a_N| + |a_N - K| < \epsilon + \epsilon = 2\epsilon = |L - K|$$

which is absurd (i.e. $x < x$ is false for all $x \in \mathbb{R}$). Thus we have obtained a contradiction so it must be the case that $L = K$. ∎

Remark 2.1.13. It should be noted the idea of taking $N = \max\{N_1, N_2\}$ will be quite prevalent in analysis proofs. This is due to the fact that if one has a condition that holds for $n \geq N_1$ and another condition that holds for $n \geq N_2$, then one can ensure both conditions hold for all $n \geq \max\{N_1, N_2\}$. Note the same holds for any finite number $N_1, \ldots, N_k$, but need not hold for an infinite number $N_1, N_2, \ldots$ since there need not be any natural numbers n such that $n \geq N_k$ for all $k \in \mathbb{N}$.

2.1.4 Equivalent Formulations of the Limit

To conclude this section, we note the following that demonstrates that $|a_n - L| < \epsilon$ may be replaced with $|a_n - L| \leq \epsilon$ in the definition of the limit of a sequence. This can be useful on occasion and also establishes an important idea in handling limits: ϵ is simply a constant and may be modified. In particular, the proof of the following simply comes down to dealing with quantifies.

Proposition 2.1.14. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers, let $k > 0$, and let $L \in \mathbb{R}$. Then $(a_n)_{n\geq 1}$ converges to L if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - L| \leq k\epsilon$ for all $n \geq N$.*

Proof. Assume that $(a_n)_{n\geq 1}$ converges to L. To see the desired result, let $\epsilon > 0$ be arbitrary. Let $\epsilon_0 = k\epsilon$. Since $\epsilon > 0$ and $k > 0$, $\epsilon_0 > 0$. Hence, by the definition of the limit, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon_0$ for all $n \geq N$. Thus $|a_n - L| \leq \epsilon_0 = k\epsilon$ for all $n \geq N$. Therefore, as $\epsilon > 0$ was arbitrary, one direction of the proof is complete.

For the other direction, assume that $(a_n)_{n\geq 1}$ and L have the property listed in the statement of this proposition. To see that $(a_n)_{n\geq 1}$ converges to L, let $\epsilon > 0$ be arbitrary. Let $\epsilon_0 = \frac{\epsilon}{2k}$. Since $\epsilon > 0$ and $k > 0$, we know that $\epsilon_0 > 0$. Therefore, by the assumptions of this direction imply that there exists an $N \in \mathbb{N}$ such that $|a_n - L| \leq \epsilon_0$ for all $n \geq N$. Hence $|a_n - L| \leq \epsilon_0 < 2\epsilon_0 = k\epsilon$ for all $n \geq N$. As $\epsilon > 0$ was arbitrary, $(a_n)_{n\geq 1}$ converges to L by the definition of the limit. ∎

Remark 2.1.15. It is very important in Proposition 2.1.14 to note that the constant k CANNOT depend on n nor ϵ; k must be a fixed positive real number that exists independently of n and ϵ. Indeed, if we could choose k after we chose ϵ, we could have chose $k = \frac{1}{\epsilon}$ and thus the condition $|a_n - L| \leq k\epsilon$ would always equate to $|a_n - L| \leq 1$, which is very different from the definition of a limit.

2.2 The Monotone Convergence Theorem

With a formal definition of a limit complete, there are two main natural questions for us to ask: "Are there methods for determining certain sequences converge without appealing to the definition?" and "How can we find the limits of sequences without always appealing to the definition?" These two goals are intertwined. Let's begin with some elementary results related to the first question.

Consider the question, "Does the sequence $(n)_{n\geq 1}$ converge?" Intuitively the answer is no since this sequence does not approximate a number. In particular, the Archimedean Property (Theorem 1.3.6) implies that $\mathbb{N}$ is not bounded. This is the true reason $(n)_{n\geq 1}$ does not converge. To make this rigorous, consider the following.

Definition 2.2.1. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to be *bounded, sequence* if the set $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Before we show that convergent sequences must be bounded, we note the following useful equivalent characterization of boundedness that we will use quite often.

Lemma 2.2.2. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Then $(a_n)_{n\geq 1}$ is bounded if and only if there exists an $M \in \mathbb{R}$ such that $M > 0$ and $|a_n| \leq M$ for all $n \in \mathbb{N}$.*

Proof. First, assume that $(a_n)_{n\geq 1}$ is bounded. Hence, by the definition of a bounded sequence, $\{a_n \mid n \in \mathbb{N}\}$ has an upper and lower bound. Therefore there exists $m_1, m_2 \in \mathbb{R}$ such that $m_1 \leq a_n \leq m_2$ for all $n \in \mathbb{N}$. Hence, by taking $M = \max\{|m_1|, |m_2|, 1\}$, we obtain that $M \in \mathbb{R}$, $M > 0$, and $-M \leq m_1 \leq a_n \leq m_2 \leq M$ for all $n \in \mathbb{N}$. Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$ as desired.

Conversely, assume $M \in \mathbb{R}$ is such that $M > 0$ and $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$. Therefore $\{a_n \mid n \in \mathbb{N}\}$ is bounded below by $-M$ and bounded above by M. Hence $(a_n)_{n\geq 1}$ is bounded. ∎

Proposition 2.2.3. *Every convergent sequence is bounded.*

Proof. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers that converge to a number $L \in \mathbb{R}$. Let $\epsilon = 1$. By the definition of a limit, there exists an $N \in \mathbb{N}$ such that $|a_n - L| \leq \epsilon = 1$ for all $n \geq N$. Hence $|a_n| \leq |L| + 1$ for all $n \geq N$ by the Triangle Inequality.

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_N|, |L| + 1\}$. Clearly if $n \leq N$ then $|a_n| \leq M$ whereas if $n \geq N$ then $|a_n| \leq L + 1 \leq M$ by the above paragraph. Hence $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$ so $(a_n)_{n\geq 1}$ is bounded. ∎

Remark 2.2.4. The above shows us that boundness is a requirement for a sequence to converge. However, a bounded sequence need not converge. Indeed Example 2.1.11 shows that the sequence $((-1)^{n+1})_{n\geq 1}$ (which is clearly bounded) does not converge.

Luckily, there is a simple condition one can combine with the notion of boundedness to guarantee that certain sequences converge.

Definition 2.2.5. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to be

- *increasing* if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$,
- *non-decreasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$,
- *decreasing* if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$,
- *non-increasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$, and
- *monotone* if $(a_n)_{n\geq 1}$ is non-decreasing or non-increasing.

The following result shows a sequence being monotone and bounded is enough to guarantee the sequence converges. Indeed, if a sequence is non-decreasing and bounded, we expect the limit to be the least upper bound of the terms in the sequence by the definition of the least upper bound. Let's see why this is the case.

Theorem 2.2.6 (Monotone Convergence Theorem). *A monotone sequence $(a_n)_{n\geq 1}$ of real numbers converges if and only if $(a_n)_{n\geq 1}$ is bounded.*

Proof. By Proposition 2.2.3, if $(a_n)_{n\geq 1}$ converges, then $(a_n)_{n\geq 1}$ is bounded.

For the other direction, assume that $(a_n)_{n\geq 1}$ is a monotone sequence that is bounded. We will assume that $(a_n)_{n\geq 1}$ is a non-decreasing sequence for the remainder of the proof as the case when $(a_n)_{n\geq 1}$ is non-increasing can be proved using similar arguments.

Since $(a_n)_{n\geq 1}$ is bounded, $\{a_n \mid n \in \mathbb{N}\}$ has a least upper bounded, say α, by the Least Upper Bound Property (Theorem 1.2.35). We claim that α is the limit of $(a_n)_{n\geq 1}$. To see this, let $\epsilon > 0$ be arbitrary. Since α is the least upper bound of $\{a_n \mid n \in \mathbb{N}\}$, we know that $a_n \leq \alpha$ for all $n \in \mathbb{N}$ and $\alpha - \epsilon$ is not an upper bound of $\{a_n \mid n \in \mathbb{N}\}$. Hence there exists an $N \in \mathbb{N}$ such that $\alpha - \epsilon < a_N$. Since $(a_n)_{n\geq 1}$ is non-decreasing, we obtain for all $n \geq N$ that

$$\alpha - \epsilon < a_N \leq a_n \leq \alpha,$$

which implies $|a_n - \alpha| < \epsilon$ for all $n \geq N$. Since $\epsilon > 0$ was arbitrary, we obtain that α is the limit of $(a_n)_{n\geq 1}$ by definition. Hence $(a_n)_{n\geq 1}$ converges. ∎

The Monotone Convergence Theorem can be quite useful in showing certain sequences converge. In particular, consider the following two examples where in the second we do not even have a formula for the n^{th} term of the sequence.

Example 2.2.7. Let $0 \leq x < 1$. Consider the sequence $(x^n)_{n\geq 1}$. Since $0 \leq x < 1$, we see that $0 \leq x^{n+1} < x^n < 1$ for all $n \in \mathbb{N}$. Hence $(x^n)_{n\geq 1}$ is a bounded monotone sequence. Therefore $(x^n)_{n\geq 1}$ converges by the Monotone Convergence Theorem.

Example 2.2.8. Consider the sequence $(a_n)_{n\geq 1}$ defined recursively via $a_1 = 1$ and $a_{n+1} = \sqrt{5 + 4a_n}$ for all $n \geq 1$. We claim that $0 \leq a_n \leq a_{n_1} \leq 5$ for all $n \in \mathbb{N}$.

To see this, for each $n \in \mathbb{N}$ let P_n be the statement that "a_n and a_{n+1} are well-defined real numbers and $0 \leq a_n \leq a_{n+1} \leq 5$". (Note we need to include the "well-defined" part since we cannot take a square root of a negative number inside the real numbers and a priori we do not know that $a_{n+1} = \sqrt{5 + 4a_n}$ guarantees a_{n+1} is a real number for all n). We claim that P_n is true for all $n \in \mathbb{N}$. To show that P_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction. To do so, we must demonstrate the two conditions in Theorem 1.1.7.

Base Case: To see that P_1 is true, notice that when $n = 1$, that $a_n = a_1 = 1$ and $a_{n+1} = a_2 = \sqrt{5 + 4(1)} = \sqrt{9} = 3$ are well-defined. Moreover

$$0 \leq a_1 \leq a_2 \leq 3 \leq 5.$$

Hence P_1 is true.

Inductive Step: Assume that P_k is true; that is, assume a_k and a_{k+1} are well-defined real numbers and $0 \leq a_k \leq a_{k+1} \leq 5$. To see that P_{k+1} is true, first notice a_{k+1} is a well-defined real number. Moreover, since $a_{k+1} \geq 0$ so that $5 + 4a_{k+1} \geq 0$, we see that $a_{(k+1)+1} = \sqrt{5 + 4a_{k+1}}$ is a well-defined real number. Since we already have that $0 \leq a_{k+1}$ by the induction hypothesis, it remains to show that $a_{k+1} \leq a_{k+2}$ and $a_{k+2} \leq 5$.

Notice since $0 \leq a_k \leq a_{k+1}$ by the induction hypothesis, we have that $0 \leq 4a_k \leq 4a_{k+1}$. Hence $0 \leq 5 + 4a_k \leq 5 + 4a_{k+1}$. Therefore, by taking the square root of both sides, we obtain that

$$a_{k+1} = \sqrt{5 + 4a_k} \leq \sqrt{5 + 4a_{k+1}} \leq a_{k+2}$$

as desired. Moreover, notice since $a_{k+1} \leq 5$ that $5 + 4a_{k+1} \leq 25$. Hence, by taking square root of both sides, we obtain that

$$a_{k+2} = \sqrt{5 + 4a_{k+1}} \leq \sqrt{25} = 5$$

as desired. Hence P_{k+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the result follows by the Principle of Mathematical Induction.

Hence $(a_n)_{n\geq 1}$ is a bounded monotone sequence and thus converges by the Monotone Convergence Theorem.

Of course, although we know the sequences in Examples 2.2.7 and 2.2.8 converge, we have yet to compute their limits. Of course, we know by the proof of the Monotone Convergence Theorem that the limits are related to the greatest lower bound and least upper bounds respectively. Although we might have some luck showing that 0 is the greatest lower bound of $(x^n)_{n\geq 1}$ when $0 \leq x < 1$, if $(a_n)_{n\geq 1}$ is as in Example 2.2.8, all we know at the moment is that $\text{lub}(\{a_n \mid n \in \mathbb{N}\})$, which is at most 5. But is the answer 5 or a number less than 5? What tools do we have to help us compute limits?

2.3 Computing Limits

To answer the above questions and aid us in our computation of limits, there are several theorems we may explore to aid us.

2.3.1 Limit Arithmetic

Our first goal is to determine how limits behave with respect to the simplest operations on $\mathbb{R}$. Understanding the proofs provided below are essential understanding the definition of a limit.

Theorem 2.3.1. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers such that $L = \lim_{n\to\infty} a_n$ and $K = \lim_{n\to\infty} b_n$ exist. Then*

a) $\lim_{n\to\infty} a_n + b_n = L + K$.

b) $\lim_{n\to\infty} a_n b_n = LK$.

c) $\lim_{n\to\infty} ca_n = cL$ *for all* $c \in \mathbb{R}$.

d) If $K \neq 0$, *then there exists an* $N_0 \in \mathbb{N}$ *such that* $b_n \neq 0$ *for all* $n \geq N_0$.

e) If $K \neq 0$ *then* $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{K}$ *(where we start* b_n *at* $n = N_0$ *as in part (d)).*

f) If $K \neq 0$ *then* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$ *(where we start* b_n *at* $n = N_0$ *as in part (d)).*

Proof. To create all of these proofs, it is useful to work backwards. In particular, one should think about what one needs to show in order for a sequence to be a limit, do some computations, and then check we have the conditions based on the assumptions to make things work. Thus, when reading these proofs for the first time (if you did NOT come to class) it is useful to look a the computations involving the inequalities near the end of each portion of the proof, see what we are trying to approximate to be small, and then go back and see we could do this. We present the proof as follows instead of adding the motivation in as we go in order for the reader to see how simple and elegant these proofs are once they are put together.

a) Let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} a_n$, there exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$ for all $n \geq N_1$. Similarly, since $K = \lim_{n\to\infty} b_n$, there exists an $N_2 \in \mathbb{N}$ such that $|b_n - K| < \frac{\epsilon}{2}$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$. Hence, using the Triangle Inequality, for all $n \geq N$,

$$\begin{aligned} |(a_n + b_n) - (L + K)| &= |(a_n - L) + (b_n - K)| \\ &\leq |a_n - L| + |b_n - K| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $(a_n + b_n)_{n\geq 1}$ converges to $L + K$ by definition.

b) Let $\epsilon > 0$ be arbitrary. First note that $0 \leq |K| < |K| + 1$ so $0 \leq \frac{|K|}{|K|+1} \leq 1$ (we will use this later). Next, since $(a_n)_{n\geq 1}$ convergence, $(a_n)_{n\geq 1}$ is bounded by Proposition 2.2.3. Hence there exists an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Since $L = \lim_{n\to\infty} a_n$, there exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2(|K|+1)}$ for all $n \geq N_1$ (as $\frac{1}{2(|K|+1)} > 0$ is a constant and does not depend on ϵ nor n - see Proposition 2.1.14). Similarly, since $K = \lim_{n\to\infty} b_n$, there exists an $N_2 \in \mathbb{N}$ such that $|b_n - K| < \frac{\epsilon}{2M}$ for all $n \geq N_2$ (as $\frac{1}{2M}$ is a constant and does not depend on ϵ nor n). Let $N = \max\{N_1, N_2\}$. Hence, using the

Triangle Inequality, for all $n \geq N$,

$$\begin{aligned}
|a_n b_n - LK| &= |(a_n b_n - a_n K) + (a_n K - LK)| \\
&\leq |a_n b_n - a_n K| + |a_n K - LK| \\
&\leq |a_n||b_n - K| + |K||a_n - L| \\
&\leq M|b_n - K| + |K||a_n - L| \\
&\leq M\frac{\epsilon}{2M} + |K|\frac{\epsilon}{2(|K|+1)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence $(a_n b_n)_{n\geq 1}$ converges to LK by definition.

c) Given $c \in \mathbb{R}$, the constant sequence $(c)_{n\geq 1}$ converges to c. Hence part (c) follows from part (b) by taking $b_n = c$ for all $n \in \mathbb{N}$.

d) Assume $K \neq 0$. Let $\epsilon = \frac{|K|}{2} > 0$. Since $K = \lim_{n\to\infty} b_n$, there exists an $N_0 \in \mathbb{N}$ such that $|b_n - K| < \frac{|K|}{2}$ for all $n \geq N_0$. Therefore, by the Triangle Inequality,

$$|b_n| \geq |K| - \frac{|K|}{2} = \frac{|K|}{2} > 0$$

for all $n \geq N_0$. Hence, if $n \geq N_0$ we have that $|b_n| > 0$. Therefore $b_n \neq 0$ for all $n \geq N_0$.

e) Let $\epsilon > 0$ be arbitrary. By repeating the proof of part (d), there exists an $N_1 \in \mathbb{N}$ such that $|b_n| \geq \frac{|K|}{2}$ for all $n \geq N_1$. Thus $\frac{1}{|b_n|} \leq \frac{2}{|K|}$ for all $n \geq N_1$ (as $K \neq 0$). Since $K = \lim_{n\to\infty} b_n$, there exists an $N_2 \in \mathbb{N}$ such that $|b_n - K| < \frac{\epsilon|K|^2}{2}$ for all $n \geq N_2$ (as $\frac{|K|^2}{2} > 0$ is a constant and does not depend on ϵ nor n). Therefore, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{K}\right| &= \frac{|K - b_n|}{|b_n||K|} \\
&\leq \frac{\epsilon|K|^2}{2|b_n||K|} \\
&\leq \frac{\epsilon|K|}{2}\frac{1}{|b_n|} \\
&\leq \frac{\epsilon|K|}{2}\frac{2}{|K|} = \epsilon.
\end{aligned}$$

Hence $(\frac{1}{b_n})_{n\geq 1}$ converges to $\frac{1}{K}$ by definition.

f) By part (e), $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{K}$. Hence, as $\lim_{n\to\infty} a_n = L$, part (b) implies that $\lim_{n\to\infty} a_n \frac{1}{b_n} = L\frac{1}{K}$ completing the proof. ∎

Using Theorem 2.3.1, we can compute a lot of limits with our limited knowledge.

Example 2.3.2. For $0 \leq x < 1$, consider the sequence $(x^n)_{n\geq 1}$. In Example 2.2.7, we saw that $(x^n)_{n\geq 1}$ converges. To compute its limit, let $L = \lim_{n\to\infty} x^n$. Notice by Theorem 2.3.1 that

$$\begin{aligned} L &= \lim_{n\to\infty} x^n \\ &= \lim_{n\to\infty} x^{n+1} && \text{index shift does not change the limit} \\ &= \lim_{n\to\infty} x(x^n) \\ &= xL && \text{by Theorem 2.3.1 part (c).} \end{aligned}$$

Therefore $L(1-x) = 0$. Hence, as $1 - x \neq 0$, we obtain that $L = 0$. Thus $\lim_{n\to\infty} x^n = 0$. (Was there any surprise?)

Example 2.3.3. Consider the sequence $(a_n)_{n\geq 1}$ defined recursively via $a_1 = 1$ and $a_{n+1} = \sqrt{5 + 4a_n}$ for all $n \geq 1$. In Example 2.2.8, we used the Monotone Convergence Theorem (Theorem 2.2.6) along with the fact that $0 \leq a_n \leq a_{n+1} \leq 5$ to show that $(a_n)_{n\geq 1}$ converges. It remains to compute the limit of this sequence.

Let $L = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \sqrt{5 + 4a_n}$ for all $n \geq 1$, we have that $a_{n+1}^2 = 5 + 4a_n$ for all $n \in \mathbb{N}$. Therefore, using Theorem 2.3.1, we obtain that

$$\begin{aligned} 5 + 4L = \lim_{n\to\infty} 5 + 4a_n &= \lim_{n\to\infty} a_{n+1}^2 \\ &= \lim_{n\to\infty} a_n^2 \quad \text{index shift does not change the limit} \\ &= \left(\lim_{n\to\infty} a_n\right)^2 = L^2. \end{aligned}$$

Hence $L^2 - 4L - 5 = 0$ so $(L-5)(L+1) = 0$ so $L = 5$ or $L = -1$. However, since $-1 < 0 < 1 = a_1 \leq a_n$ for all $n \in \mathbb{N}$, $|a_n - (-1)| \geq 2$ for all $n \in \mathbb{N}$ and thus -1 cannot be the limit of $(a_n)_{n\geq 1}$ by the definition of the limit. Hence $\lim_{n\to\infty} a_n = 5$.

Moreover, we can build up our collection of useful limits.

Example 2.3.4. Let $m \in \mathbb{N}$ be fixed. We claim that the sequence $(\frac{1}{n^m})_{n\geq 1}$ converges and $\lim_{n\to\infty} \frac{1}{n^m} = 0$. Indeed this result easily follows by induction on m since $\lim_{n\to\infty} \frac{1}{n} = 0$ by Example 2.1.9 and by part (b) of Theorem 2.3.1 .

Example 2.3.5. Consider the sequence $(a_n)_{n\geq 1}$ where $a_n = \frac{5n^2+2n}{3n^2-n+4}$ for all $n \in \mathbb{N}$. Does $(a_n)_{n\geq 1}$ converge and, if so, what is its limit?

To answer this question, notice that

$$a_n = \frac{5n^2 + 2n}{3n^2 - n + 4} = \frac{n^2\left(5 + \frac{2}{n^2}\right)}{n^2\left(3 - \frac{1}{n} + \frac{4}{n^2}\right)} = \frac{5 + \frac{2}{n^2}}{3 - \frac{1}{n} + \frac{4}{n^2}}.$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ by Example 2.1.9, and since $\lim_{n\to\infty} \frac{1}{n^2} = 0$ by Example 2.3.4, we obtain that

$$\lim_{n\to\infty} 5 + \frac{2}{n^2} = 5 \quad \text{and} \quad \lim_{n\to\infty} 3 - \frac{1}{n} + \frac{4}{n^2} = 3 \quad \text{so} \quad \lim_{n\to\infty} a_n = \frac{5}{3}.$$

Remark 2.3.6. In part (f) of Theorem 2.3.1, it was required in the proof that the denominator does not converge to 0. This is due to the fact that there are many different types of behaviour that may occur when the denominator of a sequence of fractions tends to zero.

For two examples, first consider the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ where $a_n = b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then clearly $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} b_n$, and

$$\frac{a_n}{b_n} = \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.

Alternatively, consider the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ where $a_n = 1$ and $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then clearly $\lim_{n\to\infty} a_n = 1$ and $\lim_{n\to\infty} b_n = 0$, yet

$$\frac{a_n}{b_n} = \frac{1}{\left(\frac{1}{n}\right)} = n$$

does not converge as $(n)_{n\geq 1}$ is not bounded (see Proposition 2.2.3).

Thus, if $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are sequences and $\lim_{n\to\infty} b_n = 0$, it is possible that $\left(\frac{a_n}{b_n}\right)_{n\geq 1}$ does not converge. However, if $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists, then by part (b) of Theorem 2.3.1 we must have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n}{b_n} b_n = \left(\lim_{n\to\infty} \frac{a_n}{b_n}\right)\left(\lim_{n\to\infty} b_n\right) = \left(\lim_{n\to\infty} \frac{a_n}{b_n}\right)(0) = 0.$$

Thus a necessary condition for $\lim_{n\to\infty} \frac{a_n}{b_n}$ to exist when $\lim_{n\to\infty} b_n = 0$ is $\lim_{n\to\infty} a_n = 0$.

2.3.2 Diverging to Infinity

We have seen several examples of sequences that do not converge. In particular, Proposition 2.2.3 says that unbounded sequences have no chance to converge. However, it is useful to discuss specific notions of divergence for unbounded sequences.

Definition 2.3.7. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to *diverge to infinity*, denoted $\lim_{n\to\infty} a_n = \infty$, if for every $M > 0$ there exists an $N \in \mathbb{N}$ such that $a_n \geq M$ for all $n \geq N$.

Similarly, a sequence $(a_n)_{n\geq 1}$ of real numbers is said to *diverge to negative infinity*, denoted $\lim_{n\to\infty} a_n = -\infty$, if for every $M < 0$ there exists an $N \in \mathbb{N}$ such that $a_n \leq M$ for all $n \geq N$.

Example 2.3.8. It is clear that $\lim_{n\to\infty} n = \infty$.

Example 2.3.9. Consider the sequence $(a_n)_{n\geq 1}$ defined by

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}.$$

Thus $(a_n)_{n\geq 1}$ is the sequence $(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, 0, 6, \ldots)$.

We claim that $(a_n)_{n\geq 1}$ does not diverge to infinity. To see this, let $M = 1$. Then for any $N \in \mathbb{N}$ we see that $2N + 1 \geq N$ and $a_{2N+1} = 0 \not\geq M$ since $2N + 1$ is odd. Thus $(a_n)_{n\geq 1}$ does not diverge to infinity.

Using the same proof ideas as in Theorem 2.3.1, we obtain the following.

Theorem 2.3.10. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of real numbers. Suppose that $(b_n)_{n\geq 1}$ diverges to ∞ (respectively $-\infty$). Then*

a) If $(a_n)_{n\geq 1}$ is bounded below (respectively above), then $\lim_{n\to\infty} a_n + b_n = \infty$ (respectively $\lim_{n\to\infty} a_n + b_n = -\infty$).

b) If there exists a $K > 0$ such that $a_n \geq K$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n b_n = \infty$ (respectively $\lim_{n\to\infty} a_n b_n = -\infty$).

c) If $(a_n)_{n\geq 1}$ is bounded, then $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Proof. We will provide the proofs when $(b_n)_{n\geq 1}$ diverges to ∞. The proofs in the case that $(b_n)_{n\geq 1}$ diverges to $-\infty$ are similar.

Assume $(b_n)_{n\geq 1}$ diverges to ∞.

a) Assume $(a_n)_{n\geq 1}$ is bounded below. To see that $(a_n + b_n)_{n\geq 1}$ diverges to ∞, let $M > 0$ be arbitrary. Since $(a_n)_{n\geq 1}$ is bounded below, by definition there exists a $K \in \mathbb{R}$ such that $a_n \geq K$ for all $n \in \mathbb{N}$. Since $(b_n)_{n\geq 1}$ diverges to ∞, by definition there exists an $N \in \mathbb{N}$ such that $b_n \geq M - K$ for all $n \geq N$. Hence for all $n \geq N$

$$a_n + b_n \geq K + (M - K) = M.$$

Hence $(a_n + b_n)_{n\geq 1}$ diverges to ∞ by definition.

b) Assume there exists an $K > 0$ such that $a_n \geq K$ for all $n \in \mathbb{N}$. To see that $(a_n b_n)_{n\geq 1}$ diverges to ∞, let $M > 0$ be arbitrary. Since $(b_n)_{n\geq 1}$ diverges to ∞ and $K > 0$, by definition there exists an $N \in \mathbb{N}$ such that $b_n \geq \frac{M}{K}$ for all $n \geq N$. Hence for all $n \geq N$

$$a_n b_n \geq K\left(\frac{M}{K}\right) = M.$$

Hence $(a_n + b_n)_{n\geq 1}$ diverges to ∞ by definition.

c) Assume $(a_n)_{n\geq 1}$ is bounded. To see that $\left(\frac{a_n}{b_n}\right)_{n\geq 1}$ converges to 0, let $\epsilon > 0$ be arbitrary. Since $(a_n)_{n\geq 1}$ is bounded, there exists an $K > 0$ such that $|a_n| \leq K$ for all $n \in \mathbb{N}$. Let $M = \frac{K}{\epsilon}$. Clearly $M > 0$. Therefore, since $(b_n)_{n\geq 1}$ diverges to ∞, by definition there exists an $N \in \mathbb{N}$ such that $b_n \geq M = \frac{K}{\epsilon}$ for all $n \geq N$. Hence

$$\frac{1}{b_n} \leq \frac{\epsilon}{K}$$

for all $n \geq N$. Therefore, for all $n \geq N$ we have that

$$\left|\frac{a_n}{b_n} - 0\right| = |a_n| \left|\frac{1}{b_n}\right| \leq K\left(\frac{\epsilon}{K}\right) = \epsilon.$$

Hence $\left(\frac{a_n}{b_n}\right)_{n\geq 1}$ converges to 0 by definition. ∎

The above theorem aids us in computing limits of rational functions where the denominator grows faster than the numerator.

Example 2.3.11. Consider the sequence $(a_n)_{n\geq 1}$ where $a_n = \frac{2n+1}{n^2+3}$ for all $n \in \mathbb{N}$. Then

$$a_n = \frac{n\left(2 + \frac{1}{n}\right)}{n\left(n + \frac{3}{n}\right)} = \frac{2 + \frac{1}{n}}{n + \frac{3}{n}}.$$

Therefore, since $\lim_{n\to\infty} \frac{3}{n} = 0$ so $\left(\frac{3}{n}\right)_{n\geq 1}$ is bounded, and since $\lim_{n\to\infty} n = \infty$, we have $\lim_{n\to\infty} n + \frac{3}{n} = \infty$. Hence since $\lim_{n\to\infty} 2 + \frac{1}{n} = 2$ so $\left(2 + \frac{1}{n}\right)_{n\geq 1}$ is bounded, we have that $\lim_{n\to\infty} \frac{2n+1}{n^2+3} = 0$ by Theorem 2.3.10 part (c).

Example 2.3.12. Consider the sequence $(a_n)_{n\geq 1}$ where $a_n = \frac{\cos(n)}{n}$ for all $n \in \mathbb{N}$. Since $-1 \leq \cos(n) \leq 1$ for all $n \in \mathbb{N}$, the sequence $(\cos(n))_{n\geq 1}$ is bounded. Therefore, since $\lim_{n\to\infty} n = \infty$, we have that $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$ by Theorem 2.3.10 part (c).

2.3.3 The Squeeze Theorem

As an alternatively method to what was done in Example 2.3.12, we can show $\lim_{n\to\infty} \frac{\cos(n)}{n} = 0$ by noting that $-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ and by applying the following useful theorem. In fact, if we proved the following theorem first, we could use it to prove part (c) of Theorem 2.3.10.

Theorem 2.3.13 (Squeeze Theorem). *Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ be sequences of real numbers such that there exists an $N_0 \in \mathbb{N}$ such that*

$$a_n \leq b_n \leq c_n \qquad \text{for all } n \geq N_0.$$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $(b_n)_{n\geq 1}$ converges and $\lim_{n\to\infty} b_n = L$.

Proof. Let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} a_n$, there exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$. Hence $L - \epsilon < a_n$ for all $n \geq N_1$. Similarly, since $L = \lim_{n\to\infty} c_n$, there exists an $N_2 \in \mathbb{N}$ such that $|c_n - L| < \epsilon$ for all $n \geq N_2$. Hence $c_n < L + \epsilon$ for all $n \geq N_2$. Therefore, for all $n \geq \max\{N_0, N_1, N_2\}$, we have that

$$L - \epsilon < a_n \leq b_n \leq c_n \leq L + \epsilon.$$

Hence $L - \epsilon \leq b_n \leq L + \epsilon$ for all $n \geq \max\{N_0, N_1, N_2\}$, which implies $-\epsilon \leq b_n - L \leq \epsilon$ and thus $|b_n - L| < \epsilon$ for all $n \geq \max\{N_0, N_1, N_2\}$. Hence $(b_n)_{n\geq 1}$ converges and $\lim_{n\to\infty} b_n = L$ by definition. ∎

Example 2.3.14. Let $x \in \mathbb{R}$ be such that $-1 < x < 1$. We claim that $\lim_{n\to\infty} x^n = 0$. Indeed, recall from Example 2.3.2 that if $0 \leq x < 1$, then $\lim_{n\to\infty} x^n = 0$.

If $-1 < x < 0$, let $a = |x|$. Then $-a^n \leq x^n \leq a^n$ for all $n \in \mathbb{N}$ (i.e. $x^n = a^n$ if n is even and $x^n = -a^n$ if n is odd). Therefore, since $0 \leq a < 1$, we know from Example 2.3.2 that $\lim_{n\to\infty} a^n = 0$. Hence part (c) of Theorem 2.3.1 we also know that $\lim_{n\to\infty} -a^n = 0$. Thus the Squeeze Theorem implies that $\lim_{n\to\infty} x^n = 0$ as desired.

2.3.4 Limit Supremum and Limit Infimum

There are several sequences that do not converge nor diverge to $\pm\infty$. For example, the sequence $((-1)^{n+1})_{n\geq 1}$ has been shown to not converge and clearly does not diverge to $\pm\infty$ as it is bounded. Consequently, we may ask, "Is it possible to obtain some information about this sequence as n tends to infinity?"

Clearly everything we want to know about the sequence $((-1)^{n+1})_{n\geq 1}$ can be obtained by taking the least upper bound and greatest lower bound of its elements. Consequently, we extend the notions of least upper bound and greatest lower bound to include infinities.

Definition 2.3.15. Let X be a subset of the real numbers. The *supremum* of X, denoted $\sup(X)$, is defined to be

$$\sup(X) = \begin{cases} -\infty & \text{if } X = \emptyset \\ \mathrm{lub}(X) & \text{if } X \neq \emptyset \text{ and } X \text{ is bounded above} \\ \infty & \text{if } X \text{ is not bounded above} \end{cases}.$$

Similarly, the *infimum* of X, denoted $\inf(X)$, is defined to be

$$\inf(X) = \begin{cases} \infty & \text{if } X = \emptyset \\ \mathrm{glb}(X) & \text{if } X \neq \emptyset \text{ and if } X \text{ is bounded below} \\ -\infty & \text{if } X \text{ is not bounded below} \end{cases}$$

The infimum and supremum of sequences are not the objects we are after since we are more interested in the behaviour of sequences as n gets large. For example, consider the sequence

$$\left((-1)^n(1+\frac{1}{n})\right)_{n\geq 1}.$$

It is not difficult to see that $\frac{3}{2}$ is the supremum of this sequence and -2 is the infimum of this sequence. However, as n gets larger and larger, the largest values of the sequence are very close to 1 and the smallest values of the sequence are very close to -1. How can we express this notion for arbitrary sequences mathematically?

Let $(a_n)_{n\geq 1}$ be a sequence. To see how the largest values of $(a_n)_{n\geq 1}$ behave as n grows, we can take the suprumum after we ignore the first few terms. Consequently, we define a new sequence $(b_n)_{n\geq 1}$ defined by

$$b_n = \sup\{a_k \mid k \geq n\}.$$

It is not difficult to see that $b_1 \geq b_2 \geq b_3 \geq \cdots$ as the supremum may only get smaller as we remove terms from the set from which we are taking the supremum. Consequently we see that $(b_n)_{n\geq 1}$ is a monotone sequence. Since $(b_n)_{n\geq 1}$ is non-increasing, $(b_n)_{n\geq 1}$ either converges to a number (by the Monotone Convergence Theorem), diverges to $-\infty$, or $b_n = \infty$ for all n.

Applying the same idea with the sequence $(c_n)_{n\geq 1}$ where

$$c_n = \inf\{a_k \mid k \geq n\}$$

we arrive at the following.

Definition 2.3.16. The *limit supremum* of a sequence $(a_n)_{n\geq 1}$ of real numbers, denoted $\limsup_{n\to\infty} a_n$, is

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\} \in \mathbb{R} \cup \{\pm\infty\}.$$

Similarly, the *limit infimum* of a sequence $(a_n)_{n\geq 1}$ of real numbers, denoted $\liminf_{n\to\infty} a_n$, is

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\} \in \mathbb{R} \cup \{\pm\infty\}.$$

Remark 2.3.17. One huge advantage of the limit supremum and limit infimum over limits is that $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ always exist for all bounded sequences $(a_n)_{n\geq 1}$ and are real numbers whereas $(a_n)_{n\geq 1}$ need not converge! Indeed if $(a_n)_{n\geq 1}$ is bounded, then with

$$b_n = \sup\{a_k \mid k \geq n\},$$

we have that $(b_n)_{n\geq 1}$ is a bounded monotone sequence and thus converges by Monotone Convergence Theorem. Since $\lim_{n\to\infty} b_n = \limsup_{n\to\infty} a_n$ by definition, $\limsup_{n\to\infty} a_n$ exists.

Thus one can always discuss $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$, which is quite useful when one does not know $\lim_{n\to\infty} a_n$ exists.

Example 2.3.18. We claim that

$$\limsup_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right) = 1.$$

To see this, for each $n \in \mathbb{N}$ let

$$b_n = \sup\left\{ (-1)^k\left(1+\frac{1}{k}\right) \,\middle|\, k \geq n\right\}.$$

Thus $\limsup_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} b_n$ by definition.

Since $\lim_{k\to\infty}\frac{1}{k} = 0$, we see that

$$b_n = \begin{cases} 1+\frac{1}{n} & \text{if } n \text{ is even} \\ 1+\frac{1}{n+1} & \text{if } n \text{ is odd} \end{cases}.$$

Therefore $(b_n)_{n\geq 1}$ is the sequence $\left(\frac{3}{2}, \frac{3}{2}, \frac{5}{4}, \frac{5}{4}, \frac{7}{6}, \frac{7}{6}, \ldots\right)$ (i.e. the sequence $\left(1+\frac{1}{n}\right)_{n\geq 1}$ but each term is repeated an additional time in succession).

We claim that $\lim_{n\to\infty} b_n = 1$. To see this, let $\epsilon > 0$ be arbitrary. By the Archimedean Property (Theorem 1.3.7), there exists an $N \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$ for all $n \geq N$. Therefore, if $n \geq N$ and n is even, then

$$|b_n - 1| = \frac{1}{n} < \epsilon,$$

and if $n \geq N$ and n is odd, then

$$|b_n - 1| = \frac{1}{n+1} < \frac{1}{n} < \epsilon.$$

Hence $|b_n - 1| < \epsilon$ for all $n \geq N$. Hence $\lim_{n\to\infty} b_n = 1$ so

$$\limsup_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right) = 1$$

as desired.

Example 2.3.19. We claim that

$$\liminf_{n\to\infty}(-1)^n\left(1+\frac{1}{n}\right) = -1.$$

The proof is nearly identical to Example 2.3.18.

Unsurprisingly, there is a solid connection between limit supremum, limit infimum, and limit. In particular, one can use limit supremum and limit infimum to prove the limit exists when it does. To see this connection between these concepts, we require the following.

Theorem 2.3.20 (Comparison Theorem). *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be convergent sequences of real numbers. Suppose that there exists an $N_0 \in \mathbb{N}$ such that $a_n \leq b_n$ for all $n \geq N_0$. Then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.*

Proof. Let $L = \lim_{n\to\infty} a_n$ and let $K = \lim_{n\to\infty} b_n$. Suppose for the sake of a contradiction that that $K < L$. Therefore if $\epsilon = \frac{L-K}{2}$, then $\epsilon > 0$.

Since $L = \lim_{n\to\infty} a_n$, there exists an $N_1 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$. Hence $L - \epsilon < a_n$ for all $n \geq N_1$. Similarly, since $K = \lim_{n\to\infty} b_n$, there exists an $N_2 \in \mathbb{N}$ such that $|b_n - K| < \epsilon$ for all $n \geq N_2$. Hence $b_n < K + \epsilon$ for all $n \geq N_2$.

Therefore, if $n \geq \max\{N_1, N_2, N_0\}$, we obtain that

$$a_n - b_n > (L - \epsilon) - (K + \epsilon) = (L - K) - 2\epsilon = 0.$$

However, this contradicts the fact that $a_n \leq b_n$ for all $n \geq N_0$. Hence we have obtained a contradiction in the case that $K < L$ so it must be the case that $L \leq K$. ∎

Proposition 2.3.21. *Let $(a_n)_{n\geq 1}$ be a bounded sequence so that*

$$\liminf_{n\to\infty} a_n, \limsup_{n\to\infty} a_n \in \mathbb{R}.$$

Then

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

In addition, $(a_n)_{n\geq 1}$ converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. In this case

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

Proof. For the remainder of the proof, for each $n \in \mathbb{N}$ let

$$b_n = \sup\{a_k \mid k \geq n\} \in \mathbb{R} \qquad \text{and} \qquad c_n = \inf\{a_k \mid k \geq n\} \in \mathbb{R}.$$

Clearly

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} b_n, \quad \liminf_{n\to\infty} a_n = \lim_{n\to\infty} c_n, \text{ and } c_n \leq a_n \leq b_n \text{ for all } n \in \mathbb{N}.$$

Hence, since the limit supremum and limit infimum exist by Remark 2.3.17, the Comparison Theorem (Theorem 2.3.20) implies

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

Next, assume that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. Therefore, since $c_n \leq a_n \leq b_n$ for all $n \in \mathbb{N}$, we obtain that $(a_n)_{n\geq 1}$ converges and

$$\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$$

by the Squeeze Theorem (Theorem 2.3.13).

Finally, assume $L = \lim_{n\to\infty} a_n$ exists. To see that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, let $\epsilon > 0$ be arbitrary. Hence there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \in \mathbb{N}$. Thus $L - \epsilon \leq a_n \leq L + \epsilon$ for all $n \geq N$. Therefore $L - \epsilon \leq c_n \leq b_n \leq L + \epsilon$ for all $n \geq N$ by the definition of b_n and c_n. Hence, by considering constant sequences and the Comparison Theorem (Theorem 2.3.20), we have that

$$L - \epsilon \leq \lim_{n\to\infty} c_n \leq \lim_{n\to\infty} b_n \leq L + \epsilon$$

for all $\epsilon > 0$. In particular,

$$L - \frac{1}{m} \leq \lim_{n\to\infty} c_n \leq \lim_{n\to\infty} b_n \leq L + \frac{1}{m}$$

for all $m \in \mathbb{N}$. Therefore, since $\lim_{m\to\infty} \frac{1}{m} = 0$, the above is only possible (for example, by the Squeeze Theorem (Theorem 2.3.13)) if

$$L = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$ ∎

One important way to think about the limit supremum and limit infimum is that they give "approximate asymptotic bounds" on sequences as the following results show.

Proposition 2.3.22. *Let $(a_n)_{n\geq 1}$ be a bound sequence of real numbers so that*

$$L = \limsup_{n\to\infty} a_n \in \mathbb{R}.$$

Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_n < L + \epsilon$ for all $n \geq N$.

Furthermore, if $K < L$, then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $a_n \geq K + \epsilon$.

Proof. For each $n \in \mathbb{N}$ let

$$b_n = \sup\{a_k \mid k \geq n\} \in \mathbb{R}$$

so that $L = \lim_{n\to\infty} b_n$ by definition.

To see the first result, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} b_n$, by the definition of the limit there exists an $N \in \mathbb{N}$ such that $|b_n - L| < \epsilon$ for all $n \geq N$. Hence $b_n - L < \epsilon$ for all $n \geq N$ so that

$$a_n \leq b_n < L + \epsilon$$

for all $n \geq N$. Therefore, as $\epsilon > 0$ was arbitrary, the first part of the proof is complete.

For the second result, assume $K < L$. Suppose for the sake of a contradiction that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_n < K + \epsilon$ for all $n \geq N$. To obtain our contradiction, let $\epsilon = \frac{L-K}{2} > 0$. By our assumptions, there exists an $N \in \mathbb{N}$ such that $a_k < K + \epsilon$ for all $k \geq N$. Therefore, by the definition of b_n, we then have that $b_n \leq K + \epsilon$ for all $n \geq N$. Hence the Comparison Theorem (Theorem 2.3.20) implies that

$$L = \lim_{n\to\infty} b_n \leq K + \epsilon = K + \frac{L-K}{2} = \frac{L+K}{2}.$$

Thus

$$0 \leq \frac{K-L}{2}$$

so that $L \leq K$, which contradicts the fact that $K < L$. Hence we have obtained our contradiction, so the proof is complete. ∎

Proposition 2.3.23. *Let $(a_n)_{n\geq 1}$ be a bound sequence of real numbers so that*

$$L = \liminf_{n\to\infty} a_n \in \mathbb{R}.$$

Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_n > L - \epsilon$ for all $n \geq N$.

Furthermore, if $K > L$, then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $a_n \leq K - \epsilon$.

Proof. For each $n \in \mathbb{N}$ let

$$c_n = \inf\{a_k \mid k \geq n\} \in \mathbb{R}$$

so that $L = \lim_{n\to\infty} c_n$ by definition.

To see the first result, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} c_n$, by the definition of the limit there exists an $N \in \mathbb{N}$ such that $|c_n - L| < \epsilon$ for all $n \geq N$. Hence $L - c_n < \epsilon$ for all $n \geq N$ so that

$$a_n \geq c_n > L - \epsilon$$

for all $n \geq N$. Therefore, as $\epsilon > 0$ was arbitrary, the first part of the proof is complete.

For the second result, assume $K > L$. Suppose for the sake of a contradiction that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_n > K - \epsilon$ for all $n \geq N$. To obtain our contradiction, let $\epsilon = \frac{K-L}{2} > 0$. By our assumptions, there exists an $N \in \mathbb{N}$ such that $a_k > K - \epsilon$ for all $k \geq N$. Therefore, by the definition of c_n, we then have that $c_n \geq K - \epsilon$ for all $n \geq N$. Hence the Comparison Theorem (Theorem 2.3.20) implies that

$$L = \lim_{n\to\infty} c_n \geq K - \epsilon = K - \frac{K-L}{2} = \frac{L+K}{2}.$$

Thus

$$0 \geq \frac{K - L}{2}$$

so that $L \geq K$, which contradicts the fact that $K > L$. Hence we have obtained our contradiction, so the proof is complete. ∎

2.3.5 The Decimal Expansion of Real Numbers

Now that we have some knowledge of convergent sequences, we are able to demonstrate one common way to represent real numbers; via their decimal expansions. The goal of this subsection is to show that every real number in $[0, 1]$ has a decimal expansion.

Remark 2.3.24. Once we have shown that every element of $[0, 1]$ has a decimal expansion, we will automatically obtain that every real number has a decimal expansion. Indeed if $x \in \mathbb{R}$ and $x > 0$, then the Well Ordering Principle (Theorem 1.1.10) together with the Archimedean Property (Theorem 1.3.6) implies there exists a natural number n such that $x \in [n, n + 1]$. Hence, since $x - n \in [0, 1]$ has a decimal expansion by the results of this subsection, the decimal expansion of x is n plus the decimal expansion of $x - n$. Subsequently, if $x < 0$, then $-x > 0$ has a decimal expansion and the decimal expansion of x is the negative of the decimal expansion of $-x$.

To make mathematically precise what we mean by the decimal expansion of an element of $[0, 1]$, consider the following. The decimal 0.1 is the decimal representation of $\frac{1}{10}$, the decimal 0.01 is the decimal representation of $\frac{1}{100} = \frac{1}{10^2}$, the decimal 0.001 is the decimal representation of $\frac{1}{1000} = \frac{1}{10^3}$, and so on. So, by the decimal expression $0.a_1a_2a_3a_4\ldots$ where $a_k \in \{0, 1, 2, \ldots, 9\}$ for all $k \in \mathbb{N}$, we really mean

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \frac{a_4}{10^4} + \cdots.$$

Thus, the decimal expansion is most accurately represented by the infinite sum $\sum_{k=1}^{\infty} \frac{a_k}{10^k}$. However, as in the motivation for this course, we must be careful as we have not mathematically defined what we mean by an infinite sum. To focus on this specific case, our goal is to demonstrate that $x \in [0, 1]$ if and only if we can write

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{10^k}$$

for some sequence of $(a_k)_{k\geq 1}$ with $a_k \in \{0, 1, 2, \ldots, 9\}$ in which case "$x = 0.a_1a_2a_3a_4\ldots$".

To begin the proof, we first need to recall the formula for the sum of a geometric series.

Lemma 2.3.25. *If $a \in \mathbb{R}$ and $a \neq 1$, then for all $n \in \mathbb{N}$,*

$$\sum_{k=0}^{n} a^k = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1-a}.$$

Proof. For each $n \in \mathbb{N}$ let P_n be the statement that $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$. To show that P_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction.

Base Case: To see that P_1 is true, notice when $n = 1$ that

$$\frac{1-a^{n+1}}{1-a} = \frac{1-a^2}{1-a} = \frac{(1-a)(1+a)}{1-a} = 1 + a = a^0 + a^1 = \sum_{k=0}^{n} a^k.$$

Hence P_1 is true.

Inductive Step: Assume that P_n is true; that is, assume $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$. To see that P_{n+1} is true, notice that

$$\begin{aligned}
\sum_{k=0}^{n+1} a^k &= a^{n+1} + \sum_{k=0}^{n} a^k \\
&= a^{n+1} + \frac{1-a^{n+1}}{1-a} && \text{by the induction hypothesis} \\
&= \frac{(a^{n+1} - a^{n+2}) + (1 - a^{n+1})}{1-a} \\
&= \frac{1 - a^{(n+1)+1}}{1-a}.
\end{aligned}$$

Hence P_{n+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the result follows by the Principle of Mathematical Induction. ∎

To demonstrate characterization of the elements of $[0, 1]$ via their decimal expansion, we will divide the result into two parts. First we will demonstrate the easier result that "decimal expansion" define elements of $\mathbb{R}$.

Proposition 2.3.26. *Let $(a_n)_{n\geq 1}$ be a sequence with $a_n \in \{0, 1, \ldots, 9\}$ for all $n \in \mathbb{N}$ and for each $n \in \mathbb{N}$, let*

$$s_n = \sum_{k=1}^{n} \frac{a_k}{10^k}.$$

Then the sequence $(s_n)_{n\geq 1}$ converges and $\lim_{n\to\infty} s_n \in [0, 1]$.

Proof. To see that $(s_n)_{n\geq 1}$ converges, we will apply the Monotone Convergence Theorem (Theorem 2.2.6). To see that $(s_n)_{n\geq 1}$ is monotone, note by construction that for all $n \in \mathbb{N}$

$$0 \leq s_n \leq s_n + \frac{a_{n+1}}{10^{n+1}} = s_{n+1}.$$

Hence $(s_n)_{n\geq 1}$ is monotone. To see that $(s_n)_{n\geq 1}$ is bounded, we claim that $s_n \leq 1$ for all $n \in \mathbb{N}$. To see this, notice

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{a_k}{10^k} \leq \sum_{k=1}^{n} \frac{9}{10^k} &= \frac{9}{10}\sum_{k=1}^{n} \frac{1}{10^{k-1}} \\ &= \frac{9}{10}\sum_{k=0}^{n-1} \left(\frac{1}{10}\right)^k \\ &= \frac{9}{10}\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}} && \text{by Lemma 2.3.25} \\ &= \frac{9}{10}\frac{1-\left(\frac{1}{10}\right)^n}{\frac{9}{10}} = 1-\left(\frac{1}{10}\right)^n \leq 1 \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $(s_n)_{n\geq 1}$ is a bounded monotone sequence and thus converges by the Monotone Convergence Theorem (Theorem 2.2.6). Moreover, since $0 \leq s_n \leq 1$ for all $n \in \mathbb{N}$, it follows from the Comparison Theorem (Theorem 2.3.20) that $\lim_{n\to\infty} s_n \in [0,1]$ as desired. ∎

Now that we have seen every 'decimal expansion" define an element of $\mathbb{R}$ we will complete the converse and show that every element of $\mathbb{R}$ has a decimal expansion!

Theorem 2.3.27. *If $x \in [0,1]$, then there exists a sequence $(a_k)_{k\geq 1}$ such that $a_k \in \{0,1,\ldots,9\}$ for all $k \in \mathbb{N}$ such that if*

$$s_n = \sum_{k=1}^{n} \frac{a_k}{10^k}$$

for all $n \in \mathbb{N}$, then $(s_n)_{n\geq 1}$ converges to x.

Proof. Fix $x \in [0,1]$. We will construct the sequence $(a_k)_{k\geq 1}$ recursively.

Let

$$a_1 = \max\left\{k \in \{0,1,\ldots,9\} \,\middle|\, \frac{k}{10} \leq x\right\}.$$

Thus $s_1 = \frac{a_1}{10}$ has been defined. Subsequently define

$$a_2 = \max\left\{k \in \{0,1,\ldots,9\} \,\middle|\, \frac{k}{100} \leq x - s_1\right\}.$$

Thus $s_2 = \frac{a_1}{10} + \frac{a_2}{100}$ has been defined. To proceed recursively if $a_1, \ldots, a_n$ and thus $s_1, \ldots, s_n$ have been defined, we define

$$a_{n+1} = \max\left\{k \in \{0,1,2,\ldots,9\} \,\middle|\, \frac{k}{10^{n+1}} \leq x - s_n\right\}$$

and thus $s_{n+1} = \sum_{k=1}^{n+1} \frac{a_k}{10^k}$ is defined.

With $(a_k)_{k\geq 1}$ defined recursively as above, all that remains is to prove that $(s_n)_{n\geq 1}$ converges to x. To see that $\lim_{n\to\infty} s_n = x$, we claim that $s_n \leq x \leq s_n + \frac{1}{10^n}$ for all $n \in \mathbb{N}$. To see this, for each $n \in \mathbb{N}$ let P_n be the statement that $s_n \leq x \leq s_n + \frac{1}{10^n}$. To show that P_n is true for all $n \in \mathbb{N}$, we will apply the Principle of Mathematical Induction.

Base Case: To see that P_1 is true, recall that

$$a_1 = \max\left\{k \in \{0, 1, \ldots, 9\} \,\middle|\, \frac{k}{10} \leq x\right\}$$

and $s_1 = \frac{a_1}{10}$. Therefore $s_1 \leq x$ by definition. To see the other inequality, we must divide the discussion into two cases.

Case 1: $a_1 \neq 9$. Assume $a_1 \neq 9$. To see that $x \leq s_1 + \frac{1}{10}$, suppose for the sake of a contradiction that $x > s_1 + \frac{1}{10} = \frac{a_1+1}{10}$. Therefore $a_1 + 1$ is such that $a_1 + 1 \in \{0, 1, 2, \ldots, 9\}$ and $\frac{a_1+1}{10} \leq x$. Since this contradicts the definition of a_1, we have a contradiction. Hence $x \leq s_1 + \frac{1}{10}$ in this case.

Case 2: $a_1 = 9$. Assume $a_1 = 9$. Then

$$x \leq 1 = \frac{a_1}{10} + \frac{1}{10} = s_1 + \frac{1}{10}$$

as desired.

Therefore P_1 is true.

Inductive Step: Assume that P_n is true; that is, assume $s_n \leq x \leq s_n + \frac{1}{10^n}$. To see that P_{n+1} is true, recall

$$a_{n+1} = \max\left\{k \in \{0, 1, 2, \ldots, 9\} \,\middle|\, \frac{k}{10^{n+1}} \leq x - s_n\right\}$$

and $s_{n+1} = \sum_{k=1}^{n+1} \frac{a_k}{10^k}$. Hence $s_{n+1} \leq x$ by definition. To see the other inequality, we must divide the discussion into two cases.

Case 1: $a_{n+1} \neq 9$. Assume $a_{n+1} \neq 9$. To see that $x \leq s_{n+1} + \frac{a_{n+1}}{10^{n+1}}$, suppose for the sake of a contradiction that $x > s_{n+1} + \frac{1}{10^{n+1}} = s_n + \frac{a_{n+1}+1}{10}$. Therefore $a_{n+1}+1$ is such that $a_{n+1}+1 \in \{0, 1, 2, \ldots, 9\}$ and $\frac{a_{n+1}+1}{10} \leq x - s_n$. Since this contradicts the definition of a_{n+1}, we have a contradiction. Hence $x \leq s_{n+1} + \frac{1}{10^{n+1}}$ when $a_{n+1} \neq 9$.

Case 2: $a_{n+1} = 9$. Assume $a_{n+1} = 9$. Suppose for the sake of a contradiction that $x > s_{n+1} + \frac{1}{10^{n+1}}$. Then

$$x > \frac{9}{10^{n+1}} + s_n + \frac{1}{10^{n+1}} = s_n + \frac{1}{10^n}$$

which contradicts the induction hypothesis. Hence $x \leq s_{n+1} + \frac{1}{10^{n+1}}$.

Therefore P_{n+1} is true.

Therefore, as we have demonstrated the base case and the inductive step, the result follows by the Principle of Mathematical Induction.

Since $s_n \leq x \leq s_n + \frac{1}{10^n}$ for all $n \in \mathbb{N}$, we see that

$$0 \leq |s_n - x| \leq \frac{1}{10^n}$$

for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{1}{10^n} = 0$ by Example 2.3.2, for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\frac{1}{10^n} < \epsilon$ for all $n \geq N$ and thus $|s_n - x| < \epsilon$ for all $n \geq N$. Hence $x = \lim_{n\to\infty} s_n$ as desired. ∎

Remark 2.3.28. An observant reader of Proposition 2.3.26 and Theorem 2.3.27 would have notice that there was nothing special in the proofs that required the number 10 (provided we stopped the terms of the decimal expansion at 9). Therefore, by repeating these proofs, it is possible to show that $x \in [0, 1]$ if and only if there exists a sequence $(a_k)_{k\geq 1}$ with $a_k \in \{0, 1\}$ for all $k \in \mathbb{N}$ such that

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{2^k}.$$

The above representation is called the *binary representation of* x. Similarly, it can be shown that $x \in [0, 1]$ if and only if there exists a sequence $(a_k)_{k\geq 1}$ with $a_k \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$ such that

$$x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}.$$

The above representation is called the *ternary representation of* x. These and other expansions can often be useful.

2.4 The Bolzano–Weierstrass Theorem

We have seen that many sequences do not converge. The limit supremum and limit infimum do provide us with some information about how a sequence $(a_n)_{n\geq 1}$ behaves for large n. However, one question we can ask is, "If we have a sequence that does not converge, can we remove some terms from the sequence to make it converge?" Of course for convergence, our new sequence must be bounded by Proposition 2.2.3. Thus perhaps a better question is, "If we have a *bounded* sequence that does not converge, can we remove terms from the sequence to make it converge?"

2.4.1 Subsequences

To answer the above question, we must describe what we mean by 'remove terms from a sequence'. This is made precise by the following mathematical notion.

Definition 2.4.1. A *subsequence* of a sequence $(a_n)_{n\geq 1}$ of real numbers is any sequence $(b_k)_{k\geq 1}$ of real numbers such that there exists an increasing sequence of natural numbers $(n_k)_{k\geq 1}$ so that $b_k = a_{n_k}$ for all $k \in \mathbb{N}$.

It is important to note that a subsequence of $(a_n)_{n\geq 1}$ removes some terms but leaves an infinite number of terms that appear in the same order.

Example 2.4.2. If $(a_n)_{n\geq 1}$ is our favourite sequence $a_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$ and if we choose $n_k = 2k - 1$ for all $n \in \mathbb{N}$, then $(a_{n_k})_{k\geq 1}$ is the subsequence $(1, 1, 1, \ldots)$. Note $(a_n)_{n\geq 1}$ diverges whereas the subsequence $(a_{n_k})_{k\geq 1}$ converges. Hence divergent sequences can have convergent subsequences.

Example 2.4.3. If $(b_n)_{n\geq 1}$ is the sequence where $b_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ and if we choose $n_k = k^2$ for all $k \in \mathbb{N}$, then $(b_{n_k})_{k\geq 1}$ is the subsequence $(1, \frac{1}{4}, \frac{1}{9}, \ldots) = \left(\frac{1}{k^2}\right)_{k\geq 1}$. Notice that $(b_n)_{n\geq 1}$ and the subsequence $(b_{n_k})_{k\geq 1}$ both converge to 0. In fact, for convergent sequences, the subsequences must also converge to the same number as the following result show.

Proposition 2.4.4. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers that converges to L. Every subsequence of $(a_n)_{n\geq 1}$ converges to L.*

Proof. Let $(a_{n_k})_{k\geq 1}$ be a subsequence of $(a_n)_{n\geq 1}$. To see that $(a_{n_k})_{k\geq 1}$ converges to L, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} a_n$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Since $(n_k)_{k\geq 1}$ is an increasing sequence of natural numbers, there exists an $N_0 \in \mathbb{N}$ such that $n_k \geq N$ for all $k \geq N_0$. Hence $|a_{n_k} - L| < \epsilon$ for all $k \geq N_0$. Therefore, as $\epsilon > 0$ was arbitrary, we obtain that $\lim_{k\to\infty} a_{n_k} = L$ by the definition of the limit. ∎

2.4.2 The Peak Point Lemma

It is natural to ask, "Given a sequence, are there any 'nice' subsequences?" Of course 'nice' is a subjective term. However, we have seen that monotone sequences were quite nice via the Monotone Convergence Theorem. So, "Does every sequence have a monotone subsequence?" Yes they do!

Lemma 2.4.5 (The Peak Point Lemma). *Every sequence of real numbers has a monotone subsequence.*

In order to prove the above lemma (and from which it gets its name), we will use the following notion.

Definition 2.4.6. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. An index $n_0 \in \mathbb{N}$ is said to be a *peak point* for the sequence $(a_n)_{n\geq 1}$ if $a_{n_0} > a_n$ for all $n \geq n_0$.

The reason we think of n_0 as a peak point of $(a_n)_{n\geq 1}$ if $a_n \leq a_{n_0}$ for all $n \geq n_0$ is that if along a number line of the natural numbers we draw a line of height a_n at n for all $n \in \mathbb{N}$, then if one stood on top of the line at n_0 one is at a peak and can see off to infinity without another line getting in the way.

With the notion of a peak point, we can prove the Peak Point Lemma.

Proof of Lemma 2.4.5. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. The proof is divided into two cases:

Case 1: $(a_n)_{n\geq 1}$ has an infinite number of peak points. By assumption there exists indices $n_1 < n_2 < n_3 < \cdots$ such that n_k is a peak point for all $k \in \mathbb{N}$. Therefore, we have by the definition of a peak point that $a_{n_k} > a_{n_{k+1}}$ for all $k \in \mathbb{N}$. Hence $(a_{n_k})_{k\geq 1}$ is a decreasing subsequence of $(a_n)_{n\geq 1}$.

Case 2: $(a_n)_{n\geq 1}$ has a finite number (or no) peak points. Let n_0 be the largest (i.e. last) peak point of $(a_n)_{n\geq 1}$ (or $n_0 = 1$ if $(a_n)_{n\geq 1}$ has no peak points). Let $n_1 = n_0 + 1$. Thus n_1 is not a peak point of $(a_n)_{n\geq 1}$. Therefore there exists a $n_2 > n_1 = n_0 + 1$ such that $a_{n_2} \geq a_{n_1}$. Subsequently, since $n_2 > n_1 > n_0$, n_2 is not a peak point. Therefore there exists a $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. Repeating this process ad nauseum, we obtain a sequence of indices $n_1 < n_2 < n_3 < \cdots$ such that $a_{n_{k+1}} \geq a_{n_k}$ for all $k \in \mathbb{N}$. Hence $(a_{n_k})_{k\geq 1}$ is a non-decreasing subsequence of $(a_n)_{n\geq 1}$.

As in either case a monotone subsequence can be constructed, the result follows. ∎

2.4.3 The Bolzano–Weierstrass Theorem

Combining the Peak Point Lemma together with the Monotone Convergence Theorem, we easily obtain the following very useful result.

Theorem 2.4.7 (The Bolzano-Weierstrass Theorem). *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $(a_n)_{n\geq 1}$ be a bounded sequence of real numbers. By the Peak Point Lemma (Lemma 2.4.5), there exists a monotone subsequence $(a_{n_k})_{k\geq 1}$ of $(a_n)_{n\geq 1}$. Since $(a_n)_{n\geq 1}$ is bounded, $(a_{n_k})_{n\geq 1}$ is also bound and thus converges by the Monotone Convergence Theorem (Theorem 2.2.6). ∎

2.5 Completeness of the Real Numbers

In the previous sections, we have seen some methods that can help us compute limits provided we know they exist and we have seen how to show that some sequences converge. However, for a general sequence, determining whether a given sequence converges can be a difficult task as the only method we have is to verify Definition 2.1.7. The challenge with verifying the definition of a convergent sequence is that one must first guess the limit and then show the sequence converges to the limit. Thus it is natural to ask, “Is there a way to determine whether a sequence converges without having an intuition about what the limit should be?”

2.5.1 Cauchy Sequences

If a sequence was going to converge, then eventually all terms in the sequence are as close to the limit as we would like. In particular, by the Triangle Inequality, eventually all terms in the sequence are as close to each other as we would like. This leads us to the notion of a Cauchy sequence.

Heuristic Definition. A sequence $(a_n)_{n\geq 1}$ is said to be Cauchy if the terms of $(a_n)_{n\geq 1}$ are arbitrarily as close to each other as we would like as long as n is large enough.

As with the definition of the limit of a sequence, the notion of Cauchy sequence can be made mathematically precise.

Definition 2.5.1. A sequence $(a_n)_{n\geq 1}$ of real numbers is said to be *Cauchy* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.

To formalize our motivation that if “eventually all terms in the sequence are as close to the limit” then “eventually all terms in the sequence are as close to each other as we would like”, we prove the following showing that a sequence being Cauchy is required for the sequence to converge.

Theorem 2.5.2. *Every convergent sequence of real numbers is Cauchy.*

Proof. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers that converges to $L \in \mathbb{R}$. To see that $(a_n)_{n\geq 1}$ is Cauchy, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{n\to\infty} a_n$, there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \frac{\epsilon}{2}$ for all $n \geq N$. Thus, for all $n, m \geq N$,

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $(a_n)_{n\geq 1}$ is Cauchy by definition. ∎

Consequently, it is natural to ask whether the converse of Theorem 2.5.2 holds; that is, if a sequence is Cauchy, does it automatically converge? Before we tackle this question, a few examples and a remark are useful.

Remark 2.5.3. We claim that to verify $|a_n - a_m| < \epsilon$ for all $n, m \geq N$, it suffices to only verify $|a_n - a_m| < \epsilon$ for all $n, m \geq N$ with $n \geq m$. Indeed if we have that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$ with $n \geq m$, then if $n, m \geq N$ are such that $m > n$, then since

$$|a_n - a_m| = |(-1)(a_m - a_n)| = |a_m - a_n|$$

we obtain that $|a_n - a_m| \leq |a_m - a_n| < \epsilon$ as desired.

Moreover, just by repeating the same ideas as used in the proof of Proposition 2.1.14 (i.e. manipulating the “for all $\epsilon > 0$” quantifies), a sequence $(a_n)_{n\geq 1}$ is Cauchy if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| \leq \epsilon$ for all $n, m \geq N$.

Note the following example show that it can be easier to check that a sequence is Cauchy then it is to check whether the sequence converges. Thus a proof that “Every Cauchy sequence converges” would be incredibly useful.

Example 2.5.4. For each $n \in \mathbb{N}$, let $a_n = \sum_{k=1}^{n} \frac{1}{2^k} \sin(k)$. Consider the sequence $(a_n)_{n\geq 1}$. It is difficult to determine whether $(a_n)_{n\geq 1}$ converges by the definition of the limit since we have no idea what the limit should be. Moreover, since $\frac{1}{2^n}\sin(n)$ could be positive or negative (and it is difficult to know which when n is large), it could be $a_n \leq a_{n-1}$ or $a_n \geq a_{n-1}$. Hence $(a_n)_{n\geq 1}$ is not monotone so the Monotone Convergence Theorem does not help. Thus we have no techniques to determine whether or not $(a_n)_{n\geq 1}$ converges.

However, it is not too difficult to verify that $(a_n)_{n\geq 1}$ is Cauchy. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$ by Example 2.2.7, there exists an $N \in \mathbb{N}$ so that $\frac{1}{2^N} < \epsilon$. Then, for all $n, m \geq N$ with $n \geq m$, we have that

$$\begin{aligned}
|a_n - a_m| &= \left| \sum_{k=1}^{n} \frac{1}{2^k}\sin(k) - \sum_{k=1}^{m} \frac{1}{2^k}\sin(k) \right| && \text{by definition} \\
&= \left| \sum_{k=m+1}^{n} \frac{1}{2^k}\sin(k) \right| && \text{by cancellation} \\
&\leq \sum_{k=m+1}^{n} \left| \frac{1}{2^k}\sin(k) \right| && \text{by the Triangle Inequality} \\
&\leq \sum_{k=m+1}^{n} \frac{1}{2^k} && \text{as } |\sin(x)| \leq x \\
&= \left(\frac{1}{2}\right)^{m+1} \frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}} && \text{the sum of a geometric series} \\
&\leq \left(\frac{1}{2}\right)^{m+1} \frac{1 - 0}{1 - \frac{1}{2}} && \text{since } \left(\frac{1}{2}\right)^{n-m} > 0 \\
&\leq \frac{1}{2^{N+1}} \frac{1}{1 - \frac{1}{2}} && \text{as } m \geq N \\
&= \frac{1}{2^N} < \epsilon.
\end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, $(a_n)_{n\geq 1}$ is Cauchy.

Example 2.5.5. Note that it is possible that a sequence $(a_n)_{n\geq 1}$ satisfies $\lim_{n\to\infty} a_{n+1} - a_n = 0$ but is not Cauchy. Indeed if $a_n = \sum_{k=1}^{n} \frac{1}{k}$ for all

$n \in \mathbb{N}$, then $a_{n+1} - a_n = \frac{1}{n+1}$ which clearly converges to zero. However, it is possible to show that $(a_n)_{n\geq 1}$ diverges to infinity. Although we cannot prove this divergence at this time, many students will have seen series in previous courses and techniques of the Chapter 6 will enable this proof.

2.5.2 Convergence of Cauchy Sequences

As discussed in the previous subsection, a positive answer to the question "Does every Cauchy sequence converge?" would be incredibly useful as it is often easier to check a sequence is Cauchy then it is to verify the definition of the limit. One method for providing intuition to what the answer of this question is is to see if Cauchy sequences share similar properties to convergent sequences. In particular, analyzing Proposition 2.2.3 and its proof, we obtain the following.

Lemma 2.5.6. *Every Cauchy Sequence is bounded.*

Proof. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. Since $(a_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Hence, by letting $m = N$, we obtain that $|a_n| \leq |a_N| + 1$ for all $n \geq N$ by the Triangle Inequality.

Let $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$. Clearly if $n \leq N$ then $|a_n| \leq M$ whereas if $n \geq N$ then $|a_n| \leq |a_N| + 1 \leq M$ by the above paragraph. Hence $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$ so $(a_n)_{n\geq 1}$ is bounded. ∎

As further intuition towards whether all Cauchy sequence converge, recall Proposition 2.4.4 demonstrates subsequences of convergent sequence must converge. The following demonstrates the converse is true if our sequence is assumed to be Cauchy.

Lemma 2.5.7. *Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. If a subsequence of $(a_n)_{n\geq 1}$ converges, then $(a_n)_{n\geq 1}$ converges.*

Proof. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence with a convergent subsequence $(a_{n_k})_{k\geq 1}$ and let $L = \lim_{k\to\infty} a_{n_k}$. We claim that $\lim_{n\to\infty} a_n = L$. To see this, let $\epsilon > 0$ be arbitrary. Since $(a_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m \geq N$. Furthermore, since $L = \lim_{k\to\infty} a_{n_k}$, there exists an $n_j \geq N$ such that $|a_{n_j} - L| < \frac{\epsilon}{2}$. Hence, if $n \geq N$ then

$$|a_n - L| \leq |a_n - a_{n_j}| + |a_{n_j} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, as $\epsilon > 0$ was arbitrary, $(a_n)_{n\geq 1}$ is converges to L by definition. ∎

Using Lemma 2.5.7, we easily obtain the following.

Theorem 2.5.8 (Completeness of the Real Numbers). *Every Cauchy sequence of real numbers converges.*

Proof. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. By Lemma 2.5.6, $(a_n)_{n\geq 1}$ is bounded. Therefore the Bolzano-Weierstrass Theorem (Theorem 2.4.7) implies that $(a_n)_{n\geq 1}$ has a convergent subsequence. Hence Lemma 2.5.7 implies that $(a_n)_{n\geq 1}$ converges. ∎

Again, the advantage of Theorem 2.5.8 is that it is often much easier to check a sequence is Cauchy over checking the definition of a limit. Indeed, consider the following example.

Example 2.5.9. For each $n \in \mathbb{N}$, let $a_n = \sum_{k=1}^{n} \frac{1}{2^k} \sin(k)$. Recall from Example 2.5.4 that we could not deduce whether or not the sequence $(a_n)_{n\geq 1}$ converges, but we were able to show that $(a_n)_{n\geq 1}$ was Cauchy. Hence $(a_n)_{n\geq 1}$ converges by Theorem 2.5.8.

Remark 2.5.10. Theorem 2.5.8 demonstrates that the real numbers is a complete space (a space where every Cauchy sequence converges). The terminology comes from the fact that complete spaces have no 'holes' in them. In fact, the Completeness of the Real Numbers is logically equivalent to the Least Upper Bound Property (i.e. if instead of asking for the real numbers to have the Least Upper Bound Property we asked for them to be complete, we would still end up with the real numbers). In fact, some authors call the Least Upper Bound Property the "Completeness Property". We elected for our choice of terminology as "completeness" does refer to "every Cauchy sequence converges" in subsequent analysis courses whereas the Least Upper Bound Property is of lesser importance in future analysis courses.

Remark 2.5.11. Cauchy sequences have additional uses beyond verify convergence of sequences of real numbers. In particular, by using an equivalence relation on the set of all Cauchy sequences of rational numbers, it is possible to construct the real numbers. This is done in Appendix B.4 for the interested reader.

Chapter 3

An Introduction to Topology

In addition to the notion of convergent sequences, we desire to analyze analytic properties of the real numbers. In particular, by changing our perspective slightly on what it means for a sequence to converge, we arrive at an area of mathematics closely related to analysis called topology. The term topology comes from the Greek words $\tau\acute{o}\pi o\sigma$ meaning place (or space) and $\lambda\acute{o}\gamma o\sigma$ meaning study. Thus topology exactly means the study of spaces! In this chapter, we will introduce students to topology via analysis motivated topological properties of the real numbers.

3.1 Topology of the Real Numbers

Recall that a sequence $(a_n)_{n\geq 1}$ of real numbers is said to converge to $L \in \mathbb{R}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Instead, by using the notion of an open interval, we see that a sequence $(a_n)_{n\geq 1}$ of real numbers is said to converge to $L \in \mathbb{R}$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $a_n \in (L - \epsilon, L + \epsilon)$ for all $n \geq N$. Consequently, we can use open intervals to describe convergent sequences instead of distances. The whole basis for topology that we will explore in this section is to generalize the notion of an open interval.

3.1.1 Open Sets

The following is the correct generalization of open intervals to discuss further topics in analysis and to introduce the area of topology.

Definition 3.1.1. A set $U \subseteq \mathbb{R}$ is said to be *open* if whenever $x \in U$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. The set

$$\mathcal{T} = \{U \subseteq \mathbb{R} \mid U \text{ is open}\}$$

is called the *topology on* $\mathbb{R}$.

Note the letter U is traditionally used for an open set since open sets nearly the same as another topological notion known as a neighbourhood and the German word for neighbourhood is 'umgebung'.

Example 3.1.2. Unsurprisingly, each open interval is open. To see this, suppose $a, b \in \mathbb{R}$ are such that $a < b$. To see that (a, b) is open, let $x \in (a, b)$ be arbitrary. Then, if $\epsilon = \min\{x - a, b - x\}$, then $\epsilon > 0$ and $(x - \epsilon, x + \epsilon) \subseteq (a, b)$. Thus, as $x \in (a, b)$ was arbitrary, (a, b) is open.

Using similar arguments, it is possible to show that if $a = -\infty$ and/or $b = \infty$, then (a, b) is open. Consequently $(-\infty, \infty) = \mathbb{R}$ is open.

Example 3.1.3. If $a, b \in \mathbb{R}$ are such that $a < b$, then $[a, b)$ is not open. To see this, we note that $a \in [a, b)$ but we claim that $(a - \epsilon, a + \epsilon) \not\subseteq [a, b)$ for all $\epsilon > 0$. To see this, note for all $\epsilon > 0$ that $a - \frac{1}{2}\epsilon \in (a - \epsilon, a + \epsilon)$ but $a - \frac{1}{2}\epsilon \notin [a, b)$. Hence $[a, b)$ is not open is not open by definition.

Similar arguments can be used to show that $(a, b]$ and $[a, b]$ are not open.

Example 3.1.4. The empty set is open since the definition of open is vacuously true for $\emptyset$ (as there are no elements in the empty set).

To solidify our motivation for examining open subsets of $\mathbb{R}$, we prove the following.

Proposition 3.1.5. *Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. A number $L \in \mathbb{R}$ is the limit of $(a_n)_{n\geq 1}$ if and only if for every open set $U \subseteq \mathbb{R}$ such that $L \in U$ there exists an $N \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq N$.*

Proof. Assume that $L = \lim_{n\to\infty} a_n$. To see the desired property holds, let U be an arbitrary open subset of $\mathbb{R}$ such that $L \in U$. Since $L \in U$ and U is open, there exists an $\epsilon > 0$ such that $(L - \epsilon, L + \epsilon) \subseteq U$. Since $L = \lim_{n\to\infty} a_n$, the definition of the limit implies that there exists an $N \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ for all $n \geq N$. Hence $a_n \in (L - \epsilon, L + \epsilon) \subseteq U$ for all $n \geq N$. Therefore, as U was arbitrary, the desired property has been demonstrated.

Conversely, assume that every open set $U \subseteq \mathbb{R}$ such that $L \in U$ there exists an $N \in \mathbb{N}$ such that $a_n \in U$ for all $n \geq N$. To see that $(a_n)_{n\geq 1}$ converges to L, let $\epsilon > 0$ be arbitrary. Since $(L - \epsilon, L + \epsilon)$ is an open by Example 3.1.2, the assumptions of this direction imply that there exists an $N \in \mathbb{N}$ such that $a_n \in (L - \epsilon, L + \epsilon)$ for all $n \geq N$. Hence $|a_n - L| < \epsilon$ for all $n \geq N$. Therefore, since $\epsilon > 0$ was arbitrary, $(a_n)_{n\geq 1}$ converges to L by the definition of the limit. ∎

Proposition 3.1.5 yields an alternative definition for the limit of a sequence of real numbers. This definition is particularly useful in generalizing limits to abstract spaces where one defines a 'good' notion of open sets (i.e. a topology) which then determines which sequences converge. A 'good' notion of open sets must mimic the properties illustrated in the following proposition for open subsets of $\mathbb{R}$.

Proposition 3.1.6. *Let I be an non-empty set and for each $i \in I$ let U_i be an open subset of $\mathbb{R}$. Then*

a) $\bigcup_{i\in I} U_i$ is open in $\mathbb{R}$, and

b) $\bigcap_{i\in I} U_i$ is open in $\mathbb{R}$ provided I has a finite number of elements.

Proof. a) To see that $\bigcup_{i\in I} U_i$ is open, let $x \in \bigcup_{i\in I} U_i$ be arbitrary. Then $x \in U_{i_0}$ for some $i_0 \in I$. Therefore, as U_{i_0} is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U_{i_0}$. Hence

$$(x - \epsilon, x + \epsilon) \subseteq \bigcup_{i\in I} U_i.$$

Since $x \in \bigcup_{i\in I} U_i$ was arbitrary, $\bigcup_{i\in I} U_i$ is open.

b) To see that $\bigcap_{i\in I} U_i$ is open in $\mathbb{R}$ provided I has a finite number of elements, $x \in \bigcap_{i\in I} U_i$ be arbitrary. Hence $x \in U_i$ for each $i \in I$. Since U_i is open, for each $i \in I$ there exists an $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subseteq U_i$. Let

$$\epsilon = \min\{\epsilon_i \mid i \in I\}.$$

Since I has a finite number of elements, $\epsilon > 0$. Furthermore, by the definition of ϵ, $(x - \epsilon, x + \epsilon) \subseteq U_i$ for all $i \in I$. Hence

$$(x - \epsilon, x + \epsilon) \subseteq \bigcap_{i\in I} U_i.$$

Since $x \in \bigcap_{i\in I} U_i$ was arbitrary, $\bigcap_{i\in I} U_i$ is open. ∎

Remark 3.1.7. It is important to note that the conclusions of part (b) of Proposition 3.1.6 fails when I has an infinite number of elements. To see this, for each $n \in \mathbb{N}$ let $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. Then U_n is open for all $n \in \mathbb{N}$ by Example 3.1.2 but

$$\bigcap_{n\geq 1} U_n = \{0\}$$

is clearly not open as there is no open interval that is contained inside a single point.

Proposition 3.1.6 shows us that the union of any number of open intervals is an open set. In fact, our next result demonstrates that every open subset of the real numbers is a union of open intervals. To begin, we require a lemma.

Lemma 3.1.8. *Let $U \subseteq \mathbb{R}$ be a non-empty open set. Define a relation $\sim$ on U as follow: for $x, y \in U$, $x \sim y$ if and only if $[x, y] \subseteq U$ and $[y, x] \subseteq U$ (note $\emptyset \subseteq U$, $[y, x] = \emptyset$ if $y > x$, and $[x, y] = \emptyset$ if $x > y$). Then $\sim$ is an equivalence relation on U.*

Proof. To see that $\sim$ is an equivalence relation, we must check that $\sim$ satisfies three properties.

Reflexivity. To see that $\sim$ is reflexive, let $x \in U$ be arbitrary. Since $[x, x] = \{x\} \subseteq U$, $x \sim x$ by definition. Hence $\sim$ is reflexive.

Symmetry. To see that $\sim$ is symmetric, let $x, y \in U$ be such that $x \sim y$. Thus $[x, y] \subseteq U$ and $[y, x] \subseteq U$. Hence $[y, x] \subseteq U$ and $[x, y] \subseteq U$ so $y \sim x$ by definition. Thus $\sim$ is symmetric.

Transitivity. To see that $\sim$ is transitive, let $x, y, z \in U$ be such that $x \sim y$ and $y \sim z$. Since we have already verified symmetry, we may assume without loss of generality that $x \leq z$ (otherwise interchange x and z). Consequently, to show that $x \sim z$, it suffices to prove that $[x, z] \subseteq U$. We can now divide the proof into three cases.

Case 1: $y \leq x$. Since $y \sim z$, we know that $[y, z] \subseteq U$. Therefore, since $y \leq x$ in this case, we know that $[x, z] \subseteq [y, z] \subseteq U$. Thus $x \sim z$ in this case.

Case 2: $x \leq y \leq z$. Since $x \sim y$ and $y \sim z$, we know that $[x, y] \subseteq U$ and $[y, z] \subseteq U$. Therefore, since $x \leq y \leq z$, we have that

$$[x, z] = [x, y] \cup [y, z] \subseteq U \cup U = U.$$

Thus $x \sim z$ in this case.

Case 3: $z \leq y$. Since $x \sim y$, we know that $[x, y] \subseteq U$. Therefore, since $z \leq y$ in this case, we know that $[x, z] \subseteq [x, z] \subseteq U$. Thus $x \sim z$ in this case.

Therefore, since the above three cases cover all possible cases, we have that $\sim$ is transitive as desired.

Hence, as we have verified the three properties of an equivalence relation, $\sim$ is an equivalence relation. ∎

Proposition 3.1.9. *Every (non-empty) open subset of* $\mathbb{R}$ *is a union of disjoint open intervals.*

Proof. Let U be an open subset of $\mathbb{R}$. If $U = \emptyset$, then U is technically an empty union of open intervals. Thus, assume that U is non-empty. Let $\sim$ be the equivalence relation on U from Lemma 3.1.8.

For each $x \in U$, let E_x denote the equivalence class of x with respect to $\sim$. Clearly

$$U = \bigcup_{x \in U} E_x$$

as $x \in E_x$ for all $x \in U$. Moreover, by the properties of equivalence relations, if $x, y \in U$ then either $E_x = E_y$ or $E_x \cap E_y = \emptyset$. Hence if each E_x is an open interval, the proof will be complete.

Let $x \in U$ be fixed. Let

$$\alpha_x = \inf(E_x) \qquad \text{and} \qquad \beta_x = \sup(E_x).$$

We claim that $E_x = (\alpha_x, \beta_x)$.

First, we claim that $\alpha_x < \beta_x$. To see this, notice that $x \in E_x \subseteq U$. Since U is open, there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. Clearly $y \sim x$ for all $y \in (x - \epsilon, x + \epsilon)$ so

$$\alpha_x \leq x - \epsilon < x + \epsilon \leq \beta_x.$$

To see that $(\alpha_x, \beta_x) \subseteq E_x$, let $y \in (\alpha_x, \beta_x)$ be arbitrary. Since $\alpha_x < y < \beta_x$, by the definition of inf and sup there exists $z_1, z_2 \in E_x$ such that

$$\alpha_x \leq z_1 < y < z_2 \leq \beta_x.$$

Since $z_1, z_2 \in E_x$, we have $z_1 \sim x$ and $z_2 \sim x$. Thus $z_1 \sim z_2$ by transitivity so $[z_1, z_2] \subseteq U$. Hence $y \in [z_1, z_2] \subseteq U$. Therefore, as $y \in (\alpha_x, \beta_x)$ was arbitrary, $(\alpha_x, \beta_x) \subseteq E_x$.

To see that $E_x \subseteq (\alpha_x, \beta_x)$, note that

$$E_x \subseteq (\alpha_x, \beta_x) \cup \{\alpha_x, \beta_x\}$$

by the definition of α_x and β_x. Thus it suffices to show that $\alpha_x, \beta_x \notin E_x$.

Suppose for the sake of a contradiction that $\beta_x \in E_x$. Thus $\beta_x \sim x$ and

$$\beta_x \in E_x \subseteq U \subseteq \mathbb{R}.$$

Since $\beta_x \in U$ and U is open, there exists an $\epsilon > 0$ so that $(\beta_x - \epsilon, \beta_x + \epsilon) \subseteq U$. Hence

$$\beta_x + \frac{1}{2}\epsilon \sim \beta_x \sim x.$$

Thus $\beta_x + \frac{1}{2}\epsilon \in E_x$. However $\beta_x + \frac{1}{2}\epsilon > \beta_x$ so $\beta_x + \frac{1}{2}\epsilon \in E_x$ contradicts the fact that $\beta_x = \sup(E_x)$. Hence we have obtained a contradiction so $\beta_x \notin E_x$. Similar arguments show that $\alpha_x \notin E_x$. Hence $E_x = (\alpha_x, \beta_x)$ thereby completing the proof. ■

Remark 3.1.10. Considering Proposition 3.1.9, a natural question to ask is, "How many open intervals do we need in the union?" Looking at the proof of Proposition 3.1.9, we see instead of writing $U = \bigcup_{x \in U} E_x$, we can write $U = \bigcup_{x \in S} E_x$ where S contains one element from each equivalence class. Since each equivalence class is an open interval and since each open interval contains a rational number by Proposition 1.3.8, we can write $U = \bigcup_{x \in Q} E_x$ where $Q \subseteq \mathbb{Q}$. As the cardinality of the rational numbers equals the cardinality of the natural numbers (see your MATH 1200 textbook if there was not time to cover countable and uncountable sets), we can write $U = \bigcup_{n \in \mathbb{N}} I_n$ where each I_n is an open interval. In particular, every open subset of $\mathbb{R}$ is a countable union of open intervals!

3.1.2 Closed Sets

Although the notion of open sets is important in future courses, the following notion is far more important for this course.

Definition 3.1.11. A set $F \subseteq \mathbb{R}$ is said to be *closed* if $F^c = \mathbb{R} \setminus F$ is open.

Note the letter F is traditionally used for a closed set since the French word for closed is 'fermé'.

Example 3.1.12. As $\emptyset$ and $\mathbb{R}$ are open, $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$ are closed.

Example 3.1.13. For all $a, b \in \mathbb{R}$ with $a \leq b$, the closed interval $[a, b]$ is closed. Indeed notice that

$$[a, b]^c = (-\infty, a) \cup (b, \infty)$$

which is a union of open sets and thus open by Proposition 3.1.6. Hence $[a, b]$ is closed by definition.

Example 3.1.14. For all $a \in \mathbb{R}$, the sets $[a, \infty)$ and $(-\infty, a]$ are closed. To see this, notice that

$$[a, \infty)^c = (-\infty, a) \qquad \text{and} \qquad (-\infty, b]^c = (b, \infty)$$

are open. Hence $[a, \infty)$ and $(-\infty, a]$ are closed by definition.

Example 3.1.15. It is important to note that there are subsets of $\mathbb{R}$ that are not open nor closed. Indeed if $a, b \in \mathbb{R}$ are such that $a < b$, then $[a, b)$ is not open and not closed. To see this, first note that $[a, b)$ is not open by Example 3.1.3. Furthermore, notice that

$$[a, b)^c = (-\infty, a) \cup [b, \infty).$$

We claim that $[a, b)^c$ is not open. To see this, we note that $b \in [a, b)^c$ but we claim that $(b - \epsilon, b + \epsilon) \not\subseteq [a, b)^c$ for all $\epsilon > 0$. Indeed, if $\epsilon > 0$ and

$$\delta = \min\left\{\frac{1}{2}\epsilon, \frac{b-a}{4}\right\},$$

then $b - \delta \in (b - \epsilon, b + \epsilon)$ but $b - \delta \notin (-\infty, a) \cup [b, \infty) = [a, b)^c$. Therefore, by definition, $[a, b)^c$ is not open so $[a, b)$ is not closed.

Due to the nature of the complement of a set, the following trivially follows from Proposition 3.1.6.

Proposition 3.1.16. *Let I be an non-empty set and for each $i \in I$ let F_i be a closed subset of $\mathbb{R}$. Then*

- $\bigcap_{i\in I} F_i$ *is closed in* $\mathbb{R}$*, and*

- *$\bigcup_{i\in I} F_i$ is closed in $\mathbb{R}$ provided I has a finite number of element.*

Proof. Since

$$\left(\bigcap_{i\in I} F_i\right)^c = \bigcup_{i\in I} F_i^c \qquad \text{and} \qquad \left(\bigcup_{i\in I} F_i\right)^c = \bigcap_{i\in I} F_i^c$$

by de Morgan's Laws, the result follows by the definition of a closed set along with Proposition 3.1.6. ∎

The reason we are interested in closed sets is the following result that shows that closed sets contain all of their limits.

Proposition 3.1.17. *A set $F \subseteq \mathbb{R}$ is closed if and only if whenever $(a_n)_{n\geq 1}$ is a convergent sequence of real numbers with $a_n \in F$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \in F$.*

Proof. Assume $F \subseteq \mathbb{R}$ is closed. Let $(a_n)_{n\geq 1}$ be a convergent sequence of real numbers with $a_n \in F$ for all $n \in \mathbb{N}$ and let $L = \lim_{n\to\infty} a_n$. To see that $L \in F$, suppose for the sake of a contradiction that $L \notin F$. Hence $L \in F^c$. Since F is closed, F^c is open. Therefore since $L \in F^c$ and F^c is open, there exists an $\epsilon > 0$ such that $(L-\epsilon, L+\epsilon) \subseteq F^c$. However, since $L = \lim_{n\to\infty} a_n$, there exists an $N \in \mathbb{N}$ such that $a_N \in (L-\epsilon, L+\epsilon) \subseteq F^c$. Hence $a_N \in F^c$ and $a_N \in F$ which is a contradiction. Therefore it must be the case that $L \in F$.

To prove the other direction, assume F is not closed. Our goal is to construct a sequence $(a_n)_{n\geq 1}$ that converges to $L \notin F$ with $a_n \in F$ for all $n \in \mathbb{N}$. Since F is not closed, F^c is not open. Therefore there exists an $L \in F^c$ such that $(L-\epsilon, L+\epsilon) \not\subseteq F^c$ for all $\epsilon > 0$. Thus for each $n \in \mathbb{N}$ there exists a number $a_n \in (L - \frac{1}{n}, L + \frac{1}{n})$ with $a_n \notin F^c$. Hence $(a_n)_{n\geq 1}$ is a sequence of real numbers with $a_n \in F$ for all $n \in \mathbb{N}$ and

$$L - \frac{1}{n} \leq a_n \leq L + \frac{1}{n}$$

for all $n \in \mathbb{N}$. By the Squeeze Theorem (Theorem 2.3.13), $(a_n)_{n\geq 1}$ converges to L. Since $a_n \in F$ for all $n \in \mathbb{N}$ and $L \notin F$, the proof is complete. ∎

Example 3.1.18. The set

$$A = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$$

is not closed since $\frac{1}{n} \in A$ for all $n \in \mathbb{N}$ and $0 = \lim_{n\to\infty} \frac{1}{n}$ yet $0 \notin A$. However, one can show that

$$B = \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$$

is closed by showing that every convergent sequence whose elements are in B (of which there are lots) converges to an element in B. Indeed, if $(b_n)_{n\geq 1}$ is a convergent sequence with $b_n \in B$ for all $n \in \mathbb{N}$, then either $\lim_{n\to\infty} b_n = 0 \in B$, or there exists an $N, k \in \mathbb{N}$ such that $b_n = \frac{1}{k}$ for all $n \geq N$ and thus $\lim_{n\to\infty} b_n = \frac{1}{k} \in B$ in this case. Alternatively, one can use the technology of the next section (specifically Example 3.2.6 and Theorem 3.2.8) to prove that B is closed.

3.1.3 Closure of a Set

As Example 3.1.18 shows, it can be possible to take a set and add a few number of points in order to make the set closed. This can be formalized by the following result which shows there is a smallest closed set containing a given set.

Lemma 3.1.19. *Let $A \subseteq \mathbb{R}$. Then there exists a closed subset $F \subseteq \mathbb{R}$ such that $A \subseteq F$ and if $F_1 \subseteq \mathbb{R}$ is closed and $A \subseteq F_1$, then $F \subseteq F_1$. That is, there is a smallest closed subset of $\mathbb{R}$ that contains A.*

Proof. Let
$$\mathcal{F}_A = \{Y \subseteq \mathbb{R} \mid X \subseteq Y,\ Y \text{ is closed}\}$$
and let
$$F = \bigcap_{Y \in \mathcal{F}_A} Y.$$
We claim that F is the set we are looking for. To see this, first notice that F is closed by Proposition 3.1.16 as it is the intersection of closed sets. Moreover, since $A \subseteq Y$ for all $Y \in \mathcal{F}_A$, $A \subseteq F$. Thus F is a closed set that contains A.

To see that F is the smallest closed set that contains A, let $F_1 \subseteq \mathbb{R}$ be an arbitrary closed set such that $A \subseteq F_1$. Hence $F_1 \in \mathcal{F}_A$ by definition. Therefore
$$F = \bigcap_{Y \in \mathcal{F}_A} Y \subseteq F_1$$
since F_1 is one of the sets in the intersection. Therefore, as F_1 was arbitrary, F has the desired properties. ■

Due to Lemma 3.1.19, we can make the following definition.

Definition 3.1.20. The *closure* of a subset A of $\mathbb{R}$, denoted $\overline{A}$, is the smallest closed subset of $\mathbb{R}$ containing A.

Example 3.1.21. It is not difficult to see that $\overline{(0,1)} = [0,1]$. Indeed, clearly $[0,1]$ is a closed set containing $(0,1)$ and any closed set containing $(0,1)$ must contain 0 and 1 since there are sequences with elements in $(0,1)$ that converge to 0 and 1 respectively.

Example 3.1.22. It is not difficult to see that if $F \subseteq \mathbb{R}$ is closed, then $\overline{F} = F$ (i.e. the smallest closed set containing a closed set is the initial closed set).

Example 3.1.23. The closure of the rational numbers in the real numbers is the real numbers. To see this, note for each $\gamma \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists a rational number q_n such that

$$\gamma - \frac{1}{n} \leq q_n \leq \gamma + \frac{1}{n}$$

by Proposition 1.3.8. Hence, the Squeeze Theorem (Theorem 2.3.13) implies that $(q_n)_{n\geq 1}$ converges to γ. Therefore, any closed set that contains $\mathbb{Q}$ must also contain γ by Proposition 3.1.17. Hence the only closed set containing $\mathbb{Q}$ is $\mathbb{R}$ so $\overline{\mathbb{Q}} = \mathbb{R}$.

Generalizing the idea in Example 3.1.23, we obtain the following alternative characterization of the closure of a set of real numbers.

Lemma 3.1.24. *Let $A \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$. Then $x \in \overline{A}$ if and only if for all $\epsilon > 0$ there exists a $a \in A$ so that $|x - a| < \epsilon$.*

Proof. Assume $x \in \overline{A}$. To see the desired property of x, suppose for the sake of a contradiction that there exists an $\epsilon > 0$ so that $|x - a| \geq \epsilon$ for all $a \in A$. Then $(x - \epsilon, x + \epsilon) \cap A = \emptyset$. Hence $A \subseteq (-\infty, x - \epsilon] \cap [x + \epsilon, \infty)$. Since $(-\infty, x - \epsilon] \cap [x + \epsilon, \infty)$ is a closed set containing A, we have $\overline{A} \subseteq (-\infty, x - \epsilon] \cap [x + \epsilon, \infty)$ by the definition of the closure. However, $\overline{A} \subseteq (-\infty, x - \epsilon] \cap [x + \epsilon, \infty)$ contradicts the fact that $x \in \overline{A}$. Hence x has the desired property.

Conversely, assume that $x \in \mathbb{R}$ has the property that for all $\epsilon > 0$ there exists a $a \in A$ so that $|x - a| < \epsilon$. Thus, for each $n \in \mathbb{N}$ there exists an $a_n \in A$ such that $|x - a_n| < \frac{1}{n}$. Hence $(a_n)_{n\geq 1}$ is a sequence in A that converges to x. By Proposition 3.1.17, any closed set that contains A must contain a_n for all $n \in \mathbb{N}$ and thus must contain x. Hence $x \in \overline{A}$ by definition as desired. ∎

Moreover, the closure of a set can be characterized by adding all of the points to make Proposition 3.1.17 work.

Lemma 3.1.25. *Let $A \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$. Then $x \in \overline{A}$ if and only if there exists a sequence $(a_n)_{n\geq 1}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} a_n$.*

Proof. Assume $x \in \overline{A}$. To see the desired property, note by Lemma 3.1.24 that for all $n \in \mathbb{N}$ there exists an $a_n \in A$ such that $|x - a_n| < \frac{1}{n}$. Thus $(a_n)_{n\geq 1}$ is a sequence of real numbers with $a_n \in A$ for all $n \in \mathbb{N}$ and

$$x - \frac{1}{n} \leq a_n \leq x + \frac{1}{n}$$

for all $n \in \mathbb{N}$. By the Squeeze Theorem (Theorem 2.3.13), $(a_n)_{n\geq 1}$ converges to x. Therefore, as $x \in \overline{A}$ was arbitrary, one direction is complete.

To see the other direction, let $x \in \mathbb{R}$ be such that there exists a sequence $(a_n)_{n\geq 1}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} a_n$. Since $A \subseteq \overline{A}$ by definition, $a_n \in \overline{A}$ for all $n \in \mathbb{N}$. Therefore, since $\overline{A}$ is closed, Proposition 3.1.17 implies that

$$x = \lim_{n\to\infty} a_n \in \overline{A}$$

as desired. ∎

3.1.4 Limit, Interior, and Boundary Points

There are many other topological notions that describe whether or not a point is related to a set. In this subsection, we will analyze the notions of a limit point, an interior point, and a boundary point for subsets of $\mathbb{R}$.

With our previous study of closed sets and the closure of a set, we have all the tools necessary to study limit points.

Definition 3.1.26. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be a *limit point of* A if exists a sequence $(a_n)_{n\geq 1}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} a_n$.

In particular, a limit point of A is a limit of a sequence of points from A. Based on the previous subsection, we immediately have the following alternative characterization of limit points

Corollary 3.1.27. *Let $A \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$. The following are equivalent:*

- *x is a limit point of A (that is, there exists a sequence $(a_n)_{n\geq 1}$ such that $a_n \in A$ for all $n \in \mathbb{N}$ and $x = \lim_{n\to\infty} a_n$).*
- *$x \in \overline{A}$.*
- *For all $\epsilon > 0$, there exists an $a \in A$ such that $|a - x| < \epsilon$.*

Proof. This follows immediately from the definition of a limit point and Lemmata 3.1.24 and 3.1.25. ∎

Furthermore, we immediately have a characterization of closed sets via limit points.

Corollary 3.1.28. *Let $F \subseteq \mathbb{R}$. Then F is a closed subset of $\mathbb{R}$ if and only if F contains all of its limit points.*

Proof. This follows immediately Proposition 3.1.17 and Corollary 3.1.27. ∎

To define the next useful topological notion of a point related to a set, we channel the idea of an open set.

Definition 3.1.29. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *interior point of* A if exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq A$. The set of interior points of A is denoted by $\mathrm{int}(A)$.

Example 3.1.30. Consider the sets $[a, b]$, $(a, b]$, $[a, b)$, and (a, b). It is elementary to see by Definition 3.1.29 that the interior of all of these sets is (a, b). Hence the term 'interior'.

Example 3.1.31. If U is an open subset of $\mathbb{R}$, then every element of U is an interior point of U by definition.

In particular, similar to how the closure of a set A is the smallest closed set containing A, the interior of a set A actually is the largest open set contained in A. To show this, we divide the proof into a few lemmata; first showing that the interior is an open set, and then showing the interior contains all open subsets of A.

Lemma 3.1.32. *If $A \subseteq \mathbb{R}$, then $\mathrm{int}(A)$ is an open set.*

Proof. To see that $\mathrm{int}(A)$ is open, let $x \in \mathrm{int}(A)$ be arbitrary. By the definition of an interior point, there exists an $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq A$.

We claim that $(x-\frac{1}{2}\epsilon, x+\frac{1}{2}\epsilon) \subseteq \mathrm{int}(A)$. To see this, let $y \in (x-\frac{1}{2}\epsilon, x+\frac{1}{2}\epsilon)$ be arbitrary. To see that $y \in \mathrm{int}(A)$, first notice if $z \in (y - \frac{1}{2}\epsilon, y + \frac{1}{2}\epsilon)$ then

$$|z - x| \leq |z - y| + |y - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and thus $z \in (x - \epsilon, x + \epsilon) \subseteq A$. Hence $(y - \frac{1}{2}\epsilon, y + \frac{1}{2}\epsilon) \subseteq A$ so $y \in \mathrm{int}(A)$ by definition. Therefore, since $y \in (x - \frac{1}{2}\epsilon, x + \frac{1}{2}\epsilon)$ was arbitrary, $(x - \frac{1}{2}\epsilon, x + \frac{1}{2}\epsilon) \subseteq \mathrm{int}(A)$. Hence, since $x \in \mathrm{int}(A)$ was arbitrary, $\mathrm{int}(A)$ is open by the definition of an open set. ■

Lemma 3.1.33. *Let $A \subseteq \mathbb{R}$. If $U \subseteq A$ and U is an open subset of $\mathbb{R}$, then $U \subseteq \mathrm{int}(A)$.*

Proof. Assume $U \subseteq A$ and U is an open subset of $\mathbb{R}$. To see that $U \subseteq \mathrm{int}(A)$, let $x \in U$ be arbitrary. Since U is open, by the definition of an open set there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U \subseteq A$. Hence $x \in \mathrm{int}(A)$ by the definition of the interior. Therefore, since $x \in U$ was arbitrary, $U \subseteq \mathrm{int}(A)$ as desired. ■

Corollary 3.1.34. *If $A \subseteq \mathbb{R}$, then*

$$\mathrm{int}(A) = \bigcup_{U \in \Gamma} U \qquad \textit{where } \Gamma = \{U \subseteq A \mid U \textit{ is an open subset of } \mathbb{R}\}.$$

Hence $\mathrm{int}(A)$ is the largest open subset of A.

Proof. If $U \in \Gamma$ then $U \subseteq \mathrm{int}(A)$ by Lemma 3.1.33. Hence $\bigcup_{U \in \Gamma} U \subseteq \mathrm{int}(A)$.

For the other inclusion, let $x \in \mathrm{int}(A)$ be arbitrary. By the definition of the interior there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq A$. Since $(x - \epsilon, x + \epsilon)$ is an open set and thus $(x - \epsilon, x + \epsilon) \in \Gamma$, we obtain that

$$x \in (x - \epsilon, x + \epsilon) \subseteq \bigcup_{U \in \Gamma} U.$$

Therefore, since $x \in \mathrm{int}(A)$ was arbitrary, we obtain that $\mathrm{int}(A) \subseteq \bigcup_{U \in \Gamma} U$ as desired. ∎

For our final type of point based on a set, there are many equivalent characterizations. We will adopt the following as the definition as it mimics the definition of the interior point and we will prove the equivalent characterizations in due course.

Definition 3.1.35. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is said to be an *boundary point of* A if for all $\epsilon > 0$ there exists an $a \in A$ and a $b \in \mathbb{R} \setminus A$ such that $a, b \in (x - \epsilon, x + \epsilon)$. The set of boundary points of A is denoted by $\mathrm{bdy}(A)$.

As promised, there are many equivalent characterizations of boundary points.

Proposition 3.1.36. *Let $A \subseteq \mathbb{R}$ and let $x \in \mathbb{R}$. The following are equivalent:*

(i) x *is a boundary point of* A *(that is,* $\epsilon > 0$ *there exists an* $a \in A$ *and a* $b \in \mathbb{R} \setminus A$ *such that* $a, b \in (x - \epsilon, x + \epsilon)$*).*

(ii) There exists sequences $(a_n)_{n \geq 1}$ *and* $(b_n)_{n \geq 1}$ *such that* $a_n \in A$ *and* $b_n \in \mathbb{R} \setminus A$ *for all* $n \in \mathbb{N}$ *and* $x = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$*.*

(iii) $x \in \overline{A} \cap \overline{(\mathbb{R} \setminus A)}$*.*

(iv) $x \in \overline{A} \setminus \mathrm{int}(A)$*.*

(v) $x \notin \mathrm{int}(A) \cup \mathrm{int}(\mathbb{R} \setminus A)$*.*

Proof. To begin, note that (i), (ii), and (iii) are equivalent by Corollary 3.1.27. To complete the proof, we will show that (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i).

To see that (iii) $\Longrightarrow$ (iv), assume $x \in \overline{A} \cap \overline{(\mathbb{R} \setminus A)}$. Hence $x \in \overline{A}$. Therefore, to show that $x \in \overline{A} \setminus \mathrm{int}(A)$, it remains only to show that $x \notin \mathrm{int}(A)$. To see this, suppose for the sake of a contradiction that $x \in \mathrm{int}(A)$. By the definition of the interior, this implies that there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq A$. Hence $(x - \epsilon, x + \epsilon) \cap (\mathbb{R} \setminus A) = \emptyset$. Therefore Lemma 3.1.24 implies that $x \notin \overline{(\mathbb{R} \setminus A)}$. Since this contradicts the fact that $x \in \overline{A} \cap \overline{(\mathbb{R} \setminus A)}$, it follows that $x \notin \mathrm{int}(A)$. Hence (iii) $\Longrightarrow$ (iv).

To see that (iv) $\Longrightarrow$ (v), assume $x \in \overline{A} \setminus \mathrm{int}(A)$. Therefore $x \notin \mathrm{int}(A)$. Therefore, to show that $x \notin \mathrm{int}(A) \cup \mathrm{int}(\mathbb{R} \setminus A)$, it remains only to show that $x \notin \text{int}(\mathbb{R} \setminus A)$. To see this, suppose for the sake of a contradiction that $x \in \text{int}(\mathbb{R} \setminus \mathbb{R})$. By the definition of the interior, this implies that there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq \mathbb{R} \setminus \mathbb{R}$. Hence $(x - \epsilon, x + \epsilon) \cap A = \emptyset$. Therefore Lemma 3.1.24 implies that $x \notin \overline{A}$. Since this contradicts the fact that $x \in \overline{A} \setminus \text{int}(A)$, it follows that $x \notin \text{int}(\mathbb{R} \setminus A)$. Hence (iv) $\implies$ (v).

To see that (v) $\implies$ (i), assume $x \notin \text{int}(A) \cup \text{int}(\mathbb{R} \setminus A)$. Since $x \notin \text{int}(A)$, by the definition of the interior we know for all $\epsilon > 0$ that $(x - \epsilon, x + \epsilon) \not\subseteq A$. Hence for all $\epsilon > 0$ there exists a $b \in \mathbb{R} \setminus A$ such that $b \in (x - \epsilon, x + \epsilon)$. Similarly, since $x \notin \text{int}(\mathbb{R} \setminus A)$, by the definition of the interior we know for all $\epsilon > 0$ that $(x - \epsilon, x + \epsilon) \not\subseteq \mathbb{R} \setminus A)$. Hence for all $\epsilon > 0$ there exists an $a \in A$ such that $a \in (x - \epsilon, x + \epsilon)$. Hence (i) holds so (v) $\implies$ (i). ∎

To motivate the terminology of the boundary point, consider the following examples.

Example 3.1.37. Consider the sets $[a, b]$, $(a, b]$, $[a, b)$, and (a, b). We claim that the set of boundary points of all of these sets is $\{a, b\}$. This follows immediately since the closure of these sets are $[a, b]$ and since the closure of the complements of these sets are $(-\infty, a] \cup [b, \infty)$. Hence the boundary points of these sets are truly "on the boundary.

In fact, we boundary points are related to one of the initial concepts in this course.

Example 3.1.38. Let A be a non-empty subset of $\mathbb{R}$ that is bounded above. We claim that $\alpha = \text{lub}(A)$ is a boundary point of A. Hence all of the characterizations from Proposition 3.1.36 hold for α.

To see that α is a boundary point of A, let $\epsilon > 0$ be arbitrary. We desire to show that there exists an $a \in A$ and a $b \in \mathbb{R} \setminus A$ such that $a, b \in (\alpha - \epsilon, \alpha + \epsilon)$. To begin, let $b = \alpha + \frac{1}{2}\epsilon$. Therefore, since $b > \alpha$ and α is the least upper bound of A, it follows that $b \notin A$. Hence $b \in \mathbb{R} \setminus A$ and $b \in (\alpha - \epsilon, \alpha + \epsilon)$. Therefore, it remains only to show the existence of a.

To see that there exists an $a \in A$ such that $a \in (\alpha - \epsilon, \alpha + \epsilon)$, suppose for the sake of a contradiction that $A \cap (\alpha - \epsilon, \alpha + \epsilon) = \emptyset$. Let $\gamma = \alpha - \frac{1}{2}\epsilon$. Clearly $\gamma < \alpha$. Moreover, since $a \leq \alpha$ for all $a \in A$ and since $A \cap (\alpha - \epsilon, \alpha + \epsilon) = \emptyset$, it follows that $a \leq \gamma$ for all $a \in A$. Therefore γ is an upper bound of A that is less than α. Since this contradicts the fact that α is the least upper bound of A, we have a contradiction. Hence α is a boundary point of A.

Remark 3.1.39. By combining Example 3.1.38 together with Proposition 3.1.36, it follows that if A is a non-empty subset of $\mathbb{R}$ that is bounded above, then exists sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ such that $a_n \in A$ and $b_n \in \mathbb{R} \setminus A$ for all $n \in \mathbb{N}$ and $\text{lub}(A) = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. This not only is a useful property of the least upper bound of a set, but it also shows that students' original intuition when it comes to computing least upper bounds is correct: the least upper bound of a set is, in essence, the "largest element of the set" (of course, it does not need to be in the set, but it almost is).

3.1.5 The Cantor Set

In order to have one non-trivial example of for interior and boundary points, we consider an example of a subset of $\mathbb{R}$ that is very important example in analysis.

Definition 3.1.40. Let $P_0 = [0, 1]$. Construct P_1 from P_0 by removing the open interval of length $\frac{1}{3}$ from the middle of P_0 (i.e. $P_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$). Then construct P_2 from P_1 by removing the open intervals of length $\frac{1}{3^2}$ from the middle of each closed subinterval of P_1. Subsequently, having constructed P_n, construct P_{n+1} by removing the open intervals of length $\frac{1}{3^{n+1}}$ from the middle of each of the 2^n closed subintervals of P_n. The set

$$\mathcal{C} = \bigcap_{n \geq 1} P_n$$

is known as the *Cantor set*.

The Cantor set has many interesting properties. First, note that Proposition 3.1.16 implies the Cantor set is closed being the intersection of closed sets. The following gives an alternate characterization of the Cantor set using the ternary expansion of real numbers as described in Remark 2.3.28.

Lemma 3.1.41. *Let $x \in \mathbb{R}$. Then $x \in \mathcal{C}$ if and only if there is a sequence $(a_n)_{n\geq 1}$ with $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ (i.e. $x \in [0, 1]$ and x has a ternary expansion using only 0s and 2s).*

Proof. To begin, assume $x \in \mathcal{C}$. Hence $x \in P_n$ for all $n \in \mathbb{N}$ by the definition of $\mathcal{C}$. Furthermore, by the recursive construction of the P_n, there exists a sequent $(a_n)_{n\geq 1}$ such that $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ and

$$x \in \left[\sum_{k=1}^{n} \frac{a_k}{3^k}, \frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right] \subseteq P_n$$

for all $n \in \mathbb{N}$.

To see that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$, notice that

$$\left|x - \sum_{k=1}^{n} \frac{a_k}{3^k}\right| \leq \left|\left(\frac{1}{3^n} + \sum_{k=1}^{n} \frac{a_k}{3^k}\right) - \sum_{k=1}^{n} \frac{a_k}{3^k}\right| = \frac{1}{3^n}.$$

Hence

$$x - \frac{1}{3^n} \leq \sum_{k=1}^{n} \frac{a_k}{3^k} \leq x + \frac{1}{3^n}$$

for all $n \in \mathbb{N}$. Therefore, since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, the Squeeze Theorem (Theorem 2.3.13) implies that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$ as desired.

For the converse direction, assume $x \in \mathbb{R}$ is such that there exists a sequence $(a_n)_{n\geq 1}$ with $a_n \in \{0, 2\}$ for all $n \in \mathbb{N}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_k}{3^k}$.

For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} \frac{a_k}{3^k}$. By the description of P_n, we obtain that $s_n \in P_n$ for all $n \in \mathbb{N}$. In fact, we claim that $s_m \in P_n$ whenever $m \geq n$. To see this, notice if $m \geq n$ then

$$\begin{aligned}\sum_{k=1}^{n} \frac{a_k}{3^k} \leq \sum_{k=1}^{m} \frac{a_k}{3^k} = s_m &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \sum_{k=n+1}^{m} \frac{2}{3^k} \\ &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{2}{3^{n+1}} \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{1 - \frac{1}{3}} \\ &= \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1 - \left(\frac{1}{3}\right)^{m-n}}{3^n} \\ &\leq \sum_{k=1}^{n} \frac{a_k}{3^k} + \frac{1}{3^n}\end{aligned}$$

so that $s_m \in P_n$ as claimed.

Since each P_n is a closed set, since $x = \lim_{m\to\infty} s_m$, and since $s_m \in P_n$ whenever $m \geq n$, we obtain that $x \in P_n$ for each $n \in \mathbb{N}$ by Proposition 3.1.17. Hence $x \in \bigcap_{n\geq 1} P_n = \mathcal{C}$ as desired. ∎

Lemma 3.1.41 enables us to demonstrate the following which shows the set of interior and boundary points of a set may behave far differently than what was seen for intervals.

Corollary 3.1.42. *For the Cantor set $\mathcal{C}$, $\mathrm{int}(\mathcal{C}) = \emptyset$ and $\mathrm{bdy}(\mathcal{C}) = \mathcal{C}$.*

Proof. To see that $\mathrm{int}(\mathcal{C}) = \emptyset$, suppose for the sake of a contradiction that there exists an $x \in \mathrm{int}(\mathcal{C})$. By the definition of the interior, this implies that there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq \mathcal{C} = \bigcap_{n\geq 1} P_n$. Hence $(x - \epsilon, x + \epsilon) \subseteq P_n$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{1}{3^n} = 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{3^N} < \epsilon$. By the construction of P_N we see that P_N does not contain an open interval of length more that $\frac{1}{3^N}$. Therefore, it is impossible that $(x - \epsilon, x + \epsilon) \subseteq P_N$. Hence we have obtained a contradiction so it must be the case that $\mathrm{int}(\mathcal{C}) = \emptyset$.

To see that $\mathrm{bdy}(\mathcal{C}) = \mathcal{C}$, note $\mathrm{bdy}(\mathcal{C}) = \overline{\mathcal{C}} \setminus \mathrm{int}(\mathcal{C})$ by Proposition 3.1.36. Since $\mathcal{C}$ is closed, we know that $\overline{\mathcal{C}} = \mathcal{C}$. Therefore, since we have shown that $\mathrm{int}(\mathcal{C}) = \emptyset$, we obtain that

$$\mathrm{bdy}(\mathcal{C}) = \overline{\mathcal{C}} \setminus \mathrm{int}(\mathcal{C}) = \mathcal{C} \setminus \emptyset = \mathcal{C}$$

as desired. ∎

The intriguing thing about the Cantor set being a closed set with empty interior is that Cantor set is equinumerous with the real numbers; that is, the Cantor set and the real numbers have the same cardinality (the same number of elements). For those that saw cardinality in MATH 1200, we include the following.

Corollary 3.1.43. $|\mathcal{C}| = |\mathbb{R}|$.

Proof. To see that $\mathcal{C}$ is uncountable, let

$$X = \{(b_n)_{n\geq 1} \mid, b_n \in \{0,1\} \text{ for all } n \in \mathbb{N}\}.$$

Define $f : X\{0,1\} \to \mathcal{C}$ by

$$f((b_n)_{n\geq 1}) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2b_k}{3^k}$$

for all $(b_n)_{n\geq 1} \in X$. Clearly f is a well-defined injective function so $|\mathcal{C}| \geq 2^{|\mathbb{N}|} = |\mathbb{R}|$. Therefore, since $\mathcal{C} \subseteq \mathbb{R}$, we obtain that $|\mathcal{C}| = |\mathbb{R}|$ as desired. ∎

3.2 Compact Sets

Although the topological notion of a closed set is nice due to Proposition 3.1.17, there is a far more useful topological notion. One essential idea in analysis is the ability to reduce infinity objects into finite objects as the latter are easier to handle. Since the topology on $\mathbb{R}$ determines the analytic structure on subsets of $\mathbb{R}$ and it would be nice when given a subset of $\mathbb{R}$ to be able to reduce to a finite number of open sets. This notion is known as a compact set. Although for the real numbers there is not much difference between closed and compact sets, it is really the notion of compactness that makes many results in analysis work. In particular, many of the most important theorems in this course fundamentally rely on the fact that every closed interval is compact.

3.2.1 Definition of a Compact Set

It turns out that there are many equivalent notions of a compact set for subsets of the real numbers and many of these notions extend to more general settings. However, the most general notion of a compact set is based on the following.

Definition 3.2.1. Let $A \subseteq \mathbb{R}$. A collection $\{U_i \mid i \in I\}$ of subsets of $\mathbb{R}$ is said to be an *open cover* of A if U_i is open for all $i \in I$ and $A \subseteq \bigcup_{i\in I} U_i$.

Example 3.2.2. If for each $n \in \mathbb{N}$ we let $U_n = (-n, n)$, then $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ (and any subset of $\mathbb{R}$).

Example 3.2.3. If for each $n \in \mathbb{N}$ we let $U_n = \left(\frac{1}{n}, 1\right)$, then $\left\{ \left(\frac{1}{n}, 1\right) \,\middle|\, n \in \mathbb{N}\right\}$ is an open cover of $(0, 1)$.

The most general definition of a compact set is as follows.

Definition 3.2.4. A set $K \subseteq \mathbb{R}$ is said to be *compact* if every open cover of K has a finite subcover; that is, if $\{U_i \mid i \in I\}$ is an open cover of K, then there exists an $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that $K \subseteq \bigcup_{k=1}^n U_{i_k}$.

Note the letter K is traditionally used for a compact set since the German word for compact is 'kompakt'.

Remark 3.2.5. The rationale for why compact sets are easy to work with is that it is much easier to deal with a finite number of objects than it is to deal with an infinite number (e.g. taking a maximum instead of a supremum).

Example 3.2.6. Let

$$K = \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}.$$

We claim that K is a compact set.

To see this, let $\{U_i \mid i \in I\}$ be any open cover of K. Since $0 \in \bigcup_{i \in I} U_i$, there exists an $i_0 \in I$ such that $0 \in U_{i_0}$. Therefore, since U_{i_0} is open, there exists an $\epsilon > 0$ so that $(-\epsilon, \epsilon) \subseteq U_{i_0}$.

Since $\lim_{n\to\infty} \frac{1}{n} = 0$ there exists an $N \in \mathbb{N}$ such that $\frac{1}{n} \in (-\epsilon, \epsilon) \subseteq U_{i_0}$ for all $n \geq N$. Furthermore, since $K \subseteq \bigcup_{i \in I} U_i$, for each $n < N$ we may chose an $i_n \in I$ such that $\frac{1}{n} \in U_{i_n}$. Hence, by construction

$$K \subseteq \bigcup_{k=0}^{N-1} U_{i_k}$$

so $\{U_{i_0}, \ldots, U_{i_{N-1}}\}$ is a finite open subcover of K. Hence, since $\{U_i \mid i \in I\}$ was an arbitrary open cover of K, K is compact by definition.

It is natural to ask whether $\mathbb{R}$ is compact. Since the open cover $\{(-n, n) \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ clearly has no finite subcovers, we see that $\mathbb{R}$ is not compact. More generally, using the same open cover, we note the following.

Theorem 3.2.7. *If $K \subseteq \mathbb{R}$ is compact, then K is bounded.*

Proof. Let $K \subseteq \mathbb{R}$ be compact. For each $n \in \mathbb{N}$, let $U_n = (-n, n)$. Therefore, since $\bigcup_{n\geq 1} U_n = \mathbb{R}$, we have that $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of K. Since K is compact, there exists numbers $k_1, \ldots, k_m \in \mathbb{N}$ such that $\{U_{k_1}, \ldots, U_{k_m}\}$ is an open cover of K. Therefore, if $M = \max\{k_1, \ldots, k_m\}$, then

$$K \subseteq \bigcup_{j=1}^m U_{k_j} = (-M, M).$$

Hence K is bounded. ∎

By analyzing the open cover $\left\{ \left(\frac{1}{n}, 1\right) \,\middle|\, n \in \mathbb{N}\right\}$ of $(0,1)$, we see that any finite subcover does not contain an interval of the form $(0, \epsilon)$ for some $\epsilon > 0$. Hence this open cover has no finite subcovers so $(0,1)$ is not compact. More generally, using similar ideas, we obtain the following.

Theorem 3.2.8. *If $K \subseteq \mathbb{R}$ is compact, then K is closed.*

Proof. Let $K \subseteq \mathbb{R}$ be compact. Suppose for the sake of a contradiction that K is not closed. By Proposition 3.1.17 there exists a convergent sequence $(a_n)_{n\geq 1}$ such that $a_n \in K$ for all $n \in \mathbb{N}$ yet $L = \lim_{n\to\infty} a_n \notin K$. We will use the sequence $(a_n)_{n\geq 1}$ to construct an open cover of K that has no finite subcovers thereby contradicting the fact that K is compact.

For each $n \in \mathbb{N}$ let

$$U_n = \left(-\infty, L - \frac{1}{n}\right) \cup \left(L + \frac{1}{n}, \infty\right).$$

Notice that each U_n is open and

$$\bigcup_{n\geq 1} U_n = \mathbb{R} \setminus \{L\}.$$

Hence, as $L \notin K$, $\{U_n \mid n \in \mathbb{N}\}$ is an open cover of K.

Since K is compact, $\{U_n \mid n \in \mathbb{N}\}$ has a finite subcover of K. Thus there exists $k_1, \ldots, k_m \in \mathbb{N}$ such that $K \subseteq \bigcup_{j=1}^m U_{k_j}$. Let $M = \max\{k_1, \ldots, k_m\}$. Then

$$K \subseteq \bigcup_{j=1}^m U_{k_j} \subseteq \left(-\infty, L - \frac{1}{M}\right) \cup \left(L + \frac{1}{M}, \infty\right).$$

However, since $a_n \in K$ for all $n \in \mathbb{N}$, we see that $|a_n - L| \geq \frac{1}{M}$ for all $n \in \mathbb{N}$. Since this contradicts the fact that $L = \lim_{n\to\infty} a_n$ we have obtained a contradiction. Hence K is closed. ∎

3.2.2 The Heine-Borel Theorem

By combining Theorems 3.2.7 and 3.2.8, we see that every compact subset of $\mathbb{R}$ is closed and bounded. In fact, the following theorem shows these are the only compact subsets of $\mathbb{R}$. Before the proof, it should be noted that the notion of a compact set is very different and much more important than the notion of a closed and bound set in future courses.

Theorem 3.2.9 (The Heine-Borel Theorem). *A set $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded.*

Proof. If K is a compact subset of $\mathbb{R}$, then K is bounded and closed by Theorems 3.2.7 and 3.2.8 respectively.

Assume $K \subseteq \mathbb{R}$ is closed and bounded. To see that K is compact, let $\{U_i \mid i \in I\}$ be an arbitrary an open cover of K. We claim that $\{U_i \mid i \in I\}$

has a finite subcover of K. To see this, suppose for the sake of a contradiction that $\{U_i \mid i \in I\}$ does not have a finite subcover of K.

Before we proceed, perhaps a little intuition on where this proof is going is required. We will use the supposition that $\{U_i \mid i \in I\}$ does not have a finite subcover of K to construct a sequence of decreasing closed intervals I_n with lengths tending to 0 so that $K \cap I_n$ does not have a finite subcover for all $n \in \mathbb{N}$. We will then use the concepts of Cauchy sequences and closed sets to prove that there is a point x that is in all of the $K \cap I_n$. This point x must then be in U_{i_0} for some i_0. However, as U_{i_0} is open, there is an interval around x that is contained in U_{i_0}. Since the length of I_n tends to 0 and I_n contains x for all $n \in \mathbb{N}$, this will force $I_N \subseteq U_{i_0}$ for some $N \in \mathbb{N}$ thereby contradicting the fact that $K \cap I_N$ does not have a finite subcover. Now, let us formalize this argument.

Since K is bounded, there exists an $M \in \mathbb{R}$ such that $K \subseteq [-M, M]$. Since $\{U_i \mid i \in I\}$ is an open cover that does not have a finite subcover of K, it must be the case that

$$K \cap [-M, 0] \quad \text{or} \quad K \cap [0, M]$$

does not have a finite subcover (as if each had a finite subcover, then combining the two finite subcovers would yield a finite subcover of K). Choose $I_1 = [a_1, b_1]$ from $\{[-M, 0], [0, M]\}$ so that $K \cap I_1$ does not have a finite subcover. Note that $|b_1 - a_1| = M$.

Next, since $\{U_i \mid i \in I\}$ is an open cover that does not have a finite subcover of $K \cap I_1$, it must be the case that

$$K \cap \left[a_1, \frac{a_1 + b_1}{2}\right] \quad \text{or} \quad K \cap \left[\frac{a_1 + b_1}{2}, b_1\right]$$

does not have a finite subcover (as if each had a finite subcover, then combining the two finite subcovers would yield a finite subcover of $K \cap I_1$). Choose $I_2 = [a_2, b_2]$ from

$$\left\{\left[a_1, \frac{a_1 + b_1}{2}\right], \left[\frac{a_1 + b_1}{2}, b_1\right]\right\}$$

so that $K \cap I_2$ does not have a finite subcover. Note that $|b_2 - a_2| = \frac{1}{2}|b_1 - a_1| = \frac{1}{2}M$.

Using the same idea in the previous paragraph, there must exist closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ such that $K \cap I_n$ does not have a finite subcover for all $n \in \mathbb{N}$ and if $I_n = [a_n, b_n]$, then $|b_n - a_n| = \frac{1}{2^{n-1}}M$. Since $K \cap I_n$ does not have a finite subcover for all $n \in \mathbb{N}$, $K \cap I_n \neq \emptyset$ for all $n \in \mathbb{N}$ (since the empty set clearly has a finite subcover). Hence, for each $n \in \mathbb{N}$, we can choose a $c_n \in K \cap I_n$.

We claim that the sequence $(c_n)_{n \geq 1}$ is Cauchy. To see this, let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$ by Example 2.2.7, there exists an $N \in \mathbb{N}$

such that $\frac{1}{2^{N-1}}M < \epsilon$. Therefore, if $n, m \geq N$, then $c_n \in I_n \subseteq I_N$ and $c_m \in I_m \subseteq I_N$ so

$$|c_n - c_m| \leq |b_N - a_N| = \frac{1}{2^{N-1}}M < \epsilon.$$

Hence, as $\epsilon > 0$ was arbitrary, $(c_n)_{n\geq 1}$ is Cauchy. Hence $L = \lim_{n\to\infty} c_n$ exists by the Completeness of the Real Numbers (Theorem 2.5.8).

Since K is closed by assumption and $c_n \in K$ for all $n \in \mathbb{N}$, $L \in K$ by Proposition 3.1.17. Moreover, note that $L \in I_n$ for all $n \in \mathbb{N}$ by Proposition 3.1.17 since I_n is closed and $c_m \in I_n$ for all $m \geq n$ (as we can start the sequence at n instead of at 1).

Since $\{U_i \mid i \in I\}$ is an open cover of K and $L \in K$, there exists an $i_0 \in I$ so that $L \in U_{i_0}$. Since U_{i_0} is open and $L \in U_{i_0}$, there exists an $\epsilon > 0$ so that $(L - \epsilon, L + \epsilon) \subseteq U_{i_0}$.

Since $\lim_{n\to\infty} \frac{1}{2^n} = 0$, there exists an $N \in \mathbb{N}$ such that $|b_N - a_N| = \frac{1}{2^{N-1}}M < \epsilon$. Hence, since $L \in I_N = [a_N, b_N]$, we have that $a_N \leq L \leq b_N$. Therefore, since $|b_N - a_N| < \epsilon$ and $a_N \leq L \leq b_N$, we must have that

$$L - \epsilon < a_N \leq L \leq b_N < L + \epsilon.$$

Hence

$$I_N = [a_N, b_N] \subseteq (L - \epsilon, L + \epsilon) \subseteq U_{i_0}.$$

Thus U_{i_0} is a finite open cover of $K \cap I_N$. However, this contradicts the fact that $K \cap I_n$ does not admit a finite subcover. Hence we have obtained a contradiction so $\{U_i \mid i \in I\}$ must admit a finite subcover of K. Since $\{U_i \mid i \in I\}$ was an arbitrary an open cover of K, K is compact by definition. ∎

3.2.3 Sequential Compactness

In general topological spaces, there are other notions of compactness. The following is another notion of compactness notion that can be quite useful.

Definition 3.2.10. A set $K \subseteq \mathbb{R}$ is said to be *sequentially compact* if whenever $(a_n)_{n\geq 1}$ is a sequence of real numbers with $a_n \in K$ for all $n \in \mathbb{N}$ there exist a subsequence of $(a_n)_{n\geq 1}$ that converges to an element of K.

We have seen that bounded sequences in $\mathbb{R}$ have convergent subsequences by the Bolzano-Weirstrass Theorem (Theorem 2.4.7) and this has already been of use for us in this course to show that every Cauchy sequence in $\mathbb{R}$ converges. This is one of the many reasons why one might be interested in sequentially compact sets.

Perhaps (unsurprisingly), sequentially compact and compact are the same notion for subsets of real numbers.

Theorem 3.2.11. *A set $K \subseteq \mathbb{R}$ is sequentially compact if and only if K is compact.*

Proof. Assume K is compact. Thus K is closed and bounded by the Heine-Borel Theorem (Theorem 3.2.9). To see that K is sequentially compact, let $(a_n)_{n\geq 1}$ be an arbitrary sequence of real numbers with $a_n \in K$ for all $n \in \mathbb{N}$. Thus $(a_n)_{n\geq 1}$ must be bounded as K is bounded. Hence $(a_n)_{n\geq 1}$ has a convergent subsequence $(a_{n_k})_{k\geq 1}$ by the Bolzano-Weierstrass Theorem (Theorem 2.4.7). Since $a_{n_k} \in K$ for all $k \in \mathbb{N}$ and since K is closed, the limit of $(a_{n_k})_{k\geq 1}$ must be in K by Proposition 3.1.17. Hence, as $(a_n)_{n\geq 1}$ was arbitrary, K is sequentially compact by definition.

Assume K is sequentially compact. To see that K is compact, we will show that K is closed and bounded in order to invoke the Heine-Borel Theorem (Theorem 3.2.9.

To see that K is bounded above, suppose for the sake of a contradiction that K is not bounded above. Thus for all $n \in \mathbb{N}$ there exists a $a_n \in K$ such that $a_n \geq n$. Therefore, since every subsequence of $(a_n)_{n\geq 1}$ is unbounded and thus cannot converge by Proposition 2.2.3, $(a_n)_{n\geq 1}$ does not have a convergent subsequence. As this contradicts the fact that K is sequentially compact, we must have that K is above bounded. As a similar argument shows that K is bounded below, K is bounded as desired.

To see that K is closed, suppose for the sake of a contradiction that K is not closed. Thus Proposition 3.1.17 implies that there exists a convergent sequence $(a_n)_{n\geq 1}$ such that $a_n \in K$ for all $n \in \mathbb{N}$ yet if $L = \lim_{n\to\infty} a_n$ then $L \notin K$. Therefore Proposition 2.4.4 implies that every subsequence of $(a_n)_{n\geq 1}$ converges to $L \notin K$. As this contradicts the fact that K is sequentially compact, we must have that K is closed.

Hence K is closed and bounded. Hence the Heine-Borel Theorem (Theorem 3.2.9) implies that K is compact. ∎

3.2.4 The Finite Intersection Property

To describe our final equivalent definition of compactness, we require the following definition.

Definition 3.2.12. A collection $\{A_i \mid i \in I\}$ of subsets of $\mathbb{R}$ is said to have the *finite intersection property* if whenever $J \subseteq I$ has a finite number of elements, $\bigcap_{j\in J} A_j \neq \emptyset$. That is, any finite intersection involving the sets from $\{A_i \mid i \in I\}$ must be non-empty.

Proposition 3.2.13. *Let $K \subseteq \mathbb{R}$ be closed. Then $K \subseteq \mathbb{R}$ is compact if and only if whenever*

$$\{F_i \mid i \in I\}$$

is a collection of closed subsets of K with the finite intersection property, then $\bigcap_{i\in I} F_i \neq \emptyset$.

Proof. Assume K is a compact subset of $\mathbb{R}$. Let $\{F_i \mid i \in I\}$ be a collection of closed subsets of K with the finite intersection property. We must show that $\bigcap_{i\in I} F_i \neq \emptyset$. To see this, suppose for the sake of a contradiction that $\bigcap_{i\in I} F_i = \emptyset$. For each $i \in I$, let $U_i = F_i^c$. Thus each U_i is an open set for each $i \in I$. Moreover

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I} F_i^c = \left(\bigcap_{i\in I} F_i\right)^c = \emptyset^c = \mathbb{R}$$

by de Morgan's Laws. Hence $\{U_i \mid i \in I\}$ is an open subcover of K. Therefore, since K is compact, by definition there exists an $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that

$$K \subseteq \bigcup_{m=1}^{n} U_{i_m}.$$

Hence

$$\bigcap_{m=1}^{n} F_{i_m} = \bigcap_{m=1}^{n} U_{i_m}^c = \left(\bigcup_{m=1}^{n} U_{i_m}\right)^c \subseteq K^c.$$

However, since $F_{i_m} \subseteq K$ for all m, we have that $\bigcap_{m=1}^{n} F_{i_m} \subseteq K$. Hence

$$\bigcap_{m=1}^{n} F_{i_m} \subseteq K \cap K^c = \emptyset$$

so $\bigcap_{m=1}^{n} F_{i_m} = \emptyset$. As this contradicts the finite intersection property, we have a contradiction. Hence $\bigcap_{i\in I} F_i \neq \emptyset$.

For the other direction, to see that K is compact, let $\{U_i \mid i \in I\}$ be any open cover of K. To see that $\{U_i \mid i \in I\}$ has a finite subcover of K, suppose for the sake of a contradiction that $\{U_i \mid i \in I\}$ does not have a finite subcover of K. For each $i \in I$, let $F_i = U_i^c \cap K$. Thus F_i is closed for all $i \in I$ by Proposition 3.1.16 being the intersection of closed sets. Moreover, clearly $F_i \subseteq K$ for all $i \in I$.

We claim that $\{F_i \mid i \in I\}$ has the finite intersection property. To see this, fix $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$. Since $\{U_i \mid i \in I\}$ does not have a finite subcover of K, we know that

$$K \nsubseteq \bigcup_{m=1}^{n} U_{i_m}.$$

Hence there exists an $x \in K$ such that $x \notin U_{i_m}$ for all $m \in \{1, \ldots, n\}$. Hence $x \in K$ and $x \in U_{i_m}^c$ for all $m \in \{1, \ldots, n\}$ so $x \in K \cap U_{i_m}^c = F_{i_m}$ for all $m \in \{1, \ldots, n\}$. Thus $x \in \bigcap_{m=1}^{n} F_{i_m}$. Thus, as $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ were arbitrary, $\{F_i \mid i \in I\}$ has the finite intersection property.

Since $\{F_i \mid i \in I\}$ are closed subsets of K with the finite intersection property, the assumptions of this direction imply that $\bigcap_{i\in I} F_i \neq \emptyset$. Let

$y \in \bigcap_{i \in I} F_i$. Then $y \in K$ as each F_i is a subset of K. Moreover $y \in F_i \subseteq U_i^c$ for all $i \in I$. Hence $y \notin U_i$ for all $i \in I$ so that

$$y \notin \bigcup_{i \in I} U_i.$$

However, as $y \in K$, this contradicts the fact that $\{U_i \mid i \in I\}$ is an open cover of K. Therefore $\{U_i \mid i \in I\}$ must have a finite subcover of K. Therefore, since $\{U_i \mid i \in I\}$ was arbitrary, K is compact by definition. ∎

Chapter 4

Continuity

So far we have examined the analytical properties of the real numbers via sequences and open sets. Although sequences are quite useful in analysis, they are more of a discrete structure on the real numbers. If we desire to be able to deal with a continuum of real numbers at once, we should turn our attention to functions on the real numbers. In particular, we need to upgrade our notion of a limit to functions. Subsequently, we can examine one if not the most important concept in analysis: continuity.

4.1 Limits of Functions

Given a function $f : \mathbb{R} \to \mathbb{R}$ and a point $\alpha \in \mathbb{R}$, our goal is to upgrade the notion of a limit of a sequence to define the limit of f at α. In particular, we desire to describe the behaviour of $f(x)$ as x gets 'closer and closer' to α. However, $f(\alpha)$ exists, so this concept might seem weird; that is, why do we want to know how f behaves as x gets 'closer and closer' to α since we know $f(\alpha)$? The short answer is that $f(x)$ may behave very differently as x gets 'closer and closer' to α than it does at $x = \alpha$. This leads us to the following heuristic concept.

Heuristic Definition. A number L is said to be the limit of a function f as x tends to α if the values of $f(x)$ approximate L provided that x is arbitrarily close to but not equal to a.

Of course, since we only care about the behaviour of f at points close to α, we do not need that f is be defined on all of $\mathbb{R}$. Although it is possible to write down the notion of a limit of a function defined on an arbitrary subset of the real numbers, we will focus our attention on functions that are defined on the following sets as we do not feel this takes away the spirit of what is being done.

Definition 4.1.1. A *finite interval* is any interval of the form (a, b), $(a, b]$,

$[a, b)$, or $[a, b]$ where $a, b \in \mathbb{R}$ are such that $a < b$. The point a the *left endpoint* of I and the point b the *right endpoint* of the interval.

4.1.1 Definition of a Limit

By formalizing our heuristic definition of the limit of a function as we did with sequences, we arrive at our definition of a limit (and the reason this course is often called a first course in ϵ-δ).

Definition 4.1.2. Let I be a finite interval, let α be an element or endpoint of I, and let $f : I \to \mathbb{R}$. A number $L \in \mathbb{R}$ is said to be the *limit* of f as x tends to α if for every $\epsilon > 0$ there exists a $\delta > 0$ (which depends on ϵ) such that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon$.

If L is the limit of f as x tends to α, we say the limit of $f(x)$ as x tends to α *exists* and write $L = \lim_{x\to\alpha} f(x)$. If no $L \in \mathbb{R}$ is the limit of f as x tends to α, we say that the limit of $f(x)$ as x tends to α *does not exist.*

In the case that $L = \lim_{x\to\alpha} f(x)$ and α is the left endpoint of I, we say that f *converges to L as x approaches α from above* or that L is the *right-sided limit* of f as x approaches α and write $L = \lim_{x\to\alpha+} f(x)$.

In the case that $L = \lim_{x\to\alpha} f(x)$ and α is the right endpoint of I, we say that f *converges to L as x approaches α from below* or that L is the *left-sided limit* of f as x approaches α and write $L = \lim_{x\to\alpha-} f(x)$.

Remark 4.1.3. Note the assumptions that α is an element or endpoint of I and $f : I \to \mathbb{R}$ are necessary to ensure for all $\delta > 0$ there are $x \in I$ such that $0 < |x - \alpha| < \delta$ and $f(x)$ is define (i.e. there are points where we can evaluate f at).

Moreover, since we are only interested in the behaviour of f as x tends to α, it is not necessary that $f(\alpha)$ is defined. Of course, since the value of $f(\alpha)$ does not effect any portion of the definition of the limit, if $f(\alpha)$ is not defined, we can just set $f(\alpha) = 0$ and use the definition.

Remark 4.1.4. If I is a finite interval, $\alpha \in I$ is not an endpoint, and $f : I \to \mathbb{R}$, then $\lim_{x\to\alpha+} f(x)$ and $\lim_{x\to\alpha-} f(x)$ can still be discussed by restricting f to the intervals $I \cap (\alpha, \infty)$ and $I \cap (-\infty, \alpha)$ respectively.

As it took some time for to get use to the ϵ-N definition of a limit of a sequence, some examples of using the ϵ-δ definition of a limit are warranted.

Example 4.1.5. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$ for all $x \in \mathbb{R}$. We claim for all $\alpha \in \mathbb{R}$ that $\lim_{x\to\alpha} f(x) = c$. To see this, fix $\alpha \in \mathbb{R}$ and let $\epsilon > 0$ be arbitrary. Then, if $\delta = 1$ we have that $\delta > 0$. Moreover, if $x \in \mathbb{R}$ and $0 < |x - \alpha| < \delta$, then

$$|f(x) - c| = |c - c| = 0 < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\lim_{x\to\alpha} f(x) = c$ by definition.

Unsurprisingly, changing the value of the function at a single point does not change the limit. The following is an example of how to show this.

Example 4.1.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \pi & \text{if } x \neq 0 \\ -\pi & \text{if } x = 0 \end{cases}.$$

We claim for all $\alpha \in \mathbb{R}$ that $\lim_{x\to\alpha} f(x) = \pi$. To see this, we will first deal with the case that $\alpha = 0$.

To see that $\lim_{x\to 0} f(x) = \pi$, let $\epsilon > 0$ be arbitrary. Then, if $\delta = 1$ we have that $\delta > 0$. Moreover, if $x \in \mathbb{R}$ and $0 < |x - 0| < \delta$, then $x \neq 0$ so $f(x) = \pi$ and thus

$$|f(x) - \pi| = |\pi - \pi| = 0 < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\lim_{x\to 0} f(x) = \pi$ by definition.

Now, suppose $\alpha \in \mathbb{R} \setminus \{0\}$. To see that $\lim_{x\to 0} f(x) = \pi$, let $\epsilon > 0$ be arbitrary. Let $\delta = |\alpha|$. Thus $\delta > 0$. Moreover, if $x \in \mathbb{R}$ and $0 < |x - \alpha| < \delta$, then $x \neq 0$ so $f(x) = \pi$ and thus

$$|f(x) - \pi| = |\pi - \pi| = 0 < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\lim_{x\to\pi} f(x) = c$ by definition.

Of course, it is easy to see the limit of a specific function.

Example 4.1.7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$. We claim for all $\alpha \in \mathbb{R}$ that $\lim_{x\to\alpha} f(x) = \alpha$. To see this, fix $\alpha \in \mathbb{R}$ and let $\epsilon > 0$ be arbitrary. Then, if $\delta = \epsilon$ we have that $\delta > 0$. Moreover, if $x \in \mathbb{R}$ and $0 < |x - \alpha| < \delta$, then

$$|f(x) - \alpha| = |x - \alpha| < \delta < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\lim_{x\to\alpha} f(x) = \alpha$ by definition.

As when working with the definition of the limit of a sequence, it is often useful to see what one needs in order to obtain $|f(x) - L| < \epsilon$ and then pick the appropriate δ. Indeed, in the following example, look at the computation we do for $|f(x) - L|$ first and then see how and why we chose the δ we did.

Example 4.1.8. We claim that $\lim_{x\to 3} x^2 = 9$. To see this, let $\epsilon > 0$ be arbitrary. Let

$$\delta = \min\left\{\frac{\epsilon}{7}, 1\right\}.$$

Clearly $\delta > 0$. Moreover, if $0 < |x - 3| < \delta$, then $|x - 3| < 1$ so $2 < x < 4$ and thus $5 < x + 3 < 7$. Hence $0 < |x - 3| < \delta$ implies $|x + 3| < 7$. Hence, if $0 < |x - 3| < \delta$ then

$$|x^2 - 9| = |(x + 3)(x - 3)| = |x + 3||x - 3| < 7\delta < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $\lim_{x\to 3} x^2 = 9$ by definition.

Remark 4.1.9. Assume I is a finite interval, α is an element or endpoint of I, and $f : I \to \mathbb{R}$. By negating Definition 4.1.2, f does not converge to L as x tends to α if there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in I$ with $0 < |x - \alpha| < \delta$ such that $|f(x) - \alpha| \geq \epsilon$.

Thus, the limit of f as x tends to α does not exist if for all $L \in \mathbb{R}$ there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in I$ with $0 < |x - \alpha| < \delta$ such that $|f(x) - \alpha| \geq \epsilon$.

Example 4.1.10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

We claim that $\lim_{x\to 0} f(x)$ does not exist. To see this, suppose for the sake of a contradiction that $\lim_{x\to 0} f(x) = L$. Let $\epsilon = 1$. Since $\lim_{x\to 0} f(x) = L$, by the definition of the limit there exists a $\delta > 0$ so that if $x \in \mathbb{R}$ and $0 < |x - 0| < \delta$, then $|f(x) - L| < \epsilon$.

Let $x_1 = \frac{1}{2}\delta$. Therefore $0 < |x_1 - 0| < \delta$ and thus

$$|1 - L| = |f(x_1) - L| < \epsilon = 1.$$

Hence $0 < L < 2$ so $L > 0$. Similarly, let $x_2 = -\frac{1}{2}\delta$. Therefore $0 < |x_2 - 0| < \delta$ and thus

$$|-1 - L| = |f(x_2) - L| < \epsilon = 1.$$

Hence $-2 < L < 0$ so $L < 0$. As $L > 0$ and $L < 0$ are impossible, we have a contradiction. Thus $\lim_{x\to 0} f(x)$ does not exist.

Of course the real problem with Example 4.1.10 is that $\lim_{x\to 0+} f(x) = 1 \neq -1 = \lim_{x\to 0-} f(x)$.

Theorem 4.1.11. *Let I be a finite interval, let $\alpha \in I$ not be an endpoint, and let $f : I \to \mathbb{R}$. Then $\lim_{x\to\alpha} f(x)$ exists if and only if $\lim_{x\to\alpha+} f(x)$ and $\lim_{x\to\alpha-} f(x)$ exist and $\lim_{x\to\alpha+} f(x) = \lim_{x\to\alpha-} f(x)$. Furthermore, if $\lim_{x\to\alpha} f(x)$ exists, then*

$$\lim_{x\to\alpha} f(x) = \lim_{x\to\alpha+} f(x) = \lim_{x\to\alpha-} f(x).$$

Proof. First assume $\lim_{x\to\alpha} f(x)$ exists and let $L = \lim_{x\to\alpha} f(x)$. To see that $\lim_{x\to\alpha+} f(x)$ and $\lim_{x\to\alpha-} f(x)$ exist and are both equal to L, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\alpha} f(x)$, there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$, then $|f(x) - L| < \epsilon$. Hence if $x \in I \cap (\alpha, \infty)$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, by the definition of the right-sided limit $\lim_{x\to\alpha+} f(x)$ exists and equals L. Similarly if $x \in I \cap (-\infty, \alpha)$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon$. Thus, as $\epsilon > 0$ was arbitrary, by the definition of the left-sided limit $\lim_{x\to\alpha-} f(x)$ exists and equals L. Hence this direction of the proof is complete.

For the other direction, assume $\lim_{x\to\alpha+} f(x)$ and $\lim_{x\to\alpha-} f(x)$ exist and $L = \lim_{x\to\alpha+} f(x) = \lim_{x\to\alpha-} f(x)$. To see that $\lim_{x\to\alpha} f(x)$ exists and equals L, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\alpha+} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I\cap(\alpha,\infty)$ and $0 < |x-\alpha| < \delta_1$, then $|f(x)-L| < \epsilon$. Similarly, since $L = \lim_{x\to\alpha-} f(x)$, there exists a $\delta_2 > 0$ such that if $x \in I \cap (-\infty, \alpha)$ and $0 < |x - \alpha| < \delta_2$, then $|f(x) - L| < \epsilon$. Therefore, if $\delta = \min\{\delta_1, \delta_2\} > 0$, then $x \in I$ and $0 < |x - \alpha| < \delta$ implies $|f(x) - L| < \epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, we have by the definition of the limit that $\lim_{x\to\alpha} f(x)$ exists and equals L. ∎

4.1.2 Uniqueness of the Limit

Of course, just with sequences, there can be only one limit. Thus the use of the word 'the' in the definition of 'the' limit. Note the proof is very similar to the proof used in Proposition 2.1.12.

Proposition 4.1.12. *Let I be a finite interval, let α be an element or endpoint of I, and let $f : I \to \mathbb{R}$. If L and K are limits of f as x tends to α, then $L = K$.*

Proof. We will provide two different (but basically the same) proofs of this fact.

For the first, we will provide a direct proof. Assume L and K are limits of f as x tends to a. To see that $L = K$, let $\epsilon > 0$ be arbitrary. Since L is a limit of f as x tends to a, we know by the definition of a limit that there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_1$ then $|f(x) - L| < \epsilon$. Similarly, since K is a limit of f as x tends to a, we know by the definition of a limit that there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_1$ then $|f(x) - K| < \epsilon$

Let $\delta_1 = \min\{\delta_1, \delta_2\} > 0$. By the above paragraph, we have that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon$ and $|f(x) - K| < \epsilon$. Choose $x_0 \in I$ such that $0 < |x_0 - \alpha| < \delta$ (such an x_0 exists since I is an interval and α be an element or endpoint of I). Hence by the Triangle Inequality

$$|L - K| \leq |L - f(x_0)| + |f(x_0) - K| < \epsilon + \epsilon = 2\epsilon.$$

Therefore, we have obtained that $|L - K| < 2\epsilon$ for all $\epsilon > 0$. Hence, by the same argument used in the proof of Proposition 2.1.12, $|L - K| = 0$ so $L = K$ as desired.

For the second, we will provide an indirect proof. Suppose for the sake of a contradiction that $L \neq K$. Let $\epsilon = \frac{|L-K|}{2}$. Since $L \neq K$, we know that $\epsilon > 0$.

Since L is a limit of f as x approaches α, we know by the definition of a limit that there exists a $\delta_1 > 0$ such that if $0 < |x - \alpha| < \delta_1$ then $|f(x) - L| < \epsilon$. Similarly, since K is a limit of f as x approaches α, we know by the definition of a limit that there exists a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $|f(x) - K| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\} > 0$. By the above paragraph, we have that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon$ and $|f(x) - K| < \epsilon$. Choose $x_0 \in I$ such that $0 < |x_0 - \alpha| < \delta$ (such an x_0 exists since I is an interval and α be an element or endpoint of I). Hence by the Triangle Inequality

$$|L - K| \leq |L - f(x_0)| + |f(x_0) - K| < \epsilon + \epsilon = 2\epsilon = |L - K|$$

which is absurd (i.e. $x < x$ is false for all $x \in \mathbb{R}$). Thus we have obtained a contradiction so it must be the case that $L = K$. ■

4.1.3 Equivalent Definitions of a Limit

As with sequences, there are alternative definitions one could take for the limit of a function. First off, recall from Proposition 2.1.14 that it is possible to change the '$<$' in the definition of a limit to '$\leq$'.

Proposition 4.1.13. *Let I be a finite interval, let α be an element or endpoint of I, let $f : I \to \mathbb{R}$, let $L \in \mathbb{R}$, and let $k > 0$. Then $L = \lim_{x\to\alpha} f(x)$ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| \leq k\epsilon$.*

Proof. Assume that $L = \lim_{x\to\alpha} f(x)$. To see the desired result, let $\epsilon > 0$ be arbitrary. Let $\epsilon_0 = k\epsilon$. Since $\epsilon > 0$ and $k > 0$, $\epsilon_0 > 0$. Hence, by the definition of the limit, there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| < \epsilon_0$. Thus $|f(x) - L| \leq \epsilon_0 = k\epsilon$ for all $x \in I$ with $0 < |x - \alpha| < \delta$. Therefore, as $\epsilon > 0$ was arbitrary, one direction of the proof is complete.

For the other direction, assume that f and L have the property listed in the statement of this proposition. To see that $L = \lim_{x\to\alpha} f(x)$, let $\epsilon > 0$ be arbitrary. Let $\epsilon_0 = \frac{\epsilon}{2k}$. Since $\epsilon > 0$ and $k > 0$, we know that $\epsilon_0 > 0$. Therefore, by the assumptions of this direction imply that there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$ then $|f(x) - L| \leq k\epsilon_0$. Hence $|f(x) - L| \leq \epsilon_0 < 2\epsilon_0 = k\epsilon$ for all $x \in I$ with $0 < |x - \alpha| < \delta$. As $\epsilon > 0$ was arbitrary, $L = \lim_{x\to\alpha} f(x)$ by the definition of the limit. ■

Remark 4.1.14. As with sequences, in Proposition 4.1.13 it is vital that the constant k used does not depend on ϵ. Indeed if we could choose k after we chose ϵ, we could have chose $k = \frac{1}{\epsilon}$ and thus the condition $|f(x) - L| \leq k\epsilon$ would always equate to $|f(x) - L| \leq 1$, which is very different than the definition of the limit.

Remark 4.1.15. By similar arguments, it is not difficult to see that one can change the condition '$0 < |x - \alpha| < \delta$' to the conditions '$0 < |x - \alpha| \leq \delta$' in Definition 4.1.2 and Proposition 4.1.13. Indeed clearly if the conclusion $|f(x) - L| < \epsilon$ holds for '$0 < |x - \alpha| \leq \delta$', then it holds for '$0 < |x - \alpha| < \delta$'. Conversely, if the conclusion $|f(x) - L| < \epsilon$ holds for '$0 < |x - \alpha| < \delta_0$', then it holds for '$0 < |x - \alpha| < \frac{1}{2}\delta \leq \delta$' and we let $\delta_0 = \frac{1}{2}\delta$.

As with sequences, there is a topological definition of the limit. However, there is also a sequential definition of the limit that is quite useful considering our study and mastery of convergent sequences.

Theorem 4.1.16 (Characterizations of Limits). *Let I be a finite interval, let α be an element or endpoint of I, and let $f : I \to \mathbb{R}$. Then the following are equivalent:*

*1) **(ϵ-δ Definition)** $L = \lim_{x\to\alpha} f(x)$.*

*2) **(Topological Definition)** If U is an open set containing L, there exists an open set V containing α so that if $x \in V \cap I$ and $x \neq \alpha$ then $f(x) \in U$.*

*3) **(Sequential Definition)** If $(x_n)_{n\geq 1}$ is a sequence such that $x_n \in I \setminus \{\alpha\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$, then $\lim_{n\to\infty} f(x_n) = L$.*

Proof. To see that 1) implies 2), assume that $L = \lim_{x\to\alpha} f(x)$. To see that 2) is true, let U be an open set containing L. By the definition of an open set, there exists an $\epsilon > 0$ so that

$$(L - \epsilon, L + \epsilon) \subseteq U.$$

Since $L = \lim_{x\to\alpha} f(x)$, by the definition of the limit there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$, then $|f(x) - L| < \epsilon$. Let

$$V = (\alpha - \delta, \alpha + \delta).$$

Hence V is an open set containing α. Moreover, if $x \in V \cap I$ and $x \neq \alpha$ then $x \in I$ and $0 < |x - \alpha| < \delta$ so $|f(x) - L| < \epsilon$ and thus

$$f(x) \in (L - \epsilon, L + \epsilon) \subseteq U.$$

Therefore, as U was arbitrary, 2) holds.

To see that 2) implies 3), assume that 2) is true. To see that 3) is true, let $(x_n)_{n\geq 1}$ be such that $x_n \in I \setminus \{\alpha\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$. We desire to show that $\lim_{n\to\infty} f(x_n) = L$.

To see that $\lim_{n\to\infty} f(x_n) = L$, let $\epsilon > 0$ be arbitrary. Let

$$U = (L - \epsilon, L + \epsilon).$$

Clearly U is an open set containing L. Hence, by the assumption of 2), there exists an open set V containing α so that if $x \in V \cap I$ and $x \neq \alpha$ then $f(x) \in U$.

Since $\lim_{n\to\infty} x_n = \alpha$, Proposition 3.1.5 implies there exists an $N \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N$. Hence $x_n \in V \cap I$ for all $n \geq N$ so

$$f(x_n) \in U = (L - \epsilon, L + \epsilon)$$

for all $n \geq N$. Thus $|f(x_n) - L| < \epsilon$ for all $n \geq N$. Hence, as $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} f(x_n) = L$.

To see that 3) implies 1), assume 3) is true. To see that 1) is true, suppose for the sake of a contradiction that f does not converge to L as x tends to α. Thus there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in I$ such that $0 < |x - \alpha| < \delta$ yet $|f(x) - L| \geq \epsilon$. For each $n \in \mathbb{N}$, choose $x_n \in I$ such that $0 < |x_n - \alpha| < \frac{1}{n}$ yet $|f(x_n) - L| \geq \epsilon$. Then $(x_n)_{n\geq 1}$ is a sequence with the property that $x_n \neq \alpha$ for all $n \in \mathbb{N}$. Furthermore, since $0 < |x_n - \alpha| < \frac{1}{n}$ for all $n \in \mathbb{N}$, we obtain that $\lim_{n\to\infty} x_n = \alpha$. However, since $|f(x_n) - L| \geq \epsilon$ for all $n \in \mathbb{N}$, we see that $(f(x_n))_{n\geq 1}$ does not converge to L thereby contradicting 3). Therefore 3) implies 1) as desired. ∎

The sequential definition of the limit of a function is particularly useful in showing that limits of functions do not exists. Indeed to show that the limit of a function does not exist, we need only construct to two sequences that $x = \alpha$ but have different limits once f is applied to them.

Example 4.1.17. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}$$

has no limit as x tends to 0. To see this, suppose for the sake of a contradiction that $\lim_{x\to 0} f(x)$ exists.

Consider the sequence $(a_n)_{n\geq 1}$ where $a_n = \frac{2}{\pi(4n+1)}$for all $n \in \mathbb{N}$. Clearly $\lim_{n\to\infty} a_n = 0$ and

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1.$$

Therefore Theorem 4.1.16 implies that $\lim_{x\to 0} f(x) = 1$.

Consider the sequence $(b_n)_{n\geq 1}$ where $b_n = \frac{2}{\pi(4n-1)}$ for all $n \in \mathbb{N}$. Clearly $\lim_{n\to\infty} b_n = 0$and

$$\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} -1 = -1.$$

Therefore Theorem 4.1.16 implies that $\lim_{x\to 0} f(x) = -1$. However as $1 \neq -1$, we have a contradiction. Hence $\lim_{x\to 0} f(x)$ does not exist.

Although the sequential definition of a limit will be quite useful as we have built up our theory of limits of sequences, the ϵ-δ definition will be equally useful for applications in the pages to come. In particular, consider the following example where we can show limits exist using the ϵ-δ definition whereas it would be quite difficult to use the sequential definition of a limit.

Example 4.1.18. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{b} & \text{if } x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \setminus \{0\},\, b \in \mathbb{N}, \text{ and } \gcd(a,b) = 1 \end{cases}.$$

We claim if $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ then $\lim_{x\to\gamma} f(x) = 0$. To see this, fix $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ and let $\epsilon > 0$ be arbitrary. By the Archimedean Property (Theorem 1.3.7) there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

By the Well-Ordering Principle (Theorem 1.1.10), for each $n < N$ there exists an $m_n \in \mathbb{Z}$ such that

$$\frac{m_n}{n} < \gamma < \frac{m_n + 1}{n}$$

(i.e. if $\gamma > 0$, take $m_n + 1$ to be the least natural number such that $\frac{m_n+1}{n} > \gamma$, and if $\gamma < 0$, repeat with $-\gamma$). Let

$$\delta = \min\left(\{|\gamma|\} \cup \left\{\gamma - \frac{m_n}{n}, \frac{m_n+1}{n} - \gamma\right\}_{n=1}^{N-1}\right).$$

Note that $\delta > 0$ by construction.

To see that δ works for this ϵ in the definition of the limit, let $x \in \mathbb{R}$ be such that $0 < |x - \gamma| < \delta$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then

$$|f(x) - 0| = |0 - 0| = 0 < \epsilon$$

as desired. Otherwise, if $x \in \mathbb{Q}$, then we can write $x = \frac{a}{b}$ where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $\gcd(a,b) = 1$. Since $0 < |x - \gamma| < \delta \leq |\gamma|$, we see that $x \neq 0$ so $z \neq 0$. We claim since $|x - \gamma| < \delta$ that $b \geq N$. To see this, suppose for the sake of a contradiction that $b = n < N$. Notice that

$$\left|\frac{a}{n} - \gamma\right| < \min\left(\gamma - \frac{m_n}{n}, \frac{m_n+1}{n}\right).$$

However, if $a \leq m_n$ then $|\frac{a}{n} - \gamma| \geq \gamma - \frac{m_n}{n}$ whereas if $a \geq m_n + 1$ then $|\frac{a}{n} - \gamma| \geq \frac{m_n+1}{n}$. Hence we have a contradiction so $b \geq N$. Therefore

$$|f(x) - 0| = \left|\frac{1}{b} - 0\right| = \frac{1}{b} \leq \frac{1}{N} < \epsilon.$$

Therefore, $|f(x) - 0| < \epsilon$ for all $x \in \mathbb{R}$ such that $0 < |x - \gamma| < \delta$. Therefore, since $\epsilon > 0$ was arbitrary, $\lim_{x\to\gamma} f(x) = 0$ by definition.

4.1.4 Limit Theorems for Functions

Using Theorem 4.1.16, we easily import results from Chapter 2 to deal with the limit of functions. However, we will also include the ϵ-δ proofs to aid the reader in the comprehension of using the ϵ-δ definition.

Theorem 4.1.19. *Let I be a finite interval, let α be an element or endpoint of I, and let $f, g : I \to \mathbb{R}$. If $L = \lim_{x\to\alpha} f(x)$ and $K = \lim_{x\to\alpha} g(x)$, then*

a) $\lim_{x\to\alpha} f(x) + g(x) = L + K$.

b) $\lim_{x\to\alpha} f(x)g(x) = LK$.

c) $\lim_{x\to\alpha} cf(x) = cL$ *for all* $c \in \mathbb{R}$.

d) $\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \frac{L}{K}$ *whenever* $K \neq 0$.

Proof. To see this result using the sequential definition of the limit, assume $(x_n)_{n\geq 1}$ is a sequence such that $x_n \in I \setminus \{\alpha\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$. By Theorem 4.1.16 we know that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = K$. Hence Theorem 2.3.1 implies that

- $\lim_{n\to\infty} f(x_n) + g(x_n) = L + K$,
- $\lim_{n\to\infty} f(x_n)g(x_n) = LK$,
- $\lim_{n\to\infty} cf(x_n) = cL$ for all $c \in \mathbb{R}$, and
- $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{L}{K}$ whenever $K \neq 0$.

Therefore, Theorem 4.1.16 implies that a), b), c), and d) hold as $(x_n)_{n\geq 1}$ was arbitrary.

[Note: There is technically a caveat here in that for part d) we need to know that $\frac{1}{g(x)}$ is well-defined on a set of the form $J \setminus \{\alpha\}$ where J is an interval and α is an element or endpoint of J. That is, if $K = \lim_{x\to\alpha} g(x)$ and $K \neq 0$, then $g(x)$ is non-zero when x is sufficiently close to α. The formal proof is shown below.]

To prove these results using the ϵ-δ definition of the limit, we follow a very similar patter to the proofs to show Theorem 2.3.1 where N is replaced with δ.

a) Let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\alpha} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_2$, then $|f(x) - L| < \frac{\epsilon}{2}$. Similarly, since $K = \lim_{x\to\alpha} g(x)$, there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x-\alpha| < \delta_2$ then $|f(x) - K| < \frac{\epsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Hence $\delta > 0$. Moreover, if $x \in I$

and $0 < |x - \alpha| < \delta$, then $|x - \alpha| < \delta_1$ and $|x - \alpha| < \delta_2$ so

$$\begin{aligned} |(f(x) + g(x)) - (L + K)| &= |(f(x) - L) + (g(x) - K)| \\ &\leq |f(x) - L| + |g(x) - K| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $\lim_{x\to\alpha} f(x) + g(x) = L + K$ by definition.

b) Let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\alpha} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_1$ then $|f(x) - L| < \frac{\epsilon}{2(|K|+1)}$. Moreover, there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x-\alpha| < \delta_2$ then $|f(x)-L| < 1$. Thus $|f(x)| \leq |L| + 1$ for all $x \in I$ with $0 < |x - \alpha| < \delta_2$. Furthermore, since $K = \lim_{x\to\alpha} g(x)$, there exists a $\delta_3 > 0$ such that if $x \in I$ and $0 < |x-\alpha| < \delta_3$ then $|g(x) - K| < \frac{\epsilon}{2(|L|+1)}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Hence $\delta > 0$. Moreover, if $x \in I$ and $0 < |x - \alpha| < \delta$, then

$$\begin{aligned} |f(x)g(x) - LK| &= |(f(x)g(x) - f(x)K) + (f(x)K - LK)| \\ &\leq |f(x)g(x) - f(x)K| + |f(x)K - LK| \\ &\leq |f(x)||g(x) - K| + |K||f(x) - L| \\ &\leq (|L| + 1)|b_n - K| + |K||f(x) - L| \\ &\leq (|L| + 1)\frac{\epsilon}{2(|L| + 1)} + |K|\frac{\epsilon}{2(|K| + 1)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} f(x)g(x) = LK$ by definition.

c) Given $c \in \mathbb{R}$, we have that $\lim_{x\to\alpha} c = c$ by Example 4.1.5. Hence part (c) follows from part (b) by taking $g(x) = c$ for all $x \in I$.

d) To prove part d), it suffices by part b) to prove that $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{K}$ whenever $K \neq 0$.

Assume $K \neq 0$. First, we claim that there exists a $\delta_0 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_0$, then $|g(x)| \geq \frac{|K|}{2} > 0$. To see this, let $\epsilon_0 = \frac{|K|}{2} > 0$. Since $K = \lim_{x\to\alpha} g(x)$, there exists a $\delta_0 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_0$ then $|g(x) - K| < \epsilon_0 = \frac{|K|}{2}$. Therefore, by the Triangle Inequality, for all $x \in I$ such that $0 < |x - \alpha| < \delta_0$ we have

$$|g(x)| \geq |K| - \frac{|K|}{2} = \frac{|K|}{2} > 0$$

as desired. In particular, if $x \in I$ and $0 < |x - \alpha| < \delta_0$, then $\frac{1}{g(x)}$ is well-defined.

To see that $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{K}$, let $\epsilon > 0$ be arbitrary. Since $K = \lim_{x\to\alpha} g(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_1$

then $|g(x) - K| < \frac{\epsilon|K|^2}{2}$ (as $|K| neq 0$). Let $\delta = \min\{\delta_0, \delta_1\}$. Therefore, for all $x \in I$ with $0 < |x - \alpha| < \delta$, we have that $g(x) \neq 0$ and

$$\begin{aligned}\left|\frac{1}{g(x)} - \frac{1}{K}\right| &= \frac{|K - g(x)|}{|g(x)||K|} \\ &\leq \frac{\epsilon|K|^2}{2|g(x)||K|} \\ &\leq \frac{\epsilon|K|}{2}\frac{1}{|g(x)|} \\ &\leq \frac{\epsilon|K|}{2}\frac{2}{|K|} = \epsilon.\end{aligned}$$

Hence $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{K}$ by definition. ∎

Using Theorem 4.1.19 it is possible to obtain many limits from our known limits.

Example 4.1.20. Recall from Examples 4.1.5 and 4.1.7 that for each $c, \alpha \in \mathbb{R}$, we have $\lim_{x\to\alpha} c = c$ and $\lim_{x\to\alpha} x = \alpha$. Hence Theorem 4.1.19 implies that $\lim_{x\to\alpha} cx^n = c\alpha^n$ for all $n \in \mathbb{N}$ and all $c \in \mathbb{R}$. Therefore Theorem 4.1.19 again implies that $\lim_{x\to a} p(x) = p(a)$ for all polynomials p.

Example 4.1.21. Let $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials where q is not the zero polynomial. Such a function is said to be a *rational function*. If $\alpha \in \mathbb{R}$ is such that $q(\alpha) \neq 0$, then Theorem 4.1.19 again implies that $\lim_{x\to\alpha} f(x) = f(\alpha)$.

Remark 4.1.22. As with sequences, given two functions f and g such that $\lim_{x\to a} g(x) = 0$, one may ask whether $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists. Clearly if $\lim_{x\to a} \frac{f(x)}{g(x)} = L$ and $\lim_{x\to a} g(x) = 0$ then Theorem 4.1.19 implies $\lim_{x\to a} f(x)$ exists and

$$\lim_{x\to a} f(x) = \left(\lim_{x\to a} \frac{f(x)}{g(x)}\right)\left(\lim_{x\to a} g(x)\right) = L(0) = 0.$$

Like with sequences, if $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x) = 0$ there are many possible behaviours, some of which we will examine in the next section.

Of course, some of our most important theorems for limits of sequences carry forward to limits of functions. We provide the proofs using both the sequential and ϵ-δ definitions of the limit.

Theorem 4.1.23 (Squeeze Theorem). *Let I be a finite interval, let α be an element or endpoint of I, and let $f, g, h : I \to \mathbb{R}$. Suppose for each $x \in I \setminus \{\alpha\}$ that*

$$g(x) \leq f(x) \leq h(x).$$

If $\lim_{x\to\alpha} g(x)$ and $\lim_{x\to\alpha} h(x)$ exist and $L = \lim_{x\to\alpha} g(x) = \lim_{x\to\alpha} h(x)$, then $\lim_{x\to\alpha} f(x)$ exists and $\lim_{x\to\alpha} f(x) = L$.

Proof. To see this result using the sequential definition of the limit, assume $(x_n)_{n\geq 1}$ is a sequence such that $x_n \in I \setminus \{\alpha\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$. By Theorem 4.1.16 we know that $\lim_{n\to\infty} g(x_n) = L$ and $\lim_{n\to\infty} h(x_n) = L$. Moreover, we have that

$$g(x_n) \leq f(x_n) \leq h(x_n)$$

for all $n \in \mathbb{N}$. Hence the Squeeze Theorem (Theorem 2.3.13) implies that $\lim_{n\to\infty} f(x_n) = L$. Hence, Theorem 4.1.16 implies that $\lim_{x\to\alpha} f(x) = L$ as $(x_n)_{n\geq 1}$ was arbitrary.

To prove these results using the ϵ-δ definition of the limit, we follow a very similar patter to the proof of the Squeeze Theorem (Theorem 2.3.13) where N is replaced with δ.

Let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\alpha} g(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x-\alpha| < \delta_1$ then $|g(x) - L| < \epsilon$. Hence $L - \epsilon < g(x)$ for all $x \in I$ with $0 < |x - \alpha| < \delta_1$. Similarly, since $L = \lim_{x\to\alpha} h(x)$, there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x-\alpha| < \delta_2$ then $|h(x) - L| < \epsilon$. Hence $h(x) < L + \epsilon$ for all $x \in I$ with $0 < |x - \alpha| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Clearly $\delta > 0$ and if $x \in I$ is such that $0 < |x - \alpha| < \delta$, then

$$L - \epsilon < g(x) \leq f(x) \leq h(x) \leq L + \epsilon.$$

Hence $L - \epsilon \leq f(x) \leq L + \epsilon$ for all $x \in I$ such that $0 < |x - \alpha| < \delta$, which implies $-\epsilon \leq f(x) - L \leq \epsilon$ and thus $|f(x) - L| < \epsilon$ for all $x \in I$ such that $0 < |x - \alpha| < \delta$. Hence $\lim_{x\to\alpha} f(x) = L$ by definition. ∎

Again, the Squeeze Theorem has its uses when dealing with difficult functions that may be compared to simple ones.

Example 4.1.24. Consider the function

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

In Example 4.1.17 we saw that $\lim_{x\to 0} \frac{1}{x} f(x)$ did not exist. However, since

$$-|x| \leq f(x) \leq |x| \quad \text{as} \quad -1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \text{ for all } x \in \mathbb{R} \setminus \{0\},$$

and since $\lim_{x\to 0} |x| = \lim_{x\to 0} -|x| = 0$, we see that $\lim_{x\to 0} f(x) = 0$ by the Squeeze Theorem.

Finally, the Comparison Theorem is also useful when comparing limits of functions.

Theorem 4.1.25 (Comparison Theorem). *Let I be a finite interval, let α be an element or endpoint of I, and let $f, g : I \to \mathbb{R}$. Suppose for each $x \in I \setminus \{\alpha\}$ that*

$$g(x) \leq f(x).$$

If $L = \lim_{x\to\alpha} f(x)$ and $K = \lim_{x\to\alpha} g(x)$ exist, then $K \leq L$.

Proof. To see this result using the sequential definition of the limit, assume $(x_n)_{n\geq 1}$ is a sequence such that $x_n \in I\setminus\{\alpha\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$. By Theorem 4.1.16 we know that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = k$. Moreover, we have that

$$g(x_n) \leq f(x_n)$$

for all $n \in \mathbb{N}$. Hence the Comparison Theorem (Theorem 2.3.20) implies that $K \leq L$ as desired.

Let $L = \lim_{x\to\alpha} f(x)$ and $K = \lim_{x\to\alpha} g(x)$. Suppose for the sake of a contradiction that that $L < K$. Therefore if $\epsilon = \frac{K-L}{2}$, then $\epsilon > 0$.

Since $L = \lim_{x\to\alpha} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_1$ then $|f(x) - L| < \epsilon$. Hence $f(x) < L + \epsilon$ for all $x \in I$ such that $0 < |x - \alpha| < \delta_1$. Similarly, since $K = \lim_{x\to\alpha} g(x)$, there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_2$ then $|g(x) - K| < \epsilon$ Hence $K - \epsilon < g(x)$ for all $x \in I$ such that $0 < |x - \alpha| < \delta_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Thus $\delta > 0$. Moreover, since I is an interval and α be an element or endpoint of I, there exists an $x_0 \in I$ such that $0 < |x - \alpha| < \delta$. Thus

$$g(x_0) - f(x_0) > (K - \epsilon) - (L + \epsilon) = (K - L) - 2\epsilon = 0.$$

However, this contradicts the fact that $g(x_0) \leq f(x_0)$. Hence we have obtained a contradiction in the case that $K < L$ so it must be the case that $L \leq K$. ∎

4.1.5 Limits at and to Infinity

There are many more types of limits we could examine. Much of the theory follows along the same lines as the previous results in this section, so we will only summarize the definitions and results, and provided a few examples.

First, instead of requiring $\alpha \in \mathbb{R}$, we may ask for limits as x tends to $\pm\infty$.

Definition 4.1.26. Let f be a function define on an interval (c, ∞). A number $L \in \mathbb{R}$ is said to be the *limit* of f as x tends to ∞ if for every $\epsilon > 0$ there exists an $M > c$ (which depends on ϵ) such that if $x \geq M$ then $|f(x) - L| < \epsilon$. In this case, we say that $f(x)$ *converges to L as x tends to* ∞ and write $L = \lim_{x\to\infty} f(x)$.

Definition 4.1.27. Let f be a function define on an interval $(-\infty, c)$. A number $L \in \mathbb{R}$ is said to be the *limit* of f as x tends to $-\infty$ if for every $\epsilon > 0$ there exists an $M < c$ (which depends on ϵ) such that if $x \leq M$ then $|f(x) - L| < \epsilon$. In this case, we say that $f(x)$ *converges to* L *as* x *tends to* $-\infty$ and write $L = \lim_{x\to-\infty} f(x)$.

By replacing '$x \in I$ and $0 < |x - \alpha| < \delta$' with '$x \geq M$' (respectively '$x \leq M$') in the proofs done in this section, we obtain the following results.

Proposition 4.1.28. *Let $f : (c, \infty) \to \mathbb{R}$ (respectively $f : (-\infty, c) \to \mathbb{R}$). If L and K are limits of f as x tends to ∞ (respectively $-\infty$), then $L = K$.*

Proof. We will prove the result for limits to ∞ as the proof for limits to $-\infty$ can be obtained by replacing all '$\geq$', '$>$', and 'max' with '$\leq$', '$<$', and 'min' respectively.

Suppose for the sake of a contradiction that $L \neq K$. Let $\epsilon = \frac{|L-K|}{2}$. Since $L \neq K$, we know that $\epsilon > 0$.

Since K is a limit of f as x approaches ∞, we know by definition that there exists a $M_1 > c$ such that if $x \geq M_1$ then $|f(x) - L| < \epsilon$. Similarly, since K is a limit of f as x approaches ∞, we know by definition that there exists a $M_2 > 0$ such that if $x \geq M_2$ then $|f(x) - K| < \epsilon$.

Let $M = \max\{M_1, M_2\} > c$. By the above paragraph, we have that

$$|L - K| \leq |L - f(M)| + |f(M) - K| < \epsilon + \epsilon = 2\epsilon = |L - K|$$

which is absurd (i.e. $x < x$ is false for all $x \in \mathbb{R}$). Thus we have obtained a contradiction so it must be the case that $L = K$. ∎

Moreover, by repeating the ideas of the proof of Theorem 4.1.16, we can prove the following and reduce our study of limits to ∞ to sequences.

Proposition 4.1.29. *Let $f : (c, \infty) \to \mathbb{R}$ (respectively $f : (-\infty, c) \to \mathbb{R}$). Then $L = \lim_{x\to\infty} f(x)$ if and only if whenever $(x_n)_{n\geq1}$ has the properties that $x_n > c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} f(x_n) = L$.*

Proof. We will prove the result for limits to ∞ as the proof for limits to $-\infty$ can be obtained by replacing all '$\geq$', '$>$', and 'max' with '$\leq$', '$<$', and 'min' respectively.

Assume $L = \lim_{x\to\infty} f(x)$. To see the desired result, let $(x_n)_{n\geq1}$ be such that $x_n > c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$. We desire to show that $\lim_{n\to\infty} f(x_n) = L$.

To see that $\lim_{n\to\infty} f(x_n) = L$, let $\epsilon > 0$ be arbitrary. Since $L = \lim_{x\to\infty} f(x)$, there exists an $M > c$ such that if $x \geq M$ then $|f(x) - L| < \epsilon$. Since $\lim_{n\to\infty} x_n = \infty$, there exists an $N \in \mathbb{N}$ such that $x_n \geq M$ for all $n \geq N$. Hence for all $n \geq N$ we have that $x_n \geq M$ and thus $|f(x_n) - L| < \epsilon$. Hence, as $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} f(x_n) = L$.

To prove the converse, assume if $(x_n)_{n\geq 1}$ has the properties that $x_n > c$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$, then $\lim_{n\to\infty} f(x_n) = L$. To see that $L = \lim_{x\to\infty} f(x)$, suppose for the sake of a contradiction that f does not converge to L as x tends to ∞. Thus there exists an $\epsilon > 0$ such that for all $M > c$ there exists an $x \geq M$ such that $|f(x) - L| \geq \epsilon$. For each $n \in \mathbb{N}$ with $n > c$, choose $x_n \geq n$ such that $|f(x_n) - L| \geq \epsilon$. Then $(x_n)_{n>c}$ is a sequence with the property that $x_n \geq n$ for all $n > c$ and $|f(x_n) - L| \geq \epsilon$ for all $n >$. Since $x_n \geq n$ for all $n > c$ implies that $\lim_{n\to\infty} x_n = \infty$, the assumptions of this direction imply that $\lim_{n\to\infty} f(x_n) = L$. However, since $|f(x_n) - L| \geq \epsilon$ for all $n > c$, it is impossible that $\lim_{n\to\infty} f(x_n) = L$. Hence we have a contradiction so the result follows. ∎

By replacing sequences that converge to α with sequences that converge to $\pm\infty$, the following result holds by using Proposition 4.1.29 instead of Theorem 4.1.16.

Corollary 4.1.30. *The conclusions of Theorems 4.1.19, 4.1.23, and 4.1.25 when $a = \pm\infty$ (under the necessary modifications to the hypotheses).*

Example 4.1.31. It is not difficult to verify based on definitions that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Example 4.1.32. Let $f(x) = \frac{3x^2-2x+1}{2x^2+5x-2}$. Then, for sufficiently large x,

$$f(x) = \frac{(x^2)(3 - \frac{2}{x} + \frac{1}{x^2})}{(x^2)(2 + \frac{5}{x} - \frac{2}{x^2})} = \frac{3 - \frac{2}{x} + \frac{1}{x^2}}{2 + \frac{5}{x} - \frac{2}{x^2}}.$$

Hence

$$\lim_{x\to\infty} f(x) = \frac{3 - 2(0) - (0)(0)}{2 + 5(0) - 2(0)(0)} = \frac{3}{2}.$$

Example 4.1.33. We claim that $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$. Indeed, since $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$, we see that

$$-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$$

for all $x > 0$. Hence, since $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} -\frac{1}{x} = 0$, we obtain that $\lim_{x\to\infty} \frac{\sin(x)}{x} = 0$ by the Squeeze Theorem

Like with sequences, we can also discuss functions diverging to $\pm\infty$ for both limits as x tends to $\alpha \in \mathbb{R}$ and as x tends to $\pm\infty$.

Definition 4.1.34. Let I be a finite interval, let α be an element or endpoint of I, and let $f : I \to \mathbb{R}$. The function f is said to *diverge to infinity* (*negative infinity*) as x tends to α if for every $M > 0$ there exists a $\delta > 0$ (which depends on M) such that if $0 < |x - \alpha| < \delta$ then $f(x) \geq M$ ($f(x) \leq -M$). In this case we write $\lim_{x\to\alpha} f(x) = \infty$ ($\lim_{x\to\alpha} f(x) = -\infty$).

Definition 4.1.35. Let f be a function define on an interval (c, ∞). The function f is said to *diverge to infinity* (*negative infinity*) as x tends to ∞ if for every $M > 0$ there exists a $K > c$ (which depends on M) such that if $x \geq K$ then $f(x) \geq M$ ($f(x) \leq -M$). In this case we write $\lim_{x\to\infty} f(x) = \infty$ ($\lim_{x\to\infty} f(x) = -\infty$).

Definition 4.1.36. Let f be a function define on an interval $(-\infty, c)$. The function f is said to *diverge to infinity* (*negative infinity*) as x tends to $-\infty$ if for every $M > 0$ there exists a $K < c$ (which depends on M) such that if $x \leq K$ then $f(x) \geq M$ ($f(x) \leq -M$). In this case we write $\lim_{x\to-\infty} f(x) = \infty$ ($\lim_{x\to-\infty} f(x) = -\infty$).

Example 4.1.37. Notice that $\lim_{x\to\infty} x = \infty$ and $\lim_{x\to-\infty} x = -\infty$.

Example 4.1.38. Notice that $\lim_{x\to 0} \frac{1}{|x|} = \infty$. Indeed if $M > 0$, and $0 < |x| < \frac{1}{M}$, then $\frac{1}{|x|} > M$. However, $\lim_{x\to 0} \frac{1}{x} \neq \infty$ since if $x < 0$, then $\frac{1}{x} < 0 < M$.

4.2 Continuity of Functions

With our discussion of limits complete, we may move onto studying debatably the most important concept in analysis: the notion of continuity.

4.2.1 Equivalent Definitions of Continuity

In high school mathematics courses, a function is often described to be continuous if "the graph of the function is a single unbroken curve that you could draw without lifting your stylus from the surface". This definition is incredibly heuristic and, as with all mathematics, needs to be made precise. Using limits, the formal notion of continuity is easy to define.

Definition 4.2.1. Let I be a finite interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$. It is said that f is *continuous* at α if $\lim_{x\to\alpha} f(x)$ exists and $\lim_{x\to\alpha} f(x) = f(\alpha)$.

Furthermore, it is said that f is *continuous on* I if f is continuous at each point in I.

Of course, using our Characterizations of Limits (Theorem 4.1.16), there are multiple ways of characterizing when a function is continuous at a point. Of course this immediately characterizes functions that are continuous on an interval by verifying continuity at each point.

Theorem 4.2.2 (Characterizations of Continuity). *Let I be a finite interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$. Then the following are equivalent:*

1) f is continuous at α.

2) *(ε-δ **Definition**) For all* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* $x \in I$ *and* $|x - \alpha| < \delta$*, then* $|f(x) - f(\alpha)| < \epsilon$*.*

3) *(**Topological Definition**) If* U *is an open set containing* $f(\alpha)$*, there exists an open set* V *containing* α *so that if* $x \in V \cap I$ *then* $f(x) \in U$*.*

4) *(**Sequential Definition**) If* $(x_n)_{n\geq 1}$ *is a sequence such that* $x_n \in I$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = \alpha$*, then* $\lim_{n\to\infty} f(x_n) = f(\alpha)$*.*

Proof. The proof of this result follows immediately from repeating the proof of the Characterizations of Limits (Theorem 4.1.16) once one realizes the condition $\lim_{x\to\alpha} f(x) = f(\alpha)$ allows one to replace '$0 < |x - \alpha| < \delta$' with '$|x - \alpha| < \delta$', '$x \in V \cap I$ and $x \neq \alpha$' with '$x \in V \cap I$', and '$x_n \in I \setminus \{\alpha\}$' with '$x_n \in I$'. The precise details are included below.

To see that 1) implies 2), assume f is continuous at α. Therefore $\lim_{x\to\alpha} f(x)$ exists and $\lim_{x\to\alpha} f(x) = f(\alpha)$. To see that 2) is true, let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to\alpha} f(x)$ exists and $\lim_{x\to\alpha} f(x) = f(\alpha)$, the definition of the limit implies there exists a $\delta > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta$, then $|f(x) - f(\alpha)| < \epsilon$. Since $|f(\alpha) - f(\alpha)| = 0 < \epsilon$, we see that if $x \in I$ and $|x - \alpha| < \delta$, then $|f(x) - f(\alpha)| < \epsilon$. Therefore, as $\epsilon > 0$ was arbitrary, 2) holds.

To see that 2) implies 3), assume that 2) is true. To see that 3) is true, let U be an open set containing $f(\alpha)$. By the definition of an open set, there exists an $\epsilon > 0$ so that

$$(f(\alpha) - \epsilon, f(\alpha) + \epsilon) \subseteq U.$$

By the assumption of 2), there exists a $\delta > 0$ such that if $x \in I$ and $|x-\alpha| < \delta$, then $|f(x) - f(\alpha)| < \epsilon$. Let

$$V = (\alpha - \delta, \alpha + \delta).$$

Hence V is an open set containing α. Moreover, if $x \in V \cap I$ then $x \in I$ and $|x - \alpha| < \delta$ so $|f(x) - f(\alpha)| < \epsilon$ and thus

$$f(x) \in (f(\alpha) - \epsilon, f(\alpha) + \epsilon) \subseteq U.$$

Therefore, as U was arbitrary, 3) holds.

To see that 3) implies 4), assume that 3) is true. To see that 4) is true, let $(x_n)_{n\geq 1}$ be such that $x_n \in I$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \alpha$. We desire to show that $\lim_{n\to\infty} f(x_n) = f(\alpha)$.

To see that $\lim_{n\to\infty} f(x_n) = f(\alpha)$, let $\epsilon > 0$ be arbitrary. Let

$$U = (f(\alpha) - \epsilon, f(\alpha) + \epsilon).$$

Clearly U is an open set containing $f(\alpha)$. Hence, by the assumption of 2), there exists an open set V containing α so that if $x \in V \cap I$ then $f(x) \in U$.

Since $\lim_{n\to\infty} x_n = \alpha$, Proposition 3.1.5 implies there exists an $N \in \mathbb{N}$ such that $x_n \in V$ for all $n \geq N$. Hence $x_n \in V \cap I$ for all $n \geq N$ so

$$f(x_n) \in U = (f(\alpha) - \epsilon, f(\alpha) + \epsilon)$$

for all $n \geq N$. Thus $|f(x_n) - f(\alpha)| < \epsilon$ for all $n \geq N$. Hence, as $\epsilon > 0$ was arbitrary, $\lim_{n\to\infty} f(x_n) = f(\alpha)$.

To see that 4) implies 1), assume 4) is true. To see that 1) is true, suppose for the sake of a contradiction that f does not converge to $f(\alpha)$ as x tends to α. Thus there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists an $x \in I$ such that $|x - \alpha| < \delta$ yet $|f(x) - f(\alpha)| \geq \epsilon$. For each $n \in \mathbb{N}$, choose $x_n \in I$ such that $|x_n - \alpha| < \frac{1}{n}$ yet $|f(x_n) - f(\alpha)| \geq \epsilon$. Then $(x_n)_{n\geq 1}$ is a sequence with the property that $\lim_{n\to\infty} x_n = \alpha$ since $|x_n - \alpha| < \frac{1}{n}$ for all $n \in \mathbb{N}$. However, since $|f(x_n) - f(\alpha)| \geq \epsilon$ for all $n \in \mathbb{N}$, we see that $(f(x_n))_{n\geq 1}$ does not converge to $f(\alpha)$ thereby contradicting 4). Therefore 4) implies 1) as desired. ∎

Remark 4.2.3. As with Proposition 2.1.14 for sequences and Proposition 4.1.13 for limits of functions, one can modify the ϵ-δ Characterization of Continuity (Theorem 4.2.2) to replace '$|f(x)-f(\alpha)| < \epsilon$' with '$|f(x)-f(\alpha)| \leq k\epsilon$' for a previously fixed constant k. Similarly, one can replace the condition '$|x - \alpha| < \delta$' with '$|x - \alpha| \leq \delta$'.

In addition to the above Characterizations of Continuity (Theorem 4.2.2), there is an interesting characterization of a continuous function on $\mathbb{R}$ that is the basis for continuity in future courses on topology.

Theorem 4.2.4. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ if and only if $f^{-1}(U)$ is open for all open subsets $U \subseteq \mathbb{R}$.*

Proof. Assume f is continuous. To see the desired conclusion, let $U \subseteq \mathbb{R}$ be an arbitrary open set. To see that $f^{-1}(U)$ is open, assume $\alpha \in f^{-1}(U)$. Since f is continuous at α, Characterizations of Continuity (Theorem 4.2.2) implies there exists an open set V such that $\alpha \in V$ and if $x \in V$ then $f(x) \in U$. Hence $V \subseteq f^{-1}(U)$. Since V is open and $\alpha \in V$, there exists an $\epsilon > 0$ such that

$$(\alpha - \epsilon, \alpha + \epsilon) \subseteq V \subseteq f^{-1}(U).$$

Therefore, since $\alpha \in f^{-1}(U)$ was arbitrary, $f^{-1}(U)$ is open by definition. Hence, as U was an arbitrary open set, this direction of the proof is complete.

Assume $f^{-1}(U)$ is open for all open subsets $U \subseteq \mathbb{R}$. To see that f is continuous on $\mathbb{R}$, let $\alpha \in \mathbb{R}$ be arbitrary. To see that f is continuous at α, let U be an open subset of $\mathbb{R}$ such that $f(\alpha) \in U$. Thus $V = f^{-1}(U)$ is open set by assumption. Moreover, since $f(\alpha) \in U$, we have that $\alpha \in f^{-1}(U) = V$. Therefore, since for all $x \in V$ we have $f(x) \in U$, f is continuous at α by the Characterizations of Continuity (Theorem 4.2.2). Hence, since $\alpha \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$. ∎

Of course, there are many useful and natural examples of continuous functions.

Example 4.2.5. Using Example 4.1.20, we see that if $p(x)$ is a polynomial, then $p(x)$ is continuous on $\mathbb{R}$ Similarly, using Example 4.1.21, we see that if $p(x)$ and $q(x)$ are polynomials, then $\frac{p(x)}{q(x)}$ is continuous at α provided $q(\alpha) \neq 0$.

Example 4.2.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$. To see this, let $\alpha \in \mathbb{R}$ be arbitrary. To see that f is continuous at α, let $\epsilon > 0$ be arbitrary. Let $\delta = \epsilon > 0$. Thus, if $x \in \mathbb{R}$ is such that $|x - \alpha| < \delta$, then, by the reverse triangle inequality, we have that

$$|f(x) - f(\alpha)| = ||x| - |\alpha|| \leq |x - \alpha| < \delta = \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, f is continuous at α. Therefore, as $\alpha > 0$ was arbitrary, f is continuous on $\mathbb{R}$ as desired.

Example 4.2.7. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{b} & \text{if } x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \setminus \{0\},\, b \in \mathbb{N}, \text{ and } \gcd(a,b) = 1 \end{cases}.$$

Recall from Example 4.1.18 that if $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ then $\lim_{x \to \gamma} f(x) = 0 = f(\gamma)$. Hence f is continuous at every irrational number.

We claim that f is not continuous at every rational number. To see this, let $r \in \mathbb{Q}$. By Proposition 1.3.9, for all $n \in \mathbb{N}$ there exists a $\gamma_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$r < \gamma_n < r + \frac{1}{n}.$$

Hence $(\gamma_n)_{n \geq 1}$ is a sequence of irrational numbers such that $\lim_{n \to \infty} \gamma_n = r$. However, since $f(\gamma_n) = 0$ for all $n \in \mathbb{N}$ yet $f(r) > 0$, we see that $\lim_{n \to \infty} f(\gamma_n) \neq f(r)$. Hence the Sequential Characterization of Continuity (Theorem 4.2.2) implies that f is not continuous at r.

Remark 4.2.8. The best way to define and show continuity of the functions $\sin(x), \cos(x)$, and e^x is via series of functions. In particular, for $x \in \mathbb{R}$, one defines

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

It is possible to show that these series *converge uniformly* and thus define continuous functions. However, the discussion of this material is more appropriately placed in MATH 3001 (Series of Functions). Consequently, we will assume throughout the remainder of the course that $\sin(x), \cos(x)$, and e^x are continuous functions on $\mathbb{R}$.

4.2.2 Operations Preserving Continuity

As the same arithmetic operations that behave well with respect to limits immediately transfer to continuous functions, we have the following operations that preserve continuity.

Theorem 4.2.9. *Let I be a finite interval, let $\alpha \in I$, and let $f, g : I \to \mathbb{R}$. If f and g are continuous at α, then*

a) $f + g$ is continuous at α.

b) fg is continuous at α.

c) cf is continuous at α for all $c \in \mathbb{R}$.

d) $\frac{f}{g}$ is continuous at α provided $g(\alpha) \neq 0$.

Proof. Apply Theorem 4.1.19 together with the definition of continuity. For those that want to see the direct ϵ-δ proofs, they are provided as follows. Note the proofs are nearly identical to those used in Theorem 4.1.19.

a) Let $\epsilon > 0$ be arbitrary. Since $f(\alpha) = \lim_{x\to\alpha} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_2$, then $|f(x) - L| < \frac{\epsilon}{2}$. Similarly, since $g(\alpha) = \lim_{x\to\alpha} g(x)$, there exists a $\delta_2 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_2$ then $|f(x) - K| < \frac{\epsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$. Hence $\delta > 0$. Moreover, if $x \in I$ and $|x - \alpha| < \delta$, then $|x - \alpha| < \delta_1$ and $|x - \alpha| < \delta_2$ so

$$\begin{aligned} |(f(x) + g(x)) - (f(\alpha) + g(\alpha))| &= |(f(x) - L) + (g(x) - K)| \\ &\leq |f(x) - f(\alpha)| + |g(x) - g(\alpha)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence $\lim_{x\to\alpha} f(x) + g(x) = f(\alpha) + g(\alpha)$ by definition. Therefore $f + g$ is continuous at α.

b) Let $\epsilon > 0$ be arbitrary. Since $f(\alpha) = \lim_{x\to\alpha} f(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_1$ then $|f(x) - L| < \frac{\epsilon}{2(|K|+1)}$. Moreover, there exists a $\delta_2 > 0$ such that if $x \in I$ and $0 < |x - \alpha| < \delta_2$ then $|f(x) - f(\alpha)| < 1$. Thus $|f(x)| \leq |f(\alpha)| + 1$ for all $x \in I$ with $|x - \alpha| < \delta_2$. Furthermore, since $g(\alpha) = \lim_{x\to\alpha} g(x)$, there exists a $\delta_3 > 0$ such that if

$x \in I$ and $|x - \alpha| < \delta_3$ then $|g(x) - g(\alpha)| < \frac{\epsilon}{2(|L|+1)}$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Hence $\delta > 0$. Moreover, if $x \in I$ and $|x - \alpha| < \delta$, then

$$\begin{aligned} |f(x)g(x) - f(\alpha)g(\alpha)| &= |(f(x)g(x) - f(x)g(\alpha)) + (f(x)g(\alpha) - f(\alpha)g(\alpha))| \\ &\leq |f(x)g(x) - f(x)g(\alpha)| + |f(x)g(\alpha) - f(\alpha)g(\alpha)| \\ &\leq |f(x)||g(x) - g(\alpha)| + |g(\alpha)||f(x) - f(\alpha)| \\ &\leq (|f(\alpha)| + 1)|b_n - g(\alpha)| + |g(\alpha)||f(x) - f(\alpha)| \\ &\leq (|f(\alpha)| + 1)\frac{\epsilon}{2(|f(\alpha)| + 1)} + |g(\alpha)|\frac{\epsilon}{2(|g(\alpha)| + 1)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $\lim_{n\to\infty} f(x_n)g(x_n) = f(\alpha)g(\alpha)$ by definition. Therefore fg is continuous at α.

c) Given $c \in \mathbb{R}$, we have that $\lim_{x\to\alpha} c = c$ by Example 4.1.5. Hence part (c) follows from part (b) by taking $g(x) = c$ for all $x \in I$.

d) To prove part d), it suffices by part b) to prove that $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{g(\alpha)}$ whenever $g(\alpha) \neq 0$.

Assume $g(\alpha) \neq 0$. First, we claim that there exists a $\delta_0 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_0$, then $|g(x)| \geq \frac{|g(\alpha)|}{2} > 0$. To see this, let $\epsilon_0 = \frac{|g(\alpha)|}{2} > 0$. Since $g(\alpha) = \lim_{x\to\alpha} g(x)$, there exists a $\delta_0 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_0$ then $|g(x) - g(\alpha)| < \epsilon_0 = \frac{|g(\alpha)|}{2}$. Therefore, by the Triangle Inequality, for all $x \in I$ such that $|x - \alpha| < \delta_0$ we have

$$|g(x)| \geq |g(\alpha)| - \frac{|g(\alpha)|}{2} = \frac{|g(\alpha)|}{2} > 0$$

as desired. In particular, if $x \in I$ and $|x - \alpha| < \delta_0$, then $\frac{1}{g(x)}$ is well-defined.

To see that $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{g(\alpha)}$, let $\epsilon > 0$ be arbitrary. Since $g(\alpha) = \lim_{x\to\alpha} g(x)$, there exists a $\delta_1 > 0$ such that if $x \in I$ and $|x - \alpha| < \delta_1$ then $|g(x) - g(\alpha)| < \frac{\epsilon|g(\alpha)|^2}{2}$ (as $|g(\alpha)| \neq 0$). Let $\delta = \min\{\delta_0, \delta_1\}$. Therefore, for all $x \in I$ with $|x - \alpha| < \delta$, we have that $g(x) \neq 0$ and

$$\begin{aligned} \left|\frac{1}{g(x)} - \frac{1}{g(\alpha)}\right| &= \frac{|g(\alpha) - g(x)|}{|g(x)||g(\alpha)|} \\ &\leq \frac{\epsilon|g(\alpha)|^2}{2|g(x)||g(\alpha)|} \\ &\leq \frac{\epsilon|g(\alpha)|}{2}\frac{1}{|g(x)|} \\ &\leq \frac{\epsilon|g(\alpha)|}{2}\frac{2}{|g(\alpha)|} = \epsilon. \end{aligned}$$

Hence $\lim_{x\to\alpha} \frac{1}{g(x)} = \frac{1}{g(\alpha)}$ by definition. Therefore $\frac{1}{g}$ is continuous at α. ∎

One of the nicest operation for functions not see in Theorem 4.2.9 is the composition of functions. Indeed, if $f, g : \mathbb{R} \to \mathbb{R}$ are functions, we can consider the function $g \circ f$ and ask whether or nor $g \circ f$ is continuous at a point. Of course, we will want to extend to functions that are not defined on all of $\mathbb{R}$, so we will need to impose conditions to guarantee the composition is well-defined.

Theorem 4.2.10. *Let f be a function defined on an open interval I_1 containing a number $\alpha \in \mathbb{R}$. Let g be a function defined on an open interval I_2 which contains $f(I_1)$. If f is continuous at α and g is continuous at $f(\alpha)$, then $g \circ f$ is continuous at α.*

Proof. We will provide three proofs; one for each Characterization of Continuity (Theorem 4.2.2).

ϵ-δ Proof: To see that $g \circ f$ is continuous at α, let $\epsilon > 0$ be arbitrary. Since g is continuous at $f(\alpha)$, there exists a $\delta_1 > 0$ such that if $y \in I_2$ and $|y - f(\alpha)| < \delta_1$, then $|g(y) - g(f(\alpha))| < \epsilon$. Moreover, since $\delta_1 > 0$ and since f is continuous at α, there exists a $\delta > 0$ such that if $x \in I_1$ and $|x - \alpha| < \delta$ then $|f(x) - f(\alpha)| < \delta_1$. Hence, if $x \in I_1$ and $|x - \alpha| < \delta$ then

$$|f(x) - f(\alpha)| < \delta_1 \quad \text{and} \quad f(x) \in f(I_1) \subseteq I_2$$

so

$$|g(f(x)) - g(f(\alpha))| < \epsilon$$

(i.e. let $y = f(x)$ above). Therefore, as $\epsilon > 0$ was arbitrary, $g \circ f$ is continuous at α.

Topological Proof: To see that $g \circ f$ is continuous at α, let U be an open subset containing $g(f(\alpha))$. Since g is continuous at $f(\alpha)$, there exists an open set V_1 containing $f(\alpha)$ so that if $y \in V_1 \cap I_2$ then $g(y) \in U$. Since f is continuous at α and V_1 is an open set containing α, there exists an open set V containing α so that if $x \in V \cap I_1$ then $f(x) \in V_1$. Hence, if $x \in I_1 \cap V$ then

$$f(x) \in V_1 \quad \text{and} \quad f(x) \in f(I_1) \subseteq I_2$$

so

$$g(f(x)) \in U$$

(i.e. let $y = f(x)$ above). Therefore, as U was arbitrary, $g \circ f$ is continuous at α.

Sequential Proof: To see that $g \circ f$ is continuous at α, let $(x_n)_{n \geq 1}$ be a sequence such that $x_n \in I_1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = \alpha$. Since f is continuous at α, we know that $\lim_{n \to \infty} f(x_n) = f(\alpha)$. However, since $f(I_1) \subseteq I_2$, we have that $f(x_n) \in I_2$ for all $n \in \mathbb{N}$. Therefore, since g is continuous at $f(\alpha)$, we obtain that $\lim_{n \to \infty} g(f(x_n)) = g(f(\alpha))$. Therefore, as $(x_n)_{n \geq 1}$ was arbitrary, $g \circ f$ is continuous at α. ∎

4.3 Uniform Continuity

Before we move onto examining the properties and importance of continuous functions on the real numbers, there is actually a stronger form of continuity that we desire to examine. For a function f to be continuous on an interval I, the ϵ-δ Characterization of Continuity (Theorem 4.2.2) states that for each $\alpha \in I$ and each $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in I$ and $|x - \alpha| < \delta$, then $|f(x) - f(\alpha)| < \epsilon$. Note that this a prior δ depends not only on ϵ but the $\alpha \in I$ selected. But what if we want one δ to rule them all, one δ to find them, one δ to bring them all, and in the darkness bind them... Umm I mean what if we wanted to know given an $\epsilon > 0$ there exists a $\delta > 0$ that worked for all α simultaneously in I?

Definition 4.3.1. A function f defined on an interval I is said to be *uniformly continuous* on I if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in I$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Remark 4.3.2. It is elementary by the ϵ-δ Characterization of Continuity that if f is uniformly continuous on an interval I, then f is continuous on I.

Remark 4.3.3. As eluded to in the introduction, the benefit of having a uniform continuous function f on an interval is that for any $\epsilon > 0$ there exists a $\delta > 0$ that worked for all α simultaneously in I. This is quite useful in that we can guarantee that f does not vary too much on any small subinterval of I. This will be of particular importance for an essential result in Chapter 6.

Remark 4.3.4. As with Proposition 2.1.14 for sequences and Proposition 4.1.13 for limits of functions, one can modify the definition of uniform continuity (Definition 4.3.1) to replace '$|f(x)-f(y)| < \epsilon$' with '$|f(x)-f(y)| \leq k\epsilon$' for a previously fixed constant k. Similarly, one can replace the condition '$|x - y| < \delta$' with '$|x - y| \leq \delta$'.

Of course, it is quite easy to verify that some functions are uniformly continuous.

Example 4.3.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ for all $x \in \mathbb{R}$. We claim that f is uniformly continuous. To see this, let $\epsilon > 0$ be arbitrary. Let $\delta = \epsilon > 0$. Thus if $x, y \in I$ and $|x - y| < \delta$, then

$$|f(x) - f(y)| = |x - y| < \delta = \epsilon.$$

Hence, as $\epsilon > 0$ was arbitrary, the result follows.

However, not all continuous functions on intervals are uniformly continuous. Indeed if a continuous function "grows too fast" it is possible that the function is not uniformly continuous. Before we demonstrate such an example, we believe it is useful to the reader to formally negate Definition

4.3.1; that is, a function f on an interval I is not uniformly continuous if there exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in I$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$.

Example 4.3.6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. We claim that f is not uniformly continuous. To see this, we claim that Definition 4.3.1 fails for $\epsilon = 2$; that is, there does not exist a $\delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x - y| < \delta$ then $|f(x) - f(y)| < 2$. To see this, let $\delta > 0$ be arbitrary. By the Archimedian Property (Theorem 1.3.7), there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Let $x = n$ and $y = n + \frac{1}{n}$. Then $|x_n - y_n| = \frac{1}{n} < \delta$ yet

$$|f(x_n) - f(y_n)| = \left| n^2 - \left(n + \frac{1}{n}\right)^2 \right| = 2 + \frac{1}{n^2} \geq 2.$$

Therefore, as $\delta > 0$ was arbitrary, there does not exists a $\delta > 0$ such that if $x, y \in \mathbb{R}$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Hence f is not uniformly continuous on $\mathbb{R}$ by definition.

Of course, the notion of uniform continuity depends on the domain of definition of our function.

Example 4.3.7. Let $f : (-1, 1) \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in (-1, 1)$. We claim that f is uniformly continuous. To see this, let $\epsilon > 0$ be arbitrary. Let $\delta = \frac{\epsilon}{2} > 0$. We claim that δ works. To see this, let $x, y \in I$ be such that $|x - y| < \delta$. Since $x, y \in I$, we know that $|x|, |y| \leq 1$. Thus

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| \\ &= |x + y||x - y| \\ &\leq (|x| + |y|)|x - y| \\ &\leq 2|x - y| \\ &< 2\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, as $\epsilon > 0$ was arbitrary, f is uniformly continuous.

Also, uniform continuity does not permit functions to oscillate quickly.

Example 4.3.8. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \sin\left(\frac{1}{x}\right)$ for all $x \in (0, 1)$. We claim that f is not uniformly continuous on $(0, 1)$. To see this, we claim Definition 4.3.1 fails for $\epsilon = 1$. To see this, for $n \in \mathbb{N}$ let $x_n = \frac{1}{2\pi n + \frac{\pi}{2}} \in (0, 1)$ and $y_n = \frac{1}{2\pi n + \frac{3\pi}{2}} \in (0, 1)$. Since

$$\lim_{n\to\infty} x_n = 0 = \lim_{n\to\infty} y_n,$$

for any $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $|x_N| < \frac{1}{2}\delta$ and $|y_N| < \frac{1}{2}\delta$. Hence $|x_N - y_N| < \delta$ yet

$$|f(x_N) - f(y_N)| = |1 - (-1)| = 2 \geq 1.$$

Hence f is not uniformly continuous on $(0, 1)$.

Although the above examples seem to imply that uniform continuity is a much more difficult property to deduce and handle, it turns out that if we only consider continuous functions on closed intervals, then something special happens.

Theorem 4.3.9. *Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose for the sake of a contradiction that f is not uniformly continuous on $[a, b]$. Hence exists an $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in [a, b]$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. Thus for each $n \in \mathbb{N}$ we can choose $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ yet $|f(x_n) - f(y_n)| \geq \epsilon$.

By the Bolzano-Weierstrass Theorem (Theorem 2.4.7), there exists a subsequence $(x_{n_k})_{k\geq 1}$ of $(x_n)_{n\geq 1}$ that converges to some number $L \in [a, b]$ (i.e. $[a, b]$ is sequentially compact). Consider the subsequence $(y_{n_k})_{k\geq 1}$ of $(y_n)_{n\geq 1}$. Notice for all $k \in \mathbb{N}$ that

$$|y_{n_k} - L| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - L| \leq \frac{1}{n_k} + |x_{n_k} - L| \leq \frac{1}{k} + |x_{n_k} - L|.$$

Therefore, since $\lim_{k\to\infty} |x_{n_k} - L| = 0$ and $\lim_{k\to\infty} \frac{1}{k} = 0$, we obtain that $\lim_{k\to\infty} y_{n_k} = L$ by the Squeeze Theorem.

Since $L = \lim_{k\to\infty} x_{n_k}$ and since f is continuous at L, there exists an $N_1 \in \mathbb{N}$ such that $|f(x_{n_k}) - L| < \frac{\epsilon}{2}$ for all $k \geq N_1$. Similarly, since $L = \lim_{k\to\infty} y_{n_k}$ and since f is continuous at L, there exists an $N_2 \in \mathbb{N}$ such that $|f(y_{n_k}) - L| < \frac{\epsilon}{2}$ for all $k \geq N_2$. Therefore, if $k_0 = \max\{N_1, N_2\}$, we obtain that

$$|f(x_{n_{k_0}}) - f(y_{n_{k_0}})| \leq |f(x_{n_{k_0}}) - f(L)| + |f(L) - f(y_{n_{k_0}})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which contradicts the fact that $|f(x_{n_{k_0}}) - f(y_{n_{k_0}})| \geq \epsilon$. Hence, as we have obtained a contradiction, it must have been the case that f is uniformly continuous on $[a, b]$. ∎

Remark 4.3.10. Note that the fact that $[a, b]$ is (sequentially) compact is essential in the proof of Theorem 4.3.9. In fact, in future courses (i.e. MATH 4011), it will be observed that continuous functions on compact sets are automatically uniformly continuous. The proof is nearly identical in MATH 4011.

Using Theorem 4.3.9, we can demonstrate additional functions on $\mathbb{R}$ are uniformly continuous.

Example 4.3.11. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos(x)$ for all $x \in \mathbb{R}$. We claim that f is uniformly continuous on $\mathbb{R}$. To see this, let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous on $[-2\pi, 2\pi]$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in [-2\pi, 2\pi]$ and $|x - y| < \delta$. Due to the fact that $\cos(x + 2\pi) = \cos(x)$ for all $x \in \mathbb{R}$, it is then easy to see that if $x, y \in \mathbb{R}$ are such that $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

The following uses Theorem 4.3.9 to show that many functions on $\mathbb{R}$ are uniformly continuous. Note this does not describe all uniformly continuous functions on $\mathbb{R}$ by, for example, Example 4.3.5.

Proposition 4.3.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ exist, then f is uniformly continuous on $\mathbb{R}$.*

Proof. Exercise. ∎

Note Example 4.3.8 shows that the conclusions of Theorem 4.3.9 do not extend to continuous functions on finite open intervals. It turns out that the only reason such continuous functions are not uniformly continuous is that the one-sided limits at the boundaries do not exist.

Proposition 4.3.13. *Let $f : (a, b) \to \mathbb{R}$ where $a, b \in \mathbb{R}$. Then f is uniformly continuous on (a, b) if and only if there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in (a, b)$.*

Proof. First assume there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in (a, b)$. Thus $g(x)$ is uniformly continuous on $[a, b]$ by Theorem 4.3.9. Therefore, since $f(x) = g(x)$ for all $x \in (a, b)$, the definition of uniform continuity of g on $[a, b]$ immediately implies the uniform continuity of f on (a, b) (i.e. for $\epsilon > 0$, the δ that works for g also works for f). Hence one direction of the proof is complete.

For the other direction, assume f is uniformly continuous on (a, b). Our goal is to find a continuous $g : [a, b] \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in (a, b)$. In particular, since f is continuous on (a, b), the only way a function $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ is if

$$g(a) = \lim_{x\to a+} f(x) \qquad \text{and} \qquad g(b) = \lim_{x\to b-} f(x).$$

Hence, provided the above two one-sided limits exist, we will define $g(a)$ and $g(b)$ accordingly and the proof will be complete.

To see that $\lim_{x\to a+} f(x)$ exists, we will apply the Sequential Characterization of a Limit (Theorem 4.1.16). In particular, we must show that for every sequence $(x_n)_{n\geq1}$ with $x_n \in (a, b)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$ that $(f(x_n))_{n\geq1}$ converges AND that $\lim_{x\to a} f(x_n)$ are all equal for every choice of $(x_n)_{n\geq1}$.

For the first part of the claim, let $(x_n)_{n\geq1}$ be an arbitrary sequence such that $a < x_n < b$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$. To see that $(f(x_n))_{n\geq1}$ converges, we claim that $(f(x_n))_{n\geq1}$ is Cauchy. To see that $(f(x_n))_{n\geq1}$ is

Cauchy, let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous on (a, b) there exists a $\delta > 0$ such that if $x, y \in (a, b)$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since $\lim_{n\to\infty} x_n = a$, we know that $(x_n)_{n\geq 1}$ is Cauchy by Theorem 2.5.2. Hence there exists an $N \in \mathbb{N}$ such that $|x_n - x_m| < \delta$ for all $n, m \geq N$. Therefore, if $n, m \geq N$, we obtain that $|f(x_n) - f(x_m)| < \epsilon$ as desired. Hence, as $\epsilon > 0$ was arbitrary, $(f(x_n))_{n\geq 1}$ is Cauchy and thus converges by Theorem 2.5.8.

For the second part of the claim, let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be such that $a < x_n, y_n < b$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a$. Thus $L = \lim_{n\to\infty} f(x_n)$ and $K = \lim_{n\to\infty} f(y_n)$ exist by the previous paragraph. Our goal is to show that $L = K$.

To see that $L = K$, let $\epsilon > 0$ be arbitrary. Since f is uniformly continuous on (a, b) there exists a $\delta > 0$ such that $x, y \in (a, b)$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{3}$. Since $\lim_{n\to\infty} x_n - y_n = 0$, there exists an $N_1 \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for all $n \geq N_1$. Since $\lim_{n\to\infty} f(x_n) = L$, there exists an $N_2 \in \mathbb{N}$ such that $|f(x_n) - L| < \frac{\epsilon}{3}$ for all $n \geq N_2$. Similarly, since $\lim_{n\to\infty} f(y_n) = K$, there exists an $N_3 \in \mathbb{N}$ such that $|y_n - a| < \frac{\epsilon}{3}$ for all $n \geq N_3$. Thus, if $N = \max\{N_1, N_2, N_3\}$, then $|x_N - y_N| < \delta$ so

$$|L - K| \leq |L - f(x_N)| + |f(x_N) - f(y_N)| + |f(y_N) - K| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus, we have shown that $|L - K| < \epsilon$ for all $\epsilon > 0$. This implies $|L - K| = 0$ and thus $L = K$ as desired.

Hence we may define $g(a)$ so that $g(a) = \lim_{x\to a+} f(x)$. Similar arguments show that we may define $g(b)$ as desired thereby completing the proof. ∎

Remark 4.3.14. In the proof of the second part of the claim in Theorem 4.3.13, we have used what is known as a three-ϵ argument (i.e. we needed to make three terms small via a procedure making the middle one small first and then we could make the other two small). Three-ϵ appear often in analysis and are quite useful. For example, one can use a three-ϵ argument to show that a 'uniform limit of continuous functions is a continuous function', which is an essential result in MATH 3001.

4.4 The Intermediate Value Theorem

With our enhancement of continuity on closed intervals, let us return to looking at the properties and importance of continuous functions. In particular, in this section we will develop one of the three 'value' theorems in this course. In particular, by combining these three pieces of the Triforce, one obtains unlimited control over continuous functions on $\mathbb{R}$.

To motivate this first theorem, consider the following scenario. Consider a person walking on a straight path. Assume somewhere along the path is a waterfall that the person desires to see. If this person walks from one end of the path to the other, will they see the waterfall? Of course, logic says they must. But how can we mathematically prove said result.

Of course, specific assumptions must be made in the above problem. For example, we are assuming that position is a function of time (no time travel permitted via Delorean's and TARDISs'). Furthermore, we must assume that our functions are continuous at each point (i.e. no teleportation). Once these assumptions our made, our first Triforce theorem mathematically proves the answer. Before we obtain this piece of the Triforce, it is useful to extract a useful lemma that has already been used in the proof of Theorem 4.2.9.

Lemma 4.4.1. *Let I be an interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$.*

a) If f is continuous at α and $f(\alpha) > 0$, then there exists a $\delta > 0$ such that if $x \in I$ and $|x - \alpha| < \delta$, then $f(x) > 0$.

b) If f is continuous at α and $f(\alpha) < 0$, then there exists a $\delta > 0$ such that if $x \in I$ and $|x - \alpha| < \delta$, then $f(x) < 0$.

Proof. We will only prove a) as the proof of b) is nearly identical.

Let $f : I \to \mathbb{R}$ be continuous at α with $f(\alpha) > 0$. To see the desired result, let $\epsilon = \frac{1}{2}f(\alpha)$. Since $f(\alpha) > 0$, clearly $\epsilon > 0$. Since f is continuous at α, there exists a $\delta > 0$ such that if $x \in I$ and $|x-\alpha| < \delta$, then $|f(x)-f(\alpha)| < \epsilon$. Therefore, for all $x \in I$ with $|x - \alpha| < \delta$, we have that $f(\alpha) - f(x) < \epsilon$ so

$$f(x) > f(\alpha) - \epsilon = f(\alpha) - \frac{1}{2}f(\alpha) = \frac{1}{2}f(\alpha) > 0.$$

Hence the result is complete. ∎

Theorem 4.4.2 (The Intermediate Value Theorem). *Let I be an interval, let $a, b \in I$ be such that $a < b$, and let $f : I \to \mathbb{R}$ be continuous. If $\alpha \in \mathbb{R}$ is such that $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$, then there exists a $c \in (a, b)$ such that $f(c) = \alpha$.*

Proof. By the assumptions of the theorem, we can assume that $f : [a, b] \to \mathbb{R}$ is continuous. To prove the result, we will assume $\alpha \in \mathbb{R}$ is such that $f(a) < \alpha < f(b)$. The proof of the result when $f(b) < \alpha < f(a)$ is nearly identical and left to the reader.

Our goal is to prove there exists a $c \in (a, b)$ such that $f(c) = \alpha$. To do so, we will shift f so that we can assume $\alpha = 0$. To do this, define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - \alpha$ for all $x \in [a, b]$. Clearly g is continuous on $[a, b]$, $g(a) < 0 < g(b)$ and there exists a $c \in (a, b)$ such that $f(c) = \alpha$ if and only if there exists a $c \in (a, b)$ such that $g(c) = 0$. Therefore, to complete the proof it suffices to show that there exists a $c \in (a, b)$ such that $g(c) = 0$.

Let

$$S = \{x \in [a, b] \mid g(x) \le 0\}.$$

Since $f(a) < 0$, we see that $a \in S$ so S is non-empty. Furthermore, since $g(b) > 0$, S is bounded above by b. Hence the Least Upper Bound Property (Theorem 1.2.35) implies that $c = \text{lub}(S)$ exists. Note $c \in [a, b]$ by construction.

We claim that $g(c) = 0$. To see this, we first claim that $g(c) \geq 0$. To see this, suppose for the sake of a contradiction that $g(c) < 0$. Since $g(c) < 0$ and $g(b) > 0$ we see that $c \neq b$. Moreover, since $g(c) < 0$, Lemma 4.4.1 implies there $\delta > 0$ such that if $x \in I$ and $|x - c| < \delta$, then $g(x) < 0$. Since $c \neq b$, there exists an $d \in (c, b)$ such that $c < d < c + \delta$ (e.g. $d = c + \frac{1}{2}\min\{b - c, \delta\}$). Hence $g(d) < 0$ so $d \in S$. However, since $c < d$, this contradicts the fact that c is the least upper bound of S. Hence, it must be the case that $g(c) \geq 0$.

Now that we have established $g(c) \geq 0$, to see that $g(c) = 0$, suppose for the sake of a contradiction that $g(c) > 0$. Since $g(a) < 0$, this implies $c \neq a$ so $c \in (a, b]$. Since $g(c) > 0$ and g is continuous at c, Lemma 4.4.1 implies there exists a $\delta > 0$ such that if $x \in I$ and $|x - c| < \delta$, then $g(x) > 0$. Let $d = \max\{a, c - \delta\}$. Clearly $d \in [a, b]$. Moreover, for all x such that $d < x \leq c$, we have that $g(x) > 0$ so $x \notin S$. However, since $c = \text{lub}(S)$, we have that $s \leq c$ for all $s \in S$. Hence, for all $s \in S$ we have $s \leq c$ and, since $d < s \leq c$ is false, it must be the case that $s \leq d$. Thus d is an upper bound of S. However, since $d < c$ and $c = \text{lub}(S)$, we have a contradiction. Hence $g(c) = 0$.

Since $g(c) = 0$, $g(a) < 0$, and $g(b) > 0$, we have that $c \neq a$ and $c \neq b$ so $c \in (a, b)$ as desired. ■

The Intermediate Value Theorem has a wide range of applications. One of the most useful applications of the Intermediate Value Theorem is that it can be used to establish the solution to complicated equations.

Example 4.4.3. We claim there exists a $z \in [0, \frac{\pi}{2}]$ such that $\cos(z) = z$. To see this, consider the function $f(x) = x - \cos(x)$. Since

$$f(0) = 0 - 1 = -1 < 0 \qquad \text{and} \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 0 > 0,$$

and since f is continuous, the Intermediate Value Theorem implies there exists a $z \in (0, \frac{\pi}{2})$ such that $f(z) = 0$ (and thus $\cos(z) = z$).

It is potentially useful for future studies in analysis to note that the Intermediate Value Theorem is related to a topological notion known as connectedness. Although not as important as compactness, connectedness has its own uses. Thus, for the remainder of the section, we will introduce the notion of a connected set and see an alternative proof of the Intermediate Value Theorem using this notion.

Definition 4.4.4. A subset $A \subseteq \mathbb{R}$ is said to be *disconnected* if there exists open sets $U, V \subseteq \mathbb{R}$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$.

A subset $A \subseteq \mathbb{R}$ is said to be *connected* if A is not disconnected; that is, if there does not exists open sets $U, V \subseteq \mathbb{R}$ such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, and $U \cap V = \emptyset$.

The motivation for the above definition of a disconnected set is that one can 'divide up' our set into two disjoint portions separated in a natural topological sense with open sets. This definition is made clearer by the following example.

Example 4.4.5. Let $A = [0, 1] \cup [3, 4]$. We claim that A is a disconnected subset of $\mathbb{R}$. To see this, let $U = (-\infty, 2)$ and let $V = (2, \infty)$. Clearly U and V are open subsets of $\mathbb{R}$. Moreover, notice that $U \cap A = [0, 1] \neq \emptyset$, $V \cap A = [3, 4] \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq U \cup V$. Hence A is disconnected by definition.

In fact, the real numbers are quite nice in the sense that one can completely describe all of their connected sets as the following result shows. Note the proof of this result has the same flavour as the proof of the Intermediate Value Theorem (and consequently, some of the technical details are illustrated more fully in the proof of the Intermediate Value Theorem).

Theorem 4.4.6. *Let $A \subseteq \mathbb{R}$. Then A is connected if and only if A is an interval (singletons count as intervals here).*

Proof. First, assume that A is not an interval. To see that A is not connected, note since A is not an interval that there exists $x, y \in A$ and $z \in \mathbb{R} \setminus A$ such that $x < z < y$. Therefore, since $U = (-\infty, z)$ and $V = (z, \infty)$ are open sets such that $x \in U \cap A$ so $U \cap A \neq \emptyset$, $y \in V \cap A$ so $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $A \subseteq \mathbb{R} \setminus \{z\} \subseteq U \cup V$, A is not connected by definition.

To see the converse, let A be an interval in $\mathbb{R}$. Suppose for the sake of a contradiction that A is not connected. Hence there exists open subsets U and V of A such that $U \cap A \neq \emptyset$, $V \cap A \neq \emptyset$, $U \cap V = \emptyset$, and $U \cup V = A$. As $U \cap A$ and $V \cap A$ are non-empty, select $a \in U \cap A$ and $b \in V \cap b$. As $U \cap V \neq \emptyset$, it must be the case that $a \neq b$. By exchanging the labelling of U and V if necessary, we may assume that $a < b$. Since A is an interval, we must have that $[a, b] \subseteq A$.

Since $U \cap [a, b] \neq \emptyset$, the scalar

$$\alpha = \text{lub}(U \cap [a, b])$$

is an element of $[a, b]$. Thus as $[a, b] \subseteq A \subseteq U \cup V$, either $\alpha \in U$ or $\alpha \in V$. Let us show that both of these options lead to a contradiction.

<u>Case 1: $\alpha \in U$.</u> If $\alpha \in U$, then by the definition of an open subsets there exists an $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha + \epsilon) \subseteq U$. If $\alpha + \epsilon < b$ then $\alpha + \epsilon \in U \cap [a, b]$. However, this contradicts the fact that $\alpha = \text{lub}(U \cap [a, b])$ as $\alpha + \epsilon > \alpha$. Hence it must be the case that $b \in (\alpha - \epsilon, \alpha + \epsilon) \subseteq U$ which contradicts the fact that $b \in V$ and $U \cap V = \emptyset$.

Case 2: $\alpha \in V$. If $\alpha \in V$, then by the definition of an open subsets there exists an $\epsilon > 0$ such that $(\alpha - \epsilon, \alpha + \epsilon) \subseteq V$. Therefore, as $U \cap V = \emptyset$, it must be the case that $(\alpha - \epsilon, \alpha] \cap U = \emptyset$ thereby contradicting the fact that $\alpha = \text{lub}(U)$.

Therefore we have a contradiction. Hence A is connected. ∎

Now that Theorem 4.4.6 has shown the connected subsets of $\mathbb{R}$ are exactly the intervals, we move towards demonstrating how the Intermediate Value Theorem is connected to connected sets. In particular, the following is the true version of the Intermediate Value Theorem.

Theorem 4.4.7 (The Topological Intermediate Value Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $C \subseteq \mathbb{R}$ is connected, then $f(C)$ is connected.*

Proof. Let $C \subseteq \mathbb{R}$ be a connected set. To see that $f(C)$ is connected, suppose for the sake of a contradiction that $f(C)$ is not connected. Hence there exists open subsets $U, V \subseteq \mathbb{R}$ such that $U \cap f(C) \neq \emptyset$, $V \cap f(C) \neq \emptyset$, $U \cap V = \emptyset$, and $f(C) \subseteq U \cup V$.

Let $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$. Since f is continuous, Theorem 4.2.4 implies that U' and V' are open sets. We desire to show that U' and V' cause C to be disconnected thereby yielding our contradiction.

First we claim that $U' \cap C \neq \emptyset$. To see this, note since $U \cap f(C) \neq \emptyset$ that there exists a $x \in C$ such that $f(x) \in U$. Hence $x \in f^{-1}(U)$ and $x \in C$ so $U' \cap C \neq \emptyset$. A similar proof shows that $V' \cap C \neq \emptyset$.

Next, to see that $U' \cap V' = \emptyset$, suppose to the contrary that there exists an $x \in U' \cap V'$. Then $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. Therefore $f(x) \in U$ and $f(x) \in V$ so $f(x) \in U \cap V$ which contradicts the fact that $U \cap V = \emptyset$. Hence $U' \cap V' = \emptyset$.

Finally, we claim that $C \subseteq U' \cup V'$. To see this, let $x \in C$ be arbitrary. Then $f(x) \in f(C) \subseteq U \cup V$. Hence $f(x) \in U$ or $f(x) \in V$ which implies $x \in f^{-1}(U) = U'$ or $x \in f^{-1}(V) = V'$. Hence, as $x \in C$ was arbitrary, $C \subseteq U' \cup V'$.

Therefore, C is not connected by definition, which contradicts the fact that C is connected. Hence $f(C)$ is connected. ∎

By combining the Topological Intermediate Value Theorem together with the description of connected sets from Theorem 4.4.6, we can provide another proof of the Intermediate Value Theorem.

Proof of the Intermediate Value Theorem (Theorem 4.4.2). Let I be an interval, let $a, b \in I$ be such that $a < b$, and let $f : I \to \mathbb{R}$ be continuous. Let $\alpha \in I$ be such that $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$.

Since f is continuous on $[a, b]$, the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ f(a) & \text{if } x < a \\ f(b) & \text{if } x > b \end{cases}$$

is a continuous function on $\mathbb{R}$. Therefore, since $[a, b]$ is connected by Theorem 4.4.6 and since $g : \mathbb{R} \to \mathbb{R}$ is continuous, the Topological Intermediate Value Theorem implies that

$$g([a, b]) = f([a, b])$$

is a connected set. Hence Theorem 4.4.6 implies that $f([a, b])$ is an interval. Therefore, since $f(a), f(b) \in f([a, b])$, since $f([a, b])$ is an interval, and since $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$, it follows that $\alpha \in f([a, b])$. Hence there exists an $c \in [a, b]$ such that $f(c) = \alpha$. Since $f(a) \neq \alpha$ and $f(b) \neq \alpha$, it follows that $c \neq a$ and $x \neq b$ so $c \in (a, b)$ as desired. ■

4.5 Continuity of Inverse Functions

There is another use of the Intermediate Value Theorem that yields a important property of continuous functions we desire to examine. Specifically, we desire to use of the Intermediate Value Theorem to examine the inverse (under composition) of continuous functions. This theory is particularly useful to showing that specific operations and properties of the real numbers are as we would expect. For example, Proposition 1.3.10 provided a direct proof that the square root of a positive number exists. Whereas we would need to adapt that proof to obtain the n^{th} root of positive numbers exist, by considering the n^{th} root as the inverse of the n^{th} power, the technology of this section will not only give us that n^{th} roots exist, it will allow us to show that taking the n^{th} root defines a continuous function. Of course we could show the continuity of the n^{th} root by hand, but this technology applies to all invertible continuous functions, such as trigonometric functions on restricted intervals and the exponential function. In particular, once the exponential function is defined (see Remark 4.2.8) and shown to be invertible, the existence of the natural logarithm follows immediately from this section.

Recall that a function $f : X \to Y$ is invertible if and only if it is bijective (see Appendix A.2). As a function is always surjective once one replaces the co-domain with the range of the function, surjectivity will not be an issue for us. Thus we begin by focusing on when a function is injective. For continuous functions on the real numbers, there are certain properties that will aid us in determining when functions are injective. Note these properties for functions are parallels to the similar properties of sequences discussed in Section 2.2

Definition 4.5.1. Let I be an interval. A function $f : I \to \mathbb{R}$ is said to be

- *increasing* on I if $f(x_1) < f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$,
- *non-decreasing* on I if $f(x_1) \leq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$,
- *decreasing* on I if $f(x_1) > f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$,
- *non-increasing* on I if $f(x_1) \geq f(x_2)$ whenever $x_1, x_2 \in I$ and $x_1 < x_2$,
- *monotone* on I if f is non-decreasing or non-increasing.

In fact, for continuous functions, injectivity is directly characterized by these concepts.

Proposition 4.5.2. *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous. Then f is injective if and only if f is increasing or decreasing on I.*

Proof. Assume that f is increasing on I. To see that f is injective, let $x_1, x_2 \in I$ be such that $x_1 \neq x_2$. Thus either $x_1 < x_2$ or $x_2 < x_1$. If $x_1 < x_2$ then $f(x_1) < f(x_2)$ as f is increasing. Similarly, if $x_2 < x_1$ then $f(x_2) < f(x_1)$ as f is increasing. Hence $f(x_1) \neq f(x_2)$ in either case so f is injective by definition.

Assume that f is decreasing on I. To see that f is injective, let $x_1, x_2 \in I$ be such that $x_1 \neq x_2$. Thus either $x_1 < x_2$ or $x_2 < x_1$. If $x_1 < x_2$ then $f(x_1) > f(x_2)$ as f is decreasing. Similarly, if $x_2 < x_1$ then $f(x_2) > f(x_1)$ as f is decreasing. Hence $f(x_1) \neq f(x_2)$ in either case so f is injective by definition.

Finally, assume that f is not increasing nor decreasing on I. Therefore, there must exist three points $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$ such that either

- $f(x_1) < f(x_2)$ and $f(x_3) < f(x_2)$, or
- $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$.

Thus we divide the proof into two cases.

<u>Case 1: $x_1 < x_2 < x_3$, $f(x_1) < f(x_2)$, and $f(x_3) < f(x_2)$.</u> Let

$$\beta = \max\{f(x_1), f(x_3)\} \quad \text{and} \quad \alpha = \frac{\beta + f(x_2)}{2}.$$

Hence $\alpha \in \mathbb{R}$ is such that $f(x_1) < \alpha < f(x_2)$ and $f(x_3) < \alpha < f(x_2)$. Since f is continuous on $[x_1, x_2]$, the Intermediate Value Theorem (Theorem 4.4.2) implies there exists a $c \in (x_1, x_2)$ such that $f(c) = \alpha$. Similarly, since f is continuous on $[x_2, x_3]$, the Intermediate Value Theorem (Theorem 4.4.2) implies there exists a $d \in (x_1, x_2)$ such that $f(d) = \alpha$. Therefore, since $c < d$ and $f(c) = \alpha = f(d)$ we see that f is not injective on I.

Case 2: $x_1 < x_2 < x_3$, $f(x_1) > f(x_2)$, and $f(x_3) > f(x_2)$. By repeating the same ideas as in Case 1, we see that f is not injective on I.

Combining the above, we obtain that f is injective if and only if f is increasing or decreasing on I as desired. ∎

Remark 4.5.3. It is important to note that Proposition 4.5.2 is false if the assumption that f is continuous is removed. Indeed if we define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

then f is injective on $\mathbb{R}$ but f is neither increasing nor decreasing.

Now that we understand injective continuous functions on the real numbers, we turn our attention to showing that their inverses are continuous. To do so, we will actually prove a partial converse of the Intermediate Value Theorem (Theorem 4.4.2). The following result is only a partial inverse as we need to assume our functions are monotone.

Theorem 4.5.4. *Let I be an interval and let $f : I \to \mathbb{R}$ be monotone. Then f is continuous on I if and only if whenever $a, b \in I$ are such that $a < b$ and $\alpha \in \mathbb{R}$ is such that $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$ there exists a $c \in (a, b)$ such that $f(c) = \alpha$.*

Proof. If f is continuous on I, then whenever $a, b \in I$ are such that $a < b$ and $\alpha \in \mathbb{R}$ is such that $f(a) < \alpha < f(b)$ or $f(b) < \alpha < f(a)$ there exists a $c \in [a, b]$ such that $f(c) = \alpha$ by the Intermediate Value Theorem (Theorem 4.4.2).

To prove the converse, we will assume that f is non-decreasing as the proof when f is non-increasing will follow by similar arguments. To show that f is continuous on I, let $x_0 \in I$ be arbitrary. We will first deal with the case that x_0 is not an endpoint of I. In fact, the proof when x_0 is an endpoint of I will follow by similar arguments.

To see that f is continuous at x_0, let $\epsilon > 0$ be arbitrary. Our goal is to find $c_1, c_2 \in I$ such that $x_0 \in (c_1, c_2)$ and $|f(x) - f(x_0)| \leq \epsilon$ for all $x \in (c_1, c_2)$.

To find c_1, recall since x_0 is not the left endpoint of I that there exists an $a \in I$ such that $a < x_0$. Let

$$\alpha = \max\{f(a), f(x_0) - \epsilon\}.$$

In the case that $\alpha = f(a)$, let $c_1 = a$. Notice then that if $c_1 < x \leq x_0$, then

$$\begin{aligned} 0 &\leq f(x_0) - f(x) && \text{as } f \text{ is non-decreasing} \\ &\leq f(x_0) - f(a) && \text{as } f \text{ is non-decreasing} \\ &= f(x_0) - \alpha && \text{as } f(a) = \alpha \\ &\leq f(x_0) - (f(x_0) - \epsilon) && \text{as } f(x_0) - \epsilon \leq \alpha \\ &= \epsilon \end{aligned}$$

as desired. Otherwise we are in the case that $f(a) < \alpha = f(x_0) - \epsilon < f(x_0)$. By the assumptions of this direction of the proof, there exists a $c_1 \in (a, x_0)$ such that $f(c_1) = \alpha$. Consequently, if $c_1 < x \leq x_0$, then

$$\begin{aligned} 0 &\leq f(x_0) - f(x) && \text{as } f \text{ is non-decreasing} \\ &\leq f(x_0) - f(c_1) && \text{as } f \text{ is non-decreasing} \\ &= f(x_0) - \alpha && \text{as } f(c_1) = \alpha \\ &= f(x_0) - (f(x_0) - \epsilon) && \text{as } \alpha = f(x_0) - \epsilon \\ &= \epsilon \end{aligned}$$

as desired. Hence, in both case, there exists a $c_1 \in [a, x_0)$ such that $|f(x) - f(x_0)| \leq \epsilon$ for all $x \in (c_1, x_0]$.

To find c_2, recall since x_0 is not the right endpoint of I that there exists a $b \in I$ such that $x_0 < b$. Let

$$\beta = \min\{f(b), f(x_0) + \epsilon\}.$$

In the case that $\beta = f(b)$, let $c_2 = b$. Notice then that if $x_0 \leq x < c_2$, then

$$\begin{aligned} 0 &\leq f(x) - f(x_0) && \text{as } f \text{ is non-decreasing} \\ &\leq f(b) - f(x_0) && \text{as } f \text{ is non-decreasing} \\ &= \beta - f(x_0) && \text{as } f(b) = \beta \\ &\leq (f(x_0) + \epsilon) - f(x_0) && \text{as } \beta \leq f(x_0) + \epsilon \\ &= \epsilon \end{aligned}$$

as desired. Otherwise $f(b) > \beta = f(x_0) + \epsilon > f(x_))$. By the assumptions of this direction of the proof, there exists a $c_2 \in (x_0, b)$ such that $f(c_2) = \beta$. Consequently, if $x_0 \leq x < c_2$, then

$$\begin{aligned} 0 &\leq f(x) - f(x_0) && \text{as } f \text{ is non-decreasing} \\ &\leq f(c_2) - f(x_0) && \text{as } f \text{ is non-decreasing} \\ &= \beta - f(x_0) && \text{as } f(c_2) = \beta \\ &= (f(x_0) + \epsilon) - f(x_0) && \text{as } \beta = f(x_0) + \epsilon \\ &= \epsilon \end{aligned}$$

as desired. Hence, in both cases, there exists a $c_2 \in (x_0, b]$ such that $|f(x) - f(x_0)| \leq \epsilon$ for all $x \in [x_0, c_2)$.

Therefore, if we let $\delta = \min\{x_0 - c_1, c_2 - x_0\}$, then $\delta > 0$ and if $x \in I$ is such that $|x - x_0| < \delta$, then $|f(x) - f(x_0)| \leq \epsilon$. Hence f is continuous at x_0.

In the case that x_0 is the right endpoint of I, the above shows that for all $\epsilon > 0$ there exists a $c_1 \in I$ such that if $x \in (c_1, x_0]$ then $|f(x) - f(x_0)| \leq \epsilon$. Therefore, by letting $\delta = x_0 - c_1$, we have that $\delta > 0$ and if $x \in I$ is such that $|x - x_0| < \delta$, then $|f(x) - f(x_0)| \leq \epsilon$. Hence f is continuous at x_0.

In the case that x_0 is the left endpoint of I, the above shows that for all $\epsilon > 0$ there exists a $c_2 \in I$ such that if $x \in [x_0, c_2)$ then $|f(x) - f(x_0)| \leq \epsilon$. Therefore, by letting $\delta = c_2 - x_0$, we have that $\delta > 0$ and if $x \in I$ is such that $|x - x_0| < \delta$, then $|f(x) - f(x_0)| \leq \epsilon$. Hence f is continuous at x_0.

Hence, as we have show f is continuous at all $x_0 \in I$, f is continuous on I as desired. ∎

Using Theorem 4.5.4 we can show that inverses of continuous functions are continuous!

Corollary 4.5.5. *Let I be an interval. If $f : I \to \mathbb{R}$ is injective and continuous, then $f(I)$ is an interval and the inverse of f on its image, $f^{-1} : f(I) \to I$, is continuous.*

Proof. Assume $f : I \to \mathbb{R}$ is injective and continuous. Hence Proposition 4.5.2 implies that f is increasing or decreasing. We will assume that f is increasing as the proof that f is decreasing will follow by similar arguments.

To see that $f(I)$ is an interval, assume $y_1, y_2 \in f(I)$ are such that $y_1 < y_2$. Thus there exists $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $f(x_1) < f(x_2)$, it must be the case that $x_1 < x_2$ as f was increasing. Therefore, by the Intermediate Value Theorem (Theorem 4.4.2), we obtain that

$$[y_1, y_2] \subseteq f(I).$$

Therefore, since $y_1, y_2 \in f(I)$ were arbitrary, $f(I)$ is an interval. (If it is not clear why this property implies $f(I)$ is an interval, see the proof of Proposition 3.1.9.)

Since $f : I \to f(I)$ is injective and surjective, the inverse $f^{-1} : f(I) \to I$ exists. We claim that f^{-1} is increasing. To see this, assume $y_1, y_2 \in f(I)$ are such that $y_1 < y_2$. Thus there exists $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since $f(x_1) < f(x_2)$, it must be the case that $x_1 < x_2$ as f was increasing. Hence

$$f^{-1}(y_1) = f^{-1}(f(x_1)) = x_1 < x_2 = f^{-1}(f(x_2)) = f^{-1}(y_2).$$

Therefore, as $y_1, y_2 \in f(I)$ were arbitrary, f^{-1} is increasing.

Hence $f^{-1} : f(I) \to I$ is an increasing function such that $f^{-1}(f(I))) = I$. Therefore f^{-1} is continuous by Theorem 4.5.4 as f^{-1} satisfies the conclusions of the Intermediate Value Theorem. ∎

Example 4.5.6. Consider the functions

- for $n \in \mathbb{N}$, the function $f_n : [0,\infty) \to [0,\infty)$ defined by $f_n(x) = x^{2n}$ for all $x \in [0,\infty)$,

- for $n \in \mathbb{N}$, the function $g_n : \mathbb{R} \to \mathbb{R}$ defined by $g_n(x) = x^{2n+1}$ for all $x \in [0,\infty)$,

- the function $h : \mathbb{R} \to [0,\infty)$ defined by $h(x) = e^x$ for all $x \in \mathbb{R}$,

- the function $c : [0,\pi] \to [-1,1]$ defined by $c(x) = \cos(x)$ for all $x \in [0,\pi]$,

- the function $s : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1]$ defined by $s(x) = \sin(x)$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

- the function $t : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to (-\infty,\infty)$ defined by $t(x) = \tan(x)$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

It is possible to show that these functions are invertible. Hence Corollary 4.5.5 implies the following functions exists and are continuous:

- for $n \in \mathbb{N}$, the function $f_n^{-1} : [0,\infty) \to [0,\infty)$ defined by $f_n^{-1}(x) = \sqrt[2n]{x}$ for all $x \in [0,\infty)$,

- for $n \in \mathbb{N}$, the function $g_n^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $g_n^{-1}(x) = \sqrt[2n+1]{x}$ for all $x \in [0,\infty)$,

- the function $h^{-1} : [0,\infty) \to \mathbb{R}$ defined by $h^{-1}(x) = \ln(x)$ for all $x \in \mathbb{R}$,

- the function $c^{-1} : [-1,1] \to [0,\pi]$ defined by $c^{-1}(x) = \arccos(x)$ for all $x \in [0,\pi]$,

- the function $s^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $s^{-1}(x) = \arcsin(x)$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and

- the function $t^{-1} : (-\infty,\infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ defined by $t^{-1}(x) = \arctan(x)$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The question remains, "How do we show these functions are injective and surjective"? Of course, once we show these functions are injective (i.e. increasing or decreasing), then surjectivity follows from Corollary 4.5.5 and computing limits/values. So our real question is, "How do we show a function on an interval of $\mathbb{R}$ is increasing or decreasing?"

4.6 The Extreme Value Theorem

Before answering the question poised at the end of the previous section, we turn our attention to our second 'value' theorem and piece of the Triforce. This result, which is really about continuous functions on compact sets, allows for us to deduce certain continuous functions have maxima and minima. Our endeavour to compute the locations of theses maxima and minina will also enable us to determine when functions on intervals are increasing or decreasing thereby answering the question poised at the previous section.

Of course, in order to have maxima and minima, our functions must have the following property.

Definition 4.6.1. Let I be an interval. A function $f : I \to \mathbb{R}$ is said to be *bounded* if $f(I)$ is a bounded subset of $\mathbb{R}$.

By using the fact that closed intervals are compact, we can prove that continuous functions on closed intervals are bounded and have maxima and minima.

Theorem 4.6.2 (Extreme Value Theorem). *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. There exists points $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous. First we claim that f is bounded. To see this, suppose for the sake of a contradiction that f is not bounded. Thus for each $n \in \mathbb{N}$ there exists an $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. Consider the sequence $(x_n)_{n\geq1}$. By the Bolzano-Weierstrass Theorem (Theorem 2.4.7) there exists a subsequence $(x_{n_k})_{k\geq1}$ of $(x_n)_{n\geq1}$ that converges to some number $c \in [a, b]$ (i.e. $[a, b]$ is sequentially compact). Since f is continuous at c, $(f(x_{k_n}))_{k\geq1}$ converges to $f(c)$. Hence $(f(x_{k_n}))_{k\geq1}$ is bounded by Proposition 2.2.3. However, this contradicts the fact that $(f(x_{k_n}))_{k\geq1}$ is not bounded since $|f(x_{n_k})| \geq n_k$ for all $k \in \mathbb{N}$. Thus $f([a, b])$ is a bounded set.

To show the existence of the point x_1, note since $f([a, b])$ is bounded that $\alpha = \mathrm{glb}(f([a, b]))$ is well-defined. By the definition of α, for each $n \in \mathbb{N}$ there exists a $y_n \in [a, b]$ such that

$$\alpha \leq f(y_n) < \alpha + \frac{1}{n}.$$

Hence $\lim_{n\to\infty} f(y_n) = \alpha$ by the Squeeze Theorem.

By the Bolzano-Weierstrass Theorem (Theorem 2.4.7) there exists a subsequence $(y_{n_k})_{k\geq1}$ of $(y_n)_{n\geq1}$ that converges to some number $x_1 \in [a, b]$ (i.e. $[a, b]$ is sequentially compact). Since f is continuous on $[a, b]$,

$$f(x_1) = \lim_{k\to\infty} f(y_{x_k}) = \alpha.$$

Hence $f(x_1) \leq f(x)$ for all $x \in [a, b]$ by the definition of the greatest lower bound.

Similar arguments show that if and $\beta = \text{lub}(f([a,b]))$, then there exists an $x_2 \in [a,b]$ such that $f(x_2) = \beta$. Hence $f(x) \leq f(x_2)$ for all $x \in [a,b]$ by the definition of β. Thus the proof is complete. ∎

Remark 4.6.3. Note the Extreme Value Theorem requires continuous functions on closed, finite intervals. Indeed consider the function $f : (0,\infty) \to (0,\infty)$ defined by $f(x) = \frac{1}{x}$ for all $x \in (0,\infty)$. Then there does not exist an $x_2 \in (0,\infty)$ such that $f(x) \leq f(x_2)$ for all $x \in (0,\infty)$ since $\lim_{x\to 0+} f(x) = \infty$. Similarly, there does not exist an $x_1 \in (0,\infty)$ such that $f(x_1) \leq f(x)$ for all $x \in (0,\infty)$ since $\lim_{x\to\infty} f(x) = 0$ yet there is no $x \in (0,\infty)$ such that $f(x) = 0$.

Note the proof of the Extreme Value Theorem relied heavily on compactness via sequential compactness and the Bolzano-Weierstrass Theorem (Theorem 2.4.7). In fact, there is another proof of the Extreme Value Theorem that makes more implicit use of compactness and generalizes in future courses. In fact, the following is the true version of the Extreme Value Theorem.

Theorem 4.6.4 (The Topological Extreme Value Theorem). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $K \subseteq \mathbb{R}$ is compact, then $f(K)$ is compact.*

Proof. Let $K \subseteq \mathbb{R}$ be a compact set. To see that $f(K)$ is compact, let $\{U_i \mid i \in I\}$ be an open cover of $f(K)$. By Theorem 4.2.4, $\{f^{-1}(U_i) \mid i \in I\}$ is a collection of open sets since f is continuous. Moreover, since

$$f(K) \subseteq \bigcup_{i\in I} U_i \quad \text{implies} \quad K \subseteq \bigcup_{i\in I} f^{-1}(U_i),$$

we have that $\{f^{-1}(U_i) \mid i \in I\}$ is an open cover of K. Therefore, since K is compact, the definition of a compact set implies there exists an $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in I$ such that

$$K \subseteq \bigcup_{k=1}^{n} f^{-1}(U_{i_k}).$$

Hence

$$f(K) \subseteq \bigcup_{k=1}^{n} U_{i_k}.$$

Therefore $\{U_{i_k} \mid k \in \{1,\ldots,n\}\}$ is a finite subcover of K. Therefore, since $\{U_i \mid i \in I\}$ was arbitrary, $f(K)$ is compact. ∎

Using the Topological Extreme Value Theorem, we can derive a different (but fundamentally the same) proof of the Extreme Value Theorem.

Proof of the Extreme Value Theorem (Theorem 4.6.2). Let $f : [a, b] \to \mathbb{R}$ be continuous. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \\ f(a) & \text{if } x < a \\ f(b) & \text{if } x > b \end{cases}.$$

Since f is continuous on $[a, b]$, it follows that g is continuous on $[a, b]$.

Since g is continuous and $[a, b]$ is compact by the Heine-Borel Theorem (Theorem 3.2.9), $g([a, b]) = f([a, b])$ is compact by the Topological Extreme Value Theorem. Hence $f([a, b])$ is closed and bounded by the Heine-Borel Theorem (Theorem 3.2.9).

To show the existence of the point x_1, note since $f([a, b])$ is bounded that $\alpha = \text{glb}(f([a, b]))$ is well-defined. By the definition of α, for each $n \in \mathbb{N}$ there exists a $y_n \in [a, b]$ such that

$$\alpha \leq f(y_n) < \alpha + \frac{1}{n}.$$

Hence $\lim_{n\to\infty} f(y_n) = \alpha$ by the Squeeze Theorem.

By the Bolzano-Weierstrass Theorem (Theorem 2.4.7) there exists a subsequence $(y_{n_k})_{k\geq 1}$ of $(y_n)_{n\geq 1}$ that converges to some number $x_1 \in [a, b]$ (i.e. $[a, b]$ is sequentially compact). Since f is continuous on $[a, b]$,

$$f(x_1) = \lim_{k\to\infty} f(y_{x_k}) = \alpha.$$

Hence $f(x_1) \leq f(x)$ for all $x \in [a, b]$ by the definition of the greatest lower bound.

Similar arguments show that if and $\beta = \text{lub}(f([a, b]))$, then there exists an $x_2 \in [a, b]$ such that $f(x_2) = \beta$. Hence $f(x) \leq f(x_2)$ for all $x \in [a, b]$ by the definition of β. Thus the proof is complete. ∎

Of course, the Extreme Value Theorem says that maximum and minimum are obtain, but provides no method for computing them. How can we compute these maxima and minima?

Chapter 5

Differentiation

With the above study of continuity, we turn our attention to studying another important concept in calculus: differentiation. Constructed to be an approximation to the slope of the tangent line of the graph of a function at a point, derivatives are essential to studying the rate of changes of dynamical systems. In addition, derivatives provide answers to our outstanding questions from Chapter 4. In particular, provided the derivatives exists, we will obtain a method to compute the maxima and minima of continuous functions on closed intervals and be able to show functions are increasing or decreasing on intervals thereby obtaining that their inverse functions are continuous. Moreover, we will be able to complete our third 'value' theorem thereby assembling the full Triforce. This will enable us to prove a useful theorem to aid in computing the limits of functions and aid in approximating functions with polynomials.

5.1 The Derivative

To begin our study of the theory of differentiation, as always we require a formal definition of the derivative and need to derive the basic properties of the derivative.

5.1.1 Definition of a Derivative

Given an open interval I, an $\alpha \in I$, and function $f : I \to \mathbb{R}$, we desire the derivative of f at α to be the slope of the tangent line of the graph of f at α. As an approximation to the slope, we can pick any point $x \in I$ and compute the slope of the line from $(x, f(x))$ to $(\alpha, f(\alpha))$; namely

$$\frac{f(x) - f(\alpha)}{x - \alpha}.$$

As x gets closer and closer to α, the slope of the line from $(x, f(x))$ to $(\alpha, f(\alpha))$ should better and better approximate the slope of the tangent line to f at α. In particular, these slopes better and better approximating the instantaneous rate of change of f at α.

To make the definition of the derivative precise, we simply need to use our formal definition of the limit of a function. To do so, we note the above slope expression is not defined at $x = \alpha$, which is fine in our definition of a limit as discussed in Remark 4.1.3.

Definition 5.1.1. Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$. It is said that f is *differentiable* at α if

$$\lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha}$$

exists. If f is differentiable at α, we denote the above limit by $f'(\alpha)$.

If f is differentiable at each point x in I, then the function $f' : I \to \mathbb{R}$ whose value at x is $f'(x)$ is called the *derivative* of f on I.

Remark 5.1.2. There is another way to formally define the derivative of a function f at α. Indeed, if x is tending to α, then $x - \alpha$ tends to 0. Substituting $h = x - \alpha$, we see $x = \alpha + h$ so

$$\lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}.$$

This alternate formulation of the derivative is often useful for computations.

Luckily many of the functions we naturally consider on the real line are differentiable.

Example 5.1.3. Let $c \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$ for all $x \in \mathbb{R}$. We claim that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$. To see this, notice for all $\alpha \in \mathbb{R}$ that

$$\lim_{x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = \lim_{x \to \alpha} \frac{c - c}{x - \alpha} = \lim_{x \to \alpha} 0 = 0.$$

Hence $f'(\alpha)$ exists for all $\alpha \in \mathbb{R}$ and $f'(\alpha) = 0$ as desired.

Example 5.1.4. Let $n \in \mathbb{N}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ for all $x \in \mathbb{R}$. We claim that f is differentiable on $\mathbb{R}$ and $f'(x) = nx^{n-1}$ for all

$x \in \mathbb{R}$. To see this, notice for all $\alpha \in \mathbb{R}$ that

$$\begin{aligned}
&\lim_{h\to 0} \frac{f(\alpha+h)-f(\alpha)}{h} \\
&= \lim_{h\to 0} \frac{(h+\alpha)^n - \alpha^n}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{n} \binom{n}{k} h^k \alpha^{n-k} - \alpha^n\right) && \text{by the Binomial Theorem (Theorem 1.1.8)} \\
&= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=1}^{n} \binom{n}{k} h^k \alpha^{n-k}\right) \\
&= \lim_{h\to 0} \sum_{k=1}^{n} \binom{n}{k} h^{k-1} \alpha^{n-k} \\
&= \binom{n}{1} \alpha^{n-1} && \text{as } \lim_{h\to 0} h^{k-1} = 0 \text{ for all } k > 1 \\
&= n\alpha^{n-1}.
\end{aligned}$$

Hence $f'(\alpha)$ exists for all $\alpha \in \mathbb{R}$ and $f'(\alpha) = n\alpha^{n-1}$ as desired.

Of course, not all nice functions are differentiable at every point.

Example 5.1.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$ for all $x \in \mathbb{R}$. Since $f(x) = x$ for all $x > 0$, it follows that $f'(x) = 1$ if $x > 0$. Similarly, since $f(x) = -x$ if $x < 0$, it follows that $f'(x) = -1$ if $x < 0$.

However f is not differentiable at 0. Indeed

$$\lim_{h\to 0+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0+} \frac{h}{h} = 1$$

whereas

$$\lim_{h\to 0-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1.$$

Thus $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ does not exist. Hence f is not differentiable at 0.

Remark 5.1.6. Of course, students will recall from their previous calculus courses that if we define $f, g, h : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin(x)$, $g(x) = \cos(x)$, and $h(x) = e^x$ for all $x \in \mathbb{R}$, then f, g, and h are all differentiable on $\mathbb{R}$ with

$$f'(x) = \cos(x), \qquad g'(x) = -\sin(x), \qquad \text{and} \qquad h'(x) = e^x.$$

The question is, how do we show this?

Recall the best way to define the above functions is with series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n.$$

If one verifies we can "take the derivative term by term" in the above series, then we will have the above formulae for the derivatives. However, the discussion why this can be done is more appropriately placed in MATH 3001 (Series of Functions). Consequently, we will assume throughout the remainder of the course that $\sin(x), \cos(x)$, and e^x are differentiable on $\mathbb{R}$ with the above derivatives.

Of course, the reason we have discussed differentiable functions after continuous functions is that the differentiable functions are a subset of the continuous functions.

Theorem 5.1.7. *Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$. If f is differentiable at α, then f is continuous at α.*

Proof. Assume that $f'(\alpha)$ exists. Therefore $\lim_{x\to\alpha} \frac{f(x)-f(\alpha)}{x-\alpha}$ exists. Since $\lim_{x\to\alpha} x - \alpha = 0$ and since

$$f(x) - f(\alpha) = \left(\frac{f(x) - f(\alpha)}{x - \alpha}\right)(x - \alpha),$$

we obtain that $\lim_{x\to\alpha} f(x) - f(\alpha)$ exists by Theorem 4.1.19 and

$$\lim_{x\to\alpha} f(x) - f(\alpha) = \left(\lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha}\right)\left(\lim_{x\to\alpha} x - \alpha\right) = f'(\alpha)0 = 0.$$

Hence $\lim_{x\to\alpha} f(x) = f(\alpha)$ so f is continuous at α. ∎

Remark 5.1.8. Consequently, as Chapter 4 demonstrated a plethora of functions that were not continuous at points, there are many examples of functions that are not differentiable at points. However, note that continuity does not imply differentiability. Indeed the absolute value function is continuous on $\mathbb{R}$ but not differentiable at 0 by Example 5.1.5. In fact, there a collection of functions that are continuous on $\mathbb{R}$ but not differentiable at any point in the real numbers! Said functions are constructed via series of functions and thus are more naturally examined in MATH 3001 (Series of Functions).

5.1.2 Rules of Differentiation

With our formal definition of the derivative of a function, we can again build up knowledge of how differentiability behaves with respect to various operations. We begin with the

Proposition 5.1.9. *Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$ be differentiable at α. For each constant $c \in \mathbb{R}$ the function $cf : I \to \mathbb{R}$ defined via $(cf)(x) = cf(x)$ for all $x \in I$ is differentiable at α and*

$$(cf)'(\alpha) = cf'(\alpha).$$

Proof. Since

$$\begin{aligned}\lim_{x\to\alpha} \frac{(cf)(x) - (cf)(\alpha)}{x-\alpha} &= \lim_{x\to\alpha} \frac{c(f(x) - f(\alpha))}{x-\alpha} \\ &= c \lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x-\alpha} && \text{by Theorem 4.1.19} \\ &= cf'(\alpha),\end{aligned}$$

the proof is complete. ∎

Proposition 5.1.10. *Let I be an open interval, let $\alpha \in I$, and let $f, g : I \to \mathbb{R}$ be differentiable at α. The function $f + g : I \to \mathbb{R}$ defined via $(f + g)(x) = f(x) + g(x)$ for all $x \in I$ is differentiable at α and*

$$(f + g)'(\alpha) = f'(\alpha) + g'(\alpha).$$

Proof. Since

$$\begin{aligned}\lim_{x\to\alpha} \frac{(f+g)(x) - (f+g)(\alpha)}{x-\alpha} &= \lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x-\alpha} + \frac{g(x) - g(\alpha)}{x-\alpha} \\ &= f'(\alpha) + g'(\alpha)\end{aligned}$$

by by Theorem 4.1.19, the proof is complete. ∎

Example 5.1.11. For $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{R}$, let $p : \mathbb{R} \to \mathbb{R}$ be defined by

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for all $x \in \mathbb{R}$. By Examples 5.1.3 and 5.1.4 together with Propositions 5.1.9 and 5.1.10, it follows that p is differentiable on $\mathbb{R}$ and

$$p'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1 + 0$$

for all $x \in \mathbb{R}$.

The ability to take derivatives of products is slightly more delicate.

Proposition 5.1.12 (Product Rule). *Let I be an open interval, let $\alpha \in I$, and let $f, g : I \to \mathbb{R}$ be differentiable at α. The function $fg : I \to \mathbb{R}$ defined via $(fg)(x) = f(x)g(x)$ for all $x \in I$ is differentiable at α and*

$$(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha).$$

Proof. To begin, notice for $x \in \mathbb{R} \setminus \{\alpha\}$ that

$$\begin{aligned}\frac{(fg)(x) - (fg)(\alpha)}{x - \alpha} &= \frac{f(x)g(x) - f(\alpha)g(\alpha)}{x - \alpha} \\ &= \frac{f(x)g(x) - f(x)g(\alpha)}{x - \alpha} + \frac{f(x)g(\alpha) - f(\alpha)g(\alpha)}{x - \alpha} \\ &= f(x)\frac{g(x) - g(\alpha)}{x - \alpha} + g(\alpha)\frac{f(x) - f(\alpha)}{x - \alpha}.\end{aligned}$$

Since $f'(\alpha)$ exists, f is continuous at α by Theorem 5.1.7. Therefore $\lim_{x\to\alpha} f(x) = f(\alpha)$. Since $g'(\alpha) = \lim_{x\to\alpha} \frac{g(x)-g(\alpha)}{x-\alpha}$, we obtain by Theorem 4.1.19 that

$$\lim_{x\to\alpha} f(x)\frac{g(x) - g(\alpha)}{x - \alpha} = \left(\lim_{x\to\alpha} f(x)\right)\left(\lim_{x\to\alpha} \frac{g(x) - g(\alpha)}{x - \alpha}\right) = f(\alpha)g'(\alpha).$$

Since Theorem 4.1.19 also implies that

$$\lim_{x\to\alpha} g(\alpha)\frac{f(x) - f(\alpha)}{x - \alpha} = g(\alpha)\lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = g(\alpha)f'(\alpha),$$

we obtain by Theorem 4.1.19 that

$$\lim_{x\to\alpha} \frac{(fg)(x) - (fg)(\alpha)}{x - \alpha} = f(\alpha)g'(\alpha) + g(\alpha)f'(\alpha)$$

thereby completing the proof. ∎

Example 5.1.13. Using the Product Rule, it follows that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x\sin(x)$ for all $x \in \mathbb{R}$ is differentiable on $\mathbb{R}$ and

$$f'(x) = (1)\sin(x) + x(\cos(x)) = \sin(x) + x\cos(x)$$

for all $x \in \mathbb{R}$.

Remark 5.1.14. Note the product rule can be extended to an product of a finite number of differentiable functions. Indeed if f, g, and h are all differentiable at α, then

$$\begin{aligned}(fgh)'(\alpha) &= f'(\alpha)(gh)(\alpha) + f(\alpha)(gh)'(\alpha) \\ &= f'(\alpha)g(\alpha)h(\alpha) + f(\alpha)(g'(\alpha)h(\alpha) + g(\alpha)h'(\alpha)) \\ &= f'(\alpha)g(\alpha)h(\alpha) + f(\alpha)g'(\alpha)h(\alpha) + f(\alpha)g(\alpha)h'(\alpha).\end{aligned}$$

More generally, by the Principle of Mathematical Induction, it can be shown that if $f_1, \ldots, f_n$ are all differentiable at α, then $f_1 \cdots f_n$ is differentiable at α with

$$(f_1 \cdots f_n)'(\alpha) = \sum_{j=1}^{n} f_1(\alpha) \cdots f_{j-1}(\alpha) f_j'(\alpha) f_{j+1}(\alpha) \cdots f_n(\alpha).$$

In particular, since the derivative of x is easily seen to be 1, we can use this generalized product rule to obtain that the derivative of x^n is nx^{n-1} thereby bypassing the need to know the Binomial Theorem in Example 5.1.4.

To derive our next rule invoking quotients of functions, we begin with a subcase.

Lemma 5.1.15. *Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$ be differentiable at α. If $f(\alpha) \neq 0$, the function $h : I \to \mathbb{R}$ defined via $h(x) = \frac{1}{f(x)}$ is differentiable at α and*

$$h'(\alpha) = -\frac{f'(\alpha)}{(f(\alpha))^2}.$$

Proof. To begin, first note that the assumption that $f(\alpha) \neq 0$ does not imply that h is well-defined on all of I. However, since $f'(\alpha)$ exists, f is continuous at α by Theorem 5.1.7. Hence Lemma 4.4.1 implies there exists an open interval J such that $\alpha \in J$ and $f(x) \neq 0$ for all $x \in J$. Therefore $h : J \to \mathbb{R}$ is well-defined so it makes sense to discuss whether $h'(\alpha)$ exists (i.e. see Definition 5.1.1).

To show that h is differentiable at α, note for all $x \in J$ that

$$\begin{aligned} \frac{h(x) - h(\alpha)}{x - \alpha} &= \frac{\frac{1}{f(x)} - \frac{1}{f(\alpha)}}{x - \alpha} \\ &= \frac{\frac{f(\alpha) - f(x)}{f(\alpha) f(x)}}{x - \alpha} \\ &= -\frac{f(x) - f(\alpha)}{f(x) f(\alpha)(x - \alpha)}. \end{aligned}$$

Since $f'(\alpha)$ exists, f is continuous at α by Theorem 5.1.7. Therefore $\lim_{x\to\alpha} f(x) = f(\alpha)$. Moreover, since $f(\alpha) \neq 0$, we have that $\lim_{x\to\alpha} \frac{1}{f(x)} = \frac{1}{f(\alpha)}$ by Theorem 4.1.19. Hence Theorem 4.1.19 implies that

$$\begin{aligned} \lim_{x\to\alpha} \frac{h(x) - h(\alpha)}{x - \alpha} &= -\frac{1}{f(\alpha)} \left(\lim_{x\to\alpha} \frac{1}{f(x)} \right) \left(\lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} \right) \\ &= -\frac{1}{f(\alpha)} \left(\frac{1}{f(\alpha)} \right) (f'(\alpha)) = -\frac{f'(\alpha)}{(f(\alpha))^2} \end{aligned}$$

as desired. ∎

Combining Lemma 5.1.15 with the Product Rule, we obtain the following.

Proposition 5.1.16 (Quotient Rule). *Let I be an open interval, let $\alpha \in I$, and let $f, g : I \to \mathbb{R}$ be differentiable at α. If $g(\alpha) \neq 0$, the function h defined via $h(x) = \frac{f(x)}{g(x)}$ is differentiable at α and*

$$h'(\alpha) = \frac{f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}.$$

Proof. By the same argument as used in Lemma 5.1.15, h is well-defined on an open interval containing α so it makes sense to discuss the derivative of h at α. Moreover, by Lemma 5.1.15 and Proposition 5.1.12, we obtain that $h(x)$ is differentiable at α and

$$h'(\alpha) = f'(\alpha)\frac{1}{g(\alpha)} + f(\alpha)\left(-\frac{g'(\alpha)}{(g(\alpha))^2}\right) = \frac{f'(\alpha)g(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}$$

as desired. ∎

Example 5.1.17. Recall the tangent function $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is defined by

$$h(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By the Quotient Rule, it follows that h is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\begin{aligned} h'(x) &= \left(\frac{\sin(x)}{\cos(x)}\right)' \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} = \sec^2(x) \end{aligned}$$

The final differentiation rule of this section, the Chain Rule, is likely very familiar to those that have studied Calculus. The Chain Rule allows one to compute the derivative of the composition of functions, provided the composition makes sense and derivatives exist. However, many “proofs" of the Chain Rule seen in elementary calculus have a large flaw in them. Specifically, to show that $g \circ f$ is differentiable at α, these flawed proofs write

$$\frac{g(f(x)) - g(f(\alpha))}{x - \alpha} = \frac{g(f(x)) - g(f(\alpha))}{f(x) - f(\alpha)}\frac{f(x) - f(\alpha)}{x - \alpha}.$$

Of course, this does not make sense or work if there does not exists an open interval containing α for which $f(x) \neq f(\alpha)$ as we cannot divide by 0. Even if one attempts to say that if $f(x) = f(\alpha)$ then $g(f(x)) - g(f(\alpha)) = 0$, one runs into issues. To demonstrate that such problematic f exist, we note the following example.

Example 5.1.18. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \end{cases}.$$

Notice that $\lim_{n\to\infty} \frac{1}{\pi n} = 0$ and $f\left(\frac{1}{\pi n}\right) = 0$ for all $n \in \mathbb{N}$. Hence f has a sequence of zeros that tend to 0. Even with this, we claim that f is differentiable at 0. To see this, notice for all $x \in \mathbb{R} \setminus \{0\}$ that

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin\left(\frac{1}{x}\right) - 0}{x - 0} = x \sin\left(\frac{1}{x}\right).$$

Since

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$$

for all $x \in \mathbb{R} \setminus \{0\}$, it follows that

$$-|x| \leq \left|\frac{f(x) - f(0)}{x - 0}\right| \leq |x|$$

for all $x \in \mathbb{R} \setminus \{0\}$. Therefore, since $\lim_{x\to 0} |x| = 0$ by Example 5.1.5, the Squeeze Theorem for Functions (Theorem 4.1.23) implies that

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0.$$

Hence f is differentiable at 0 with $f'(0) = 0$.

In order to rigorously prove the Chain Rule, we begin with the following which yields an alternate definition for the derivative of a function.

Theorem 5.1.19 (Carathéodory's Theorem). *Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$. Then f is differentiable at α if and only if there exists a function $\varphi : I \to \mathbb{R}$ such that φ is continuous at α and*

$$f(x) = f(\alpha) + \varphi(x)(x - \alpha)$$

for all $x \in I$. Moreover $f'(\alpha) = \varphi(\alpha)$.

Proof. To begin, assume $\varphi : I \to \mathbb{R}$ is continuous at α and

$$f(x) = f(\alpha) + \varphi(x)(x - \alpha)$$

for all $x \in I$. To see that f is differentiable at α, notice if $x \in I \setminus \{\alpha\}$ then

$$\frac{f(x) - f(\alpha)}{x - \alpha} = \frac{\varphi(x)(x - \alpha)}{x - \alpha} = \varphi(x).$$

Therefore since φ is continuous at α, we obtain that

$$\lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = \lim_{x\to\alpha} \varphi(x) = \varphi(\alpha).$$

Hence f is differentiable at α with $f'(\alpha) = \varphi(\alpha)$.

To prove the other direction, assume that f is differentiable at α. Define $\varphi : I \to \mathbb{R}$ by

$$\varphi(x) = \begin{cases} f'(\alpha) & \text{if } x = \alpha \\ \frac{f(x)-f(\alpha)}{x-\alpha} & \text{if } x \neq \alpha \end{cases}$$

for all $x \in I$. Clearly $f(x) = f(\alpha) + \varphi(x)(x - \alpha)$ for all $x \in I$. Moreover, since

$$\lim_{x\to\alpha} \varphi(x) = \lim_{x\to\alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = f'(\alpha) = \varphi(\alpha),$$

φ is continuous at α as desired. ∎

Using Carathéodory's Theorem, the Chain Rule follows without much issue.

Theorem 5.1.20 (Chain Rule). *Let I and J be open intervals, let $g : J \to \mathbb{R}$, and let $f : I \to \mathbb{R}$ be such that $f(I) \subseteq J$. Suppose that $\alpha \in I$, f is differentiable at α, and g is differentiable at $f(\alpha)$. Then $g \circ f : I \to \mathbb{R}$ is differentiable at α and*

$$(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).$$

Proof. By Carathéodory's Theorem (Theorem 5.1.19) there exists functions $\varphi : I \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ such that

- φ is continuous at α,
- $f(x) = f(\alpha) + \varphi(x)(x - \alpha)$ for all $x \in I$,
- $f'(\alpha) = \varphi(\alpha)$,
- ψ is continuous at $f(\alpha)$,
- $g(x) = g(f(\alpha)) + \psi(x)(x - f(\alpha))$ for all $x \in J$, and
- $g'(f(\alpha)) = \psi(f(\alpha))$.

Therefore

$$g(f(x)) - g(f(\alpha)) = \psi(f(x))(f(x) - f(\alpha)) = \psi(f(x))\varphi(x)(x - \alpha).$$

Since f is differentiable at α and thus continuous at α by Theorem 5.1.7, and since ψ is continuous at $f(\alpha)$, $\psi \circ f$ is continuous at α by Theorem 4.2.10. Therefore, since φ is continuous at α, the function $h : I \to \mathbb{R}$

defined by $h(x) = \psi(f(x))\varphi(x)$ for all $x \in I$ is continuous at α. Since $g(f(x)) = g(f(\alpha)) + h(x)(x - \alpha)$, Carathéodory's Theorem (Theorem 5.1.19) implies that that $g \circ f$ is differentiable at α with derivative

$$h(\alpha) = \psi(f(\alpha))\varphi(\alpha) = g'(f(\alpha))f'(\alpha)$$

as desired. ∎

Example 5.1.21. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \cos(x^3)$. By the Chain Rule, f is differentiable on $\mathbb{R}$ with

$$f'(x) = (-\sin(x^3))(3x^2).$$

5.2 Derivatives and Extreme Values of Functions

With the above study of the differentiation, we can turn our attention to resolving some outstanding questions from Chapter 4. In this section we will focus on a method for locating the extreme values of a continuous functions, whose existence is guaranteed by the Extreme Value Theorem (Theorem 4.6.2). Of course, the results of this section only apply to differentiable functions, but most functions one wants to compute the extreme values of are differentiable.

Of course, since students have already taken MATH 1300 and know how to compute extreme values via derivatives, we will focus on verifying the results students have already been using oppose to repeating the common examples and procedures.

To discuss the computation of extreme values, it is first useful to identify various types of extreme values.

Definition 5.2.1. Let I be an interval and let $f : I \to \mathbb{R}$. It is said that f has a

- *global maximum* at c if $f(x) \leq f(c)$ for all $x \in I$.
- *global minimum* at c if $f(x) \geq f(c)$ for all $x \in I$.
- *local maximum* at c if there exists an open interval $J \subseteq I$ such that $c \in J$ and $f(x) \leq f(c)$ for all $x \in J$.
- *local minimum* at c if there exists an open interval $J \subseteq I$ such that $c \in J$ and $f(x) \geq f(c)$ for all $x \in J$.

Recall continuous functions on closed intervals automatically have a global maximum and a global minimum by the Extreme Value Theorem (Theorem 4.6.2). In order to locate these points, there is a simple method if the function is differentiable.

Proposition 5.2.2. *Let I be an interval and let $f : I \to \mathbb{R}$. If f has a local maximum or local minimum at $c \in I$ and if $f'(c)$ exists, then $f'(c) = 0$.*

Proof. Assume that f has a local maximum or local minimum at $c \in I$ and if $f'(c)$ exists. We will only prove that $f'(c) = 0$ in the case that f has a local maximum at c as the proof when f has a local minimum at c is similar.

Since f has a local maximum at c, there exists an open interval $J \subseteq I$ such that $c \in J$ and $f(x) \leq f(c)$ for all $x \in J$. If $x \in J$ and $x > c$, then

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

as both the numerator and denominator are positive. Therefore, as J is an open interval containing c,

$$f'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \geq 0.$$

Similarly, if $x \in J$ and $x < c$, then

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

as the numerator is positive whereas the denominator is negative. Therefore, as J is an open interval containing c,

$$f'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \leq 0.$$

Hence the above inequalities show $0 \leq f'(c) \leq 0$ and thus $f'(c) = 0$ as desired. ∎

Of course, students learned in MATH 1300 various tests for determining whether a critical point c where $f'(c) = 0$ is a local maxima, minima, or neither. To demonstrate the validity of such tests, we need another theoretical tool.

5.3 The Mean Value Theorem

The main theoretical tool to obtain not only the extreme value differentiation tests and many other results in this chapter is our third and final piece of the Triforce (i.e. our third 'value' theorem). This final essential theorem is motivated by the following problem: Suppose one drove from York University to the University of Waterloo in 45 minutes. Note said drive is 66 miles. Thus this person averaged 88 miles per hour. How can we prove that at some point in the journey the driver hit 88 miles per hour (and thus saw some serious shit)?

Of course there are some natural assumptions we must make. For example, we must assume that distance is a function of time (no time travel) and that distance is a continuous function of time (no teleporters). Moreover, to be able to measure the speed of the vehicle at any instant in time, we must make the assumption that the distance function must be differentiable.

Our theorem (Theorem 5.3.3) will demonstrate that there must be a point where the instantaneous speed of the vehicle is the average (or mean) value of the speed; namely 88 miles per hour in this case. Consequently, at some point in the journey, the driver saw some serious shit.

To prove the said theorem, we start with a lemma that is easier to prove and will enable us to prove the desired theorem via a simple translation. Moreover, we introduce the following terminology to simplify our assumptions.

Definition 5.3.1. Let $a, b \in \mathbb{R}$ be such that $a < b$. A function $f : (a, b) \to \mathbb{R}$ is said to be *differentiable on* (a, b) if f is differentiable at each point in (a, b). A function $f : [a, b] \to \mathbb{R}$ is said to be *differentiable on* $[a, b]$ if f is continuous on $[a, b]$ and differentiable on (a, b).

Lemma 5.3.2 (Rolle's Theorem). *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$ and $f(a) = f(b) = 0$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ with the property that $f(a) = f(b) = 0$. To demonstrate that there exists a $c \in (a, b)$ so that $f'(c) = 0$, we divide the proof into three cases.

Case 1: $f(x) = 0$ for all $x \in (a, b)$. Clearly $f'(x) = 0$ for all $x \in (a, b)$ by Example 5.1.3. Hence any $c \in (a, b)$ has the property that $f'(c) = 0$.

Case 2: There is an $x_0 \in (a, b)$ with $f(x_0) > 0$. By the Extreme Value Theorem (Theorem 4.6.2) there exists an $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. Thus $f(c) \geq f(x_0) > 0$ so $c \neq a, b$. Since $f(c) \geq f(x)$ for all $x \in [a, b]$, c must be a local maximum of f on (a, b) and thus $f'(c) = 0$ by Proposition 5.2.2.

Case 3: There is an $x_0 \in (a, b)$ with $f(x_0) < 0$. By the Extreme Value Theorem 4.6.2 there exists an $c \in [a, b]$ such that $f(c) \leq f(x)$ for all $x \in [a, b]$. Thus $f(c) \leq f(x_0) < 0$ so $c \neq a, b$. Since $f(c) \leq f(x)$ for all $x \in [a, b]$, c must be a local minimum of f on (a, b) and thus $f'(c) = 0$ by Proposition 5.2.2.

As the above three cases cover all possibilities, the result follows. ∎

Using a simple translation trick together with Rolle's Theorem, we obtain the full Triforce.

Theorem 5.3.3 (Mean Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$. Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

for all $x \in [a,b]$. Clearly g is continuous on $[a,b]$ and differentiable on (a,b) with

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Notice that

$$g(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

and

$$g(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = 0.$$

Hence Rolle's Theorem (Lemma 5.3.2) implies that there exists a $c \in (a,b)$ such that $g'(c) = 0$. Therefore

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

so the result follows. ∎

Now that our Triforce is complete, any result we wish to prove is ours!

Kidding aside, the remainder of this chapter will consist of using the Mean Value Theorem to obtain additional powerful results.

Remark 5.3.4. Note that the conclusions of the Mean Value Theorem can fail even if f is not differentiable at a single point. Indeed if $f : [-1,1] \to \mathbb{R}$ is defined by $f(x) = |x|$, then f is continuous on $[-1,1]$ and differentiable on $(-1,1) \setminus \{0\}$. However there is no point $c \in (-1,1)$ such that $f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = \frac{1-1}{-2} = 0$.

5.4 The First Derivative Test

Returning to the discussion of maxima and minima of functions, our first application of the Mean Value Theorem is the proof of the elementary result from calculus that enables one to determine whether a critical point of a differentiable function is a local maxima or a local minima.

Theorem 5.4.1 (First Derivative Test). *Let $f : (a,b) \to \mathbb{R}$ be differentiable on (a,b). Suppose $c \in (a,b)$ has the property that there exists a $\delta > 0$ such that*

- *$f'(x)$ exists and $f'(x) > 0$ for all $x \in (c, c+\delta) \subseteq (a,b)$, and*

- *$f'(x)$ exists and $f'(x) < 0$ for all $x \in (c - \delta, c) \subseteq (a, b)$.*

Then f has a local minimum at c.

Similarly, suppose $c \in (a, b)$ has the property that there exists a $\delta > 0$ such that

- *$f'(x)$ exists and $f'(x) < 0$ for all $x \in (c, c + \delta) \subseteq (a, b)$, and*
- *$f'(x)$ exists and $f'(x) > 0$ for all $x \in (c - \delta, c) \subseteq (a, b)$.*

Then f has a local maximum at c.

Proof. Let $f : (a, b) \to \mathbb{R}$ be differentiable on (a, b). Assume $c \in (a, b)$ has the property that there exists a $\delta > 0$ such that

- $f'(x)$ exists and $f'(x) > 0$ for all $x \in (c, c + \delta)$, and
- $f'(x)$ exists and $f'(x) < 0$ for all $x \in (c - \delta, c)$.

To see that f has a local minimum at c, first let $x \in (c, c + \delta) \subseteq (a, b)$ be arbitrary. Since f is continuous on $[c, x]$ and differentiable on (c, x), the Mean Value Theorem (Theorem 5.3.3) implies there exists a $d \in (c, x)$ such that

$$f'(d) = \frac{f(x) - f(c)}{x - c}.$$

Since $d \in (c, c + \delta)$, we have by assumption that $f'(d) > 0$. Hence the above equation implies $f(x) > f(c)$ for all $x \in (c, c + \delta)$. Similarly, let $x \in (c - \delta, c) \subseteq (a, b)$ be arbitrary. Since f is continuous on $[x, c]$ and differentiable on (x, c), the Mean Value Theorem (Theorem 5.3.3) implies there exists a $d \in (x, c)$ such that

$$f'(d) = \frac{f(c) - f(x)}{c - x}.$$

Since $d \in (c - \delta, c)$, we have by assumption that $f'(d) < 0$. Hence the above equation implies $f(x) > f(c)$ for all $x \in (c - \delta, c)$. Therefore, f has a local minimum at c by definition.

The proof of the second portion of this result is similar to the first. ∎

5.5 The Inverse Function Theorem

For our next application of the Mean Value Theorem, we can return to our question of demonstrating the functions from Example 4.5.6 are increasing or decreasing on the intervals of definition and thus are invertible with continuous inverses. To do so, we will demonstrate a connection between increasing and decreasing differentiable functions and the values of their derivatives. We will also prove the Inverse Function Theorem which will show that the inverse of differentiable functions are differentiable and permit the computation of their derivatives. Although the proof of the Inverse Function Theorem does not require the Mean Value Theorem, it is included here being its most natural place in the course.

To use the Mean Value Theorem to determine when functions are increasing or decreasing via their derivatives is an easy task.

Theorem 5.5.1 (Increasing Function Theorem). *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is non-decreasing on $[a, b]$. Similarly, if $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$*

Proof. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. Assume $f'(x) \geq 0$ for all $x \in (a, b)$. To see that f is non-decreasing on $[a, b]$, let $x_1, x_2 \in [a, b]$ be such that $x_1 < x_2$. Since f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), the Mean Value Theorem (Theorem 5.3.3) implies that there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

By the assumptions on f, $f'(c) \geq 0$. Therefore, since $x_1 < x_2$, we must have that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0$$

Hence f must be non-decreasing on $[a, b]$ as desired.

The proof in the case that $f'(x) > 0$ for all $x \in (a, b)$ follows by replacing '$\geq$' with '$>$' in the above proof. ∎

A similar proof shows the following.

Theorem 5.5.2 (Decreasing Function Theorem). *Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$. If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is non-increasing on $[a, b]$. Similarly, if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.*

Before we return to the functions in Example 4.5.6 to show the functions are indeed increasing or decreasing on their domains and thus define invertible functions with continuous inverses, we will demonstrate the following useful theorem which both implies the inverse functions are differentiable and enables the computation of their derivatives.

Theorem 5.5.3 (Inverse Function Theorem). *Let I be an interval, let $f : I \to \mathbb{R}$ be injective and continuous, and let $g : f(I) \to I$ be the inverse of f on its image. If $c \in I$ is not an endpoint of I and f is differentiable at c with $f'(c) \neq 0$, then g is differentiable at $f(c)$ and*

$$g'(f(c)) = \frac{1}{f'(c)}.$$

Proof. First, since f is injective and continuous, $f(I)$ is an interval and g is continuous on $f(I)$ by Corollary 4.5.5. Moreover, Proposition 4.5.2 implies f is increasing or decreasing on I so that $f(c)$ is not an endpoint of $f(I)$. Hence it makes sense to consider the derivative of g at $f(c)$.

To see that g is differentiable at $f(c)$ and that $g'(f(c)) = \frac{1}{f'(c)}$, we must show that

$$\lim_{x \to f(c)} \frac{g(x) - g(f(c))}{x - f(c)} = \frac{1}{f'(c)}.$$

To see this, we will invoke the Sequential Characterization of the Limit (Theorem 4.1.16).

Assume $(x_n)_{n\geq 1}$ is a sequence such that $x_n \in f(I) \setminus \{f(c)\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = f(c)$. Let $y_n = g(x_n) \in I$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} x_n = f(c)$ and since g is continuous at $f(c)$, we obtain that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} g(x_n) = g(f(c)) = c.$$

Since $f(y_n) = x_n \neq f(c)$ and since f is injective, $y_n \neq c$ for all $n \in \mathbb{N}$. Therefore

$$\frac{g(x_n) - g(f(c))}{x_n - f(c)} = \frac{y_n - c}{f(y_n) - f(c)} = \frac{1}{\frac{f(y_n)-f(c)}{y_n - c}}$$

for all $n \in \mathbb{N}$. Hence, since $f'(c)$ exists and $f'(c) \neq 0$, we have that

$$\lim_{n\to\infty} \frac{g(x_n) - g(f(c))}{x_n - f(c)} = \lim_{n\to\infty} \frac{1}{\frac{f(y_n)-f(c)}{y_n - c}} = \frac{1}{f'(c)}$$

by Theorem 4.1.19. Therefore, since $(x_n)_{n\geq 1}$ was arbitrary, we obtain that $g'(f(c))$ exists and equals $\frac{1}{f'(c)}$ by Sequential Characterization of the Limit (Theorem 4.1.16). ∎

With the Inverse Function Theorem in hand, we can return to Example 4.5.6 to demonstrate said functions are invertible, with continuous differentiable inverses. In particular, the following examples show the plethora of results we required to demonstrate these inverse functions exist, are continuous, and are differentiable!

Example 5.5.4. For each $n \in \mathbb{N}$, let $f_n : [0, \infty) \to [0, \infty)$ be defined by $f_n(x) = x^{2n}$ for all $x \in [0, \infty)$. Since f_n is a polynomial, f_n is continuous on $[0, \infty)$ and differentiable on $(0, \infty)$.

Notice by Example 5.1.4 that

$$f_n'(x) = 2nx^{2n-1} > 0$$

for all $x \in (0, \infty)$. Hence the Increasing Function Theorem (Theorem 5.5.1) implies that f_n is increasing on $[0, b]$ for all $b > 0$ and thus f_n is increasing on $[0, \infty)$. Therefore, since $f_n(0) = 0$ and $\lim_{x\to\infty} f_n(x) = \infty$, the Intermediate

Value Theorem implies that the range of f_n is $[0, \infty)$. Moreover f_n is injective by Proposition 4.5.2 and thus $f_n^{-1} : [0, \infty) \to [0, \infty)$ exists and is continuous by Corollary 4.5.5.

Since $f_n'(x) \neq 0$ for all $x \in (0, \infty)$, the Inverse Function Theorem (Theorem 5.5.3) implies that f_n^{-1} is differentiable on $(0, \infty)$ and

$$(f_n^{-1})'(c^{2n}) = \frac{1}{f_n'(c)} = \frac{1}{2nc^{2n-1}} = \frac{c}{2nc^{2n}}$$

for all $c \in (0, \infty)$. Therefore, by letting $x = c^{2n} = f_n(c)$, we see that

$$(f_n^{-1})'(x) = \frac{c}{2nc^{2n}} = \frac{f_n^{-1}(x)}{2nx}$$

for all $x \in (0, \infty)$.

For $x \in [0, \infty)$, we call $f_n^{-1}(x)$ the $2n^{\text{th}}$-*root of* x and write $f_n^{-1}(x) = \sqrt[2n]{x}$. Thus the derivative of the $2n^{\text{th}}$-root functions is

$$(f_n^{-1})'(x) = \frac{1}{2n}\frac{\sqrt[2n]{x}}{x} = \frac{1}{2n}\frac{\sqrt[2n]{x}}{\left(\sqrt[2n]{x}\right)^{2n}} = \frac{1}{2n}\frac{1}{\left(\sqrt[2n]{x}\right)^{2n-1}}$$

(just as one would expect from Calculus).

Example 5.5.5. For each $n \in \mathbb{N}$, let $g_n : \mathbb{R} \to \mathbb{R}$ be defined by $g_n(x) = x^{2n+1}$ for all $x \in \mathbb{R}$. Since g_n is a polynomial, g_n is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$.

Notice by Example 5.1.4 that

$$g_n'(x) = (2n+1)x^{2n}$$

for all $x \in \mathbb{R}$. Since $g_n'(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$, the Increasing Function Theorem (Theorem 5.5.1) implies that g_n is increasing on $[0, b]$ and $[-b, 0]$ for all $b > 0$ and thus g_n is increasing on $\mathbb{R}$. Therefore, since $\lim_{x\to\infty} g_n(x) = \infty$ and $\lim_{x\to-\infty} g_n(x) = -\infty$, the Intermediate Value Theorem implies that the range of g_n is $\mathbb{R}$. Moreover g_n is injective by Proposition 4.5.2 and thus $g_n^{-1} : \mathbb{R} \to \mathbb{R}$ exists and is continuous by Corollary 4.5.5.

Since $g_n'(x) \neq 0$ for all $x \in \mathbb{R} \setminus \{0\}$, the Inverse Function Theorem (Theorem 5.5.3) implies that g_n^{-1} is differentiable on $\mathbb{R} \setminus \{0\}$ and

$$(g_n^{-1})'(c^{2n+1}) = \frac{1}{g_n'(c)} = \frac{1}{(2n+1)c^{2n}} = \frac{c}{(2n+1)c^{2n+1}}$$

for all $c \in \mathbb{R} \setminus \{0\}$. Therefore, by letting $x = c^{2n+1} = g_n(c)$, we see that

$$(fg_n^{-1})'(x) = \frac{c}{(2n+1)c^{2n+1}} = \frac{g_n^{-1}(x)}{(2n+1)x}$$

for all $x \in \mathbb{R} \setminus \{0\}$.

For $x \in \mathbb{R}$, we call $g_n^{-1}(x)$ the $(2n+1)^{\text{st}}$-*root of* x and write $g_n^{-1}(x) = \sqrt[2n+1]{x}$. Thus the derivative of the $(2n+1)^{\text{st}}$-root functions is

$$(g_n^{-1})'(x) = \frac{1}{2n+1}\frac{\sqrt[2n+1]{x}}{x} = \frac{1}{2n+1}\frac{\sqrt[2n+1]{x}}{(\sqrt[2n+1]{x})^{2n+1}} = \frac{1}{2n+1}\frac{1}{(\sqrt[2n+1]{x})^{2n}}$$

(just as one would expect from Calculus).

Example 5.5.6. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = e^x$ for all $x \in \mathbb{R}$. In MATH 3001, it will be show that $e^x > 0$ for all $x \in \mathbb{R}$, $e^{-x} = \frac{1}{e^x}$ for all $x \in \mathbb{R}$, h is continuous, and h is differentiable on $\mathbb{R}$ with

$$h'(x) = e^x > 0$$

for all $x \in \mathbb{R}$.

By the Increasing Function Theorem (Theorem 5.5.1), h is increasing on $[-b, b]$ for all $b > 0$ and thus h is increasing on $\mathbb{R}$. Therefore, since it can be shown in MATH 3001 that $\lim_{x\to\infty} h(x) = \infty$ and since $e^{-x} = \frac{1}{e^x}$ for all $x \in \mathbb{R}$ implies that $\lim_{x\to-\infty} h(x) = 0$, the Intermediate Value Theorem implies that the range of h is $(0, \infty)$. Moreover h is injective by Proposition 4.5.2 and thus $h^{-1} : (0, \infty) \to (0, \infty)$ exists and is continuous by Corollary 4.5.5.

Since $h'(x) \neq 0$ for all $x \in \mathbb{R}$, the Inverse Function Theorem (Theorem 5.5.3) implies that h^{-1} is differentiable on $(0, \infty)$ and

$$(h^{-1})'(e^c) = \frac{1}{h'(c)} = \frac{1}{e^c}$$

for all $c \in \mathbb{R}$. Therefore, by letting $x = e^c$, we see that

$$(h^{-1})'(x) = \frac{1}{x}$$

for all $x \in (0, \infty)$.

For $x \in (0, \infty)$, we call $h^{-1}(x)$ the *natural logarithm of* x and write $h^{-1}(x) = \ln(x)$. Thus

$$(\ln)'(x) = \frac{1}{x}$$

(just as one would expect from Calculus).

Example 5.5.7. Let $c : [0, \pi] \to [-1, 1]$ be defined by $c(x) = \cos(x)$ for all $x \in [0, \pi]$. In MATH 3001, it will be show that c is differentiable on $[0, \pi]$ with

$$c'(x) = -\sin(x) < 0$$

for all $x \in (0, \pi)$.

By the Decreasing Function Theorem (Theorem 5.5.2), c is decreasing on $(0, \pi)$. Therefore, since $c(0) = 1$ and $c(\pi) = -1$, the Intermediate Value Theorem implies that the range of c is $[-1, 1]$. Moreover c is injective by Proposition 4.5.2 and thus $c^{-1} : [-1, 1] \to [0, \pi]$ exists and is continuous by Corollary 4.5.5.

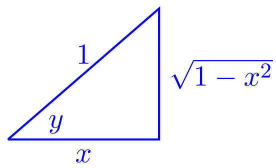
Since $c'(x) \neq 0$ for all $x \in (0, \pi)$, the Inverse Function Theorem (Theorem 5.5.3) implies that c^{-1} is differentiable on $(-1, 1)$ and

$$(c^{-1})'(\cos(y)) = \frac{1}{c'(y)} = \frac{1}{-\sin(y)}$$

for all $y \in (0, \pi)$. Therefore, by letting $x = \cos(y)$,

$$(c^{-1})'(x) = -\frac{1}{\sqrt{1 - x^2}}$$

where we have used the following triangle:



For $x \in [-1, 1]$, we call $c^{-1}(x)$ the *arccosine of x* and write $c^{-1}(x) = \arccos(x)$. Thus

$$(\arccos)'(x) = -\frac{1}{\sqrt{1 - x^2}}$$

(just as one would expect from Calculus).

Example 5.5.8. Let $s : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ be defined by $s(x) = \sin(x)$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In MATH 3001, it will be show that s is differentiable on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with

$$s'(x) = \cos(x) > 0$$

for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

By the Increasing Function Theorem (Theorem 5.5.1), s is increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, since $s\left(-\frac{\pi}{2}\right) = -1$ and $s\left(\frac{\pi}{2}\right) = 1$, the Intermediate Value Theorem implies that the range of s is $[-1, 1]$. Moreover s is injective by Proposition 4.5.2 and thus $s^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ exists and is continuous by Corollary 4.5.5.

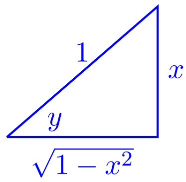
Since $s'(x) \neq 0$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the Inverse Function Theorem (Theorem 5.5.3) implies that s^{-1} is differentiable on $(-1, 1)$ and

$$(s^{-1})'(\sin(y)) = \frac{1}{s'(y)} = \frac{1}{\cos(y)}$$

for all $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, by letting $x = \sin(y)$,

$$(s^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$$

where we have used the following triangle:



For $x \in [-1,1]$, we call $s^{-1}(x)$ the *arcsine of* x and write $s^{-1}(x) = \arcsin(x)$. Thus

$$(\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}$$

(just as one would expect from Calculus).

Example 5.5.9. Let $t : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ be defined by $t(x) = \tan(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Since $t(x) = \frac{\sin(x)}{\cos(x)}$ and $\cos(x) \neq 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we see by the Quotient Rule (Proposition 5.1.16) that t is differentiable on $(-\frac{\pi}{2}, \frac{\pi}{2})$ with

$$t'(x) = \frac{\cos^2(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} > 0$$

for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

By the Increasing Function Theorem (Theorem 5.5.1), t is increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, since $\lim_{x\to\frac{\pi}{2}} t(x) = \infty$ and $\lim_{x\to-\frac{\pi}{2}} t(x) = -\infty$, the Intermediate Value Theorem implies that the range of t is $\mathbb{R}$. Moreover t is injective by Proposition 4.5.2 and thus $t^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ exists and is continuous by Corollary 4.5.5.

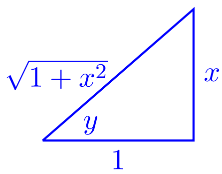
Since $t'(x) \neq 0$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the Inverse Function Theorem (Theorem 5.5.3) implies that t^{-1} is differentiable on $\mathbb{R}$ and

$$(t^{-1})'(\tan(y)) = \frac{1}{t'(y)} = \cos^2(y)$$

for all $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Therefore, by letting $x = \tan(y)$,

$$(t^{-1})'(x) = \frac{1}{1+x^2}$$

where we have used the following triangle:



For $x \in \mathbb{R}$, we call $t^{-1}(x)$ the *arctangent of* x and write $t^{-1}(x) = \arctan(x)$. Thus

$$(\arctan)'(x) = \frac{1}{1+x^2}$$

(just as one would expect from Calculus).

5.6 L'Hôpital's Rule

For our next application of the Mean Value Theorem (Theorem 5.3.3), we will establish one of the most important techniques in Calculus for computing limits of certain indeterminate forms: L'Hôpital's Rule. As students have seen L'Hôpital's Rule in previous calculus courses, this section will focus on its formal proof. To prove L'Hôpital's Rule, we need an enhancement of the Mean Value Theorem (Theorem 5.3.3).

Theorem 5.6.1 (Cauchy's Mean Value Theorem). *If $f, g : [a, b] \to \mathbb{R}$ are differentiable on $[a, b]$ with $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists a $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

(Note: When $g(x) = x$ for all $x \in [a, b]$, this is precisely the Mean Value Theorem.)

Proof. Similar to how the Mean Value Theorem (Theorem 5.3.3) was proved using Rolle's Theorem (Lemma 5.3.2) via the use of a particular function, Cauchy's Mean Value Theorem will be proved via Rolle's Theorem (Lemma 5.3.2) via the use of a particular function.

To begin, note by the Mean Value Theorem (Theorem 5.3.3) that there exists a $d \in (a, b)$ such that

$$g'(d) = \frac{g(b) - g(a)}{b - a}.$$

Therefore, since $g'(d) \neq 0$, we obtain that $g(b) - g(a) \neq 0$.

Define $h : [a, b] \to \mathbb{R}$ by

$$h(x) = \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)) - f(x) + f(a)$$

for all $x \in [a, b]$. Note that h makes sense as $g(b) - g(a) \neq 0$. Furthermore, since f and g are differentiable on $[a, b]$, we obtain that h is differentiable on $[a, b]$ with

$$h'(x) = \frac{f(b) - f(a)}{g(b) - g(a)}g'(x) - f'(x)$$

for all $x \in (a,b)$. Moreover, notice that

$$h(a) = \frac{f(b) - f(a)}{g(b) - g(a)}(g(a) - g(a)) - f(a) + f(a) = 0$$

whereas

$$h(b) = \frac{f(b) - f(a)}{g(b) - g(a)}(g(b)-g(a))-f(b)+f(a) = (f(b)-f(a))-f(b)+f(a) = 0.$$

Hence by Rolle's Theorem (Lemma 5.3.2) or, alternatively, by the Mean Value Theorem, there exists a $c \in (a,b)$ such that $h'(c) = 0$. Hence

$$0 = \frac{f(b) - f(a)}{g(b) - g(a)}g'(c) - f'(c).$$

Therefore, since $g'(c) \neq 0$, we obtain that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

as desired. ∎

Using Cauchy's Mean Value Theorem, we can present a formal proof of L'Hôspital's Rule (which is commonly believed to be first proved by Bernoulli).

Theorem 5.6.2 (L'Hôpital's Rule). *Let $a, b \in \mathbb{R}$ and let $f, g : (a,b) \to \mathbb{R}$ be differentiable on (a,b). Suppose $g'(x) \neq 0$ for all $x \in (a,b)$ and either*

i) $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = 0$*, or*

ii) $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = \pm\infty$.

Then the following hold:

a) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$*, then* $\lim_{x\to a+} \frac{f(x)}{g(x)} = L$.

b) If $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \pm\infty$*, then* $\lim_{x\to a+} \frac{f(x)}{g(x)} = \pm\infty$.

Similarly, the result holds with $a+$ exchanged with $b-$, ∞, or $-\infty$.

Proof. To begin the proof, we claim in all cases that there exists at most one point x in (a,b) such that $g(x) = 0$. To see this, notice for all $x_1, x_2 \in (a,b)$ with $x_1 < x_2$ that g is continuous on $[x_1, x_2]$ (by Theorem 5.1.7) and differentiable on (x_1, x_2). Hence the Mean Value Theorem (Theorem 5.3.3) implies there exists a $d \in (x_1, x_2)$ such that

$$g'(d) = \frac{g(x_2) - g(x_1)}{x_2 - x_1}.$$

As $g'(d) \neq 0$, we obtain that $g(x_2) - g(x_1) \neq 0$. As this holds for all $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, there exists at most one point, say γ, in (a, b) such that $g(\gamma) = 0$.

To begin proving this result, we will begin with parts (a) and (b) in the $x \to a+$ setting.

Proof of (a); $x \to a+$: To begin the proof of part (a), assume (i) or (ii) holds and assume $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = L$. Let $\epsilon > 0$ be arbitrary. By the definition of the limit, there exists a $b' \in (a, b)$ such that

$$\left|\frac{f'(x)}{g'(x)} - L\right| < \epsilon$$

for all $x \in (a, b')$. If γ is the unique zero of g, we may assume that $b' < \gamma$ by decreasing b' if necessary.

Let α and β be arbitrary numbers such that $a < \alpha < \beta < b'$. Since f and g are continuous on $[\alpha, \beta]$, differentiable on (α, β), and $g'(x) \neq 0$ for all $x \in (\alpha, \beta)$, Cauchy's Mean Value Theorem (Theorem 5.6.1) implies there exists a $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Hence, as $c_{\alpha,\beta} \in (\alpha, \beta) \subseteq (a, b')$, we obtain that

$$\left|\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} - L\right| = \left|\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} - L\right| < \epsilon$$

for all α and β such that $a < \alpha < \beta < b'$.

To proceed, we will now need to divide the proof depending on whether we have assumption (i) or assumption (ii)

Case 1: (i) holds. Since (i) holds, we know that $\lim_{\alpha\to a+} f(\alpha) = 0 = \lim_{\alpha\to a+} g(\alpha)$. Therefore, by fixing a $\beta \in (a, b')$ and taking the limit of $\alpha \in (a, \beta)$ as α tends to a, we obtain since $g(\beta) \neq 0$ that

$$\frac{f(\beta)}{g(\beta)} = \lim_{\alpha\to a+} \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Hence for all $\beta \in (a, b')$ we have that

$$\left|\frac{f(\beta)}{g(\beta)} - L\right| \leq \epsilon$$

Since $\epsilon > 0$ was arbitrary, we obtain that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = L$$

as desired.

Case 2: (ii) holds. Since (ii) holds, we know that $\lim_{x\to a+} g(x) = \pm\infty$. Since

$$\begin{aligned}\frac{f(\beta)}{g(\alpha)} - \frac{f(\alpha)}{g(\alpha)} &= \frac{1}{g(\alpha)}(f(\beta) - f(\alpha)) \\ &= \frac{1}{g(\alpha)}(g(\beta) - g(\alpha))\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} \\ &= \frac{g(\beta)}{g(\alpha)}\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} - \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}\end{aligned}$$

we obtain that

$$\frac{f(\alpha)}{g(\alpha)} = \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} + \frac{f(\beta)}{g(\alpha)} - \frac{g(\beta)}{g(\alpha)}\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}.$$

Hence

$$\begin{aligned}\left|\frac{f(\alpha)}{g(\alpha)} - L\right| &\leq \left|\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} - L\right| + \left|\frac{f(\beta)}{g(\alpha)}\right| + \left|\frac{g(\beta)}{g(\alpha)}\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}\right| \\ &\leq \epsilon + \left|\frac{f(\beta)}{g(\alpha)}\right| + \left|\frac{g(\beta)}{g(\alpha)}\right|(L+\epsilon)\end{aligned}$$

for all $\beta \in (a, b')$ and for all $\alpha \in (a, \beta)$. However, for any fixed β, we know since $\lim_{\alpha\to a+} g(\alpha) = \pm\infty$ that

$$\lim_{\alpha\to a+} \left|\frac{f(\beta)}{g(\alpha)}\right| + \left|\frac{g(\beta)}{g(\alpha)}\right|(L+\epsilon) = 0.$$

Hence there exists a $\delta > 0$ such that if $a < \alpha < a + \delta$, then

$$0 \leq \left|\frac{f(\beta)}{g(\alpha)}\right| + \left|\frac{g(\beta)}{g(\alpha)}\right|(L+\epsilon) < \epsilon.$$

Therefore, if $a < \alpha < a + \delta$, we have that

$$\left|\frac{f(\alpha)}{g(\alpha)} - L\right| < 2\epsilon.$$

Since $\epsilon > 0$ was arbitrary, we obtain that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = L$$

as desired. (Note the assumption that $\lim_{x\to a+} f(x) = \infty$ was not needed for this case to work.)

Hence, as the above two cases cover all possible cases, part (a) has been demonstrated in the case that $x \to a+$.

Proof of (b); $x \to a+$: To begin the proof of part (b), assume (i) or (ii) holds and assume $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = \infty$ as the proof when the limit is $-\infty$ is similar. Let $M > 0$ be arbitrary. By the definition of the limit, there exists a $b' \in (a, b)$ such that

$$\frac{f'(x)}{g'(x)} > M$$

for all $x \in (a, b')$. If γ is the unique zero of g, we may assume that $b' < \gamma$ by decreasing b' if necessary.

Let α and β be arbitrary numbers such that $a < \alpha < \beta < b'$. Since f and g are continuous on $[\alpha, \beta]$, differentiable on (α, β), and $g'(x) \neq 0$ for all $x \in (\alpha, \beta)$, Cauchy's Mean Value Theorem (Theorem 5.6.1) implies there exists a $c_{\alpha,\beta} \in (\alpha, \beta)$ such that

$$\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Hence, as $c_{\alpha,\beta} \in (\alpha, \beta) \subseteq (a, b')$, we obtain that

$$\frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} > M.$$

To proceed, we will now need to divide the proof depending on whether we have assumption (i) or assumption (ii)

Case 1: (i) holds. Since (i) holds, we know that $\lim_{\alpha\to a+} f(\alpha) = 0 = \lim_{\alpha\to a+} g(\alpha)$. Therefore, by fixing a $\beta \in (a, b')$ and taking the limit of $\alpha \in (a, \beta)$ as α tends to a, we obtain since $g(\beta) \neq 0$ that

$$\frac{f(\beta)}{g(\beta)} = \lim_{\alpha\to a+} \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}.$$

Hence, for all $\beta \in (a, b')$, we have that

$$\frac{f(\beta)}{g(\beta)} \geq M.$$

Since $M > 0$ was arbitrary, we obtain that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$$

as desired.

Case 2: (ii) holds. Since (ii) holds, we know that

$$\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = \pm\infty.$$

In addition, we may repeat the computation in part (a) to obtain that

$$\frac{f(\alpha)}{g(\alpha)} = \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})} + \frac{f(\beta)}{g(\alpha)} - \frac{g(\beta)}{g(\alpha)} \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}.$$

Thus

$$\frac{f(\alpha)}{g(\alpha)} = \frac{f(\beta)}{g(\alpha)} + \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}\left(1 - \frac{g(\beta)}{g(\alpha)}\right).$$

Notice that as $\lim_{x\to a+} f(x) = \lim_{x\to a+} g(x) = \pm\infty$, we obtain there exists a $\delta_1 > 0$ such that

$$\frac{f(\beta)}{g(\alpha)} > 0 \qquad \text{and} \qquad \frac{g(\beta)}{g(\alpha)} > 0$$

whenever $a < \alpha < \beta < a+\delta_1$ (i.e. $f(\beta)$, $g(\alpha)$, and $g(\beta)$ must eventually all by the same sign). Notice for all fixed $\beta < a+\delta_1$ that since $\lim_{x\to a+} g(x) = \pm\infty$ we have that

$$\lim_{\alpha\to a+} \frac{f(\beta)}{g(\alpha)} = 0 \qquad \text{and} \qquad \lim_{\alpha\to a+} \left(1 - \frac{g(\beta)}{g(\alpha)}\right) = 1.$$

Hence there exists a $0 < \delta < \delta_1$ such that if $a < \alpha < a + \delta$, then

$$\frac{f(\beta)}{g(\alpha)} \geq -\frac{M}{4} \qquad \text{and} \qquad \left(1 - \frac{g(\beta)}{g(\alpha)}\right) > \frac{1}{2}.$$

Therefore, if $a < \alpha < a + \delta$, we have that

$$\begin{aligned} \frac{f(\alpha)}{g(\alpha)} &= \frac{f(\beta)}{g(\alpha)} + \frac{f'(c_{\alpha,\beta})}{g'(c_{\alpha,\beta})}\left(1 - \frac{g(\beta)}{g(\alpha)}\right) \\ &\geq -\frac{M}{4} + M\frac{1}{2} = \frac{M}{4}. \end{aligned}$$

Since $M > 0$ was arbitrary, we obtain that

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$$

as desired.

The proof is nearly identical when we replace $a+$ with $b-$ (change the role of α and β). To demonstrate what occurs when $a+$ is replaced with $-\infty$ and $b-$ is replaced with ∞, we will demonstrate how the proof of part (a), case (i) can be adapted when when we replace $a+$ with $-\infty$ as the adaptation to all other parts/cases are similar.

Assume $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to-\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Let $h(x) = f\left(\frac{1}{x}\right)$ and $k(x) = g\left(\frac{1}{x}\right)$ for all $x \in (-\infty, b)$. Notice

$$\lim_{x\to 0-} h(x) = \lim_{x\to-\infty} f(x) = 0 = \lim_{x\to-\infty} g(x) = \lim_{x\to 0-} k(x).$$

Also notice that h and k are differentiable on $(-\infty, b) \setminus \{0\}$ via the Chain Rule (Theorem 5.1.20) with

$$h'(x) = -\frac{1}{x^2} f'\left(\frac{1}{x}\right) \qquad \text{and} \qquad k'(x) = -\frac{1}{x^2} g'\left(\frac{1}{x}\right).$$

Therefore

$$\lim_{x\to 0-}\frac{h'(x)}{k'(x)} = \lim_{x\to 0-}\frac{f'\left(\frac{1}{x}\right)}{g'\left(\frac{1}{x}\right)} = \lim_{x\to -\infty}\frac{f'(x)}{g'(x)} = L.$$

Hence, by our previous proofs, we obtain that

$$\lim_{x\to 0-}\frac{h(x)}{k(x)} = L.$$

Since the existence of the above limit implies $\lim_{x\to-\infty}\frac{f(x)}{g(x)}$ exists and since $\lim_{x\to-\infty}\frac{f(x)}{g(x)} = \lim_{x\to 0-}\frac{h(x)}{k(x)}$, the result follows. ∎

To demonstrate some uses of L'Hôspital's rule, consider the following examples.

Example 5.6.3. Using L'Hôspital's rule we can compute $\lim_{x\to\infty}\frac{x}{e^x}$. Indeed, since $(x)' = 1$ and $(e^x)' = e^x$, and since

$$\lim_{x\to\infty}\frac{1}{e^x} = 0,$$

we obtain by L'Hôspital's rule that $\lim_{x\to\infty}\frac{x}{e^x} = 0$. Similarly, using induction, we can use L'Hôspital's rule to show that

$$\lim_{x\to\infty}\frac{x^n}{e^x} = 0$$

for all $n \in \mathbb{N}$; that is, e^x grows substantially faster than any power of x!

Example 5.6.4. Using L'Hôspital's rule, we can compute

$$\lim_{x\to 0+} x\ln(x).$$

Although it does not appear that we may apply L'Hôspital's rule, notice that $x\ln(x) = \frac{\ln(x)}{\frac{1}{x}}$. Therefore, since $\lim_{x\to 0+} -\ln(x) = \infty$ whereas $\lim_{x\to 0+}\frac{1}{x} = \infty$, the hypotheses of L'Hôspital's rule may apply to compute $\lim_{x\to 0+}\frac{-\ln(x)}{\frac{1}{x}}$, so, upto multiplying by -1, we may be able to compute the desired limit. Since $(\ln(x))' = \frac{1}{x}$ and $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, and since

$$\lim_{x\to 0+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0+} -x = 0$$

we obtain that $\lim_{x\to 0+} x\ln(x) = 0$ by L'Hôspital's rule.

Example 5.6.5. Using similar techniques, we may compute

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

First, note that for $a, x > 0$, a natural way to define a^x is by $a^x = e^{x\ln(a)}$. If so, notice that $\ln(a^x) = \ln(e^{x\ln(a)}) = x\ln(a)$ by definition. Moreover

$$a^x a^y = e^{x\ln(a)}e^{y\ln(a)} = e^{x\ln(a)+y\ln(a)} = e^{(x+y)\ln(a)} = a^{x+y}$$

where the second equality comes from properties of the exponential function that will be developed in MATH 3001.

To compute the desired limit, we will instead first compute

$$\lim_{x\to\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$$

Notice that $\lim_{x\to\infty}\frac{1}{x} = 0$ and $\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right) = \ln(1) = 0$. Furthermore, since $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$ and $\left(\ln\left(1+\frac{1}{x}\right)\right)' = \frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)$, and since

$$\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1,$$

we obtain that $\lim_{x\to\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right) = 1$ by L'Hôspital's rule. Therefore, as e^x is continuous,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{1}{x}\right)^x\right)} = e^1 = e.$$

Hence, using the sequential definition of limits, we obtain the well-known limit

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

5.7 Taylor's Theorem

Another application of the Mean Value Theorem is the ability to approximate a differentiable function f pointwise using polynomials. To go beyond a first-order (i.e. linear) approximation, we will need more than just one derivative of f.

Definition 5.7.1. Let I be an open interval and let $f : I \to \mathbb{R}$ be differentiable. If f' is differentiable at $\alpha \in I$, the derivative of f' at α is called *the second derivative of* f and is denoted $f''(\alpha)$. In particular

$$f''(\alpha) = \lim_{x\to\alpha}\frac{f'(x) - f'(\alpha)}{x - \alpha}.$$

In general, for any $n \in \mathbb{N}$, the $(n+1)^{\text{st}}$-derivative of f is

$$f^{(n+1)}(\alpha) = \lim_{x\to\alpha} \frac{f^{(n)}(x) - f^{(n)}(\alpha)}{x-\alpha}$$

provided $f^{(n)}$ exists on an open interval containing α and the above limit exists. For convenience, $f^{(0)} = f$.

When a functions f is has $n+1$ derivatives at a point, our goal is to approximate f with the following polynomials.

Definition 5.7.2. Assuming that f is n-times differentiable at α (which means it is $(n-1)$-times differentiable in an open interval containing α), the *n^{th}-degree Taylor polynomial of f centred at α* is

$$P_{f,\alpha,n}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k.$$

Example 5.7.3. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$ for all $x \in \mathbb{R}$, then $P_{f,0,n}(x) = x^2$ for all $n \geq 2$.

Example 5.7.4. If $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = e^x$ for all $x \in \mathbb{R}$, then

$$P_{f,0,n}(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k,$$

which are the polynomials that define e^x up to a limit in Remark 4.2.8.

The following shows the use of the Taylor polynomials: provided x is close to α and the $(n+1)^{\text{st}}$ derivative of f is not too large, then $f(x)$ is almost $P_{f,\alpha,n}(x)$.

Theorem 5.7.5 (Taylor's Theorem). *Let I be an open interval, let $\alpha \in I$, and let $f : I \to \mathbb{R}$ be $n+1$ times differentiable on I. If $x \in I \setminus \{\alpha\}$, then there exists a $c_x \in (\alpha, x) \cup (x, \alpha)$ such that*

$$f(x) = P_{f,\alpha,n}(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-\alpha)^{n+1}.$$

Proof. To prove this result, we will again follow the proof of the Mean Value Theorem (Theorem 5.3.3) by constructing a specific function and then use Rolle's Theorem (Lemma 5.3.2) to obtain the desired result

To begin, fix $x \in I \setminus \{\alpha\}$. Consider the function $g : I \to \mathbb{R}$ defined by

$$g(t) = f(x) - f(t) - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

for all $t \in \mathbb{R}$. Notice that

$$g(\alpha) = f(x) - P_{f,\alpha,n}(x) \qquad \text{and} \qquad g(x) = 0.$$

Moreover, since f is $(n+1)$-times differentiable on I, we see that g is continuous and differentiable on I. Since x is fixed, differentiating with respect to t yields

$$\frac{1}{k}!f^{(k)}(t)(x-t)^k \longmapsto_{\frac{\mathrm{d}}{\mathrm{d}t}} \frac{1}{k!}f^{(k+1)}(t)(x-t)^k + \frac{1}{(k-1)!}f^{(k)}(t)(x-t)^{k-1}$$

for all $0 \leq k \leq n$. Therefore

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Unfortunately, g is not the function we are looking for. The function we are looking for is $h : I \to \mathbb{R}$ which is defined by

$$h(t) = g(t) - \left(\frac{x-t}{x-\alpha}\right)^{n+1} g(\alpha)$$

for all $t \in I$. Notice that

$$h(\alpha) = g(\alpha) - g(\alpha) = 0 \qquad \text{and} \qquad h(x) = g(x) - 0 = 0.$$

Since g is differentiable on I, h is continuous on $[a,x] \cup [x,a]$ and differentiable on $(a,x) \cup (x,a)$. Hence Rolle's Theorem (Lemma 5.3.2) implies that there exists a $c \in (a,x) \cup (x,a)$ such that $h'(c) = 0$. Since

$$h'(t) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)\frac{-1}{x-\alpha}\left(\frac{x-t}{x-\alpha}\right)^n g(\alpha),$$

we obtain that

$$\frac{f^{(n+1)}(c)}{n!}(x-c)^n = (n+1)\frac{1}{x-\alpha}\left(\frac{x-c}{x-\alpha}\right)^n g(\alpha).$$

Therefore, since $c \neq x$, we obtain that

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x-\alpha)^{n+1} = g(\alpha) = f(x) - P_{f,\alpha,n}(x)$$

as desired. ∎

Remark 5.7.6. The most important use of Taylor's Theorem is when one knows bounds for $f^{(n+1)}(c_x)$. Indeed, if one knows that $|f^{(n+1)}(c)| \leq M$ for all $c \in (\alpha-\delta, \alpha+\delta)$ for some $M > 0$, then we have that

$$|f(x) - P_{f,\alpha,n}(x)| \leq \frac{M}{(n+1)!}(x-\alpha)^{n+1}$$

for all $x \in (\alpha-\delta, \alpha+\delta)$. Consequently, provided we can approximate M well, we can approximate $f(x)$ with $P_{f,\alpha,n}(x)$ on this interval! This can be quite useful as dealing with polynomials is substantially easier than dealing with an arbitrary function.

5.8 Anti-Derivatives

For our final application of the Mean Value Theorem, we will demonstrate that functions with the same derivative on an open interval must differ by a constant. To obtain this result, we begin with a lemma.

Corollary 5.8.1. *If I is an open interval and if $f : I \to \mathbb{R}$ is differentiable on I with $f'(x) = 0$ for all $x \in I$, then there exists an $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ for all $x \in I$.*

Proof. Let $a, b \in I$ be arbitrary points such that $a < b$. Since f is continuous on $[a, b]$ (by Theorem 5.1.7) and differentiable on (a, b), the Mean Value Theorem (Theorem 5.3.3) implies there exists a $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

However, since $f'(x) = 0$ for all $x \in I$, we see that $f'(c) = 0$ and thus $f(b) = f(a)$.

Fix a point $x_0 \in I$ and let $\alpha = f(x_0)$. If $x \in I$ and $x \neq x_0$, then either $x > x_0$ or $x < x_0$. In either case the above shows that $f(x) = f(x_0) = \alpha$. Hence $f(x) = \alpha$ for all $x \in I$. ∎

Corollary 5.8.2. *If I is an open interval and $f, g : I \to \mathbb{R}$ are differentiable on I with $f'(x) = g'(x)$ for all $x \in I$, then there exists an $\alpha \in \mathbb{R}$ such that $f(x) = g(x) + \alpha$ for all $x \in I$.*

Proof. Let $h : I \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$. Then h is differentiable on I and

$$h'(x) = f'(x) - g'(x) = 0$$

for all $x \in I$. Hence there exists an $\alpha \in \mathbb{R}$ such that $h(x) = \alpha$ for all $x \in I$. Hence $f(x) = g(x) + \alpha$ for all $x \in I$. ∎

Based on the above, we make the following definition.

Definition 5.8.3. Let I be an open interval and let $f : I \to \mathbb{R}$. A function $F : I \to \mathbb{R}$ is said to be an *anti-derivative* of f on I if F is differentiable on I and $F'(x) = f(x)$ for all $x \in I$.

Thus Corollary 5.8.2 implies that if F is an anti-derivative of f, then all anti-derivatives of f are of the form $F(x) + c$ for some fixed constant $c \in \mathbb{R}$. Anti-derivative are important tools for our next chapter.

Chapter 6

Integration

For our final chapter, we will study what will be shown to be the opposite of differentiation; namely integration. Integration has a wide variety of uses in calculus as it allows the computation of the area under a curve and permits the averaging of the values obtained by a function over an integral. However, as students have taken MATH 1310 and know how to integrate, the purpose of this chapter is to formal defined the Riemann integral, develop and prove the basic properties of the Riemann integral, and demonstrate and prove the connections between differentiation and integration through the Fundamental Theorems of Calculus (Theorems 6.2.2 and 6.2.4) to further enhance students' repertoire of analysis arguments.

6.1 The Riemann Integral

The formal definition of the Riemann integral is modelled on trying to approximate the area under the graph of a function. The idea of approximating this area is to divide up the interval one wants to integrate over into small bits and approximate the area under the graph via rectangles. Thus we must make such constructions formal. Once this is done, we must decide whether or not these approximations are good approximations to the area. If they are, the resulting limit will be the Riemann integral.

6.1.1 Partitions and Riemann Sums

In order to 'divide up the interval into small bits', we will use the following notion.

Definition 6.1.1. A *partition* of a closed interval $[a, b]$ is a finite list of real numbers $\{t_k\}_{k=0}^n$ such that

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

Eventually, we will want to ensure that $|t_k - t_{k-1}|$ is small for all k in order to obtain better and better approximations to the area under a graph. To obtain a lower bound for the area under a graph, we can choose our approximating rectangles to have the largest possible height while remaining completely under the graph. This leads us to the following notion.

Definition 6.1.2. Let $\mathcal{P} = \{t_k\}_{k=0}^n$ be a partition of $[a, b]$ and let $f : [a, b] \to \mathbb{R}$ be bounded. The *lower Riemann sum* of f associated to $\mathcal{P}$, denoted $L(f, \mathcal{P})$, is

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} m_k(t_k - t_{k-1})$$

where, for all $k \in \{1, \ldots, n\}$,

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Example 6.1.3. If $f : [0, 1] \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in [0, 1]$ and if $\mathcal{P} = \{t_k\}_{k=0}^n$ is a partition of $[0, 1]$, it is easy to see that

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} t_{k-1}(t_k - t_{k-1})$$

as f obtains its minimum on $[t_{k-1}, t_k]$ at t_{k-1}.

If it so happens that $t_k = \frac{k}{n}$ for all $k \in \{0, 1, \ldots, n\}$, we see that

$$\begin{aligned}
L(f, \mathcal{P}) &= \sum_{k=1}^{n} \frac{k-1}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\
&= \sum_{k=1}^{n} \frac{1}{n^2}(k-1) \\
&= \frac{1}{n^2}\left(\sum_{j=1}^{n-1} j\right) \\
&= \frac{1}{n^2}\frac{n(n-1)}{2} = \frac{1-\frac{1}{n}}{2}
\end{aligned}$$

where the fact that $\sum_{j=1}^{n-1} j = \frac{n(n-1)}{2}$ follows by an induction argument. Clearly, as n tends to infinity, $L(f, \mathcal{P})$ tends to $\frac{1}{2}$ for this particular partitions, which happens to be the area under the graph of f on $[0, 1]$.

Although lower Riemann sums accurately estimate the area under the graph of the function in the previous example, perhaps we also need an upper bound for the area under the graph. By choose our approximating rectangles to have the smallest possible height while remaining completely above the graph, we obtain the following notion.

Definition 6.1.4. Let $\mathcal{P} = \{t_k\}_{k=0}^n$ be a partition of $[a,b]$ and let $f : [a,b] \to \mathbb{R}$ be bounded. The *upper Riemann sum* of f associated to $\mathcal{P}$, denoted $U(f,\mathcal{P})$, is

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} M_k(t_k - t_{k-1})$$

where, for all $k \in \{1,\ldots,n\}$,

$$M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Example 6.1.5. If $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = x$ for all $x \in [0,1]$ and if $\mathcal{P} = \{t_k\}_{k=0}^n$ is a partition of $[0,1]$, it is easy to see that

$$U(f,\mathcal{P}) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})$$

as f obtains its maximum on $[t_{k-1}, t_k]$ at t_k.

If it so happens that $t_k = \frac{k}{n}$ for all $k \in \{0,1,\ldots,n\}$, we see that

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=1}^{n} \frac{k}{n}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{n^2} k \\ &= \frac{1}{n^2}\left(\sum_{k=1}^{n} k\right) \\ &= \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1+\frac{1}{n}}{2} \end{aligned}$$

where the fact that $\sum_{k=1}^n k = \frac{n(n+1)}{2}$ follows by an induction argument. Clearly, as n tends to infinity, $U(f,\mathcal{P})$ tends to $\frac{1}{2}$ for this particular partitions, which happens to be the area under the graph of f on $[0,1]$.

Although we have been able to approximate the area under the graph of $f(x) = x$ using upper and lower Riemann sums, how do we know whether we can accurate do so for other functions? To analyze this question, we must first decide whether we can compare the upper and lower Riemann sums of a function. Clearly we have that $L(f,\mathcal{P}) \leq U(f,\mathcal{P})$ for any bounded function $f : [a,b] \to \mathbb{R}$ and any partition $\mathcal{P}$ of $[a,b]$. However, if $\mathcal{Q}$ is another partition of $[a,b]$, is it the case that $L(f,\mathcal{Q}) \leq U(f,\mathcal{P})$? Of course our intuition using 'areas under a graph' says this should be so, but how do we prove it?

To answer the above question and provide some 'sequence-like' structure to partitions, we define an ordering on the set of partitions.

Definition 6.1.6. Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a, b]$. It is said that $\mathcal{Q}$ is a *refinement* of $\mathcal{P}$, denoted $\mathcal{P} \leq \mathcal{Q}$, if $\mathcal{P} \subseteq \mathcal{Q}$; that is $\mathcal{Q}$ has all of the points that $\mathcal{P}$ has, and possibly more.

It is not difficult to check that refinement defines a partial ordering (Definition 1.2.8) on the set of all partitions of $[a, b]$ (see Example 1.2.10). Furthermore, the following says that if $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then we should have better upper and lower bounds for the area under the graph of a function if we use $\mathcal{Q}$ instead of $\mathcal{P}$.

Lemma 6.1.7. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $[a, b]$ and let $f : [a, b] \to \mathbb{R}$ be bounded. If $\mathcal{Q}$ is a refinement of $\mathcal{P}$, then*

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P}).$$

Proof. Note the inequality $L(f, \mathcal{Q}) \leq U(f, \mathcal{Q})$ is clear. Thus it remains only to show that $L(f, \mathcal{P}) \leq L(f, \mathcal{Q})$ and $U(f, \mathcal{Q}) \leq U(f, \mathcal{P})$. Write $\mathcal{P} = \{t_k\}_{k=0}^n$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

To show the desired inequalities, we will first show that adding a single point to $\mathcal{P}$ does not decrease the lower Riemann sum and does not increase the upper Riemann sum. As there are only a finite number of points one needs to add to $\mathcal{P}$ to obtain $\mathcal{Q}$, the proof will follow.

To implement the above strategy, assume $\mathcal{Q} = \mathcal{P} \cup \{t'\}$ where $t' \in [a, b]$ is such that $t_{q-1} < t' < t_q$ for some $q \in \{1, \ldots, n\}$. For all $k \in \{1, \ldots, n\}$, let

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Therefore

$$L(f, \mathcal{P}) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}) \qquad \text{and} \qquad U(f, \mathcal{P}) = \sum_{k=1}^{n} M_k(t_k - t_{k-1}).$$

Moreover, if we define

$$\begin{aligned} m'_q &= \inf\{f(x) \mid x \in [t_{q-1}, t']\}, \\ m''_q &= \inf\{f(x) \mid x \in [t', t_q]\}, \\ M'_q &= \sup\{f(x) \mid x \in [t_{q-1}, t']\}, \text{ and} \\ M''_q &= \sup\{f(x) \mid x \in [t', t_q]\}, \end{aligned}$$

then we easily see that $m_q \leq m'_q, m''_q$, that $M'_q, M''_q \leq M_q$, and that

$$\begin{aligned} L(f, \mathcal{Q}) &= m'_q(t' - t_{q-1}) + m''_q(t_q - t') + \sum_{\substack{k=1 \\ k \neq q}}^{n} m_k(t_k - t_{k-1}), \quad \text{and} \\ U(f, \mathcal{Q}) &= M'_q(t' - t_{q-1}) + M''_q(t_q - t') + \sum_{\substack{k=1 \\ k \neq q}}^{n} M_k(t_k - t_{k-1}). \end{aligned}$$

Therefore

$$\begin{aligned}L(f,\mathcal{Q})-L(f,\mathcal{P}) &= m_q'(t'-t_{q-1})+m_q''(t_q-t')-m_q(t_q-t_{q-1})\\ &\geq m_q(t'-t_{q-1})+m_q(t_q-t')-m_q(t_q-t_{q-1})=0\end{aligned}$$

so $L(f,\mathcal{P})\leq L(f,\mathcal{Q})$. Similarly

$$\begin{aligned}U(f,\mathcal{Q})-U(f,\mathcal{P}) &= M_q'(t'-t_{q-1})+M_q''(t_q-t')-M_q(t_q-t_{q-1})\\ &\leq M_q(t'-t_{q-1})+M_q(t_q-t')-M_q(t_q-t_{q-1})=0\end{aligned}$$

so $U(f,\mathcal{Q})\leq U(f,\mathcal{P})$. Hence the result follows when $\mathcal{Q}=\mathcal{P}\cup\{t'\}$.

To complete the proof, let $\mathcal{Q}$ be an arbitrary refinement of $\mathcal{P}$. Hence we can write $\mathcal{Q}=\mathcal{P}\cup\{t_k'\}_{k=1}^m$ for some $\{t_k'\}_{k=1}^m\subseteq(a,b)$. Thus, by adding a single point at a time, we obtain that

$$L(f,\mathcal{P})\leq L(f,\mathcal{P}\cup\{t_1'\})\leq L(f,\mathcal{P}\cup\{t_1',t_2'\})\leq\cdots\leq L(f,\mathcal{Q})$$

and

$$U(f,\mathcal{P})\geq U(f,\mathcal{P}\cup\{t_1'\})\geq U(f,\mathcal{P}\cup\{t_1',t_2'\})\geq\cdots\geq U(f,\mathcal{Q}),$$

which completes the proof. ∎

In order to answer our question of whether $L(f,\mathcal{Q})\leq U(f,\mathcal{P})$ for all partitions $\mathcal{P}$ and $\mathcal{Q}$, we can use Lemma 6.1.7 provided we have a partition that is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$: that is, there is a least upper bound of $\mathcal{P}$ and $\mathcal{Q}$.

Definition 6.1.8. Given two partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a,b]$, the *common refinement* of $\mathcal{P}$ and $\mathcal{Q}$ is the partition $\mathcal{P}\cup\mathcal{Q}$ of $[a,b]$.

Remark 6.1.9. Clearly, given two partitions $\mathcal{P}$ and $\mathcal{Q}$, $\mathcal{P}\cup\mathcal{Q}$ is a partition that is a refinement of both $\mathcal{P}$ and $\mathcal{Q}$. Consequently, if $f:[a,b]\to\mathbb{R}$ is bounded, then Lemma 6.1.7 implies that

$$L(f,\mathcal{P})\leq L(f,\mathcal{P}\cup\mathcal{Q})\leq U(f,\mathcal{P}\cup\mathcal{Q})\leq U(f,\mathcal{Q}).$$

Hence any lower bound for the area under a curve is smaller than any upper bound for the area under a curve.

Remark 6.1.10. Before moving on, we note the above shows that the partially ordered set of partitions of a closed interval $[a,b]$ is a *direct set* (that is, a partially ordered set with the property that if $\mathcal{P}$ and $\mathcal{Q}$ are elements of the partially ordered set, then there exists an element $\mathcal{R}$ such that $\mathcal{P}\leq\mathcal{R}$ and $\mathcal{Q}\leq\mathcal{R}$). A set of real numbers indexed by a direct set is called a *net* and one can discuss the convergences of nets in $\mathbb{R}$ as we did with sequences. It turns out nothing new is gained by using nets instead of sequences and we can avoid the discussion of nets in our discussion of integrals (although they exist in the background). However, in later courses (e.g. MATH 4081 - Topology) it is necessary to replace sequences with nets. Thus Riemann integration serves the additional purpose of giving students their first fundamental example of convergence of a net.

6.1.2 Definition of the Riemann Integral

In order to define the Riemann integral of a bounded function on a closed interval, we desire that the upper and lower Riemann sums both better and better approximate a single number. Using the above observations, we notice that if $f : [a, b] \to \mathbb{R}$ is bounded, then

$$\begin{aligned}\sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ \leq \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.\end{aligned}$$

Therefore, in order for there to be no reasonable discrepancy between our approximations, we will like an equality in the above inequality, in which case the value obtained should be the area under the graph. Unfortunately, this is not always the case.

Example 6.1.11. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

for all $x \in [0, 1]$. Since each open interval always contains at least one element from each of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ by Propositions 1.3.8 and 1.3.9, we easily see that $L(f, \mathcal{P}) = 0$ and $U(f, \mathcal{P}) = 1$ for all partitions $\mathcal{P}$ of $[0, 1]$. Hence

$$\begin{aligned}\sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [0, 1]\} \\ \neq \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [0, 1]\}.\end{aligned}$$

So what should be the area under the graph of this function?

Instead of focusing on correcting our notion of the integral to remove Example 6.1.11 (something that will be done in MATH 4012), we will instead simply just restrict our attention to the following type of functions.

Definition 6.1.12. Let $f : [a, b] \to \mathbb{R}$ be bounded. It is said that f is *Riemann integrable* on $[a, b]$ if

$$\begin{aligned}\sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ = \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.\end{aligned}$$

If f is Riemann integrable on $[a, b]$, the *Riemann integral of f from a to b*, denoted $\int_a^b f(x)\,dx$, is defined to be

$$\begin{aligned}\int_a^b f(x)\,dx &= \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ &= \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.\end{aligned}$$

Remark 6.1.13. Notice that if f is Riemann integrable on $[a,b]$, then

$$L(f,\mathcal{P}) \leq \int_b^a f(x)\,dx \leq U(f,\mathcal{P})$$

for every partition $\mathcal{P}$ of $[a,b]$ by the definition of the Riemann integral.

Clearly the function f in Example 6.1.11 is not Riemann integrable. However, which types of function are Riemann integrable and how can we compute the value of the integral? To illustrate the definition, we note the following simple examples (note if the first example did not work out the way it does, we clearly would not have a well-defined notion of area under a graph using Riemann integrals).

Example 6.1.14. Let $c \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = c$ for all $x \in [a,b]$. If $\mathcal{P} = \{t_k\}_{k=0}^n$ is a partition of $[a,b]$, we see that

$$L(f,\mathcal{P}) = U(f,\mathcal{P}) = \sum_{k=1}^n c(t_k - t_{k-1}) = c\sum_{k=1}^n t_k - t_{k-1} = c(t_n - t_0) = c(b-a).$$

Hence f is Riemann integrable and $\int_a^b f(x)\,dx = c(b-a)$. (Was there any doubt?)

Example 6.1.15. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x$ for all $x \in [0,1]$. For each $n \in \mathbb{N}$, note Example 6.1.3 demonstrates the existence of a partition $\mathcal{P}_n$ such that $L(f,\mathcal{P}_n) = \frac{1-\frac{1}{n}}{2}$. Hence

$$\sup\{L(f,\mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a,b]\} \geq \limsup_{n\to\infty} \frac{1-\frac{1}{n}}{2} = \frac{1}{2}.$$

Similarly, for each $n \in \mathbb{N}$, Example 6.1.5 demonstrates the existence of a partition $\mathcal{Q}_n$ such that $U(f,\mathcal{Q}_n) = \frac{1+\frac{1}{n}}{2}$. Hence

$$\inf\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a,b]\} \leq \liminf_{n\to\infty} \frac{1+\frac{1}{n}}{2} = \frac{1}{2}.$$

Therefore, since

$$\begin{aligned}\sup\{L(f,\mathcal{P}) \mid &\mathcal{P} \text{ a partition of } [a,b]\} \\ &\leq \inf\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a,b]\},\end{aligned}$$

the above computations show both the inf and sup must be $\frac{1}{2}$. Hence f is Riemann integrable on $[0,1]$ and $\int_0^1 x\,dx = \frac{1}{2}$.

Example 6.1.16. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in [0,1]$. We claim that f is Riemann integrable on $[0,1]$ and $\int_0^1 x^2\, dx = \frac{1}{3}$. To see this, let $n \in \mathbb{N}$ and let $\mathcal{P}_n = \{t_k\}_{k=1}^n$ be the partition of $[0,1]$ such that $t_k = \frac{k}{n}$ for all $n \in \mathbb{N}$. Then, by an induction argument to compute the value of the sums,

$$\begin{aligned} L(f,\mathcal{P}) &= \sum_{k=1}^{n} \frac{(k-1)^2}{n^2}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{n^3}(k-1)^2 \\ &= \frac{1}{n^3}\left(\sum_{j=1}^{n-1} j^2\right) \\ &= \frac{1}{n^3}\frac{(n-1)(n)(2(n-1)+1)}{6} = \frac{2n^3 - 3n^2 + n}{6n^3} \end{aligned}$$

and

$$\begin{aligned} U(f,\mathcal{P}) &= \sum_{k=1}^{n} \frac{k^2}{n^2}\left(\frac{k}{n} - \frac{k-1}{n}\right) \\ &= \sum_{k=1}^{n} \frac{1}{n^3}k^2 \\ &= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2\right) \\ &= \frac{1}{n^3}\frac{n(n+1)(2n+1)}{2} = \frac{2n^3 + 3n^2 + n}{6n^3}. \end{aligned}$$

Hence, since $\lim_{n\to\infty} \frac{2n^3-3n^2+1}{6n^3} = \lim_{n\to\infty} \frac{2n^3+3n^2+1}{6n^3} = \frac{1}{3}$, we see that

$$\begin{aligned} \frac{1}{3} &\leq \sup\{L(f,\mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a,b]\} \\ &\leq \inf\{U(f,\mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a,b]\} \leq \frac{1}{3}. \end{aligned}$$

Hence the inequalities must be equalities so f is Riemann integrable on $[0,1]$ by definition with $\int_0^1 x^2\, dx = \frac{1}{3}$

Note in the previous two examples, the functions were demonstrated to be Riemann integrable on $[0,1]$ via partitions $\mathcal{P}$ such that $L(f,\mathcal{P})$ and $U(f,\mathcal{P})$ were as closes as one would like. Coincidence, I think not!

Theorem 6.1.17. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a,b]$ such that*

$$0 \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \epsilon.$$

Proof. Note we must have that $0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P})$ for any partition $\mathcal{P}$ by earlier discussions.

First assume that f is Riemann integrable. Hence, with $I = \int_a^b f(x)\,dx$, we have by the definition of the integral that

$$\begin{aligned} I &= \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ &= \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. By the definition of the supremum, there exists a partition $\mathcal{P}_1$ of $[a, b]$ such that

$$I - \frac{\epsilon}{2} < L(f, \mathcal{P}_1).$$

Similarly, by the definition of the infimum, there exists a partition $\mathcal{P}_2$ of $[a, b]$ such that

$$U(f, \mathcal{P}_2) < I + \frac{\epsilon}{2}.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ which is a partition of $[a, b]$. Since $\mathcal{P}$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$, we obtain that

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq U(f, \mathcal{P}_2)$$

by Lemma 6.1.7. Hence

$$\begin{aligned} U(f, \mathcal{P}) - L(f, \mathcal{P}) &\leq U(f, \mathcal{P}_2) - L(f, \mathcal{P}_1) \\ &= (U(f, \mathcal{P}_2) - I) + (I - L(f, \mathcal{P}_1)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, this direction of the proof is complete.

For the other direction, assume for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

In particular, for each $n \in \mathbb{N}$ there exists a partition $\mathcal{P}_n$ of $[a, b]$ such that

$$0 \leq U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) < \frac{1}{n}.$$

Let

$$\begin{aligned} L &= \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \text{ and} \\ U &= \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}. \end{aligned}$$

Then $L, U \in \mathbb{R}$ are such that $L \leq U$. Moreover, for each $n \in \mathbb{N}$

$$0 \leq U - L \leq U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) < \frac{1}{n}.$$

Therefore, by the Archimedean Property (Theorem 1.3.7), it follows that $U = L$. Hence f is Riemann integrable on $[a, b]$ by definition. ∎

Remark 6.1.18. Using Theorem 6.1.17, there is an easier method for approximating the Riemann integral of a Riemann integrable function. Indeed suppose $\mathcal{P} = \{t_k\}_{k=0}^n$ is a partition of $[a, b]$ with

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$$

and let $f : [a, b] \to \mathbb{R}$ be bounded. For each k, let $x_k \in [t_{k-1}, t_k]$ and let

$$R(f, \mathcal{P}, \{x_k\}_{k=1}^n) = \sum_{k=1}^n f(x_k)(t_k - t_{k-1}).$$

The sum $R(f, \mathcal{P}, \{x_k\}_{k=1}^n)$ is called a *Riemann sum.*

Clearly

$$L(f, \mathcal{P}) \leq R(f, \mathcal{P}, \{x_k\}_{k=1}^n) \leq U(f, \mathcal{P})$$

by definitions. Hence, if f is Riemann integrable, we obtain via Theorem 6.1.17 that for any $\epsilon > 0$ there exists a partition $\mathcal{P}'$ of $[a, b]$ such that

$$L(f, \mathcal{P}') \leq \int_b^a f(x)\, dx \leq U(f, \mathcal{P}') \leq L(f, \mathcal{P})' + \epsilon$$

and thus

$$\left| \int_a^b f(x)\, dx - R(f, \mathcal{P}', \{x_k\}_{k=1}^n) \right| < \epsilon$$

for any choice of $\{x_k\}_{k=1}^n$. Consequently, if one knows that f is Riemann integrable, one may approximate $\int_a^b f(x)\, dx$ using Riemann sums oppose to lower/upper Riemann sums. This is occasionally useful as convenient choices of $\{x_n\}_{k=1}^n$ may make computing the sum much easier.

Of course, our next question is, “Which types of functions are Riemann integrable?”

6.1.3 Some Integrable Functions

If the theory of Riemann integration will be of use to us, we must have a wide variety of functions that are Riemann integrable. It is easy to show some functions are Riemann integrable.

Proposition 6.1.19. *If $f : [a, b] \to \mathbb{R}$ is monotonic and bounded, then f is Riemann integrable on $[a, b]$.*

Proof. Assume $f : [a, b] \to \mathbb{R}$ is monotone and bounded. In addition, we will assume that f is non-decreasing as the proof when f is non-increasing is similar.

Let $\epsilon > 0$. Since

$$\lim_{n\to\infty} \frac{1}{n}(b - a)(f(b) - f(a)) = 0,$$

there exists an $N \in \mathbb{N}$ such that

$$0 \leq \frac{1}{N}(b-a)(f(b)-f(a)) < \epsilon.$$

Let $\mathcal{P}_N = \{t_k\}_{k=0}^N$ be the partition such that

$$t_k = a + \frac{k}{N}(b-a)$$

for all $k \in \{0, \ldots, N\}$. Notice $t_k - t_{k-1} = \frac{1}{n}(b-a)$ for all k (and thus we call $\mathcal{P}_N$ the *uniform partition* of $[a, b]$ into N intervals). Since f is non-decreasing, if for all $k \in \{1, \ldots, N\}$

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\},$$

then

$$m_k = f(t_{k-1}) \qquad \text{and} \qquad M_k = f(t_k).$$

Hence

$$\begin{aligned}
0 &\leq U(f, \mathcal{P}_n) - L(f, \mathcal{P}_n) \\
&= \sum_{k=1}^N M_k(t_k - t_{k-1}) - \sum_{k=1}^N m_k(t_k - t_{k-1}) \\
&= \sum_{k=1}^N f(t_k)\frac{1}{N}(b-a) - \sum_{k=1}^N f(t_{k-1})\frac{1}{N}(b-a) \\
&= f(t_N)\frac{1}{N}(b-a) - f(t_0)\frac{1}{N}(b-a) \\
&= \frac{1}{N}(b-a)(f(b)-f(a)) < \epsilon.
\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, Theorem 6.1.17 implies that f is Riemann integrable on $[a, b]$. ∎

Of course, if continuous functions were not Riemann integrable, Riemann integration would be worthless to us. The fact that continuous functions on closed intervals are uniformly continuous is vital int he following proof.

Theorem 6.1.20. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$.*

Proof. Assume $f : [a, b] \to \mathbb{R}$ is continuous. Therefore f is bounded by the Extreme Value Theorem (Theorem 4.6.2). Hence it makes sense to discuss whether f is Riemann integrable.

In order to invoke Theorem 6.1.17 to show that f is Riemann integrable, let $\epsilon > 0$ be arbitrary. Since $f : [a, b] \to \mathbb{R}$ is continuous, f is uniformly continuous on $[a,b]$ by Theorem 4.3.9. Hence there exists a $\delta > 0$ such that if $x, y \in [a,b]$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \frac{\epsilon}{b-a}$.

By the Archimedean Property (Theorem 1.3.7), there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Let $\mathcal{P}$ be the uniform partition of $[a,b]$ into n intervals; that is, let $\mathcal{P} = \{t_k\}_{k=0}^n$ be the partition such that

$$t_k = a + \frac{k}{n}(b-a)$$

for all $k \in \{0, \ldots, n\}$. For all $k \in \{0, \ldots, n\}$, let

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Since $|t_k - t_{k-1}| = \frac{1}{n} < \delta$ so $|x - y| < \delta$ for all $x, y \in [t_{k-1}, t_k]$, it must be the case that $M_k - m_k = |M_k - m_k| \leq \frac{\epsilon}{b-a}$ for all $k \in \{1, \ldots, n\}$. Hence

$$\begin{aligned} 0 \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) &= \sum_{k=1}^{n} (M_k - m_k)(t_k - t_{k-1}) \\ &\leq \sum_{k=1}^{n} \frac{\epsilon}{b-a}(t_k - t_{k-1}) \\ &= \frac{\epsilon}{b-a} \sum_{k=1}^{n} t_k - t_{k-1} = \frac{\epsilon}{b-a}(b-a) = \epsilon. \end{aligned}$$

Thus, as $\epsilon > 0$ was arbitrary, f is Riemann integrable on $[a,b]$ by Theorem 6.1.17. ∎

Of course, not all functions we desire to integrate are continuous. However, many functions one sees and deals with in real-world applications are continuous at almost every point. In particular, the following shows that if our functions are piecewise continuous, then they are Riemann integrable.

Corollary 6.1.21. *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ except at a finite number of points and f is bounded on $[a,b]$, then f is Riemann integrable on $[a,b]$.*

Proof. Assume $f : [a,b] \to \mathbb{R}$ is continuous except at a finite number of points and $f([a,b])$ is bounded. Let $\{a_k\}_{k=0}^q$ contain all of the points for which f is not continuous at and be such that

$$a = a_0 < a_1 < a_2 < \cdots < a_q = b.$$

The idea of the proof is to construct a partition such that each interval of the partition contains at most one a_k, and if an interval of the partition contains an a_k, then its length is really small.

Let $\epsilon > 0$ be arbitrary. Since $f([a,b])$ is bounded, there exists a $K > 0$ such that $|f(x)| \leq K$ for all $x \in [a,b]$. Therefore, if

$$L = \sup\{f(x) - f(y) \mid x, y \in [a,b]\},$$

then $0 \leq L \leq 2K < \infty$.

Let

$$\delta = \frac{\epsilon}{2(q+1)(L+1)} > 0.$$

By taking a and b together with endpoints of intervals centred at each a_k of radius less than $\frac{\delta}{2}$, there exists a partition $\mathcal{P}' = \{t_k\}_{k=0}^{2q+1}$ with

$$a = t_0 < t_1 < t_2 < \cdots < t_{2q+1} = b$$

such that $t_{2k+1} - t_{2k} < \delta$ for all $k \in \{0, \ldots, q\}$ and $t_{2k} < a_k < t_{2k+1}$ for all $k \in \{1, \ldots, q-1\}$. For all $k \in \{1, \ldots, 2q+1\}$, let

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Thus $M_k - m_k \leq L$ for all $k \in \{1, \ldots, 2q+1\}$.

Since f is continuous on $[t_{2k-1}, t_{2k}]$ for all $k \in \{1, \ldots, q\}$, f is Riemann integrable on $[t_{2k-1}, t_{2k}]$ by Theorem 6.1.20. Hence, by the definition of Riemann integration, there exist partitions $\mathcal{P}_k$ of $[t_{2k-1}, t_{2k}]$ such that

$$0 \leq U(f, \mathcal{P}_k) - L(f, \mathcal{P}_k) < \frac{\epsilon}{2q}.$$

Let $\mathcal{P} = \mathcal{P}' \cup \left(\bigcup_{k=1}^{q} \mathcal{P}_k\right)$. Then $\mathcal{P}$ is a partition of $[a,b]$ such that

$$\begin{aligned} 0 &\leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \\ &= \sum_{k=1}^{q} (U(f,\mathcal{P}_k) - L(f,\mathcal{P}_k)) + \sum_{k=0}^{q} (M_{2k+1} - m_{2k+1})(t_{2k+1} - t_{2k}). \end{aligned}$$

(that is, on each $[t_{2k-1}, t_{2k}]$ the partition behaves like $\mathcal{P}_k$ and thus so do the sums, and the parts of the partition remaining are of the form $[t_{2k}, t_{2k+1}]$ each of which contains at most one a_j). Hence

$$\begin{aligned} 0 &\leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \\ &\leq \sum_{k=1}^{q} \frac{\epsilon}{2q} + \sum_{k=0}^{q} L\delta \\ &\leq \frac{\epsilon}{2} + (q+1)L\delta \\ &\leq \frac{\epsilon}{2} + (q+1)L\frac{\epsilon}{2(q+1)(L+1)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, as $\epsilon > 0$ was arbitrary, f is Riemann integrable on $[a,b]$ by Theorem 6.1.17. ∎

Using the similar ideas to those used to prove Corollary 6.1.21, it is possible to show that some truly bizarre functions are Riemann integrable.

Example 6.1.22. Let $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x = 0 \\ \frac{1}{b} & \text{if } x = \frac{a}{b} \text{ where } a \in \mathbb{Z} \setminus \{0\},\, b \in \mathbb{N}, \text{ and } \gcd(a, b) = 1 \end{cases}.$$

Clearly f is bounded.

We claim that f is Riemann integrable on $[0, 1]$. To see this, let $\epsilon > 0$ be arbitrary. By the Archimedian Property (Theorem 1.3.7), there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$.

By the definition of f, let $\{a_k\}_{k=0}^{q}$ be the finite set of $x \in [0, 1]$ such that $f(x) \leq \frac{1}{N}$ and

$$0 = a_0 < a_1 < a_2 < \cdots < a_q = 1.$$

Let

$$\delta = \frac{\epsilon}{2(q + 1)} > 0.$$

By taking 0 and 1 together with endpoints of intervals centred at each a_k of radius less than $\frac{\delta}{2}$, there exists a partition $\mathcal{P} = \{t_k\}_{k=0}^{2q+1}$ with

$$0 = t_0 < t_1 < t_2 < \cdots < t_{2q+1} = 1$$

such that $t_{2k+1} - t_{2k} < \delta$ for all $k \in \{0, \ldots, q\}$ and $t_{2k} < a_k < t_{2k+1}$ for all $k \in \{1, \ldots, q - 1\}$.

For all $k \in \{1, \ldots, 2q + 1\}$, let

$$m_k = \inf\{f(x) \mid x \in [t_{k-1}, t_k]\} \quad \text{and} \quad M_k = \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}.$$

Since $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$, we see that $M_k - m_k \leq 1$ for all $k \in \{1, \ldots, 2q+1\}$. Moreover, since $t_{2k} < a_k < t_{2k+1}$ for all $k \in \{1, \ldots, q-1\}$, we have that

$$M_{2k} - m_{2k} \leq \frac{1}{N} - 0 < \frac{\epsilon}{2}$$

for all $k \in \{1, \ldots, q\}$. Therefore

$$\begin{aligned}
0 &\leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \\
&= \sum_{k=1}^{q} (M_{2k} - m_{2k})(t_{2k} - t_{2k-1}) + \sum_{k=0}^{q} (M_{2k+1} - m_{2k+1})(t_{2k+1} - t_{2k}) \\
&\leq \sum_{k=1}^{q} \frac{\epsilon}{2}(t_{2k} - t_{2k-1}) + \sum_{k=0}^{q} 1\delta \\
&\leq \frac{\epsilon}{2}\left(\sum_{k=1}^{q}(t_{2k} - t_{2k-1})\right) + (q+1)\delta \\
&\leq \frac{\epsilon}{2}(1-0) + (q+1)\delta \\
&\leq \frac{\epsilon}{2} + (q+1)\frac{\epsilon}{2(q+1)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus, as $\epsilon > 0$ was arbitrary, f is Riemann integrable on $[0,1]$ by Theorem 6.1.17.

Remark 6.1.23. Notice the main idea in the proofs of Corollary 6.1.21 and Example 6.1.22 is to construct a finite number of open intervals which contain all of the 'bad' points such that the sum of the lengths of the open intervals is as small as possible. In fact, similar arguments along with the knowledge that the set of discontinuities of a function can be used to show a bounded function on is Riemann integrable if and only if its set of discontinuities has "zero length". However, this discussion is better considered in MATH 4012 (Lebesgue Measure Theory).

6.1.4 Properties of the Riemann Integral

Now that we know several functions are Riemann integrable, we desire to derive the basic properties of the Riemann integral just as we did for limits of sequences and functions. We begin with the following that enables us to divide up a closed interval into a finite number of closed subintervals when considering Riemann integration.

Proposition 6.1.24. *Let $f : [a,b] \to \mathbb{R}$ be bounded and let $c \in (a,b)$. Then f is Riemann integrable on $[a,b]$ if and only if f is Riemann integrable on $[a,c]$ and $[c,b]$. Moreover, when f is Riemann integrable on $[a,b]$, we have that*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Proof. To begin, assume that f is Riemann integrable on $[a,b]$. To see that f is Riemann integrable on $[a,c]$ and $[c,b]$, let $\epsilon > 0$ be arbitrary. Since f

is Riemann integrable on $[a, b]$, Theorem 6.1.17 implies that there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$L(f, \mathcal{P}) \leq U(f, \mathcal{P}) \leq L(f, \mathcal{P}) + \epsilon.$$

Therefore, if $\mathcal{P}_0 = \mathcal{P} \cup \{c\}$, then $\mathcal{P}_0$ is a partition of $[a, b]$ containing c that is a refinement of $\mathcal{P}$. Therefore, by Remark 6.1.13 and Lemma 6.1.7

$$\begin{aligned} L(f, \mathcal{P}_0) &\leq U(f, \mathcal{P}_0) \\ &\leq U(f, \mathcal{P}) \\ &\leq L(f, \mathcal{P}) + \epsilon \\ &\leq L(f, \mathcal{P}_0) + \epsilon. \end{aligned}$$

Let

$$\mathcal{P}_1 = \mathcal{P}_0 \cap [a, c] \qquad \text{and} \qquad \mathcal{P}_2 = \mathcal{P}_0 \cap [c, b].$$

Then $\mathcal{P}_1$ is a partition of $[a, c]$ and $\mathcal{P}_2$ is a partition of $[c, b]$. Furthermore, due to the nature of these partitions and the definitions of the upper and lower Riemann sums, we easily see that

$$L(f, \mathcal{P}_0) = L(f, \mathcal{P}_1) + L(f, \mathcal{P}_2) \qquad \text{and} \qquad U(f, \mathcal{P}_0) = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2).$$

Hence

$$0 \leq (U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)) + (U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)) = U(f, \mathcal{P}_0) - L(f, \mathcal{P}_0) \leq \epsilon.$$

Therefore, since $0 \leq U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1)$ and $0 \leq U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2)$, it must be the case that

$$0 \leq U(f, \mathcal{P}_1) - L(f, \mathcal{P}_1) \leq \epsilon \qquad \text{and} \qquad 0 \leq U(f, \mathcal{P}_2) - L(f, \mathcal{P}_2) \leq \epsilon.$$

Hence f is integrable on both $[a, c]$ and $[c, b]$ by Theorem 6.1.17.

To prove the converse and demonstrate the desired integral equation, assume that f is Riemann integrable on $[a, c]$ and $[c, b]$. To see that f is Riemann integrable on $[a, b]$, let $\epsilon > 0$ be arbitrary. Since f is Riemann integrable on $[a, c]$ and $[c, b]$, Remark 6.1.13 together with Theorem 6.1.17 imply that there exists partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of $[a, c]$ and $[c, b]$ respectively such that

$$\begin{aligned} L(f, \mathcal{P}_1) &\leq \int_a^c f(x)\, dx \leq U(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_1) + \frac{\epsilon}{2} \text{ and} \\ L(f, \mathcal{P}_2) &\leq \int_c^b f(x)\, dx \leq U(f, \mathcal{P}_2) \leq L(f, \mathcal{P}_2) + \frac{\epsilon}{2}. \end{aligned}$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. It is elementary to see that $\mathcal{P}$ is a partition of $[a, b]$. Moreover, due to the nature of these partitions and the definitions of the upper and lower Riemann sums, we easily see that

$$L(f, \mathcal{P}) = L(f, \mathcal{P}_1) + L(f, \mathcal{P}_2) \qquad \text{and} \qquad U(f, \mathcal{P}) = U(f, \mathcal{P}_1) + U(f, \mathcal{P}_2).$$

Hence

$$\begin{aligned}
0 &\leq U(f,\mathcal{P}) - L(f,\mathcal{P}) \\
&= (U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2)) + (L(f,\mathcal{P}_1) + L(f,\mathcal{P}_2)) \\
&= (U(f,\mathcal{P}_1) - L(f,\mathcal{P}_1)) + (U(f,\mathcal{P}_2) - L(f,\mathcal{P}_2)) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, f is Riemann integrable on $[a,b]$ by Theorem 6.1.17. Moreover, we have for all $\epsilon > 0$ that

$$\begin{aligned}
\int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \epsilon &\leq L(f,\mathcal{P}_1) + L(f,\mathcal{P}_2) \\
&= L(f,\mathcal{P}) \\
&\leq \int_a^b f(x)\,dx \\
&\leq U(f,\mathcal{P}) \\
&= U(f,\mathcal{P}_1) + U(f,\mathcal{P}_2) \\
&\leq \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \epsilon.
\end{aligned}$$

Hence

$$\left| \int_a^c f(x)\,dx + \int_c^b f(x)\,dx - \int_a^b f(x)\,dx \right| < \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, we obtain that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

as desired. ∎

Of course, integrals behave well with respect to many of the same arithmetic properties that limits satisfy as the following result shows. Unfortunately, notice that multiplication is absent from this result.

Proposition 6.1.25. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions on $[a,b]$. The following are true:*

a) If $\alpha \in \mathbb{R}$, then αf is Riemann integrable on $[a,b]$ and

$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

b) $f + g$ is Riemann integrable on $[a,b]$ and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

c) If $f(x) \leq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

d) If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Proof. a) Assume $f : [a, b] \to \mathbb{R}$ is a Riemann integrable function and $\alpha \in \mathbb{R}$. To see that αf is Riemann integrable, consider an arbitrary partition $\mathcal{P}$ of $[a, b]$.

Notice if $\alpha \geq 0$ then Lemma 1.3.4 implies that $\sup(\alpha A) = \alpha \sup(A)$ and $\inf(\alpha A) = \alpha \inf(A)$ for all subsets $A \subseteq \mathbb{R}$. Therefore, if $\alpha > 0$, we have that

$$L(\alpha f, \mathcal{P}) = \alpha L(f, \mathcal{P}) \qquad \text{and} \qquad U(\alpha f, \mathcal{P}) = \alpha U(f, \mathcal{P})$$

Furthermore, since if A is a bounded subset of $\mathbb{R}$ then $\inf(-A) = -\sup(A)$ by Lemma 1.3.1, it follows that if $\alpha < 0$ then

$$L(\alpha f, \mathcal{P}) = \alpha U(f, \mathcal{P}) \qquad \text{and} \qquad U(\alpha f, \mathcal{P}) = \alpha L(f, \mathcal{P})$$

Since f is Riemann integrable on $[a, b]$, we obtain by the definition of the Riemann integral that

$$\begin{aligned}\int_a^b f(x)\,dx &= \sup\{L(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ &= \inf\{U(f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.\end{aligned}$$

Therefore, the previous above computations along with Lemmas 1.3.1 and 1.3.4, we obtain that

$$\begin{aligned}\alpha \int_a^b f(x)\,dx &= \sup\{L(\alpha f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\} \\ &= \inf\{U(\alpha f, \mathcal{P}) \mid \mathcal{P} \text{ a partition of } [a, b]\}.\end{aligned}$$

Hence αf is Riemann integrable on $[a, b]$ with

$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

b) Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable. To begin the proof, consider an arbitrary partition $\mathcal{P}$ of $[a, b]$. Since

$$\sup\{f(x)+g(x) \mid x \in [c, d]\} \leq \sup\{f(x) \mid x \in [c, d]\}+\sup\{g(x) \mid x \in [c, d]\}$$

and

$$\inf\{f(x)+g(x) \mid x\in[c,d]\} \geq \inf\{f(x) \mid x\in[c,d]\} + \inf\{g(x) \mid x\in[c,d]\}$$

for all $c, d \in [a, b]$ with $c < d$, we obtain that

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leq L(f+g,\mathcal{P}) \leq U(f+g,\mathcal{P}) \leq U(f,\mathcal{P}) + U(g,\mathcal{P})$$

by the definition of the Riemann sums.

To prove that $f + g$ is Riemann integrable and obtain the desired integral equation, let $\epsilon > 0$ be arbitrary. Since f is Riemann integrable on $[a, b]$, Remark 6.1.13 together with Theorem 6.1.17 imply that there exists a partition $\mathcal{P}_1$ of $[a, b]$ such that

$$L(f,\mathcal{P}_1) \leq \int_a^b f(x)\,dx \leq U(f,\mathcal{P}_1) \leq L(f,\mathcal{P}_1) + \frac{\epsilon}{2}.$$

Similarly, since g is Riemann integrable on $[a, b]$, Remark 6.1.13 together with Theorem 6.1.17 imply that there exists a partition $\mathcal{P}_2$ of $[a, b]$ such that

$$L(g,\mathcal{P}_2) \leq \int_a^b g(x)\,dx \leq U(g,\mathcal{P}_2) \leq L(g,\mathcal{P}_2) + \frac{\epsilon}{2}.$$

Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Then $\mathcal{P}$ is a partition of $[a, b]$ that is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. Therefore, Remark 6.1.13 together with Lemma 6.1.7 imply that

$$\begin{aligned} L(f,\mathcal{P}) \leq \int_a^b f(x)\,dx &\leq U(f,\mathcal{P}) \\ &\leq U(f,\mathcal{P}_1) \\ &\leq L(f,\mathcal{P}_1) \\ &\leq L(f,\mathcal{P}) + \frac{\epsilon}{2} \end{aligned}$$

and similarly

$$L(g,\mathcal{P}) \leq \int_a^b g(x)\,dx \leq U(g,\mathcal{P}) \leq L(g,\mathcal{P}) + \frac{\epsilon}{2}.$$

Hence, since we know that

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leq L(f+g,\mathcal{P}) \leq U(f+g,\mathcal{P}) \leq U(f,\mathcal{P}) + U(g,\mathcal{P})$$

we obtain that

$$L(f,\mathcal{P}) + L(g,\mathcal{P}) \leq L(f+g,\mathcal{P}) \leq U(f+g,\mathcal{P}) \leq L(f,\mathcal{P}) + L(g,\mathcal{P}) + \epsilon.$$

Hence $0 \leq U(f + g, \mathcal{P}) - L(f + g, \mathcal{P}) < \epsilon$. Therefore, since ϵ was arbitrary, Theorem 6.1.17 implies that $f + g$ is Riemann integrable on $[a, b]$. Moreover, by repeating the above now knowing that $f+g$ is Riemann integrable on $[a,b]$, we obtain that for all $\epsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$\begin{aligned}\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \epsilon &\leq L(f,\mathcal{P}) + L(g,\mathcal{P}) \\ &\leq L(f+g,\mathcal{P}) \\ &\int_a^b (f+g)(x)\,dx \\ &\leq U(f+g,\mathcal{P}) \\ &\leq U(f,\mathcal{P}) + U(g,\mathcal{P}) \\ &\leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \epsilon.\end{aligned}$$

Hence

$$\left|\int_a^b f(x)\,dx + \int_a^b g(x)\,dx - \int_a^b (f+g)(x)\,dx\right| \leq \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, we obtain that

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

as desired.

c) Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable and assume $f(x) \leq g(x)$ for all $x \in [a,b]$. To see the desired result, let $\epsilon > 0$ be arbitrary. Remark 6.1.13 together with Theorem 6.1.17 imply that there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$L(f,\mathcal{P}) \leq \int_a^b f(x)\,dx \leq U(f,\mathcal{P}) \leq L(f,\mathcal{P}) + \epsilon.$$

However, since $f(x) \leq g(x)$ for all $x \in [a,b]$, we know that

$$\inf\{f(x) \mid x \in [c,d]\} \leq \inf\{g(x) \mid x \in [c,d]\}$$

for all $c, d \in [a,b]$ with $c < d$. Therefore $L(f,\mathcal{P}) \leq L(g,\mathcal{P})$. Hence

$$\int_a^b f(x)\,dx - \epsilon \leq L(f,\mathcal{P}) \leq L(g,\mathcal{P}) \leq \int_a^b g(x)\,dx.$$

Hence, for all $\epsilon > 0$, we have that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx + \epsilon.$$

Therefore, we have ("by sending ϵ to 0") that

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

as desired.

d) By part c) and Example 6.1.14, we have that

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$$

as desired. ∎

Remark 6.1.26. Note that Proposition 6.1.25 does not produce a formula for the Riemann integral of the product of Riemann integrable functions. Indeed it is almost always the case that $\int_a^b (fg)(x)\,dx \neq \left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right)$. For example, using Examples 6.1.15 and 6.1.16, we see that

$$\int_0^1 x^2\,dx = \frac{1}{3} \quad \text{whereas} \quad \left(\int_0^1 x\,dx\right)^2 = \frac{1}{4}.$$

In lieu of the above remark, it is still possible to show that if f and g are Riemann integrable on $[a, b]$, then fg is Riemann integrable on $[a, b]$. To begin this proof, we first must deal with the case that $f = g$.

Lemma 6.1.27. *Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function on $[a, b]$. The function $f^2 : [a, b] \to \mathbb{R}$ defined by $f^2(x) = (f(x))^2$ for all $x \in [a, b]$ is Riemann integrable on $[a, b]$.*

Proof. Since f is bounded by the definition of Riemann integrable,

$$K = \sup\{|f(x)| \mid x \in [a, b]\} < \infty.$$

To see that f^2 is Riemann integrable, let $\epsilon > 0$ be arbitrary. Since f is Riemann integrable on $[a, b]$, Theorem 6.1.17 implies that there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \frac{1}{2(K+1)}\epsilon.$$

Write $\mathcal{P} = \{t_k\}_{k=0}^n$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

For each $k \in \{1, \ldots, n\}$ let

$$\begin{aligned} m_k(f) &= \inf\{f(x) \mid x \in [t_{k-1}, t_k]\}, \\ M_k(f) &= \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}, \\ m_k(f^2) &= \inf\{(f(x))^2 \mid x \in [t_{k-1}, t_k]\}, \text{ and} \\ M_k(f^2) &= \sup\{(f(x))^2 \mid x \in [t_{k-1}, t_k]\}. \end{aligned}$$

Notice for all $x, y \in [a,b]$ we have that

$$\begin{aligned} |(f(x))^2 - (f(y))^2| &= |f(x) + f(y)||f(x) - f(y)| \\ &\leq (|f(x)| + |f(y)|)|f(x) - f(y)| \\ &\leq (K + K)|f(x) - f(y)| = 2K|f(x) - f(y)|. \end{aligned}$$

Hence we obtain that

$$M_k(f^2) - m_k(f^2) \leq 2K(M_k(f) - m_k(f))$$

for all $k \in \{1, \ldots, n\}$. Therefore

$$0 \leq U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) \leq 2K(U(f, \mathcal{P}) - L(f, \mathcal{P})) \leq 2K\frac{1}{2(K+1)}\epsilon < \epsilon.$$

Hence f^2 is Riemann integrable by Proposition 6.1.29. ∎

Using the above and a clever decomposition of functions, we obtain the product of Riemann integrable functions is Riemann integrable.

Proposition 6.1.28. *Let $f, g : [a,b] \to \mathbb{R}$ be Riemann integrable functions on $[a,b]$. Then $fg : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$.*

Proof. Since

$$f(x)g(x) = \frac{1}{2}\left((f(x) + g(x))^2 - f(x)^2 - g(x)^2\right)$$

and since $f + g, f^2, g^2$, and $(f + g)^2$ are Riemann integrable by Proposition 6.1.25 and Lemma 6.1.27, it follows by Proposition 6.1.25 that fg is Riemann integrable. ∎

To complete our section on the properties of the Riemann integral, we have one more useful result. The main reason why this result is useful in analysis is that it plays the same role for integrals as the triangle inequality plays for sums.

Proposition 6.1.29. *Let $f : [a,b] \to \mathbb{R}$ a Riemann integrable function on $[a,b]$. Then the function $|f| : [a,b] \to \mathbb{R}$ defined by $|f|(x) = |f(x)|$ for all $x \in [a,b]$ is Riemann integrable on $[a,b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Proof. Let $\epsilon > 0$ be arbitrary. By Theorem 6.1.17, there exists a partition $\mathcal{P}$ of $[a,b]$ such that

$$0 \leq U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

Write $\mathcal{P} = \{t_k\}_{k=0}^n$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

For each $k \in \{1, \ldots, n\}$ let

$$\begin{aligned} m_k(f) &= \inf\{f(x) \mid x \in [t_{k-1}, t_k]\}, \\ M_k(f) &= \sup\{f(x) \mid x \in [t_{k-1}, t_k]\}, \\ m_k(|f|) &= \inf\{|f(x)| \mid x \in [t_{k-1}, t_k]\}, \text{ and} \\ M_k(|f|) &= \sup\{|f(x)| \mid x \in [t_{k-1}, t_k]\}. \end{aligned}$$

We claim that

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f)$$

for all $k \in \{1, \ldots, n\}$. Indeed notice if $x, y \in [t_{k-1}, t_k]$ are such that:

- $f(x), f(y) \geq 0$, then
$$|f(x)| - |f(y)| = f(x) - f(y) \leq M_k(f) - m_k(f).$$
- $f(x) \geq 0 \geq f(y)$, then
$$|f(x)| - |f(y)| \leq f(x) - f(y) \leq M_k(f) - m_k(f).$$
- $f(y) \geq 0 \geq f(x)$, then
$$|f(x)| - |f(y)| \leq f(y) - f(x) \leq M_k(f) - m_k(f).$$
- $f(x), f(y) \leq 0$, then
$$|f(x)| - |f(y)| = f(y) - f(x) \leq M_k(f) - m_k(f).$$

Using Lemma 1.3.5, by considering the supreme of the above equations over x followed by the infimum of the above equations over y, we obtain that

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f).$$

Hence

$$\begin{aligned} U(|f|, \mathcal{P}) - L(|f|, \mathcal{P}) &= \sum_{k=1}^n (M_k(|f|) - m_k(|f|))(t_k - t_{k-1}) \\ &\leq \sum_{k=1}^n (M_k(f) - m_k(f))(t_k - t_{k-1}) \\ &= U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $|f|$ is Riemann integrable on $[a, b]$ by Theorem 6.1.17.

Since $|f|$ is Riemann integrable, Proposition 6.1.25 implies that $-|f|$ is Riemann integrable. Moreover, since

$$-|f(x)| \leq f(x) \leq |f(x)|$$

for all $x \in [a, b]$, Proposition 6.1.25 also implies that

$$-\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

Hence

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

which completes the proof. ∎

6.2 The Fundamental Theorems of Calculus

For our final section of the course, note that although we have developed the Riemann integral and its properties, we still lack a simple way to compute the integral of even some of the most basic functions. Indeed the only integrals we have actually computed were Examples 6.1.15 and 6.1.16 where specific sums were used.

The goal of this final section is to prove what is know as the Fundamental Theorems of Calculus. Said theorems are named as such since they provide the ultimate connection between integration and differentiation via anti-derivatives as introduced in Section 5.8. To study these theorems, we will need to define some functions based on integrals.

To begin, assume $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$. For simplicity, let us define

$$\int_a^a f(x)\,dx = 0.$$

Therefore, if we define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt$$

for all $x \in [a, b]$, we see that F is a well-defined since f is Riemann integrable on $[a, x]$ by Proposition 6.1.24.

Lemma 6.2.1. *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ be defined by*

$$F(x) = \int_a^x f(t)\,dt$$

for all $x \in [a, b]$. Then F is continuous on $[a, b]$.

Proof. To show that F is continuous on $[a, b]$, we will show that F is uniformly continuous on $[a, b]$. To do this, let $\epsilon > 0$ be arbitrary.

Since f is bounded,

$$0 \leq M = \max\{|f(x)| \mid x \in [a, b]\} < \infty.$$

Let $\delta = \frac{\epsilon}{M+1}$. Clearly $\delta > 0$.

To see that δ works for ϵ in the definition of uniform continuity, notice if $x_1, x_2 \in [a, b]$ are such that $x_1 < x_2$, then f is Riemann integrable on $[x_1, x_2]$ by Proposition 6.1.24 and by Proposition 6.1.25 we have that

$$\left| \int_{x_1}^{x_2} f(t)\, dt \right| \leq \int_{x_1}^{x_2} |f(t)|\, dt \leq M|x_2 - x_1|.$$

Therefore, since for all $x_1 < x_2$

$$\begin{aligned} F(x_2) - F(x_1) &= \int_a^{x_2} f(t)\, dt - \int_a^{x_1} f(t)\, dt \\ &= \left(\int_a^{x_1} f(t)\, dt + \int_{x_1}^{x_2} f(t)\, dt \right) - \int_a^{x_1} f(t)\, dt \\ &= \int_{x_1}^{x_2} f(t)\, dt \end{aligned}$$

by Proposition 6.1.24, it easily follows

$$|F(x_2) - F(x_1)| \leq M|x_2 - x_1|$$

for all $x_1, x_2 \in [a, b]$ with $x_1 < x_2$. Hence, if $x_1, x_2 \in [a, b]$ are such that $x_1 < x_2$ and $|x_2 - x_1| < \delta$, then

$$|F(x_2) - F(x_1)| \leq M|x_2 - x_1| \leq M\frac{\epsilon}{M+1} \leq \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, F is uniformly continuous on $[a, b]$. ∎

Since the function F from Lemma 6.2.1 is continuous, it is possible that F is differentiable. The First Fundamental Theorem of Calculus shows this is indeed the case provided f is continuous and enables us to compute the derivative. In fact, the following shows that if we integrate a function f to obtain F, then F is an anti-derivative of f. Hence differentiation undoes integration!

Theorem 6.2.2 (The Fundamental Theorem of Calculus, I). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and let $F : [a, b] \to \mathbb{R}$ be defined by*

$$F(x) = \int_a^x f(t)\, dt$$

for all $x \in [a, b]$. Then F is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous and fix an $x \in (a, b)$. To see that F is differentiable at x and $F'(x) = f(x)$, we must show that

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

To see this, let $\epsilon > 0$ be arbitrary. Since f is continuous at x, there exists a $\delta > 0$ such that if $t \in [a, b]$ and $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$. Notice if $0 < h < \delta$ then

$$\begin{aligned}
&\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \\
&= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)1\right| && \text{by Proposition 6.1.24} \\
&= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - f(x)\left(\frac{1}{h}\int_x^{x+h} 1\,dt\right)\right| && \text{since } \frac{1}{h}\int_x^{x+h} 1\,dt = 1 \\
&= \left|\frac{1}{h}\int_x^{x+h} f(t)\,dt - \frac{1}{h}\int_x^{x+h} f(x)\,dt\right| && \text{by Propostion 6.1.25} \\
&= \left|\frac{1}{h}\int_x^{x+h} f(t) - f(x)\,dt\right| && \text{by Propostion 6.1.25} \\
&\leq \frac{1}{h}\int_x^{x+h} |f(t) - f(x)|\,dt && \text{by Propostion 6.1.29} \\
&\leq \frac{1}{h}\int_x^{x+h} \epsilon\,dt && {\scriptstyle \text{since } |t-x|\leq\delta \atop \text{for all } t\in[x,x+h]} \\
&= \frac{1}{h}(h\epsilon) = \epsilon.
\end{aligned}$$

Similarly, notice if $-\delta < h < 0$, then

$$\begin{aligned}
&\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \\
&= \left|-\frac{1}{h}\int_{x+h}^{x} f(t)\,dt - f(x)1\right| && \text{by Proposition 6.1.24} \\
&= \left|-\frac{1}{h}\int_{x+h}^{x} f(t)\,dt - f(x)\left(-\frac{1}{h}\int_{x+h}^{x} 1\,dt\right)\right| && \text{since } -\frac{1}{h}\int\int_{x+h}^{x} 1\,dt = 1 \\
&= \left|\frac{1}{h}\int_{x+h}^{x} f(t)\,dt + \frac{1}{h}\int_{x+h}^{x} f(x)\,dt\right| && \text{by Propostion 6.1.25} \\
&= \left|-\frac{1}{h}\int_{x+h}^{x} f(t) - f(x)\,dt\right| && \text{by Propostion 6.1.25} \\
&\leq -\frac{1}{h}\int_{x+h}^{x} |f(t) - f(x)|\,dt && \text{by Propostion 6.1.29} \\
&\leq -\frac{1}{h}\int_{x+h}^{x} \epsilon\,dt && {\scriptstyle \text{since } |t-x|\leq\delta \atop \text{for all } t\in[x+h,x]} \\
&= \frac{1}{h}(h\epsilon) = \epsilon.
\end{aligned}$$

Hence, for all h with $0 < |h| < \delta$,

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \epsilon.$$

Therefore, as ϵ was arbitrary, the definition of the limit implies that

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Hence $F'(x)$ exists and $F'(x) = f(x)$ as desired. ∎

Remark 6.2.3. It is important to note that we cannot replace "f is continuous" with "f is Riemann integrable" in the statement of the First Fundamental Theorem of Calculus. Indeed if we define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases},$$

then f is Riemann integral by Corollary 6.1.21. However, we see that if $F : [-1, 1] \to \mathbb{R}$ is defined by

$$F(x) = \int_{-1}^{x} f(t)\, dt,$$

then

$$F(x) = \begin{cases} -x - 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases} = |x| - 1$$

so F is not differentiable at 0.

In contrast to how the First Fundamental Theorem of Calculus shows that derivatives undo integration, the Second Fundamental Theorem of Calculus shows that integration undoes derivatives. In particular, the Second Fundamental Theorem of Calculus shows us that if we know the antiderivative of a function f, then we can compute the Riemann integral of f. Note we will provide two proofs of the Second Fundamental Theorem of Calculus; one that assumes f is continuous and is simpler, and one that makes no assumptions on f other than that f is Riemann integrable.

Theorem 6.2.4 (The Fundamental Theorem of Calculus, II). *Let $f, g : [a, b] \to \mathbb{R}$ be such that f is Riemann integrable on $[a, b]$, g is differentiable on $[a, b]$, and $g'(x) = f(x)$ for all $x \in (a, b)$. Then*

$$\int_{a}^{b} f(t)\, dt = g(b) - g(a).$$

Proof of Theorem 6.2.4 when f is continuous. Assume f is continuous. Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\, dt$$

for all $x \in [a, b]$. Since f is Riemann integrable, F is well-defined by Proposition 6.1.24.

Since f is continuous, the First Fundamental Theorem of Calculus (Theorem 6.2.2) implies that F is differentiable on (a, b) with

$$F'(x) = f(x) = g'(x)$$

for all $x \in (a, b)$. Hence Corollary 5.8.2 implies that there exists a constant $\alpha \in \mathbb{R}$ such that $F(x) = g(x) + \alpha$ for all $x \in (a, b)$. Since F is continuous on $[a, b]$ by Lemma 6.2.1 and since g is continuous on $[a, b]$ by assumption, we have that $F(x) = g(x) + \alpha$ for all $x \in [a, b]$. Hence

$$\begin{aligned}\int_a^b f(t)\, dt &= F(b) - 0 \\ &= F(b) - F(a) \\ &= (g(b) + \alpha) - (g(a) + \alpha) = g(b) - g(a).\end{aligned}$$

■

Proof of Theorem 6.2.4, no additional assumptions. Let $f, g : [a, b] \to \mathbb{R}$ be such that f is Riemann integrable on $[a, b]$, g is differentiable on $[a, b]$, and $g'(x) = f(x)$ for all $x \in (a, b)$. Note g is continuous on $[a, b]$ by definition.

Let $\epsilon > 0$ be arbitrary. By Remark 6.1.13 and Theorem 6.1.17 there exists a partition $\mathcal{P}$ of $[a, b]$ such that

$$L(f, \mathcal{P}) \leq \int_a^b f(t)\, dt \leq U(f, \mathcal{P}) \leq L(f, \mathcal{P}) + \epsilon.$$

Write $\mathcal{P} = \{t_k\}_{k=0}^n$ where

$$a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b.$$

Since g is differentiable on $[a, b]$, the Mean Value Theorem (Theorem 5.3.3) implies for each $k \in \{1, \ldots, n\}$ that there exists a $x_k \in (t_{k-1}, t_k)$ such that

$$\frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}} = g'(x_k) = f(x_k).$$

Thus

$$g(t_k) - g(t_{k-1}) = f(x_k)(t_k - t_{k-1})$$

for all $k \in \{1, \ldots, n\}$.

Notice that

$$\sum_{k=1}^n f(x_k)(t_k - t_{k-1}) = \sum_{k=1}^n g(t_k) - g(t_{k-1}) = g(t_n) - g(t_0) = g(b) - g(a).$$

Moreover, by the definition of the upper and lower Riemann sums, we know that

$$L(f,\mathcal{P}) \leq \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) \leq U(f,\mathcal{P}) \leq L(f,\mathcal{P}) + \epsilon.$$

Hence, we obtain that

$$L(f,\mathcal{P}) \leq g(b) - g(a) \leq L(f,\mathcal{P}) + \epsilon.$$

Since

$$L(f,\mathcal{P}) \leq \int_a^b f(t)\,dt \leq U(f,\mathcal{P}) \leq L(f,\mathcal{P}) + \epsilon,$$

we obtain that

$$\left| g(b) - g(a) - \int_a^b f(t)\,dt \right| \leq \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary,

$$\int_a^b f(t)\,dt = g(b) - g(a)$$

as desired. ∎

Before discussing the uses of the second Fundamental Theorem of Calculus, it is useful to have an example on why we cannot simply assume f is continuous in order to use the easier proof.

Example 6.2.5. Let $f, g : [-1,1] \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} x^2\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 2x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for all $x \in [-1,1]$.

Note

$$\left| 2x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) \right| \leq 2|x|\left|\cos\left(\frac{1}{x}\right)\right| + \left|\sin\left(\frac{1}{x}\right)\right| \leq 3$$

for all $x \in [-1,1]$. Therefore f is bounded on $[-1,1]$. Moreover, f is continuous on $[-1,0)\cup(0,1]$ since x, $\frac{1}{x}$, cos, and sin are continuous functions. However, f is not continuous at 0 since

$$\lim_{x\to 0} 2x\cos\left(\frac{1}{x}\right) = 0$$

by the Squeeze Theorem (Theorem 4.1.23), but $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist. Hence f is not continuous on $[-1,1]$. However, since f is bounded and continuous except at $x = 0$, f is Riemann integrable on $[-1,1]$ by Corollary 6.1.21.

It follows by the Chain Rule (Theorem 5.1.20) that

$$g'(x) = 2x\cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) = f(x)$$

for all $x \in (-1,0) \cup (0,1)$. Moreover, notice

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} x\cos\left(\frac{1}{x}\right) = 0 = f(0)$$

by the Squeeze Theorem (Theorem 4.1.23). Therefore, since g is continuous at $x = 1$ and $x = -1$, we obtain that g is differentiable on $[-1,1]$ with $g'(x) = f(x)$ for all $x \in (-1,1)$. Therefore

$$\int_{-1}^{1} f(x)\,dx = g(1) - g(-1)$$

by the second Fundamental Theorem of Calculus, even though f is not continuous.

Remark 6.2.6. Using the second Fundamental Theorem of Calculus, our knowledge of derivatives from Subsection 5.1.2 and derivatives of inverse functions from Section 5.5, we easily can compute some integrals:

$$\begin{aligned}
\int_0^x t^n\,dt &= \frac{x^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} = \frac{x^{n+1}}{n+1} \\
\int_0^x e^t\,dt &= e^x - e^0 = e^x - 1 \\
\int_0^x \sin(t)\,dt &= -\cos(x) - (-\cos(0)) = -\cos(x) + 1 \\
\int_0^x \cos(t)\,dt &= \sin(x) - (\sin(0)) = \sin(x) \\
\int_0^x \sec^2(t)\,dt &= \tan(x) - \tan(0) = \tan(x) \\
\int_1^x \frac{1}{t}\,dt &= \ln(x) - \ln(1) = \ln(x) \\
\int_0^x \frac{1}{\sqrt{1-t^2}}\,dt &= \arcsin(x) - \arcsin(0) = \arcsin(x) \\
\int_0^x -\frac{1}{\sqrt{1-t^2}}\,dt &= \arccos(x) - \arccos(0) = \arccos(x) - \frac{\pi}{2} \\
\int_0^x \frac{1}{1+t^2}\,dt &= \arctan(x) - \arctan(0) = \arctan(x).
\end{aligned}$$

To complete our course, we demonstrate that the Fundamental Theorems of Calculus immediately give us two common methods used to compute the value of an integral from calculus: Integration by Substitution and Integration by Parts. To prove the Integration by Substitution result, we first need a lemma.

Lemma 6.2.7. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f([a, b])$ is a closed interval.*

Proof. Since $f : [a, b] \to \mathbb{R}$ is continuous, the Extreme Value Theorem (Theorem 4.6.2) implies there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$. Hence $f([a, b]) \subseteq [f(x_1), f(x_2)]$. Since $x_1, x_2 \in [a, b]$, the Intermediate Value Theorem (Theorem 4.4.2) implies that for all $c \in (f(x_1), f(x_2))$ there exists an $x \in (x_1, x_2) \cup (x_2, x_1)$ such that $f(x) = c$. Therefore, since $f(x_1), f(x_2) \in f([a, b])$, we obtain that $[f(x_1), f(x_2)] \subseteq f([a, b])$. Therefore $f([a, b]) = [f(x_1), f(x_2)]$ so $f([a, b])$ is a closed interval. ∎

Corollary 6.2.8 (Integration by Substitution). *Let $a, b, c, d \in \mathbb{R}$ be such that $c < a < b < d$ so that $[a, b] \subseteq (c, d)$. Let $g : (c, d) \to \mathbb{R}$ be differentiable at each point in $[a, b]$ so that g' is continuous on $[a, b]$. Note $I = g([a, b])$ is an interval by Lemma 6.2.7. If $f : I \to \mathbb{R}$ be continuous, then $(f \circ g)g'$ is Riemann integrable on $[a, b]$ with*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

(Note if $g(b) \leq g(a)$ then $\int_{g(a)}^{g(b)} f(u)\,du$ is defined to equal $-\int_{g(b)}^{g(a)} f(u)\,du$.)

Proof. To begin, we must check that the functions we are considering are Riemann integrable. Since f is continuous on I, f is Riemann integrable on I by Theorem 6.1.20. Since g' is continuous on $[a, b]$, g is continuous on $[a, b]$. Therefore $f \circ g$ is continuous on $[a, b]$ by Theorem 4.2.10. Hence $(f \circ g)g'$ is continuous on $[a, b]$ and thus Riemann integrable on $[a, b]$ by Theorem 6.1.20.

Thus it remains only to show the desired integral equation. To see this, first note that $I = [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ so we can extend f to a continuous function h on $\mathbb{R}$ by

$$h(x) = \begin{cases} f(x) & \text{if } x \in [\alpha, \beta] \\ f(\alpha) & \text{if } x \leq \alpha \\ f(\beta) & \text{if } x \geq \beta \end{cases}.$$

Therefore, by the First Fundamental Theorem of Calculus (Theorem 6.2.2), if $F : [\alpha - 1, \beta + 1] \to \mathbb{R}$ is defined by

$$F(x) = \int_{\alpha-1}^{x} h(t)\,dt,$$

then F is differentiable on $(\alpha - 1, \beta + 1)$ with $F'(x) = h(x) = f(x)$ for all $x \in [\alpha, \beta] = I$.

Since $g : (c, d) \to I$ is continuous on $[a, b]$ so that $g((a - \delta, b + \delta)) \subseteq (\alpha - 1, \beta + 1)$ for some $\delta > 0$, the Chain Rule (Theorem 5.1.20) implies that $F \circ g$ is differentiable on $[a, b]$ with

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$$

for all $x \in [a,b]$. Therefore, we obtain that

$$\begin{aligned}\int_a^b f(g(x))g'(x)\,dx &= \int_a^b (F\circ g)'(x)\,dx \\ &= (F\circ g)(b) - (F\circ g)(a) && \text{by the Second Fundamental Theorem of Calculus} \\ &= F(g(b)) - F(g(a)) \\ &= \int_{g(a)}^{g(b)} f(x)\,dx && \text{by the Second Fundamental Theorem of Calculus}\end{aligned}$$

as desired. ∎

Corollary 6.2.9 (Integration by Parts). *Let $a,b,c,d \in \mathbb{R}$ be such that $c < a < b < d$ so that $[a,b] \subseteq (c,d)$. If $f,g : (c,d) \to \mathbb{R}$ are continuous and differentiable at each point in $[a,b]$ and $f',g' : [a,b] \to \mathbb{R}$ are Riemann integrable, then $f'g$ and fg' are Riemann integrable on $[a,b]$ with*

$$\int_a^b f(x)g'(x)\,dx = f(b)g(g) - f(a)g(a) - \int_a^b f'(x)g(x)\,dx.$$

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by

$$h(x) = f'(x)g(x) + f(x)g'(x).$$

Since $f'g$, fg', and h are Riemann integrable on $[a,b]$ by Theorem 6.1.20 and Propositions 6.1.25 and 6.1.28, and since fg is differentiable on (c,d) with $(fg)'(x) = h(x)$ for all $x \in [a,b]$ by the Product Rule (Proposition 5.1.12), we obtain by the Second Fundamental Theorem of Calculus (Theorem 6.2.4), and by Propositions 6.1.25 and 6.1.28 that

$$\begin{aligned}f(b)g(b) - f(a)g(a) &= \int_a^b h(x)\,dx \\ &= \int_a^b f'(x)g(x) + f(x)g'(x)\,dx \\ &= \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx\end{aligned}$$

as desired. ∎

Appendix A

MATH 1200 Background

In this appendix chapter, we will quickly review the basic background material from MATH 1200 that students should be familiar with when entering this course.

A.1 Set Notation

All mathematics must contain some notation in order for one to adequately describe the objects of study. As such, we begin by developing the notation surrounding one of the most basic objects in mathematics.

Heuristic Definition. A *set* is a collection of distinct objects.

To utilize sets, we must first develop notation to adequately describe sets and symbols to adequetly describe operations on sets. First we begin with how to write an explicit set.

Notation A.1.1. There are two notations commonly used to describe a set: namely

$$\{\text{objects}\}$$

and

$$\{\text{objects} \mid \text{conditions on the objects}\}.$$

The following are some examples of how one can use set notation to describe a set.

Example A.1.2. The set of natural numbers, denoted $\mathbb{N}$, is the set

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Example A.1.3. The set of integers, denoted $\mathbb{Z}$, is the set

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}.$$

Example A.1.4. The set of rational numbers, denoted $\mathbb{Q}$, is the set

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a, b \text{ are integers with } b \neq 0 \right\}.$$

Considering the above, it is useful to have some terminology and notation to determine when an object is in a given set.

Definition A.1.5. Given a set X and an object x, we say that x is an *element* of X if x is one of the objects that make up X. We denote that "x is an element of X" by $x \in X$ and we use $x \notin X$ to denoted when x is not an element of X.

Example A.1.6. It is clear based on the above definitions that $\frac{1}{2} \in \mathbb{Q}$ yet $\frac{1}{2} \notin \mathbb{Z}$. Similarly $0 \in \mathbb{Z}$ but $0 \notin \mathbb{N}$.

It is also useful to have terminology and notation to describe when one set contains another.

Definition A.1.7. Given a set A, a *subset* of A is any set B such that if $b \in B$ then $b \in A$. We denote "B is a subset of A" by $B \subseteq A$ and we use $B \nsubseteq A$ when B is not a subset of A.

Example A.1.8. It is clear based on the above definitions that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$, $\mathbb{Q} \nsubseteq \mathbb{Z}$, and $\mathbb{Z} \nsubseteq \mathbb{N}$.

There is one special set that is a subset of every set and is quite useful to describe.

Definition A.1.9. The *empty set*, denoted $\emptyset$, is the set with no elements.

Remark A.1.10. If A is any set, then it is vacuously true that if $x \in \emptyset$ then $x \in A$ since there are no objects x so that $x \in \emptyset$. Hence $\emptyset \subseteq A$ for all sets A.

Of course, the notion of when two sets are equal should be obvious.

Definition A.1.11. Two sets A and B are said to be *equal* if A and B have precisely the same elements. We write $A = B$ to denote that A and B are equal.

Remark A.1.12. If one is trying to prove two sets A and B are equal, one needs to demonstrate that $x \in A$ if and only if $x \in B$. If we divide this bi-conditional statement into its two components, we need to prove "if $x \in A$ then $x \in B$" and "if $x \in B$ then $x \in A$". Theses conditional statements are asking us to prove $A \subseteq B$ and $B \subseteq A$. Hence $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

When trying to prove two sets are equal, keep the above in mind as these ideas are the most common techniques to show that two sets are equal.

Now that the basics of sets have been established, we can start to construct new, larger sets from other sets. The following is a generalization of something students have seen in high school.

Definition A.1.13. Given an $n \in \mathbb{N}$ and sets $A_1, A_2, \ldots, A_n$, the *Cartesian Product of* $A_1, A_2, \ldots, A_n$ is the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_k \in A_k \text{ for all } k \in \{1, 2, \ldots, n\}\}.$$

Remark A.1.14. The most common Cartesian Product students have seen and are familiar with is $\mathbb{R}^n$ (where $\mathbb{R}$ denotes the set of real numbers). Indeed $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ where we have taken the Cartesian Product of n copies of $\mathbb{R}$. For example

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

and

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

More generally,

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

is one of the main objects of study in MATH 1021.

Instead of combining sets in pairs or, more generally, n-tuples, there is another common way to combine sets.

Definition A.1.15. Let I be a non-empty set and for each $\alpha \in I$, let A_α be a set. The *union of* $\{A_\alpha \mid \alpha \in I\}$, denoted $\bigcup_{\alpha \in I} A_\alpha$, is the set

$$\bigcup_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in I\}.$$

Example A.1.16. For two examples, if A denotes the set of all odd natural numbers and B denotes the set of all even natural numbers, then $\mathbb{N} = A \cup B$. Furthermore

$$\mathbb{N} = \bigcup_{n=1}^{\infty} \{2n - 1, 2n\}.$$

Instead of taking the set that contains all of the elements of a collection of sets, we can take the set of elements that are common to each set.

Definition A.1.17. Let I be a non-empty set and for each $\alpha \in I$, let A_α be a set. The *intersection of* $\{A_\alpha \mid \alpha \in I\}$, denoted $\bigcap_{\alpha \in I} A_\alpha$, is the set

$$\bigcap_{\alpha \in I} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in I\}.$$

Example A.1.18. For example, $\{1\} = \bigcap_{n=1}^{\infty} \{1, n, n + 1, \ldots\}$ as the number 1 is the only element of each set.

Furthermore, it is possible to 'take away' one set from another.

Definition A.1.19. Given two sets A and B, the *set difference of A by B*, denoted $A \setminus B$, is the set

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

To summarize the above set operations, consider the following example.

Example A.1.20. If $X = \{1, 2, 3\}$ and $Y = \{2, 4, 6\}$, then

$$X \cup Y = \{1, 2, 3, 4, 6\}, \quad X \cap Y = \{2\}, \quad \text{and} \quad X \setminus Y = \{1, 3\}.$$

Remark A.1.21. In this course, we will often have a set X (usually $\mathbb{R}$) and will be considering subsets of X. Consequently, given a subset Y of X, the set difference $X \setminus Y$ will be called the *complement* of Y (in X) and will be denoted Y^c for convenience.

It turns out that the operation of taking the complement of a set turns unions into complements and vice versa as the following result shows.

Theorem A.1.22 (De Morgan's Laws). *Let X and I be non-empty sets and for each $\alpha \in I$ let X_α be a subset of X. Then*

$$\left(\bigcup_{\alpha \in I} X_\alpha\right)^c = \bigcap_{\alpha \in I} X_\alpha^c \qquad \textit{and} \qquad \left(\bigcap_{\alpha \in I} X_\alpha\right)^c = \bigcup_{\alpha \in I} X_\alpha^c.$$

Proof. Notice that

$$\begin{aligned} x \in \left(\bigcup_{\alpha \in I} X_\alpha\right)^c & \text{ if and only if } x \notin \bigcup_{\alpha \in I} X_\alpha \\ & \text{ if and only if } x \notin X_\alpha \text{ for all } \alpha \in I \\ & \text{ if and only if } x \in X_i^c \text{ for all } \alpha \in I \\ & \text{ if and only if } x \in \bigcap_{\alpha \in I} X_\alpha^c \end{aligned}$$

which completes the proof of the first equation.

It is possible to repeat the same proof technique to show that the other equation holds. Alternatively, it is possible to use the first result to prove the second. To do this, we must first claim that that if $Y \subseteq X$ and $Z = Y^c$, then $Z^c = Y$; that is, the complement of the complement is the original set. Indeed notice $x \in Z^c$ if and only if $x \notin X$ if and only if $x \notin Y^c$ if and only if $x \in Y$. Hence $Z^c = Y$.

To prove the second equality using the first, for each $\alpha \in I$ let $Y_\alpha = X_\alpha^c$. By applying the first equation using the Y_α's instead of the X_α's, we obtain that

$$\left(\bigcup_{\alpha \in I} Y_\alpha\right)^c = \bigcap_{\alpha \in I} Y_\alpha^c.$$

Since $Y_\alpha = X_\alpha^c$ so $Y_\alpha^c = X_\alpha$ for all $\alpha \in I$, we have that

$$\left(\bigcup_{\alpha\in I} Y_\alpha^c\right)^c = \bigcap_{\alpha\in I} Y_\alpha.$$

Hence

$$\bigcup_{\alpha\in I} Y_\alpha^c = \left(\bigcap_{\alpha\in I} Y_\alpha\right)^c$$

by taking the complement of both sides. ∎

Sets will play an important role in this course. However, one important question that has not been addressed is, "What exactly is a set?" This questions must be asked as we have not provided a rigorous definition of a set. This leads to some interesting questions, such as, "Does the collection of all sets form a set?"

To consider these questions, let us assume that there is a set of all sets; that is the set

$$Z = \{X \mid X \text{ is a set}\}$$

makes sense. Note Z has the interesting property that $Z \in Z$. Since Z is a set, we would think that

$$Y = \{X \mid X \text{ is a set and } X \notin X\}$$

is a valid subset of Z and thus a set. Considering Y, there are two disjoint possibilities: either $Y \in Y$ or $Y \notin Y$.

If it were the case that $Y \in Y$, then the definition of Y implies $Y \notin Y$ which is a contradiction since we cannot have both $Y \in Y$ and $Y \notin Y$. Thus, as $Y \in Y$ must be false, then it must be the case that $Y \notin Y$.

However, $Y \notin Y$ implies by the definition of Y that $Y \in Y$. Again this is a contradiction since we cannot have both $Y \notin Y$ and $Y \in Y$. Therefore, if Y is a set, we would have reached a logical inconsistency in mathematics.

The above argument is known as Russell's Paradox and demonstrates that there cannot be a set of all sets. Russell's Paradox illustrates the necessity of a rigorous definition of a set. However, said definition takes us beyond the study of this class.

A.2 Functions

With our knowledge of sets, we turn next to the morphisms between sets: functions. In order to formally define what a function is and for future use in the next section, we begin with the following more general object.

Definition A.2.1. Let X and Y be sets. A *relation from X to Y* is any subset R of $X \times Y$. For $x \in X$ and $y \in Y$, we write xRy if $(x, y) \in R$ and we write $x\not{R}y$ if $(x, y) \notin R$. In the case that $Y = X$, we say that R is a relation on X.

For a natural example of a relation and to see where the notation xRy comes from, consider the following.

Example A.2.2. For example

$$R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$$

is a relation on $\mathbb{R}$ that we normally denote as $\leq$. Consequently, one sees that the notation 'xRy' for '$(x, y) \in R$' makes sense for this relation since $x \leq y$ exactly when $(x, y) \in R$.

The most formal definition of a function is that functions are specific types of relations.

Definition A.2.3. Let X and Y be sets. A *function f from X to Y* is a relation from X to Y such that if $x \in X$ then there exists a unique $y \in Y$ such that $(x, y) \in f$. We write $f : X \to Y$ to denote that f is a function from X to Y and for $x \in X$ we write $f(x)$ for the unique $y \in Y$ such that $(x, y) \in f$. The set X is called the *domain* of f and the set Y is called the *codomain* of f.

Functions go far beyond what one considers in calculus. For example, consider the following.

Example A.2.4. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Define $f : \mathbb{N} \to \mathbb{R}$ by $f(n) = a_n$ for all $n \in \mathbb{N}$. Then f is a function with domain $\mathbb{N}$ and range $\mathbb{R}$.

Remark A.2.5. Given $f : X \to Y$, it is important to remember that f is the function whereas $f(x)$ is not the function; $f(x)$ is the value of the function f at the point $x \in X$ and thus $f(x)$ is a single element of Y.

As functions are really subsets of a Cartesian Product and we have a notion for when two sets are equal, we have a notion for when two functions are equal.

Definition A.2.6. Let $f : X \to Y$ and let $g : A \to B$. We say that f *equals* g, denoted $f = g$, if

- $X = A$, and
- $f(x) = g(x)$ for all $x \in X$.

That is, two functions are equal if they have the same domain and the same value on each element of the domain.

There are many ways to construct new functions from other functions depending on the circumstances. The following is one common and useful way to construct new functions.

Definition A.2.7. Let X, Y, and Z be sets and let $f : X \to Y$ and let $g : Y \to Z$. The *composition of f and g* is the function $g \circ f : X \to Z$ such that

$$(g \circ f)(x) = g(f(x))$$

for all $x \in X$.

Example A.2.8. Consider the functions $g : \mathbb{R} \to \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2 + 2x - 1$ for all $x \in \mathbb{R}$ and $f(x) = \sqrt{x} + 1$ for all $x \in [0, \infty)$. The function $g \circ f : [0, \infty) \to \mathbb{R}$ is well-defined. Moreover, for all $x \in [0, \infty)$, we have that

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(\sqrt{x} + 1) \\ &= (\sqrt{x} + 1)^2 + 2(\sqrt{x} + 1) - 1 \\ &= x + 4\sqrt{x} - 1\end{aligned}$$

for all $x \in [0, \infty)$.

There are many properties and information one may want to describe about a function. We begin with the following.

Definition A.2.9. Let $f : X \to Y$ and let $Z \subseteq X$. The *image of Z under f*, denoted $f(Z)$, is the set

$$f(Z) = \{f(z) \mid z \in Z\} \subseteq Y.$$

The *range* (or *image*) *of f* is the set

$$\text{Range}(f) = f(X) = \{f(x) \mid x \in X\}.$$

Example A.2.10. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x} + 1$ for all $x \in [0, \infty)$. Assuming a knowledge of calculus, we can show that $\text{Range}(f) = [1, \infty)$. Furthermore $f([4, 9]) = [3, 4]$.

Definition A.2.11. A function $f : X \to Y$ is said to be *surjective* (or *onto*) if $f(X) = Y$; that is, for all $y \in Y$ there exists an $x \in X$ such that $f(x) = y$.

Example A.2.12. Clearly the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x} + 1$ for all $x \in [0, \infty)$ is not surjective since $\text{Range}(f) = [1, \infty) \neq \mathbb{R}$.

However, the function $g : [0, \infty) \to [1, \infty)$ defined by $g(x) = \sqrt{x} + 1$ for all $x \in [0, \infty)$ is surjective since $\text{Range}(g) = [1, \infty)$. Thus the notion of when a function is surjective or not is really a consideration of what one is thinking of for the codomain of the function. We can always decrease the codomain of a function to be equal to its range thereby making the function surjective.

One way to think of a surjective function is that it is a function that obtains all possible outputs. There is another property one might want to consider of a function: when does the function yield unique outputs given distinct inputs.

Definition A.2.13. A function $f : X \to Y$ is said to be *injective* (or *one-to-one*) if for all $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have that $f(x_1) \neq f(x_2)$.

Remark A.2.14. Note by taking the contrapositive of the definition of an injective function, we immediately see that a function $f : X \to Y$ is injective if whenever $x_1, x_2 \in X$ are such that $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Example A.2.15. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 2x + 1 = (x-1)^2$ for all $x \in \mathbb{R}$ is not injective since $0 \neq 2$ yet $f(0) = (0-1)^2 = (-1)^2 = 1$ and $f(2) = (2-1)^2 = 1^2 = 1 = f(0)$.

However, it is possible using calculus to show that the function $g : [1, \infty) \to \mathbb{R}$ defined by $g(x) = (x-1)^2$ for all $x \in [1, \infty)$ is injective. Thus the notion of when a function is injective or not depends on the domain of the function.

Of course, we can combine the notions of injective and surjective.

Definition A.2.16. A function $f : X \to Y$ is said to be *bijective* if f is injective and surjective.

Example A.2.17. The function $f : [0, \infty) \to [1, \infty)$ defined by $f(x) = \sqrt{x}+1$ for all $x \in [0, \infty)$ has already been seen to be surjective. As it is possible using calculus to show that f is injective, we obtain that f is a bijective function.

Similarly, the function $g : [1, \infty) \to [0, \infty)$ defined by $g(x) = (x-1)^2$ for all $x \in [1, \infty)$ has already be seen to be injective with the given domain. With the codomain written, it is possible using calculus to show that g is surjective. Hence g is a bijective function.

With f and g as in the above example, notice for all $x \in [0, \infty)$ that

$$g(f(x)) = g\left(\sqrt{x} + 1\right) = \left(\left(\sqrt{x} + 1\right) - 1\right)^2 = \sqrt{x}^2 = x$$

and for all $y \in [1, \infty)$ that

$$f(g(y)) = f\left((y-1)^2\right) = \sqrt{(y-1)^2} + 1 = (y-1) + 1 = y.$$

Thus perhaps there is a connection between bijective functions and the following type of function.

Definition A.2.18. A function $f : X \to Y$ is said to be *invertible* if there exists a function $g : Y \to X$ such that

- $(g \circ f)(x) = g(f(x)) = x$ for all $x \in X$, and
- $(f \circ g)(y) = f(g(y)) = y$ for all $y \in Y$.

The function g is called an *inverse of f*.

Remark A.2.19. Note the two conditions required for f to be an invertible function make this a 'two-sided inverse'. This is similar to how wants to be able to multiple the inverse of a matrix A on either side of A and still get the identity.

Notice in Definition A.2.18 that we called g 'a' inverse of f and not 'the' inverse of f. This is because, for all we know, it might be possible that f has multiple inverses. The following shows this is not the case.

Lemma A.2.20. *Let $f : X \to Y$ be invertible. If g_1 and g_2 are inverse of f, then $g_1 = g_2$.*

Proof. Assume g_1 and g_2 are both inverses of f. Then for all $y \in Y$, we see by the defining properties of an inverse that

$$\begin{aligned} g_1(y) &= (g_2 \circ f)(g_1(y)) \\ &= g_2(f(g_1(y))) \\ &= g_2((f \circ g_1)(y)) \\ &= g_2(y). \end{aligned}$$

Hence $g_1 = g_2$ as desired. ∎

As Lemma A.2.20 demonstrates that there can be at most one inverse of an invertible function, we desire some notation to denote this function.

Notation A.2.21. If $f : X \to Y$ is an invertible function, the inverse of f is denoted by f^{-1}.

To culminate our exploration of bijective and invertible functions, we prove the following.

Theorem A.2.22. *Let $f : X \to Y$. Then f is invertible if and only if f is bijective.*

Proof. First, assume f is invertible. Thus f^{-1} exists. To see that f is bijective, we must show that f is injective and surjective.

<u>f is injective.</u> To see that f is injective, let $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. Then

$$x_1 = g(f(x_1)) = g(f(x_2)) = x_2$$

where the middle equality follows since $f(x_1) = f(x_2)$. Therefore f is injective by definition.

f is surjective. To see that f is surjective, let $y \in Y$ be arbitrary. Let $x = g(y) \in X$. Then

$$f(x) = f(g(y)) = y.$$

Hence $y \in f(X)$. Therefore, since $y \in Y$ was arbitrary, $f(X) = Y$. Hence f is surjective by definition.

Since f is injective and surjective, f is bijective as desired.

To see the converse direction, assume $f : X \to Y$ is bijective. To show that f is invertible, we must construct an inverse of f. To do this, let $y \in Y$ be arbitrary. Since f is surjective, we know that there exists an $x_1 \in X$ such that $f(x_1) = y$. Moreover, since f is injective, we know that if $x_2 \in X$ such that $f(x_2) = y$, then $f(x_2) = f(x_1)$ so that $x_2 = x_1$. Hence, for each $y \in Y$ there exists a unique element of x, which we will denoted by x_y, such that $f(x_y) = y$.

Consider the function $g : Y \to X$ defined by $g(y) = x_y$ for. Note g is well-defined by the above paragraph. We claim that g is an inverse of f. To see this, first note for all $y \in Y$ that

$$f(g(y)) = f(x_y) = y$$

as desired. To see that $g(f(x)) = x$ for all $x \in X$, let $x \in X$ be arbitrary. Since $f(x) = y$, we have that $x = x_y$ by the definition of x_y being the unique element of X that f sends to y. Hence

$$g(f(x)) = g(y) = x_y = x$$

as desired. Thus g is the inverse of f thereby completing the proof. ■

To complete our introduction to functions, there is one more set based on functions we desire to study.

Definition A.2.23. Given $f : X \to Y$ and a subset $Z \subseteq Y$, the *preimage of Z under f* (or *inverse image*), denoted $f^{-1}(Z)$, is the set

$$f^{-1}(Z) = \{x \in X \mid f(x) \in Z\}.$$

Example A.2.24. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{R}$. Then

$$f^{-1}(\{1\}) = \{1, -1\}$$
$$f^{-1}([4, \infty)) = (-\infty, -2] \cup [2, \infty)$$
$$f^{-1}((-\infty, 0)) = \emptyset.$$

Remark A.2.25. It is important to note that although the notation $f^{-1}(Z)$ is used for the preimage of a set Z under f, the preimage of a set has nothing to do with invertibility and one does not need a function to be invertible to consider the preimage.

However, when $f : X \to Y$ is invertible and $Z \subseteq Y$, then, because f is bijective, it is possible to show that the preimage of Z under f is the equal to the image of Z under f^{-1}. To see this, assume $f : X \to Y$ is invertible and let $Z \subseteq Y$. As we normally use $f^{-1}(Z)$ to denote both sets we are interested in, let

$$A = \{x \in X \mid f(x) \in Z\} \qquad \text{(i.e. the preimage of } Z \text{ under } f)$$
$$B = \{f^{-1}(z) \mid z \in Z\} \qquad \text{(i.e. the image of } Z \text{ under } f^{-1}).$$

To see that $A = B$, we will show that $A \subseteq B$ and $B \subseteq A$.

To see that $A \subseteq B$, let $a \in A$ be arbitrary. By the definition of A, this implies $f(a) \in Z$. Therefore, by the definition of B, $f^{-1}(f(a)) \in B$. Since $a = f^{-1}(f(a))$, we obtain that $a \in B$. Therefore, since $a \in A$ was arbitrary, $A \subseteq B$.

To see that $B \subseteq A$, let $b \in B$ be arbitrary. By the definition of B, there exists a $z \in Z$ such that $f^{-1}(z) = b$. Hence

$$f(b) = f(f^{-1}(z)) = z \in Z.$$

Therefore $b \in A$ by the definition of A. Hence, as $b \in B$ was arbitrary, $B \subseteq A$.

Hence $A = B$ so, when f is invertible, the preimage of Z under f is the equal to the image of Z under f^{-1}.

To prove some results in this course, it is helpful to know how the preimage of sets behave under unions and intersections.

Proposition A.2.26. *Let $f : X \to Y$, let I be a non-empty set, and for each $\alpha \in I$ let $Z_\alpha \subseteq Y$. Then*

$$f^{-1}\left(\bigcup_{\alpha \in I} Z_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(Z_\alpha) \qquad \textit{and} \qquad f^{-1}\left(\bigcap_{\alpha \in I} Z_\alpha\right) = \bigcap_{\alpha \in I} f^{-1}(Z_\alpha).$$

Proof. The proof of this result is very similar to the proof of De Morgan's Laws.

Notice that

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{\alpha \in I} Z_\alpha\right) & \text{if and only if } f(x) \in \bigcup_{\alpha \in I} Z_\alpha \\ & \text{if and only if } f(x) \in Z_\alpha \text{ for at least one } \alpha \in I \\ & \text{if and only if } x \in f^{-1}(Z_\alpha) \text{ for at least one } \alpha \in I \\ & \text{if and only if } x \in \bigcup_{\alpha \in I} f^{-1}(Z_\alpha) \end{aligned}$$

which completes the proof of the first equation.

Similarly, notice that

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{\alpha \in I} Z_\alpha\right) &\text{if and only if } f(x) \in \bigcap_{\alpha \in I} Z_\alpha \\ &\text{if and only if } f(x) \in Z_\alpha \text{ for all } \alpha \in I \\ &\text{if and only if } x \in f^{-1}(Z_\alpha) \text{ for all } \alpha \in I \\ &\text{if and only if } x \in \bigcap_{\alpha \in I} f^{-1}(Z_\alpha) \end{aligned}$$

which completes the proof of the second equation. ∎

A.3 Equivalence Relations

Functions, although the most prevalent type of relation, are not the only special type of relation that is useful in analysis. Specifically, this section will focus on another type of relation that mimics the basic properties of equality.

Definition A.3.1. A relation R on a set X is said to be an *equivalence relation* if R has the following three properties:

- (*reflexive*) xRx for all $x \in X$.
- (*symmetric*) If $x, y \in X$ are such that xRy, then yRx.
- (*transitive*) If $x, y, z \in X$ are such that xRy and yRz, then xRz.

Remark A.3.2. It is common in mathematics to denote an equivalence relation by $\sim$.

Example A.3.3. Let $R = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$. We claim that R is an equivalence relation. To see this, we must show that the three properties of an equivalence relation hold.

Reflexivity. To see that R is reflexive, let $x \in \mathbb{R}$ be arbitrary. Since $x = x$, we have by the definition of R that xRx. Hence, since $x \in \mathbb{R}$ was arbitrary, R is reflexive.

Symmetry. To see that R is symmetric, let $x, y \in \mathbb{R}$ be such that xRy. Thus the definition of R implies that $x = y$. Hence $y = x$ so that yRx. Therefore, since $x, y \in \mathbb{R}$ were arbitrary, R is symmetric.

Transitivity. To see that R is transitive, let $x, y, z \in \mathbb{R}$ be such that xRy and yRz. Therefore $x = y$ and $y = z$ by definition. Hence $x = z$ so xRz. Therefore, since $x, y, z \in \mathbb{R}$ were arbitrary, R is transitive.

Therefore, since R is reflexive, symmetric, and transitive, R is an equivalence relation.

Example A.3.4. Let $R = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}$. To determine whether R is an equivalence relation, we will examine whether the three properties of an equivalence relation hold. If all three hold, then R is an equivalence relation. However, if at least one property fails, then R is not an equivalence relation.

Reflexivity. We claim that R is reflexive. To see this, let $x \in \mathbb{R}$ be arbitrary. Since $x \leq x$, we have that xRx. Therefore, since $x \in \mathbb{R}$ was arbitrary, R is reflexive.

Symmetry. We claim that R is not symmetric. To see this, note that $2R3$ since $2 \leq 3$, but $3\not R2$ since $2 \not\leq 3$. Hence R is not symmetric.

Transitivity. We claim that R is transitive. To see this, let $x, y, z \in \mathbb{R}$ be such that xRy and yRz. Therefore $x \leq y$ and $y \leq z$ by definition. Hence $x \leq z$ so xRz. Therefore, since $x, y, z \in \mathbb{R}$ were arbitrary, R is transitive.

Since R is not symmetric, R is not an equivalence relation.

Example A.3.5. Recall given $m, n \in \mathbb{Z}$, it is said that n *divides* m, denoted $n|m$ if there exists a $d \in \mathbb{Z}$ such that $dn = m$.

Fix $n \in \mathbb{N}$ and consider the relation

$$R = \{(m, k) \in \mathbb{Z}^2 \mid n|(m - k)\}.$$

We claim that R is an equivalence relation on $\mathbb{Z}$. To see this, we must show that the three properties of an equivalence relation hold.

Reflexivity. To see that R is reflexive, let $m \in \mathbb{Z}$ be arbitrary. Since $0n = 0$ and $0 \in \mathbb{Z}$, we have by definition that $n|0$. Hence, since $m - m = 0$, we have that $n|(m - m)$ so mRm by definition. Therefore, since $m \in \mathbb{Z}$ was arbitrary, R is reflexive.

Symmetry. To see that R is symmetric, let $m, k \in \mathbb{Z}$ be such that mRk. Hence $n|(m - k)$ by the definition of R. By the definition of divides, this implies there exists a $d \in \mathbb{Z}$ such that $dn = m - k$. Therefore, since

$$(-d)n = -dn = -(m - k) = k - m$$

and since $-d \in \mathbb{Z}$, we have that $n|(k - m)$ by the definition of divides and thus kRm. Hence, since $m, k \in \mathbb{Z}$ was arbitrary, R is symmetric.

Transitivity. To see that R is transitive, let $m, k, \ell \in \mathbb{Z}$ be such that mRk and $kR\ell$. Hence $n|(m - k)$ and $n|(k - \ell)$ by definition. By the definition of divides, this implies there exist $d, b \in \mathbb{Z}$ such that $dn = m - k$ and $bn = k - \ell$. Therefore, since

$$(d + b)n = dn + bn = (m - k) + (k - \ell) = m - \ell$$

and since $d + b \in \mathbb{Z}$, we have that $n|(m - \ell)$ by the definition of divides and thus $mR\ell$. Hence, since $m, k, \ell \in \mathbb{Z}$ was arbitrary, R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation. The equivalence relation R is called the *equivalence modulo n* equivalence relation. Moreover, if mRk, we say that *m is equivalent to k modulo n* and write $m \equiv k \mod n$.

Equivalence relations related to analysis can be seen in Sections B.1 and B.2, and in Lemma 3.1.8. For now, we turn to trying to think of equivalence relations as a form of equality. Of course there can exist objects that are equivalent with respect to an equivalence relation that are not equal. However, if we combine all elements that are equivalence to a given element, we get sets that are quite useful. These sets are some form of 'modding out by the equivalence relation' to construct objects that are equal based on the equivalence relation. To begin, let's be formal about these sets of equivalent elements.

Definition A.3.6. Let $\sim$ be an equivalence relation on a set X. For each $a \in X$, the *equivalence class of a with respect to $\sim$* is

$$[a] = \{x \in X \mid x \sim a\}.$$

Example A.3.7. Consider the equivalence modulo 2 equivalence relation. Then

$$[1] = \{\text{odd integers}\} \quad \text{and} \quad [0] = \{\text{even integers}\}.$$

To show that elements being equivalent under an equivalence relation relates to equality of the equivalence classes, we prove the following.

Lemma A.3.8. *Let $\sim$ be an equivalence relation on a set X and let $a, b \in X$. Then $a \sim b$ if and only if $[a] = [b]$.*

Proof. To begin, assume that $a \sim b$. To show that $[a] = [b]$, we will show that $[a] \subseteq [b]$ and $[b] \subseteq [a]$.

To see that $[a] \subseteq [b]$, let $x \in [a]$ be arbitrary. Hence $x \sim a$ by the definition of an equivalence class. Since equivalence relations are transitive, $x \sim a$ and $a \sim b$ implies that $x \sim b$. Hence $x \in [b]$ by the definition of an equivalence class. Therefore, since $x \in [a]$ was arbitrary, $[a] \subseteq [b]$.

To see that $[b] \subseteq [a]$, note since equivalence relations are symmetric that $a \sim b$ implies $b \sim a$. Therefore by interchanging a and b in the previous paragraph, we obtain that $[b] \subseteq [a]$. Hence $[a] = [b]$ as desired.

To see the converse direction, assume $[a] = [b]$. Since equivalence relations are reflexive, we know that $a \sim a$ so $a \in [a]$ by the definition of an equivalence class. Therefore $a \in [a] = [b]$. Since $a \in [b]$, the definition of an equivalence class implies that $a \sim b$ as desired. ■

The main benefit of considering equivalence classes is that they provide a partition of the entire space into sets of equivalent elements. The following result formalize this.

Proposition A.3.9. *Let $\sim$ be an equivalence relation on a set X and let*

$$\mathcal{C} = \{[a] \mid a \in X\}.$$

Then

a) for all $A \in \mathcal{C}$, $A \neq \emptyset$,

b) for all $x \in X$ *there exists an* $A \in \mathcal{C}$ *such that* $x \in A$, *and*

c) if $A, B \in \mathcal{C}$ *and* $A \neq B$, *then* $A \cap B = \emptyset$.

Proof. a) Let $A \in \mathcal{C}$ be arbitrary. By the definition of $\mathcal{C}$, $A = [a]$ for some $a \in X$. Since equivalence relations are reflexive, we know that $a \sim a$ so $a \in [a] = A$ by the definition of an equivalence class. Since $a \in A$, $A \neq \emptyset$. Therefore, since $A \in \mathcal{C}$ was arbitrary, the result holds.

b) Let $x \in X$ be arbitrary. Let $A = [x]$ so that $A \in \mathcal{C}$. Since equivalence relations are reflexive, we know that $x \sim x$ so $x \in [x] = A$ by the definition of an equivalence class. Therefore, since $x \in X$ was arbitrary, the result follows.

c) Let $A, B \in \mathcal{C}$ be such that $A \neq B$. Since $A, B \in \mathcal{C}$, the definition of $\mathcal{C}$ implies that there exists $a, b \in X$ such that $A = [a]$ and $B = [b]$.

Suppose for the sake of a contradiction that $A \cap B \neq \emptyset$. Hence there exists an $x \in X$ such that $x \in A \cap B$. Thus $x \in A = [a]$ and $x \in B = [b]$. Since $x \in [a]$ and $x \in [b]$, we have by the definition of an equivalence class that $x \sim a$ and $x \sim b$. Since equivalence relations are symmetric, $x \sim a$ implies $a \sim b$. Therefore, since equivalence relations are transitive and since $a \sim x$ and $x \sim b$, we obtain that $a \sim b$. Hence Lemma A.3.8 implies that $A = [a] = [b] = B$. Hence we have a contradiction to the fact that $A \neq B$ so $A \cap B \neq \emptyset$ as desired. ∎

One reason equivalence classes are important will be see in Sections B.1 and B.2 where equivalence classes are used to construct the integers and rational numbers from the natural numbers. Moreover, equivalence classes are used in Proposition 3.1.9 to describe all of the open subsets of the real numbers.

Appendix B

Constructing Number Systems

This appendix chapter is completely devoted to proving the existence and uniqueness of a totally ordered field with the Least Upper Bound Property; that is, the existence and uniqueness of the real numbers.

To begin, Section B.1 will construct and verifying the necessary properties of the integers from the natural numbers and Peaon's Axioms (Definition 1.1.1). Of course, since the existence and properties of the integers may seem elementary to the reader, so it possible to skip this section. However, we have included it to be completely formal in our axiomatic construction of the real numbers, to demonstrate the techniques that will be used in the subsequent section, and as it will simplify one argument in Section B.5.

Section B.2 will construct the rational numbers from the integers. In particular, it will be demonstrated that the rational numbers are a totally ordered field. From the rational numbers, we will then construct the real numbers in two ways.

Section B.3 will derive the real numbers from the rational numbers in a set theoretic way. In particular, the real numbers will be defined to be certain subsets of the rational numbers known as Dedekind cuts. It will be demonstrated that there are operations on Dedekind cuts that turn the real numbers into a field, although this is a colossal pain. The benefits of this approach are that it requires absolutely no knowledge of real analysis and that the total ordering and Least Upper Bound Property of the real numbers is very natural and are easily proved.

Section B.4 will derive the real numbers from the rational numbers in a more analytic way. Specifically, the real numbers will be defined to be equivalence classes of Cauchy sequences of rational numbers. It will easily follow that there are operations on these equivalence classes that turn the real numbers into a field. However, with this approach, the total ordering and Least Upper Bound Property are more complicated. The benefits of this approach are that it is generally viewed to be as an easier approach, it can be generalized to other objects (see MATH 4011), and it is much more thematic with this course on real analysis as it utilizes many of the techniques and ideas learnt throughout the course.

Section B.5 will demonstrate that every totally ordered field with the Least Upper Bound Property must be the real numbers. By this we mean any two ordered fields with the Least Upper Bound Property must be 'isomorphic' in the appropriate sense. In particular, the two objects that were constructed in Section B.3 and Section B.4 are the same even though they look different. To prove the desired result, we will first show that every totally ordered field with the Least Upper Bound Property contain the rational numbers (with the same ordering). Then the Least Upper Bound Property to show that any two totally ordered field with the Least Upper Bound Property must be isomorphic and thus equal to the real numbers.

B.1 Integers

In this section, we will construct the integers from the natural numbers. To do this, we will be assuming the existence of the natural numbers, the operations of addition and multiplication on the natural numbers, the total ordering on the natural numbers, and that the total ordering on the natural numbers has the additive property. All of these assumptions follow from Peaon's Axioms (Definition 1.1.1) once think about them for long enough.

To construct the integers, the idea is that we want to close the natural numbers under subtraction; that is, for all $m, n \in \mathbb{N}$ we want $m - n$ to make sense. Of course we can consider the set of all pairs (m, n) and think of this pair as representing $m - n$. However, as $3 - 1 = 5 - 3$, we want to be able to identify two pairs (m_1, n_1) and (m_2, n_2) as equal via a property of the natural numbers. Note that $m_1 - n_1 = m_2 - n_2$ in what we think of as the integers if and only if $m_1 + n_2 = m_2 + n_1$. Since the latter only involves natural numbers, we can define an equivalence relation on pairs of natural numbers that will lead to the integers:

Lemma B.1.1. *Let $X_\mathbb{Z} = \{(m, n) \mid m, n \in \mathbb{N}\}$ and define a relation $\sim_\mathbb{Z}$ on $X_\mathbb{Z}$ by $(m_1, n_1) \sim_\mathbb{Z} (m_2, n_2)$ if and only if $m_1 + n_2 = m_2 + n_1$. Then $\sim_\mathbb{Z}$ is an equivalent relation.*

Proof. To see that $\sim_\mathbb{Z}$ is an equivalence relation, we need to show that $\sim_\mathbb{Z}$ is reflexive, symmetric, and transitive.

Reflexive: To see that $\sim_\mathbb{Z}$ is reflexive, let $(m, n) \in X_\mathbb{Z}$. Since $m + n = m + n$, we see that $(m, n) \sim_\mathbb{Z} (m, n)$ by the definition of $\sim_\mathbb{Z}$ as desired.

Symmetric: To see that $\sim_\mathbb{Z}$ is symmetric, let $(m_1, n_1), (m_2, n_2) \in X_\mathbb{Z}$ be such that $(m_1, n_1) \sim_\mathbb{Z} (m_2, n_2)$. Hence $m_1 + n_2 = m_2 + n_1$. Thus $m_2 + n_1 = m_1 + n_2$ so that $(m_2, n_2) \sim_\mathbb{Z} (m_1, n_1)$. Hence $\sim_\mathbb{Z}$ is symmetric.

Transitive: To see that $\sim_{\mathbb{Z}}$ is transitive, let $(m_1, n_1), (m_2, n_2), (m_3, n_3) \in X_{\mathbb{Z}}$ be such that $(m_1, n_1) \sim_{\mathbb{Z}} (m_2, n_2)$ and $(m_2, n_2) \sim_{\mathbb{Z}} (m_3, n_3)$. Hence $m_1 + n_2 = m_2 + n_1$ and $m_2 + n_3 = m_3 + n_2$. By adding these two equations, we obtain that

$$m_1 + n_2 + m_2 + n_3 = m_2 + n_1 + m_3 + n_2.$$

Hence

$$m_1 + n_3 + (n_2 + m_2) = m_3 + n_1 + (n_2 + m_2).$$

Therefore, by the properties of the natural numbers, we obtain that $m_1 + n_3 = m_3 + n_1$. Hence $(m_1, n_1) \sim_{\mathbb{Z}} (m_3, n_3)$. Thus $\sim_{\mathbb{Z}}$ is transitive.

Therefore, since all three properties have been verified, $\sim_{\mathbb{Z}}$ is an equivalence relation. ■

By taking the equivalence class of the equivalence relation in Lemma B.1.1, we have constructed the integers.

Definition B.1.2. Let $X_{\mathbb{Z}}$ and $\sim_{\mathbb{Z}}$ be as in Lemma B.1.1. The *integers*, denoted $\mathbb{Z}$, are the set of equivalence classes of $\sim_{\mathbb{Z}}$; that is,

$$\mathbb{Z} = \{[(m, n)] \mid m, n \in \mathbb{N}\}$$

where $[x]$ denotes the equivalence class of x with respect to $\sim_{\mathbb{Z}}$.

Now that the integers have been constructed, we desire to extend the notions of addition and multiplication from the natural numbers to the integers. Later we will extend the partial ordering on the natural numbers to the integers.

Clearly if (m, n) represents $m - n$, we want to define $(m_1, n_1) + (m_2, n_2)$ and $(m_1, n_1) \cdot (m_2, n_2)$ as we would expect; namely

$$(m_1 - n_1) + (m_2 - n_2) = (m_1 + m_2) - (n_1 + n_2)$$
$$(m_1 - n_1) \cdot (m_2 - n_2) = (m_1 m_2 + n_1 n_2) - (m_1 n_2 + m_2 n_1).$$

The only possible problem with this is that the integers have been constructed as a set of equivalence classes. This means that there are more than one representative for each equivalence class. Therefore, to make sure that we have a well-defined definition for addition and multiplication that we can use without worrying whether it depends on the representative of the equivalence class, we prove the following.

Lemma B.1.3. *Let $(m_1, n_1), (m_2, n_2), (m_1', n_1'), (m_2', n_2') \in X_{\mathbb{Z}}$ be such that $(m_1, n_1) \sim_{\mathbb{Z}} (m_1', n_1')$ and $(m_2, n_2) \sim_{\mathbb{Z}} (m_2', n_2')$. Then*

a) $(m_1 + m_2, n_1 + n_2) \sim_{\mathbb{Z}} (m_1' + m_2', n_1' + n_2')$.

b) $(m_1 m_2 + n_1 n_2, m_1 n_2 + m_2 n_1) \sim_{\mathbb{Z}} (m_1' m_2' + n_1' n_2', m_1' n_2' + m_2' n_1')$.

Proof. Since $(m_1, n_1) \sim_{\mathbb{Z}} (m_1', n_1')$ and $(m_2, n_2) \sim_{\mathbb{Z}} (m_2', n_2')$, we know that $m_1 + n_1' = m_1' + n_1$ and $m_2 + n_2' = m_2' + n_2$.

a) To see that $(m_1+m_2, n_1+n_2) \sim_{\mathbb{Z}} (m_1'+m_2', n_1'+n_2')$, notice by adding the two equations above that we obtain

$$(m_1 + m_2) + (n_1' + n_2') = (m_1' + m_2') + (n_1 + n_2).$$

Hence $(m_1 + m_2, n_1 + n_2) \sim_{\mathbb{Z}} (m_1' + m_2', n_1' + n_2')$ by definition.

b) To see that

$$(m_1 m_2 + n_1 n_2, m_1 n_2 + m_2 n_1) \sim_{\mathbb{Z}} (m_1' m_2' + n_1' n_2', m_1' n_2' + m_2' n_1'),$$

notice by multiplying $m_1 + n_1' = m_1' + n_1$ by m_2 and n_2, we obtain that

$$\begin{aligned} m_1 m_2 + m_2 n_1' &= m_1' m_2 + m_2 n_1 \text{ and} \\ m_1 n_2 + n_1' n_2 &= m_1' n_2 + n_1 n_2. \end{aligned}$$

Hence

$$(m_1 m_2 + m_2 n_1') + (m_1' n_2 + n_1 n_2) = (m_1' m_2 + m_2 n_1) + (m_1 n_2 + n_1' n_2).$$

Similarly, by multiplying $m_2 + n_2' = m_2' + n_2$ by m_1' and n_1', we obtain that

$$\begin{aligned} m_2 m_1' + m_1' n_2' &= m_1' m_2' + m_1' n_2 \text{ and} \\ m_2 n_1' + n_1' n_2' &= m_2' n_1' + n_1' n_2. \end{aligned}$$

Hence

$$(m_2 m_1' + m_1' n_2') + (m_2' n_1' + n_1' n_2) = (m_1' m_2' + m_1' n_2) + (m_2 n_1' + n_1' n_2').$$

By adding these two large equations together, we obtain that

$$\begin{aligned} &((m_1 m_2 + n_1 n_2) + (m_1' n_2' + m_2' n_1')) + (m_2 n_1' + m_1' n_2 + m_2 m_1' + n_1' n_2) \\ &= ((m_1' m_2' + n_1' n_2') + (m_1 n_2 + m_2 n_1)) + (m_1' m_2 + n_1' n_2 + m_1' n_2 + m_2 n_1') . \end{aligned}$$

Hence, by the properties of the natural numbers, we obtain that

$$(m_1 m_2 + n_1 n_2) + (m_1' n_2' + m_2' n_1') = (m_1' m_2' + n_1' n_2') + (m_1 n_2 + m_2 n_1).$$

Thus

$$(m_1 m_2 + n_1 n_2, m_1 n_2 + m_2 n_1) \sim_{\mathbb{Z}} (m_1' m_2' + n_1' n_2', m_1' n_2' + m_2' n_1'),$$

by definition. ∎

Due to Lemma B.1.3, the following operations on $\mathbb{Z}$ are now well-defined.

Definition B.1.4. The operations of $+$ and $\cdot$ on $\mathbb{Z}$ are defined by

$$\begin{aligned}[(m_1, n_1)] + [(m_2, n_2)] &= [(m_1 + m_2, n_1 + n_2)] \\ [(m_1, n_1)] \cdot [(m_2, n_2)] &= [(m_1 m_2 + n_1 n_2, m_1 n_2 + m_2 n_1)].\end{aligned}$$

Since we have just constructed $\mathbb{Z}$ from $\mathbb{N}$, we must verify that addition and multiplication on $\mathbb{Z}$ satisfy the natural desired properties.

Theorem B.1.5. *For all* $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$*, the following hold:*

a) (Commutativity of +) $[(m_1, n_1)] + [(m_2, n_2)] = [(m_2, n_2)] + [(m_1, n_1)]$.

b) (Commutativity of $\cdot$) $[(m_1, n_1)] \cdot [(m_2, n_2)] = [(m_2, n_2)] \cdot [(m_1, n_1)]$.

c) (Associativity of +)

$$([(m_1, n_1)] + [(m_2, n_2)]) + [(m_3, n_3)] = [(m_1, n_1)] + ([(m_2, n_2)] + [(m_3, n_3)]).$$

d) (Associativity of $\cdot$)

$$([(m_1, n_1)] \cdot [(m_2, n_2)]) \cdot [(m_3, n_3)] = [(m_1, n_1)] \cdot ([(m_2, n_2)] \cdot [(m_3, n_3)]).$$

e) (Distributivity)

$$[(m_1, n_1)] \cdot ([(m_2, n_2)] + [(m_3, n_3)]) = ([(m_1, n_1)] \cdot [(m_2, n_2)]) + ([(m_1, n_1)] \cdot [(m_3, n_3)]).$$

f) (Additive Unit) $[(m_1, n_1)] + [(1, 1)] = [(m_1, n_1)]$.

g) (Multiplicative Unit) $[(m_1, n_1)] \cdot [(2, 1)] = [(m_1, n_1)]$ *and* $[(2, 1)] \neq [(1, 1)]$.

h) (Additive Unit) $[(m_1, n_1)] + [(n_1, m_1)] = [(1, 1)]$

Proof. a) Notice that

$$\begin{aligned}[(m_1, n_1)] + [(m_2, n_2)] &= [(m_1 + m_2, n_1 + n_2)] \\ &= [(m_2 + m_1, n_2 + n_1)] \\ &= [(m_2, n_2)] + [(m_1, n_1)]\end{aligned}$$

as desired.

b) Notice that

$$\begin{aligned}[(m_1, n_1)] \cdot [(m_2, n_2)] &= [(m_1 m_2 + n_1 n_2, m_1 n_2 + m_2 n_1)] \\ &= [(m_2 m_1 + n_2 n_1, m_2 n_1 + m_1 n_2)] \\ &= [(m_2, n_2)] \cdot [(m_1, n_1)]\end{aligned}$$

as desired.

c) Notice that

$$\begin{aligned}
&([(m_1, n_1)] + [(m_2, n_2)]) + [(m_3, n_3)] \\
&= [(m_1 + m_2, n_1 + n_2)] + [(m_3, n_3)] \\
&= [((m_1 + m_2) + m_3, (n_1 + n_2) + n_2)] \\
&= [(m + 1 + (m_2 + m_3), n_1 + (n_2 + n_3)] \\
&= [(m_1, n_1)] + [(m_2 + m_3, n_2 + n_3)] \\
&= [(m_1, n_1)] + ([(m_2, n_2)] + [(m_3, n_3)])
\end{aligned}$$

as desired.

d) Notice that

$$\begin{aligned}
&([(m_1, n_1)] \cdot [(m_2, n_2)]) \cdot [(m_3, n_3)] \\
&= [(m_1m_2 + n_1n_2, m_1n_2 + m_2n_1)] \cdot [(m_3, n_3)] \\
&= [((m_1m_2 + n_1n_2)m_3 + (m_1n_2 + m_2n_1)n_3, (m_1m_2 + n_1n_2)n_3 + m_3(m_1n_2 + m_2n_1))] \\
&= [(m_1m_2m_3 + n_1n_2m_3 + m_1n_2n_3 + n_1m_2n_3, m_1m_2n_3 + n_1n_2n_3 + m_1n_2m_3 + n_1m_2m_3)] \\
&= [(m_1(m_2m_3 + n_2n_3) + n_1(m_2n_3 + m_3n_2), m_1(m_2n_3 + m_3n_2) + (m_2m_3 + n_2n_3)n_1)] \\
&= [(m_1, n_1)] \cdot ([(m_2m_3 + n_2n_3, m_2n_3 + m_3n_2)]) \\
&= [(m_1, n_1)] \cdot ([(m_2, n_2)] \cdot [(m_3, n_3)])
\end{aligned}$$

as desired.

e) Notice that

$$\begin{aligned}
&[(m_1, n_1)] \cdot ([(m_2, n_2)] + [(m_3, n_3)]) \\
&= [(m_1, n_1)] \cdot [(m_2 + m_3, n_2 + n_3)] \\
&= [(m_1(m_2 + m_3) + n_1(n_2 + n_3), m_1(n_2 + n_3) + (m_2 + m_3)n_1)] \\
&= [((m_1m_2 + n_1n_2) + (m_1m_3 + n_1n_3), (m_1n_2 + m_2n_1) + (m_1n_3 + m_3n_1))] \\
&= [(m_1m_2 + n_1n_2, m_1n_2 + m_2n_1)] + [(m_1m_3 + n_1n_3, m_1n_3 + m_3n_1)] \\
&= ([(m_1, n_1)] \cdot [(m_2, n_2)]) + ([(m_1, n_1)] \cdot [(m_3, n_3)])
\end{aligned}$$

as desired.

f) First note that $2 + 1 = 3 \neq 2 = 1 + 1$ so $(2, 1) \not\sim_{\mathbb{Z}} (1, 1)$ so $[(2, 1)] \neq [(1, 1)]$. Next notice that

$$[(m_1, n_1)] + [(1, 1)] = [(m_1 + 1, n_1 + 1)]$$

Since

$$(m_1 + 1) + n_1 = m_1 + (n_1 + 1),$$

we see that

$$(m_1 + 1, n_1 + 1) \sim_{\mathbb{Z}} (m_1, n_1)$$

and thus

$$[(m_1, n_1)] + [(1, 1)] = [(m_1 + 1, n_1 + 1)] = [(m_1, n_1)]$$

as desired.

g) Notice that

$$[(m_1, n_1)] \cdot [(2, 1)] = [(2m_1 + n_1, m_1 + 2n_1)].$$

Since

$$(2m_1 + n_1) + n_1 = m_1 + (m_1 + 2n_1),$$

we see that

$$(2m_1 + n_1, m_1 + 2n_1) \sim_{\mathbb{Z}} (m_1, n_1)$$

and thus

$$[(m_1, n_1)] \cdot [(2, 1)] = [(2m_1 + n_1, m_1 + 2n_1)] = [(m_1, n_1)]$$

as desired.

h) Finally, notice that

$$[(m_1, n_1)] + [(n_1, m_1)] = [(m_1 + n_1, m_1 + n_1)].$$

Since

$$(m_1 + n_1) + 1 = 1 + (m_1 + n_1),$$

we see that

$$(m_1 + n_1, m_1 + n_1) \sim_{\mathbb{Z}} (1, 1)$$

and thus

$$[(m_1, n_1)] + [(n_1, m_1)] = [(m_1 + n_1, m_1 + n_1)] = [(1, 1)]$$

as desired. ∎

Remark B.1.6. Note Theorem B.1.5 shows some important properties when it comes to viewing $\mathbb{Z}$ via these equivalence classes. First, Theorem B.1.5 shows that $[(1, 1)]$ is the additive unit of $\mathbb{Z}$ (after all $1 - 1 = 0$). Thus we will use 0 to denote $[(1, 1)]$. Moreover, for all $[(m, n)] \in \mathbb{Z}$, we see that $-[(m, n)]$ (the additive inverse of $[(m, n)]$) is $[(n, m)]$ (after all $-(m - n) = n - m$). Furthermore Theorem B.1.5 shows that $[(2, 1)]$ is the multiplicative unit of $\mathbb{Z}$ (after all $2 - 1 = 1$). Thus we will use 1 to denote $[(2, 1)]$. Notice that $-1 = [(1, 2)]$ by these discussions. Moreover we see from part f) that $0 \neq 1$.

Before extending the partial order on $\mathbb{N}$ to $\mathbb{Z}$, we will note some properties of the integers relating to addition and multiplication that are required in Section B.2. We begin with the following.

Lemma B.1.7. *For all* $[(m, n)] \in \mathbb{Z}$, $[(m, n)] = 0$ *if and only if* $m = n$.

Proof. Notice that $[(m, n)] = 0 = [(1, 1)]$ if and only if $(m, n) \sim_{\mathbb{Z}} (1, 1)$ if and only if $m + 1 = n + 1$ if and only if $m = n$ as desired. ∎

It will be useful for us to know that the additive inverse of an element is equal to -1 times the element. This is not immediate from the properties demonstrated for $\mathbb{Z}$ and needs to be proved.

Lemma B.1.8. *For all* $[(m,n)] \in \mathbb{Z}$, $-[(m,n)] = (-1)\cdot[(m,n)]$.

Proof. Notice that

$$\begin{aligned}(-1)\cdot[(m,n)] &= [(1,2)]\cdot[(m,n)]\\ &= [(1(m)+2(n), 1(n)+2(m))]\\ &= [(m+2n, n+2m)]\\ &= [(m+n, m+n)] + [(n,m)]\\ &= 0 + [(n,m)] && \text{by Lemma B.1.7}\\ &= [(n,m)] = -[(m,n)]\end{aligned}$$

as desired. ∎

The following shows that our natural view that

$$\mathbb{Z} = \{n \mid n \in \mathbb{N}\} \cup \{-n \mid n \in \mathbb{N}\} \cup \{0\}$$

can give a nice representation of the equivalence relations.

Lemma B.1.9. *For all* $[(m,n)] \in \mathbb{Z}$ *there exists a* $k \in \mathbb{N}$ *such that* $[(m,n)] = [(k,1)]$ *or* $[(m,n)] = [(1,k)]$.

Proof. Let $[(m,n)] \in \mathbb{Z}$. To prove that there exists a $k \in \mathbb{N}$ such that $[(m,n)] = [(k,1)]$ or $[(m,n)] = [(1,k)]$, we divide the proof into three cases.

Case 1: $m = n$. In this case, we claim that $[(m,n)] = [(1,1)]$. To see this, notice since $m = n$ that

$$m + 1 = n + 1.$$

Hence $(m,n) \sim_{\mathbb{Z}} (1,1)$ so that $[(m,n)] = [(1,1)]$ as desired.

Case 2: $m > n$. In this case, there exists a $\ell \in \mathbb{N}$ such that $n + \ell = m$. Let $k = \ell + 1$ so that $k \in \mathbb{N}$ and $k \neq 1$. We claim that $[(m,n)] = [(k,1)]$. To see this, notice that

$$m + 1 = (n + \ell) + 1 = n + k.$$

Hence $(m,n) \sim_{\mathbb{Z}} (k,1)$ so that $[(m,n)] = [(k,1)]$ as desired.

Case 3: $m < n$. In this case, there exists a $\ell \in \mathbb{N}$ such that $m + \ell = n$. Let $k = \ell + 1$ so that $k \in \mathbb{N}$ and $k \neq 1$. We claim that $[(m,n)] = [(1,k)]$. To see this, notice that

$$m + k = (m + \ell) + 1 = n + 1.$$

Hence $(m,n) \sim_{\mathbb{Z}} (1,k)$ so that $[(m,n)] = [(1,k)]$ as desired.

As the three cases cover all possibilities for $(m,n) \in X_{\mathbb{Z}}$, the proof is complete. ∎

The next property we desire is to know what elements of $\mathbb{Z}$ when multiplied together can yield zero. Of course this is obvious to us considering out knowledge of the integers, but it needs to be verified based on our definition.

Lemma B.1.10. *Let* $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$. *Then* $[(m_1, n_1)] \cdot [(m_2, n_2)] = 0$ *if and only if* $[(m_1, n_1)] = 0$ *or* $[(m_2, n_2)] = 0$.

Proof. To begin, assume $[(m_1, n_1)] = 0$. Thus Lemma B.1.7 implies that $m_1 = n_1$. Therefore, Lemma B.1.7 implies that

$$\begin{aligned}[(m_1, n_1)] \cdot [(m_2, n_2)] &= [(m_1m_2 + n_1n_2, m_1n_2 + m_2n_1)] \\ &= [(m_1m_2 + m_1n_2, m_1n_2 + m_2m_1)] = 0\end{aligned}$$

as desired.

Similarly if $[(m_2, n_2)] = 0$ then $[(m_1, n_1)] \cdot [(m_2, n_2)] = 0$.

To complete the proof, assume $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ are such that $[(m_1, n_1)] \neq 0$ and $[(m_2, n_2)] \neq 0$. Our goal is to show that $[(m_1, n_1)] \cdot [(m_2, n_2)] \neq 0$. By Lemma B.1.9, we can divide the proof into four cases.

Case 1: $[(m_1, n_1)] = [(k_1, 1)]$, $[(m_2, n_2)] = [(k_2, 1)]$ for $k_1, k_2 \in \mathbb{N} \setminus \{1\}$. Notice in this case that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(k_1k_2 + 1, k_1 + k_2)].$$

Notice since $k_1, k_2 \in \mathbb{N} \setminus \{1\}$ that

$$k_1k_2 \geq 2\max\{k_1, k_2\} \geq k_1 + k_2.$$

Hence $k_1k_2 + 1 > k_1 + k_2$ so Lemma B.1.7 implies that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(k_1k_2 + 1, k_1 + k_2)] \neq 0$$

as desired.

Case 2: $[(m_1, n_1)] = [(1, k_1)]$, $[(m_2, n_2)] = [(k_2, 1)]$ for $k_1, k_2 \in \mathbb{N} \setminus \{1\}$. Notice in this case that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(k_1 + k_2, 1 + k_1k_2)].$$

Notice since $k_1, k_2 \in \mathbb{N} \setminus \{1\}$ that

$$k_1k_2 \geq 2\max\{k_1, k_2\} \geq k_1 + k_2.$$

Hence $1 + k_1k_2 > k_1 + k_2$ so Lemma B.1.7 implies that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(k_1 + k_2, 1 + k_1k_2)] \neq 0$$

as desired.

Case 3: $[(m_1, n_1)] = [(k_1, 1)]$, $[(m_2, n_2)] = [(1, k_2)]$ for $k_1, k_2 \in \mathbb{N} \setminus \{1\}$. This follows by Case 2 and commutativity of $\cdot$.

Case 4: $[(m_1, n_1)] = [(1, k_1)]$, $[(m_2, n_2)] = [(1, k_2)]$ for $k_1, k_2 \in \mathbb{N} \setminus \{1\}$. Notice in this case that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(1 + k_1k_2, k_1 + k_2)].$$

Notice since $k_1, k_2 \in \mathbb{N} \setminus \{1\}$ that

$$k_1k_2 \geq 2\max\{k_1, k_2\} \geq k_1 + k_2.$$

Hence $1 + k_1k_2 > k_1 + k_2$ so Lemma B.1.7 implies that

$$[(m_1, n_1)] \cdot [(m_2, n_2)] = [(k_1 + k_2, 1 + k_1k_2)] \neq 0$$

as desired.

Hence the proof is complete. ■

The benefit of Lemma B.1.10 is that it enables one to cancel off multiplication as the following result shows.

Corollary B.1.11. *If* $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$ *are such that* $[(m_1, n_1)] \neq 0$ *and* $[(m_1, n_1)] \cdot [(m_2, n_2)] = [(m_1, n_1)] \cdot [(m_3, n_3)]$*, then* $[(m_2, n_2)] = [(m_3, n_3)]$

Proof. Suppose $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$ are such that $[(m_1, n_1)] \neq 0$ and $[(m_1, n_1)] \cdot [(m_2, n_2)] = [(m_1, n_1)] \cdot [(m_3, n_3)]$. To begin, we claim that

$$([(m_1, n_1)] \cdot [(m_3, n_3)]) + ([(m_1, n_1)] \cdot [(n_3, m_3)]) = 0.$$

To see this, notice that

$$\begin{aligned}
&([(m_1, n_1)] \cdot [(m_3, n_3)]) + ([(m_1, n_1)] \cdot [(n_3, m_3)]) \\
&= [(m_1m_3 + n_1n_3, m_1n_3 + m_3n_1)] + [(m_1n_3 + n_1m_3, m_1m_3 + n_1n_3)] \\
&= [((m_1m_3 + n_1n_3) + (m_1n_3 + n_1m_3), (m_1n_3 + m_3n_1) + (m_1m_3 + n_1n_3))] \\
&= 0
\end{aligned}$$

by Lemma B.1.7. Therefore

$$\begin{aligned}
0 &= ([(m_1, n_1)] \cdot [(m_3, n_3)]) + ([(m_1, n_1)] \cdot [(n_3, m_3)]) \\
&= ([(m_1, n_1)] \cdot [(m_2, n_2)]) + ([(m_1, n_1)] \cdot [(n_3, m_3)]) \\
&= [(m_1, n_1)] \cdot ([(m_2, n_2)] + [(n_3, m_3)]).
\end{aligned}$$

Hence, since $[(m_1, n_1)] \neq 0$, Lemma B.1.10 implies that

$$[(m_2, n_2)] + [(n_3, m_3)] = 0.$$

Therefore, since $[(m_3, n_3)] + [(n_3, m_3)] = 0$ by Theorem B.1.5 part h), we obtain that

$$\begin{aligned}
[(m_3, n_3)] &= [(m_3, n_3)] + 0 && \text{Theorem B.1.5 part f)} \\
&= [(m_3, n_3)] + ([(m_2, n_2)] + [(n_3, m_3)]) \\
&= [(m_2, n_2)] + ([(m_3, n_3)] + [(n_3, m_3)]) && \text{by associativity and commutivity} \\
&= [(m_2, n_2)] + 0 \\
&= [(m_2, n_2)] && \text{Theorem B.1.5 part f)}
\end{aligned}$$

as desired. ∎

With the above technical details out of the way, we can turn our attention to defining the partial ordering on $\mathbb{Z}$. Clearly if (m, n) represents $m - n$, we want to define $(m_1, n_1) \leq (m_2, n_2)$ so that $m_1 - n_1 \leq m_2 - n_2$. Thinking of how we want the order on $\mathbb{Z}$ to behave, we can reduce this to the equivalent characterization $m_1 + n_2 \leq m_2 + n_1$ which only the ordering on $\mathbb{N}$. As with addition and multiplication, the only possible problem with this is that the integers have been constructed as a set of equivalence classes. This means that there are more than one representative for each equivalence class. Therefore, to make sure that we have a well-defined definition for the partial ordering that we can use without worrying whether it depends on the representative of the equivalence class, we prove the following.

Lemma B.1.12. *Let $(m_1, n_1), (m_2, n_2), (m_1', n_1'), (m_2', n_2') \in X_{\mathbb{Z}}$ be such that $(m_1, n_1) \sim_{\mathbb{Z}} (m_1', n_1')$ and $(m_2, n_2) \sim_{\mathbb{Z}} (m_2', n_2')$. Then $m_1 + n_2 \leq m_2 + n_1$ if and only if $m_1' + n_2' \leq m_2' + n_1'$.*

Proof. Since $(m_1, n_1) \sim_{\mathbb{Z}} (m_1', n_1')$ and $(m_2, n_2) \sim_{\mathbb{Z}} (m_2', n_2')$, we know that $m_1 + n_1' = m_1' + n_1$ and $m_2 + n_2' = m_2' + n_2$. Therefore, by the properties of the natural numbers,

$$\begin{aligned}
m_1 + n_2 &\leq m_2 + n_1 && \text{if and only if} \\
(m_1 + n_2) + (n_1' + n_2') &\leq (m_2 + n_1) + (n_1' + n_2') && \text{if and only if} \\
(m_1 + n_1') + n_2 + n_2' &\leq (m_2 + n_2') + n_1 + n_1' && \text{if and only if} \\
(m_1' + n_1) + n_2 + n_2' &\leq (m_2' + n_2) + n_1 + n_1' && \text{if and only if} \\
(m_1' + n_2') + (n_1 + n_2) &\leq (m_2' + n_1') + (n_1 + n_2) && \text{if and only if} \\
m_1' + n_2' &\leq m_2' + n_1'
\end{aligned}$$

as desired. ∎

By Lemma B.1.12, the following is well-defined.

Definition B.1.13. The relation of $\leq$ in $\mathbb{Z}$ is defined by $[(m_1, n_1)] \leq [(m_2, n_2)]$ if and only if $m_1 + n_2 \leq m_2 + n_1$.

Of course it is necessary to prove that Definition B.1.13 does yield a partial ordering.

Theorem B.1.14. *The relation of $\leq$ on $\mathbb{Z}$ is a partial ordering on $\mathbb{Z}$. That is, for all $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$,*

a) (Reflexivity) $[(m_1, n_1)] \leq [(m_1, n_1)]$.

b) (Antisymmetry) If $[(m_1, n_1)] \leq [(m_2, n_2)]$ and $[(m_2, n_2)] \leq [(m_1, n_1)]$, then $[(m_1, n_1)] = [(m_2, n_2)]$.

c) (Transitivity) If $[(m_1, n_1)] \leq [(m_2, n_2)]$ and $[(m_2, n_2)] \leq [(m_3, n_3)]$, then $[(m_1, n_1)] \leq [(m_3, n_3)]$.

Proof. a) To see that $[(m_1, n_1)] \leq [(m_1, n_1)]$, notice that $m_1 + n_1 \leq m_1 + n_1$ and thus $[(m_1, n_1)] \leq [(m_1, n_1)]$ by definition.

b) Assume that $[(m_1, n_1)] \leq [(m_2, n_2)]$ and $[(m_2, n_2)] \leq [(m_1, n_1)]$. Therefore $m_1 + n_2 \leq m_2 + n_1$ and $m_2 + n_1 \leq m_1 + n_2$. Therefore, by the properties of $\leq$ on $\mathbb{N}$, $m_1 + n_2 = m_2 + n_1$. Thus $(m_1, n_1) \sim_{\mathbb{Z}} (m_2, n_2)$ by definition so that $[(m_1, n_1)] = [(m_2, n_2)]$ as desired.

c) Assume that $[(m_1, n_1)] \leq [(m_2, n_2)]$ and $[(m_2, n_2)] \leq [(m_3, n_3)]$. Therefore $m_1 + n_2 \leq m_2 + n_1$ and $m_2 + n_3 \leq m_3 + n_2$. Notice that

$$\begin{aligned}
(m_1 + n_3) + n_2 &= (m_1 + n_2) + n_3 \\
&\leq (m_2 + n_1) + n_3 \\
&= (m_2 + n_3) + n_1 \\
&\leq (m_3 + n_2) + n_1 \\
&= (m_3 + n_1) + n_2.
\end{aligned}$$

Therefore, by the properties of $\leq$ on $\mathbb{N}$, we obtain that $m_1 + n_3 \leq m_3 + n_1$. Hence $[(m_1, n_1)] \leq [(m_3, n_3)]$ as desired. ∎

Since we will want to show that the ordering on the real numbers is a total ordering with the additive and multiplicative properties, it is important for us to prove the following as a first step.

Lemma B.1.15. *The partial ordering $\leq$ on $\mathbb{Z}$ has the following additional properties:*

a) $\leq$ is a total ordering; that is, if $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$, then $[(m_1, n_1)] \leq [(m_2, n_2)]$ or $[(m_2, n_2)] \leq [(m_1, n_1)]$.

b) $\leq$ has the additive property; that is, if $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$ and $[(m_1, n_1)] \leq [(m_2, n_2)]$ then $[(m_1, n_1)] + [(m_3, n_3)] \leq [(m_2, n_2)] + [(m_3, n_3)]$.

c) $\leq$ has the multiplicative property; that is, if $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ are such that $0 \leq [(m_1, n_1)]$ and $0 \leq [(m_2, n_2)]$, then $0 \leq [(m_1, n_1)] \cdot [(m_2, n_2)]$.

Proof. a) To see that $\leq$ is a total ordering, let $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ be arbitrary. Note by the properties of the ordering on the natural numbers that $m_1 + n_2 \leq m_2 + n_1$ or $m_2 + n_1 \leq m_1 + n_2$. Hence $[(m_1, n_1)] \leq [(m_2, n_2)]$ or $[(m_2, n_2)] \leq [(m_1, n_1)]$ as desired.

b) Assume $[(m_1, n_1)], [(m_2, n_2)], [(m_3, n_3)] \in \mathbb{Z}$ are such that $[(m_1, n_1)] \leq [(m_2, n_2)]$. To see that $[(m_1, n_1)] + [(m_3, n_3)] \leq [(m_2, n_2)] + [(m_3, n_3)]$, note $[(m_1, n_1)] \leq [(m_2, n_2)]$ implies that $m_1 + n_2 \leq m_2 + n_1$. Therefore

$$\begin{aligned}(m_1 + m_3) + (n_2 + n_3) &= (m_1 + n_2) + (m_3 + n_3) \\ &\leq (m_2 + n_1) + (m_3 + n_3) \\ &= (m_2 + m_3) + (n_1 + n_3).\end{aligned}$$

Hence

$$\begin{aligned}[(m_1, n_1)] + [(m_3, n_3)] &= [(m_1 + m_3, n_1 + n_3)] \\ &\leq [(m_2 + m_3, n_2 + n_3)] \\ &= [(m_2, n_2)] + [(m_3, n_3)]\end{aligned}$$

as desired.

c) Assume $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ are such that $0 \leq [(m_1, n_1)]$ and $0 \leq [(m_2, n_2)]$. By Lemma B.1.9, there exists $k, \ell \in \mathbb{N}$ so that $[(m_1, n_1)] = [(k, 1)]$ or $[(m_1, n_1)] = [(1, k)]$, and $[(m_2, n_2)] = [(\ell, 1)]$ or $[(m_2, n_2)] = [(1, \ell)]$. Notice if $[(m_1, n_1)] = [(1, k)]$, then the facts that $0 \leq [(m_1, n_1)]$ and $0 = [(1, 1)]$ imply that $1 + k \leq 1 + 1$ so that $k = 1$ by the properties of the ordering on the natural numbers and thus $[(m, n)] = [(1, 1)]$. Therefore, without loss of generality $[(m_1, n_1)] = [(k, 1)]$. Similarly we can assume without loss of generality that $[(m_2, n_2)] = [(\ell, 1)]$.

To complete the proof, we consider three cases.

<u>Case 1: $k = 1$.</u> Assume $k = 1$. Then $[(m_1, n_1)] = [(1, 1)]$. Therefore, since $\leq$ is a partial ordering,

$$[(1, 1)] \leq [(1, 1)] = [(m_2 + n_2, n_2 + m_2)] = [(m_1, n_1)] \cdot [(m_2, n_2)]$$

as desired.

<u>Case 2: $\ell = 1$.</u> Assume $\ell = 1$. Then $[(m_2, n_2)] = [(1, 1)]$. Therefore, since $\leq$ is a partial ordering,

$$[(1, 1)] \leq [(1, 1)] = [(m_1 + n_1, n_1 + m_1)] = [(m_1, n_1)] \cdot [(m_2, n_2)]$$

as desired.

<u>Case 2: $k \neq 1$ and $\ell \neq 1$.</u> Assume $k \neq 1$ and $\ell \neq 1$. Therefore, by the properties of the ordering on the natural numbers, we know that $k + \ell \leq k\ell$. Hence $1 + (k + \ell) \leq (k\ell + 1) + 1$. Therefore, by the definition of $\leq$, we obtain that

$$0 = [(1, 1)] \leq [(k\ell + 1, k + \ell)] = [(k, 1)] \cdot [(\ell, 1)] = [(m_1, n_1)] \cdot [(m_2, n_2)]$$

as desired.

Therefore, as the above three cases cover all possibilities, the result follows. ∎

To conclude this section, it is useful to note that the integers are indeed an extension of the natural numbers. After all, because of the way we defined the integers as equivalence relations of pairs of natural numbers, this is by no means clear. However, there is a natural way the natural numbers embed into the integers. Of course, since $(n + 1) - 1 = n$, we should be able to represent n as $[(n + 1, 1)]$ for all $n \in \mathbb{N}$.

Lemma B.1.16. *There exists a map $f : \mathbb{N} \to \mathbb{Z}$ such that*

- *f is injective,*
- *$f(1)$ is the multiplicative unit of $\mathbb{Z}$,*
- *$f(n + m) = f(n) + f(m)$ for all $n, m \in \mathbb{N}$,*
- *$f(nm) = f(n) \cdot f(m)$ for all $n, m \in \mathbb{N}$, and*
- *for $n, m \in \mathbb{N}$, $n \leq m$ if and only if $f(n) \leq f(m)$.*

Proof. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) = [(n + 1, 1)]$$

for all $n \in \mathbb{N}$. We claim that f is the map we are looking for.

To see that f is injective, assume $n, m \in \mathbb{N}$ are such that $f(n) = f(m)$. Therefore $[(n + 1, 1)] = [(m + 1, 1)]$ so that $(n + 1, 1) \sim_{\mathbb{Z}} (m + 1, 1)$ and thus $(n + 1) + 1 = (m + 1) + 1$. Thus $n + 2 = m + 2$ so, by the properties of the natural numbers, we have that $n = m$. Therefore, as n and m were arbitrary, f is injective.

Clearly $f(1) = [(2, 1)]$ is the multiplicative unit of $\mathbb{Z}$. Next, assume $n, m \in \mathbb{N}$. Then

$$f(n) + f(m) = [(n + 1, 1)] + [(m + 1, 1)] = [(n + m + 2, 2)].$$

Therefore, since $(n+m+2)+1 = (n+m+1)+2$, we have that $(n+m+2, 2) \sim_{\mathbb{Z}} (n + m + 1, 1)$ so that

$$f(n) + f(m) = [(n + m + 2, 2)] = [(n + m + 1, 1)] = f(n + m)$$

as desired.

Next, assume $n, m \in \mathbb{N}$. Then

$$\begin{aligned} f(n) \cdot f(m) &= [(n + 1, 1)] \cdot [(m + 1, 1)] \\ &= [((n + 1)(m + 1) + (1)(1), (n + 1)(1) + (m + 1)(1)] \\ &= [(nm + n + m + 2, n + m + 2)]. \end{aligned}$$

Therefore since

$$(nm + n + m + 2) + 1 = (nm + 1) + (n + m + 2),$$

we have that

$$(nm + n + m + 2, n + m + 2) \sim_{\mathbb{Z}} (nm + 1, 1)$$

and thus

$$f(n) \cdot f(m) = [(nm + n + m + 2, n + m + 2)] = [(nm + 1, 1)] = f(nm)$$

as desired.

Finally, for $n, m \in \mathbb{N}$, we notice that

$$\begin{aligned} f(n) \leq f(m) & \qquad \text{if and only if} \\ [(n+1, 1)] \leq [(m+1, 1)] & \qquad \text{if and only if} \\ (n+1) + 1 \leq (m+1) + 1 & \qquad \text{if and only if} \\ n + 2 \leq m_2 & \qquad \text{if and only if} \\ n \leq m & \end{aligned}$$

as desired.

Hence f has all of the desired properties so $\mathbb{N}$ naturally lies inside of $\mathbb{Z}$. ∎

B.2 Rational Numbers

With the construction of the integers complete, we will turn to constructing the rational numbers. Due to Section B.1, we know that $\mathbb{Z}$ has all the necessary properties to prove the results in this section. Moreover, we will be reverting to our usual notation for the integers.

To construct the rational, the idea is that we want to close the integers under division by non-zero numbers; that is, for all $a, b \in \mathbb{Z}$ with $b \neq 0$ we want $\frac{a}{b}$ to make sense. Of course we can consider the set of all pairs (a, b) and think of this pair as representing $\frac{a}{b}$. However, as $\frac{3}{6} = \frac{2}{4}$, we want to be able to identify two pairs (a_1, b_1) and (a_2, b_2) as equal via a property of the integers. Note that $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ in what we think of as the rational numbers if and only if $a_1 b_2 = a_2 b_1$. Since the latter only involves integers, we can define an equivalence relation on pairs of integers that will lead to the rational number:

Lemma B.2.1. *Let*

$$X_{\mathbb{Q}} = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$$

and define a relation $\sim_{\mathbb{Q}}$ *on* $X_{\mathbb{Q}}$ *by* $(a_1, b_1) \sim_{\mathbb{Q}} (a_2, b_2)$ *if and only if* $a_1 b_2 = a_2 b_1$. *Then* $\sim_{\mathbb{Q}}$ *is an equivalent relation.*

Proof. Exercise. ■

By taking the equivalence class of the equivalence relation in Lemma B.2.1, we have constructed the integers.

Definition B.2.2. Let $X_{\mathbb{Q}}$ and $\sim_{\mathbb{Q}}$ be as in Lemma B.2.1. The *rational numbers*, denoted $\mathbb{Q}$, are the set of equivalence classes of $\sim_{\mathbb{Q}}$; that is,

$$\mathbb{Q} = \{[(a, b)] \mid a, b \in \mathbb{Z}, b \neq 0\}$$

where $[x]$ denotes the equivalence class of x with respect to $\sim_{\mathbb{Q}}$.

Now that the rational numbers have been constructed, we desire to extend the notions of addition and multiplication from the integers to the rational numbers. Later we will extend the partial ordering on the integers to the rational numbers.

Clearly if (a, b) represents $\frac{a}{b}$, we want to define $(a_1, b_1) + (a_2, b_2)$ and $(a_1, b_1) \cdot (a_2, b_2)$ as we would expect; namely

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1b_2 + a_2b_1}{b_1b_2}$$
$$\frac{a_1}{b_1} \cdot \frac{a_2}{b_2} = \frac{a_1a_2}{b_1b_2}.$$

The only possible problem with this is that the rational numbers have been constructed as a set of equivalence classes. This means that there are more than one representative for each equivalence class. Therefore, to make sure that we have a well-defined definition for addition and multiplication that we can use without worrying whether it depends on the representative of the equivalence class, we prove the following. (Note Lemma B.1.10 implies if $b_1, b_2 \in \mathbb{Z} \setminus \{0\}$, then $b_1b_2 \neq 0$ so all of the objects in the following are well-defined.)

Lemma B.2.3. *Let $(a_1, b_1), (a_2, b_2), (a_1', b_1'), (a_2', b_2') \in X_{\mathbb{Q}}$ be such that $(a_1, b_1) \sim_{\mathbb{Q}} (a_1', b_1')$ and $(a_2, b_2) \sim_{\mathbb{Q}} (a_2', b_2')$. Then*

a) $(a_1b_2 + a_2b_1, b_1b_2) \sim_{\mathbb{Q}} (a_1'b_2' + a_2'b_1', b_1'b_2')$ and

b) $(a_1a_2, b_1b_2) \sim_{\mathbb{Q}} (a_1'a_2', b_1'b_2')$

Proof. Exercise. ■

Due to Lemma B.2.3, the following operations on $\mathbb{Q}$ are now well-defined.

Definition B.2.4. The operations of $+$ and $\cdot$ on $\mathbb{Q}$ are defined by

$$[(a_1, b_1)] + [(a_2, b_2)] = [(a_1b_2 + a_2b_1, b_1b_2)]$$
$$[(a_1, b_1)] \cdot [(a_2, b_2)] = [(a_1a_2, b_1b_2)].$$

Since we have just constructed $\mathbb{Q}$ from $\mathbb{Z}$, we must verify that addition and multiplication on $\mathbb{Q}$ satisfy the natural desired properties. In particular, we desire to show that $\mathbb{Q}$ is a field.

Theorem B.2.5. *The rational numbers $\mathbb{Q}$ together with the operations $+$ and $\cdot$ are a field. That is, for all $[(a_1, b_1)], [(a_2, b_2)], [(a_3, b_3)] \in \mathbb{Q}$, the following hold:*

a) (Commutativity of +) $[(a_1, b_1)] + [(a_2, b_2)] = [(a_2, b_2)] + [(a_1, b_1)]$.

b) (Commutativity of ·) $[(a_1, b_1)] \cdot [(a_2, b_2)] = [(a_2, b_2)] \cdot [(a_1, b_1)]$.

c) (Associativity of +) $([(a_1, b_1)]+[(a_2, b_2)])+[(a_3, b_3)] = [(a_1, b_1)]+([(a_2, b_2)]+[(a_3, b_3)])$.

d) (Associativity of ·) $([(a_1, b_1)] \cdot [(a_2, b_2)]) \cdot [(a_3, b_3)] = [(a_1, b_1)] \cdot ([(a_2, b_2)] \cdot [(a_3, b_3)])$.

e) (Additive Unit) $[(a_1, b_1)] + [(0, 1)] = [(a_1, b_1)]$.

f) (Multiplicative Unit) $[(a_1, b_1)] \cdot [(1, 1)] = [(a_1, b_1)]$ *with* $[(1, 1)] \neq [(0, 1)]$.

g) (Distributivity) $[(a_1, b_1)] \cdot ([(a_2, b_2)] + [(a_3, b_3)]) = ([(a_1, b_1)] \cdot [(a_2, b_2)]) + ([(a_1, b_1)] \cdot [(a_3, b_3)])$.

h) (Additive Inverse) $[(a_1, b_1)] + [(-a_1, b_1)] = [(0, 1)]$.

i) (Multiplicative Inverse) if $[(a_1, b_1)] \neq [(0, 1)]$*, then* $(b_1, a_1) \in X_{\mathbb{Q}}$ *and* $[(b_1, a_1)] \cdot [(a_1, b_1)] = [(1, 1)]$.

Proof. a) Notice that

$$\begin{aligned}[(a_1, b_1)] + [(a_2, b_2)] &= [(a_1b_2 + a_2b_1, b_1b_2)]\\ &= [(a_2b_1 + a_1b_2, b_2b_1)]\\ &= [(a_2, b_2)] + [(a_1, b_1)]\end{aligned}$$

as desired.

b) Notice that

$$\begin{aligned}[(a_1, b_1)] \cdot [(a_2, b_2)] &= [(a_1a_2, b_1b_2)]\\ &= [(a_2a_1, b_2b_1)]\\ &= [(a_2, b_2)] \cdot [(a_1, b_1)]\end{aligned}$$

as desired.

c) Notice that

$$\begin{aligned}
&([(a_1, b_1)] + [(a_2, b_2)]) + [(a_3, b_3)] \\
&= [(a_1b_2 + a_2b_1, b_1b_2)] + [(a_3, b_3)] \\
&= [((a_1b_2 + a_2b_1)b_3 + a_3(b_1b_2), (b_1b_2)b_3)] \\
&= [(a_1(b_2b_3) + (a_2b_3 + a_3b_2)b_1, b_1(b_2b_3))] \\
&= [(a_1, b_1)] + [(a_2b_3 + a_3b_2, b_2b_3)] \\
&= [(a_1, b_1)] + ([(a_2, b_2)] + [(a_3, b_3)])
\end{aligned}$$

as desired.

d) Notice that

$$\begin{aligned}
([(a_1, b_1)] \cdot [(a_2, b_2)]) \cdot [(a_3, b_3)] &= [(a_1a_2, b_1b_2)] \cdot [(a_3, b_3)] \\
&= [((a_1a_2)a_3, (b_1b_2))b_3] \\
&= [(a_1(a_2a_3), b_1(b_2b_3))] \\
&= [(a_1, b_1)] \cdot [(a_2a_3, b_2b_3)] \\
&= [(a_1, b_1)] \cdot ([(a_2, b_2)] \cdot [(a_3, b_3)])
\end{aligned}$$

as desired.

e) Notice that

$$[(a_1, b_1)] + [(0, 1)] = [(a_1(1) + 0(b_1), b_11)] = [(a_1 + 0, b_1)] = [(a_1, b_1)]$$

f) Notice that

$$[(a_1, b_1)] \cdot [(1, 1)] = [(a_1(1), b_1(1))] = [(a_1, b_1)].$$

Moreover, since $1(1) = 1 \neq 0 = 0(1)$, we see that $(1, 1) \nsim_{\mathbb{Q}} (0, 1)$ so $[(1, 1)] \neq [(0, 1)]$.

g) To begin, first notice if $b \in \mathbb{Z} \setminus \{0\}$, then $b(1) = 1(b)$ so that $(b, b) \sim_{\mathbb{Q}} (1, 1)$ and thus $[(b, b)] = [(1, 1)]$. Therefore, by using part f), we obtain that

$$\begin{aligned}
&[(a_1, b_1)] \cdot ([(a_2, b_2)] + [(a_3, b_3)]) \\
&= [(a_1, b_1)] \cdot [(a_2b_3 + a_3b_2, b_2b_3)] \\
&= [(a_1(a_2b_3 + a_3b_2), b_1(b_2b_3)] \\
&= [((a_1a_2)b_3 + (a_1a_3)b_2, b_1b_2b_3)] \\
&= [((a_1a_2)b_3 + (a_1a_3)b_2, b_1b_2b_3)] \cdot [(1, 1)] \\
&= [((a_1a_2)b_3 + (a_1a_3)b_2, b_1b_2b_3)] \cdot [(b_1, b_1)] \\
&= [(((a_1a_2)b_3 + (a_1a_3)b_2)b_1, (b_1b_2b_3)b_1)] \\
&= [((a_1a_2)(b_1b_3) + (a_1a_3)(b_1b_2), (b_1b_2)(b_1b_3)] \\
&= [(a_1a_2, b_1b_2)] + [(a_1a_3, b_1b_3)] \\
&= ([(a_1, b_1)] \cdot [(a_2, b_2)]) + ([(a_1, b_1)] \cdot [(a_3, b_3)])
\end{aligned}$$

as desired.

h) First notice since $b_1 \in \mathbb{Z} \setminus \{0\}$ that $0(1) = 0b_1^2$ so that $(0, b_1^2) \sim_{\mathbb{Q}} (0,1)$ and thus $[(0, b_1^2)] = [(0,1)]$. Therefore

$$\begin{aligned}
[(a_1, b_1)] + [(-a_1, b_1)] &= [(a_1 b_1 + (-a_1)b_1, b_1^2)] \\
&= [((a_1 + (-a_1)b_1, b_1^2)] \\
&= [(0b_1, b_1^2)] \\
&= [(0, b_1^2)] \\
&= [(0,1)]
\end{aligned}$$

as desired.

i) To complete this proof, assume $[(a_1, b_1)] \neq [(0,1)]$. We claim that this implies $a_1 \neq 0$. To see this, suppose for the sake of a contradiction that $a_1 = 0$. Then $a_1(1) = 0(1) = 0 = 0(b_1)$. Therefore $(a_1, b_1) \sim_{\mathbb{Q}} (0,1)$ so that $[(a_1, b_1)] \neq [(0,1)]$. As this is a contradiction, we obtain that $a_1 \neq 0$. Hence $(b_1, a_1) \in X_{\mathbb{Q}}$.

Finally, notice that $(b_1 a_1)(1) = (1)(a_1 b_1)$ so that $(b_1 a_1, a_1 b_1) \sim_{\mathbb{Q}} (1,1)$ and thus $[(a_1 b_1, a_1 b_1)] = [(1,1)]$. Hence

$$[(b_1, a_1)] \cdot [(a_1, b_1)] = [(b_1 a_1, a_1 b_1)] = [(1,1)]$$

as desired. ■

Remark B.2.6. Note Theorem B.2.5 shows some important properties when it comes to viewing $\mathbb{Q}$ via these equivalence classes. First, Theorem B.2.5 shows that $[(0,1)]$ is the additive unit of $\mathbb{Q}$ (after all $\frac{0}{1} = 0$). Thus we will use 0 to denote $[(0,1)]$. Moreover, for all $[(a,b)] \in \mathbb{Q}$, we see that $-[(a,b)]$ (the additive inverse of $[(a,b)]$) is $[(-a,b)]$ (after all $-\frac{a}{b} = \frac{-a}{b}$). Furthermore Theorem B.2.5 shows that $[(1,1)]$ is the multiplicative unit of $\mathbb{Q}$ (after all $\frac{1}{1} = 1$). Thus we will use 1 to denote $[(1,1)]$. Finally, for all $[(a,b)] \in \mathbb{Q}$ with $[(a,b)] \neq [(0,1)]$, we see that $[(a,b)]^{-1} = [(b,a)]$ (after all $\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$).

It is also useful to note since $\mathbb{Q}$ is a field that all the field properties from Lemma 1.2.1 hold. Specifically, we know that 0 is the unique additive unit, 1 is the unique multiplicative unit, $-x = (-1) \cdot x$ for all $x \in \mathbb{Q}$, and $0 \cdot x = 0$ for all $x \in \mathbb{Q}$. This immediately implies which element of $\mathbb{Q}$ can multiple to give 0.

Lemma B.2.7. *Let $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$. Then $[(a_1, b_1)] \cdot [(a_2, b_2)] = 0$ if and only if $[(a_1, b_1)] = 0$ or $[(a_2, b_2)] = 0$.*

Proof. By Lemma 1.2.1, if $[(a_1, b_1)] = 0$ or $[(a_2, b_2)] = 0$ then $[(a_1, b_1)] \cdot [(a_2, b_2)] = 0$.

To complete the proof, assume $[(a_1,b_1)], [(a_2,b_2)] \in \mathbb{Q}$ are such that $[(a_1,b_1)] \neq 0$ and $[(a_2,b_2)] \neq 0$. Suppose for the sake of a contradiction that $[(a_1,b_1)] \cdot [(a_2,b_2)] = 0$. Therefore

$$\begin{aligned}0 &= [(a_1,b_1)]^{-1} \cdot 0 \\ &= [(a_1,b_1)]^{-1} \cdot ([(a_1,b_1)] \cdot [(a_2,b_2)]) \\ &= \left([(a_1,b_1)]^{-1} \cdot [(a_1,b_1)]\right) \cdot [(a_2,b_2)] \\ &= 1 \cdot [(a_2,b_2)] \\ &= [(a_2,b_2)].\end{aligned}$$

Since this contradicts the fact that $[(a_2,b_2)] \neq 0$, we have a contradiction. Thus the result follows. ∎

Before moving on to constructing the partial ordering on $\mathbb{Q}$, we note the following descriptions of the additive and multiplicative identities in $\mathbb{Q}$.

Lemma B.2.8. *For all $[(a,b)] \in \mathbb{Q}$, $[(a,b)] = 0$ if and only if $a = 0$.*

Proof. Notice that $[(a,b)] = 0$ if and only if $(a,b) \sim_{\mathbb{Q}} (0,1)$ if and only if $a(1) = 0(b)$ if and only if $a = 0$ as desired. ∎

Lemma B.2.9. *For all $[(a,b)] \in \mathbb{Q}$, $[(a,b)] = 1$ if and only if $a = b$.*

Proof. Notice that $[(a,b)] = 1$ if and only if $(a,b) \sim_{\mathbb{Q}} (1,1)$ if and only if $a(1) = 1(b)$ if and only if $a = b$ as desired. ∎

To construct the partial ordering on $\mathbb{Q}$, it is first useful to have a better representation of the equivalence class defining $\mathbb{Q}$. In particular, thinking about $\mathbb{Q}$ as we normally would, if we are given $\frac{a}{b}$ for $a, b \in \mathbb{Z}$ with $b \neq 0$ then we can always force $b > 0$ by multiplying the top and bottom by -1 if needed. To make this formal with the actual definition of $\mathbb{Q}$, we prove the following.

Lemma B.2.10. *For all $[(a,b)] \in \mathbb{Q}$, there exists $a', b' \in \mathbb{Z}$ such that $b' > 0$ and $[(a,b)] = [(a',b')]$.*

Proof. Let $[(a,b)] \in \mathbb{Q}$. Since this implies $b \in \mathbb{Z} \setminus \{0\}$, we know that $b > 0$ or $b < 0$. Thus we divide the proof into two case.

Case 1: $b > 0$. In this case, we simply let $a' = a$ and $b' = b$. Clearly $b' > 0$ and $[(a,b)] = [(a',b')]$ as desired.

Case 2: $b < 0$. In this case, let $a' = -a$ and $b' = -b$. Since $b < 0$, we know by Lemma B.1.15 that $b + (-b) < 0 + (-b)$ and thus $0 < b'$. Moreover, since $-a = (-1)a$ and $-b = (-1)b$ by Lemma B.1.8, we see that

$$a'b = (-a)b = (-1)(ab) = a(-1)(b) = a(-b) = ab'.$$

Hence $(a',b') \sim_{\mathbb{Z}} (a,b)$ so $[(a,b)] = [(a',b')]$ as desired.

Therefore, as the above two cases cover all possibilities, the result follows. ∎

When constructing the partial order on $\mathbb{Q}$, by Lemma B.2.10 we only need consider elements $[(a,b)] \in \mathbb{Q}$ with $b > 0$. This enables us to greatly simplify the definition of the partial ordering.

With the above technical details out of the way, we can turn our attention to defining the partial ordering on $\mathbb{Q}$. Clearly if (a,b) represents $\frac{a}{b}$, we want to define $(a_1,b_1) \leq (a_2,b_2)$ so that $\frac{a_1}{b_1} \leq \frac{a_2}{b_2}$. Thinking of how we want the order on $\mathbb{Q}$ to behave and by assuming $b_1, b_2 > 0$, we can reduce this to the equivalent characterization $a_1b_2 \leq a-2b_1$ which only the ordering on $\mathbb{Z}$. As with addition and multiplication, the only possible problem with this is that the rational numbers have been constructed as a set of equivalence classes. This means that there are more than one representative for each equivalence class. Therefore, to make sure that we have a well-defined definition for the partial ordering that we can use without worrying whether it depends on the representative of the equivalence class, we prove the following.

Lemma B.2.11. *Let $(a_1,b_1),(a_2,b_2),(a_1',b_1'),(a_1',b_2') \in X_{\mathbb{Q}}$ be such that $b_1,b_2,b_1',b_2' > 0$, $(a_1,b_1) \sim_{\mathbb{Q}} (a_1',b_1')$, and $(a_2,b_2) \sim_{\mathbb{Q}} (a_2',b_2')$. Then $a_1b_2 \leq a_2b_1$ if and only if $a_1'b_2' \leq a_2'b_1'$.*

Proof. Since $(a_1,b_1) \sim_{\mathbb{Q}} (a_1',b_1')$, and $(a_2,b_2) \sim_{\mathbb{Q}} (a_2',b_2')$, we know that $a_1b_1' = a_1'b_1$ and $a_2b_2' = a_2'b_2$. Note by the properties of the ordering of the integers, we have by the same proof of Lemma 1.2.18 that if $a,b,c \in \mathbb{Z}$ are such that $b \geq 0$ then $a \leq c$ if and only if $ab \leq cb$. Therefore, we have that

$$\begin{aligned}
a_1b_2 &\leq a_2b_1 && \text{if and only if}\\
(a_1b_2)(b_1'b_2') &\leq (a_2b_1)(b_1'b_2') && \text{if and only if}\\
(a_1b_1')(b_2b_2') &\leq (b_1b_1')(a_2b_2') && \text{if and only if}\\
(a_1'b_1)(b_2b_2') &\leq (b_1b_1')(a_2'b_2) && \text{if and only if}\\
(b_1b_2)(a_1'b_2') &\leq (b_1b_2)(a_2'b_1') && \text{if and only if}\\
a_1'b_2' &\leq a_2'b_1'
\end{aligned}$$

as desired. ∎

Using Lemmata B.2.10 and B.2.11, the following relation on $\mathbb{Q}$ is well-defined.

Definition B.2.12. The relation of $\leq$ in $\mathbb{Q}$ is defined for all $[(a_1,b_1)],[(a_2,b_2)] \in \mathbb{Q}$ with $b_1,b_2 > 0$ by $[(a_1,b_1)] \leq [(a_2,b_2)]$ if and only if $a_1b_2 \leq a_2b_1$.

Of course it is necessary to prove that Definition B.2.12 does yield a partial ordering.

Lemma B.2.13. *The relation of $\leq$ on $\mathbb{Q}$ is a partial ordering on $\mathbb{Q}$. That is, for all $[(a_1,b_1)],[(a_2,b_2)],[(a_3,b_3)] \in \mathbb{Q}$ with $b_1,b_2,b_3 > 0$,*

a) (Reflexivity) $[(a_1,b_1)] \leq [(a_1,b_1)]$.

b) (Antisymmetry) If $[(a_1, b_1)] \leq [(a_2, b_2)]$ *and* $[(a_2, b_2)] \leq [(a_1, b_1)]$*, then* $[(a_1, b_1)] = [(a_2, b_2)]$.

c) (Transitivity) If $[(a_1, b_1)] \leq [(a_2, b_2)]$ *and* $[(a_2, b_2)] \leq [(a_3, b_3)]$*, then* $[(a_1, b_1)] \leq [(a_3, b_3)]$.

Proof. a) To see that $[(a_1, b_1)] \leq [(a_1, b_1)]$, notice that $a_1 b_1 \leq a_1 b_1$ and thus $[(a_1, b_1)] \leq [(a_1, b_1)]$ by definition.

b) Assume that $[(a_1, b_1)] \leq [(a_2, b_2)]$ and $[(a_2, b_2)] \leq [(a_1, b_1)]$. Therefore $a_1 b_2 \leq a_2 b_1$ and $a_2 b_1 \leq a_1 b_2$. Therefore, by the properties of $\leq$ on $\mathbb{Q}$, $a_1 b_2 = a_2 b_1$. Thus $(a_1, b_1) \sim_{\mathbb{Q}} (a_2, b_2)$ by definition so that $[(a_1, b_1)] = [(a_2, b_2)]$ as desired.

c) Assume that $[(a_1, b_1)] \leq [(a_2, b_2)]$ and $[(a_2, b_2)] \leq [(a_3, b_3)]$. Therefore $a_1 b_2 \leq a_2 b_1$ and $a_2 b_3 \leq a_3 b_2$. Recall by the properties of the ordering of the integers, we have by the same proof of Lemma 1.2.18 that if $a, b, c \in \mathbb{Z}$ are such that $b \geq 0$ then $a \leq c$ if and only if $ab \leq cb$. Therefore

$$\begin{aligned}
(a_1 b_3) b_2 &= (a_1 b_2) b_3 \\
&\leq (a_2 b_1) b_3 \\
&= (a_2 b_3) b_1 \\
&\leq (a_3 b_2) b_1 \\
&= (a_3 b_1) b_2
\end{aligned}$$

and thus $a_1 b_3 \leq a_3 b_1$. Hence $[(a_1, b_1)] \leq [(a_3, b_3)]$ as desired. ■

Since we will want to show that the ordering on the real numbers is a total ordering with the additive and multiplicative properties, it is important for us to prove the following as a first step.

Lemma B.2.14. *The partial ordering* $\leq$ *on* $\mathbb{Q}$ *has the following additional properties:*

a) $\leq$ *is a total ordering; that is, if* $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$ *with* $b_1, b_2 > 0$*, then* $[(a_1, b_1)] \leq [(a_2, b_2)]$ *or* $[(a_2, b_2)] \leq [(a_1, b_1)]$.

b) $\leq$ *has the additive property; that is, if* $[(a_1, b_1)], [(a_2, b_2)], [(a_3, b_3)] \in \mathbb{Q}$ *with* $b_1, b_2, b_3 > 0$ *and* $[(a_1, b_1)] \leq [(a_2, b_2)]$ *then* $[(a_1, b_1)] + [(a_3, b_3)] \leq [(a_2, b_2)] + [(a_3, b_3)]$.

c) $\leq$ *has the multiplicative property; that is, if* $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$ *with* $b_1, b_2 > 0$ *are such that* $0 \leq [(a_1, b_1)]$ *and* $0 \leq [(a_2, b_2)]$*, then* $0 \leq [(a_1, b_1)] \cdot [(a_2, b_2)]$.

Proof. a) To see that $\leq$ is a total ordering, let $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$ with $b_1, b_2 > 0$ be arbitrary. Since the ordering on $\mathbb{Z}$ is a total ordering, we know that $a_1 b_2 \leq a_2 b_1$ or $a_2 b_1 \leq a_1 b_2$. Hence $[(a_1, b_1)] \leq [(a_2, b_2)]$ or $[(a_2, b_2)] \leq [(a_1, b_1)]$ as desired.

b) Assume $[(a_1,b_1)], [(a_2,b_2)], [(a_3,b_3)] \in \mathbb{Q}$ with $b_1, b_2, b_3 > 0$ are such that $[(a_1,b_1)] \leq [(a_2,b_2)]$. To see that $[(a_1,b_1)] + [(a_3,b_3)] \leq [(a_2,b_2)] + [(a_3,b_3)]$, note $[(a_1,b_1)] \leq [(a_2,b_2)]$ implies that $a_1b_2 \leq a_2b_1$. Therefore, since the ordering on $\mathbb{Z}$ has both the additive and multiplicative properties, we have by the same proof of Lemma 1.2.18 that if $a, b, c \in \mathbb{Z}$ are such that $b \geq 0$ then $a \leq c$ if and only if $ab \leq cb$ and that

$$\begin{aligned}(a_1b_3 + a_3b_1)b_2b_3 &= (a_1b_2)b_3^2 + a_3b_1b_2b_3 \\ &\leq (a_2b_1)b_3^2 + a_3b_1b_2b_3 \\ &= (a_2b_3 + a_3b_2)(b_1b_3).\end{aligned}$$

Hence

$$\begin{aligned}[(a_1,b_1)] + [(a_3,b_3)] &= [(a_1b_3 + a_3b_1, b_1b_3)] \\ &\leq [(a_2b_3 + a_3b_2, b_2b_3)] \\ &= [(a_2,b_2)] + [(a_3,b_3)]\end{aligned}$$

as desired.

c) Assume $[(a_1,b_1)], [(a_2,b_2)] \in \mathbb{Q}$ with $b_1, b_2 > 0$ are such that $0 \leq [(a_1,b_1)]$ and $0 \leq [(a_2,b_2)]$. Since $0 = [(0,1)]$, this implies that $0b_1 \leq a_1(1)$ and $0b_2 \leq a_2(1)$ so that $0 \leq a_1$ and $0 \leq a_2$. Therefore $0 \leq a_1a_2$ by the multiplicative property of the ordering on $\mathbb{Z}$. Therefore $0(b_1b_2) = 0 \leq a_1a_2 = (a_1a_2)(1)$ so that $0 = [(0,1)] \leq [(a_1a_2, b_1b_2)] = [(a_1,b_1)] \cdot [(a_2,b_2)]$ as desired. ∎

In particular, we have now proved the following.

Corollary B.2.15. *The rational numbers $\mathbb{Q}$ with the operations $+$ and $\cdot$ and the total ordering $\leq$ are a total ordered field.*

Proof. This follows immediately from Theorem B.2.5 and Lemmata B.2.13, and B.2.14 ∎

In particular, $\mathbb{Q}$ has all of the properties we want $\mathbb{R}$ to have except for the Least Upper Bound Property. These properties will be quite useful in Sections B.3 and B.4 which construct the real numbers in two different ways.

To conclude this section, it is useful to note that the rational numbers are indeed an extension of the integers. After all, because of the way we defined the rational numbers as equivalence relations of pairs of integers, this is by no means clear. However, there is a natural way the integers embed into the rational numbers. Of course, since $\frac{n}{1} = n$, we should be able to represent n as $[(n,1)]$ for all $n \in \mathbb{Z}$.

Lemma B.2.16. *There exists a map $f : \mathbb{Z} \to \mathbb{Q}$ such that*

- *f is injective,*

- *$f(0)$ is the additive unit of $\mathbb{Q}$,*
- *$f(1)$ is the multiplicative unit of $\mathbb{Q}$,*
- *$f(n+m) = f(n) + f(m)$ for all $n, m \in \mathbb{Z}$,*
- *$f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{Z}$, and*
- *for $n, m \in \mathbb{N}$, $n < m$ if and only if $f(n) < f(m)$.*

Proof. Define $f : \mathbb{Z} \to \mathbb{Q}$ by

$$f(n) = [(n, 1)]$$

for all $n \in \mathbb{Z}$. We claim that f is the map we are looking for.

To see that f is injective, assume $n, m \in \mathbb{Z}$ are such that $f(n) = f(m)$. Therefore $[(n, 1)] = [(m, 1)]$ so that $(n, 1) \sim_{\mathbb{Q}} (m, 1)$ and thus $n(1) = m(1)$ so $n = m$. Therefore, as n and m were arbitrary, f is injective.

Clearly $f(0) = [(0, 1)]$ is the additive unit of $\mathbb{Q}$ and $f(1) = [(1, 1)]$ is the multiplicative unit of $\mathbb{Q}$.

Next, assume $n, m \in \mathbb{Z}$. Then

$$f(n) + f(m) = [(n, 1)] + [(m, 1)] = [(n(1) + m(1), 1(1))] = [(n + m, 1)] = f(n + m)$$

as desired. Moreover

$$f(n) \cdot f(m) = [(n, 1)] \cdot [(1, 1)] = [(nm, 1(1))] = [(nm, 1)] = f(nm)$$

as desired.

Finally, for $n, m \in \mathbb{N}$, we notice that

$$\begin{aligned} f(n) \leq f(m) & \quad \text{if and only if} \\ [(n, 1)] \leq [(m, 1)] & \quad \text{if and only if} \\ n(1) \leq m(1) & \quad \text{if and only if} \\ n \leq m & \end{aligned}$$

as desired.

Hence f has all of the desired properties so $\mathbb{Z}$ naturally lies inside of $\mathbb{Q}$. ∎

B.3 Real Numbers via Dedekind Cuts

With the construction of the rational numbers complete, we turn to the first of two ways we will construct the real numbers. By Section B.2, we know the rational numbers have all the necessary properties required in this section.

The idea of the construction this first construction is that each real number α should be uniquely identified by the set

$$\{q \in \mathbb{Q} \mid q < \alpha\}.$$

Consequently, we will construct the real numbers by taking the collection of subsets of the rational numbers with certain properties that the above set has. The real challenge (and pain in the ...) is to define the operations of addition and multiplication on these sets in order to obtain that the real numbers are a field. It turns out that this approach makes demonstrating that the real numbers are totally ordered with the Least Upper Bound Property quite easy. In addition, one needs just mathematical logic and properties of sets to complete this construction.

To begin, we define the type of sets we will be working with in this section.

Definition B.3.1. A *Dedekind cut of* $\mathbb{Q}$ is any subset $A \subseteq \mathbb{Q}$ that has all of the following four properties:

- $A \neq \emptyset$,
- $A \neq \mathbb{Q}$,
- A is *downward closed*; that is, if $a \in A$ and $q \in \mathbb{Q}$ are such that $q \leq a$, then $q \in A$, and
- no element of A is an upper bound for A; that is, if $a \in A$ there exists a $q \in A$ such that $a < q$.

One fact that will be useful in the arguments to come is that the complement of a Dedekind cut is *upward closed* as the following result shows.

Lemma B.3.2. *Let A be a Dedekind cut of $\mathbb{Q}$ and let $q \in \mathbb{Q} \setminus A$. If $r \in \mathbb{Q}$ and $r > q$, then $r \notin A$.*

Proof. Let A be a Dedekind cut of $\mathbb{Q}$ and let $q \in \mathbb{Q} \setminus A$. To see the desired result, let $r \in \mathbb{Q}$ be such that $r > q$. To see that $r \notin A$, suppose for the sake of a contradiction that $r \in A$. Since $q < r$ and since A is a Dedekind cut and thus downward closed, this implies that $q \in A$. Since this contradicts the fact that $x \in \mathbb{Q} \setminus A$, we have a contradiction. Hence $r \notin A$ as desired. ∎

We are finally ready to provide our first definition of the real numbers! Throughout this section, we will be using lower case letters for elements of $\mathbb{Q}$ and upper case letters for subsets of $\mathbb{Q}$. Therefore, since the following defines the real numbers to be certain subsets of $\mathbb{Q}$, the elements of the real numbers will be denoted by capital letters in this section.

Definition B.3.3. The *real numbers* , denoted $\mathbb{R}$, is the set

$$\mathbb{R} = \{A \subseteq \mathbb{Q} \mid A \text{ is a Dedekind cut of } \mathbb{Q}\}.$$

Although the operations of addition and multiplication on $\mathbb{R}$ are by no means clear, the partial ordering $\leq$ on $\mathbb{R}$ is quite easy to define.

Definition B.3.4. For $A, B \in \mathbb{R}$, we define $A \leq B$ if $A \subseteq B$.

It is elementary to see that $\leq$ is a partial ordering on $\mathbb{R}$ (see Example 1.2.10). In fact, proving $\leq$ is a total ordering on $\mathbb{R}$ is fairly straightforward.

Lemma B.3.5. *The partial ordering $\leq$ on $\mathbb{R}$ is a total ordering.*

Proof. To see that $\leq$ is a total order, suppose for the sake of a contradiction that $\leq$ is not a total ordering. Therefore there exists $A, B \in \mathbb{R}$ such that $A \not\leq B$ and $B \not\leq A$. Hence $A \not\subseteq B$ and $B \not\subseteq A$. Therefore there exists $a \in A$ and $b \in B$ such that $a \notin B$ and $b \notin A$. As clearly these conditions imply $a \neq b$, we divide the proof into two cases:

Case 1: $a < b$. Since $a \notin B$, $a \in \mathbb{Q} \setminus B$. Therefore, since $b > a$ in this case, Lemma B.3.2 implies that $b \notin B$. Since this contradicts the fact that $b \in B$, we have a contradiction in this case.

Case 2: $b < a$. Since $b \notin A$, $b \in \mathbb{Q} \setminus A$. Therefore, since $a > b$ in this case, Lemma B.3.2 implies that $a \notin A$. Since this contradicts the fact that $a \in A$, we have a contradiction in this case.

Since the above two cases cover all possibilities, we have a contradiction. Hence $\leq$ is a total ordering as desired. ∎

Perhaps more surprisingly considering how difficult determining what the real numbers are has been, it is quite easy to show that $\mathbb{R}$ has the Least Upper Bound Property.

Theorem B.3.6. *The real numbers have the Least Upper Bound Property; that is, every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound with respect to $\leq$.*

Proof. Let $\mathcal{X}$ be a non-empty subset of $\mathbb{R}$ that is bounded above. To see that $\mathcal{X}$ has a least upper bound, consider the set

$$L = \bigcup_{A \in \mathcal{X}} A.$$

Since $A \subseteq \mathbb{Q}$ for all $A \in \mathcal{X}$, it follows that $L \subseteq \mathbb{Q}$.

We claim that $L \in \mathbb{R}$. To prove this, we will show that L is a Dededkind cut by showing that L has the defining four properties of a Dedekind cut from Definition B.3.1.

$L \neq \emptyset$. To see that $L \neq \emptyset$, notice since $\mathcal{X}$ is non-empty there exists an $A \in \mathcal{X}$. Since A is a Dedekind cut, we know that $A \neq \emptyset$. Therefore, there exists an $a \in A$. By the construction of L, we see that $a \in A \subseteq L$. Hence $L \neq \emptyset$ as desired.

$L \neq \mathbb{Q}$. To see that $L \neq \mathbb{Q}$, recall that $\mathcal{X}$ is bounded above so there exists a $B \in \mathbb{R}$ such that $A \leq B$ for all $A \in \mathcal{X}$. Since B is a Dedekind cut, $B \neq \mathbb{Q}$ so there exists a $b \in \mathbb{Q} \setminus B$. Since $A \leq B$ and thus $A \subseteq B$ for all $A \in \mathcal{X}$, we see that $b \notin A$ for all $A \in \mathcal{X}$. Therefore $b \notin L$ by the construction of L. Hence $L \neq \mathbb{Q}$ as desired.

L is downward closed. To see that L is downward closed, assume $y \in L$ and $q \in \mathbb{Q}$ are such that $q \leq y$. Since $y \in L$, the construction of L implies there exists an $A \in \mathcal{X}$ such that $y \in A$. Since A is a Dedekind cut, A is downward closed. Hence $y \in A$ and $q \leq y$ implies that $q \in A \subseteq L$ as desired. Hence L is downward closed as desired.

No element of L is an upper bound for L. To see this, suppose for the sake of a contradiction that there exists an $q \in L$ such that q is an upper bound for L. By the construction of L, there exists an $A \in \mathcal{X}$ such that $q \in A$. Since $A \subseteq L$ and $y \leq q$ for all $y \in L$, it follows that $a \leq q$ for all $a \in A$. Hence q is an upper bound for A in $\mathbb{Q}$ that is in A. However, since A is a Dedekind cut, we have a contradiction. Hence no element of L is an upper bound of L as desired.

Therefore L is a Dedekind cut by Definition B.3.1 so $L \in \mathbb{R}$. Moreover, it is clear by the construction of L that $A \subseteq L$ and thus $A \leq L$ for all $A \in \mathcal{X}$. Hence L is an upper bound for $\mathcal{X}$.

We claim that L is the least upper bound for $\mathcal{X}$. To see this, let $B \in \mathbb{R}$ be an arbitrary upper bound for $\mathcal{X}$. Therefore $A \leq B$ so $A \subseteq B$ for all $A \in \mathcal{X}$. Hence, by the construction of L, we have that $L \subseteq B$ so $L \leq B$. Therefore, since B was arbitrary, L is a least upper bound for $\mathcal{X}$. Hence $\mathcal{X}$ has a least upper bound as desired. ∎

To demonstrate that $\mathbb{R}$ is a field, we must define the addition and multiplication operations on $\mathbb{R}$. Luckily addition is not too bad as there is a natural way to add two subsets of $\mathbb{Q}$ together. Of course, to have a well-defined operation on $\mathbb{R}$, we need to ensure that adding two Dedekind cuts together produces a Dedekind cut.

Lemma B.3.7. *If A and B are Dedekind cuts of $\mathbb{Q}$, then*

$$X = \{a + b \mid a \in A \text{ and } b \in B\}$$

is a Dedekind cut of $\mathbb{Q}$.

Proof. Let A and B be Dedekind cuts of $\mathbb{Q}$. To see that X is a Dededkind cut, we will verify the defining four properties of a Dedekind cut from Definition B.3.1.

$X \neq \emptyset$. To see that $X \neq \emptyset$, recall since A and B are Dedekind cuts that $A \neq \emptyset$ and $B \neq \emptyset$. Therefore, there exists an $a \in A$ and a $b \in B$. Hence $a + b \in X$ by definition so $X \neq \emptyset$ as desired.

$X \neq \mathbb{Q}$. To see that $X \neq \mathbb{Q}$, recall since A and B are Dedekind cuts that $A \neq \mathbb{Q}$ and $B \neq \mathbb{Q}$. Therefore there exists a $y \in \mathbb{Q} \setminus A$ and a $z \in \mathbb{Q} \setminus B$. By Lemma B.3.2 it follows that $a \leq y$ for all $a \in A$ and $b \leq z$ for all $b \in B$. Thus, as $y \in \mathbb{Q} \setminus A$ and a $z \in \mathbb{Q} \setminus B$, we have that $a < y$ for all $a \in A$ and $b < z$ for all $b \in B$. Hence $a + b < y + z$ for all $a \in A$ and $b \in B$. Therefore $x < y + z$ for all $x \in X$ so $y + z \notin X$. Hence $X \neq \mathbb{Q}$ as desired.

X is downward closed. To see that X is downward closed, assume $x \in X$ and $q \in \mathbb{Q}$ are such that $q \leq x$. By the definition of X there exists an $a \in A$ and a $b \in B$ such that $x = a + b$. Since $q \leq x = a + b$, we obtain that $q - b \leq a$. Therefore, since A is a Dedekind cut and thus downward closed, $q - b \in A$. Hence, with $a' = q - b \in A$, we have that $q = a' + b \in X$ by definition. Therefore X is downward closed as desired.

No element of X is an upper bound for X. To see this, suppose for the sake of a contradiction that there exists an $q \in X$ such that q is an upper bound for X. By the definition of X there exists an $a' \in A$ and a $b' \in B$ such that $q = a' + b'$. Since $x \leq q$ for all $x \in X$, it follows that $a + b \leq a' + b'$ for all $a \in A$ and $b \in B$. Thus, since $a' \in A$, $a' + b \leq a' + b'$ for all $b \in B$ so $b \leq b'$ for all $b \in B$. Therefore b' is an upper bound for B in $\mathbb{Q}$ such that $b' \in B$. However, since B is a Dedekind cut, we have a contradiction. Hence no element of X is an upper bound of X as desired.

Therefore X is a Dedekind cut by Definition B.3.1. ∎

By Lemma B.3.7, the following addition operation on $\mathbb{R}$ is well-defined.

Definition B.3.8. The operation $+$ is define on $\mathbb{R}$ by

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}$$

for all $A, B \in \mathbb{R}$.

Before we demonstrate that the addition operation on $\mathbb{R}$ has the desired field properties, we note that $\leq$ has the additive property as we desire.

Lemma B.3.9. *The total ordering $\leq$ on $\mathbb{R}$ has the additive property. That is, if $A, B, C \in \mathbb{R}$ are such that $A \leq B$, then $A + C \leq B + C$.*

Proof. Assume $A, B, C \in \mathbb{R}$ are such that $A \leq B$. Hence $A \subseteq B$. Therefore

$$\begin{aligned} A + C &= \{a + c \mid a \in A \text{ and } c \in C\} \\ &\subseteq \{b + c \mid b \in B \text{ and } c \in C\} \\ &= B + C \end{aligned}$$

so $A + C \leq B + C$ by definition as desired. ∎

Returning to showing that $\mathbb{R}$ is a field, we note that three of the four defining field properties that depend only on addition hold without much difficulty.

Lemma B.3.10. *The addition operator on the real numbers* $\mathbb{R}$ *has the following properties:*

a) (Commutativity of $+$*)* $A + B = B + A$ *for all* $A, B \in \mathbb{R}$*.*

b) (Associativity of $+$*)* $(A + B) + C = A + (B + C)$ *for all* $A, B, C \in \mathbb{R}$*.*

c) (Additive Unit) If $O = \{q \in \mathbb{Q} \mid q < 0\}$*, then* $O \in \mathbb{R}$ *and* $O + A = A$ *for all* $A \in \mathbb{R}$*.*

Proof. a) Notice that

$$\begin{aligned} A + B &= \{a + b \mid a \in A \text{ and } b \in B\} \\ &= \{b + a \mid b \in B \text{ and } a \in B\} \\ &= B + A \end{aligned}$$

as desired.

b) Notice that

$$\begin{aligned} (A + B) + C &= \{a + b \mid a \in A \text{ and } b \in B\} + C \\ &= \{(a + b) + c \mid (a \in A \text{ and } b \in B) \text{ and } c \in C\} \\ &= \{a + (b + c) \mid a \in A \text{ and } (b \in B \text{ and } c \in C)\} \\ &= A + \{b + c \mid b \in B \text{ and } c \in C\} \\ &= A + (B + C) \end{aligned}$$

as desired.

c) To begin, to show that $O \in \mathbb{R}$ we must show that O is a Dedekind cut. Clearly $O \neq \emptyset$ and $O \neq \mathbb{Q}$ by construction.

To see that O is downward closed let $q \in \mathbb{Q}$ and $r \in O$ be such that $q \leq r$. Since $r \in O$, we have that $r < 0$. Hence $q \leq r < 0$ so $q < 0$. Thus $q \in O$ as desired.

To see that O does not contain any of its upper bounds, suppose for the sake of a contradiction that there exists a $q \in O$ such that $r \leq q$ for all $r \in O$. Since $q \in O$, we know that $q < 0$. Therefore $\frac{1}{2}q < 0$ so $\frac{1}{2}q \in O$. However since $q < 0$ and since $\frac{1}{2}q \nleq q$, we have a contradiction to the fact that $r \leq q$ for all $r \in O$. Hence O does not contain any of its upper bounds. Hence O is a Dedekind cut by Definition B.3.1 so $O \in \mathbb{R}$.

To complete the proof, let $A \in \mathbb{R}$ be arbitrary. To see that $O + A = A$, we will demonstrate that $O + A \subseteq A$ and $A \subseteq O + A$.

To see that $O + A \subseteq A$, let $x \in O + A$ be arbitrary. Hence there exists a $q \in O$ and an $a \in A$ such that $x = q + a$. Since $q \in O$, we know that $q < 0$ so $x = q + a < a$. Therefore, since A is a Dedekind cut and thus downward closed, this implies that $x \in A$. Hence, since $x \in O + A$ was arbitrary, $O + A \subseteq A$.

To see that $A \subseteq O + A$, let $a \in A$ be arbitrary. Since A is a Dedekind cut, a is not an upper bound for A so there exists an $a' \in A$ such that $a < a'$. Therefore $q = a - a' < 0$ so $q \in O$. Since $a = q + a'$, we obtain that $a \in O + A$ by definition. Hence, since $a \in A$ was arbitrary, $A \subseteq O + A$.

Hence $O \in \mathbb{R}$ and $O + A = A$ as desired. ∎

Remark B.3.11. For clarification, we will use $O = \{q \in \mathbb{Q} \mid q < 0\}$ for the additive unit (i.e. zero element) for $\mathbb{R}$ throughout this section whereas we will reserve 0 for the zero element of $\mathbb{Q}$.

The remaining field property depending only on addition is to show that every element of $\mathbb{R}$ has an additive inverse. This turns out to be a non-trivial task. Before we proceed with this task, we will prove the following result for latter use. Note the proof of this result only requires the field properties we have already demonstrated.

Corollary B.3.12. *If $X \in \mathbb{R}$ is such that $A + X = A$ for all $A \in \mathbb{R}$, then $X = O$.*

Proof. Assume $X \in \mathbb{R}$ is such that $A + X = A$ for all $A \in \mathbb{R}$. Then

$$\begin{aligned} X &= X + O && \text{by Lemma B.3.10, part c)} \\ &= O + X && \text{by Lemma B.3.10, part a)} \\ &= O && \text{by assumption} \end{aligned}$$

as desired. ∎

In order to prove that every element of $\mathbb{R}$ has an additive inverse, we will require the following technical result that states if we have a Dedekind cut A, there are elements of A and $\mathbb{Q} \setminus A$ that are as close together as we want.

Lemma B.3.13. *Let $A \in \mathbb{R}$, let $a \in A$, and let $\epsilon \in \mathbb{Q}$ be such that $\epsilon > 0$. Then there exists an $a' \in A$ and a $\delta \in \mathbb{Q}$ such that $a' + \delta \in \mathbb{Q} \setminus A$, $a \leq a'$, and $0 < \delta < \epsilon$.*

Proof. To begin, recall since A is a Dedekind cut that $A \neq \mathbb{Q}$. Therefore there exists a $b_1 \in \mathbb{Q} \setminus A$. Let $a_1 = a$. Since A is a Dedekind cut and thus downward closed, $b_1 > a_1$. Therefore if $\delta_1 = b_1 - a_1 \in \mathbb{Q}$, then $\delta_1 > 0$.

Let $c_1 = \frac{1}{2}(a_1 + b_1)$. Notice that $c_1 \in A$ or $c_1 \in \mathbb{Q} \setminus A$. If $c_1 \in A$, let $a_2 = c_1 \in A$ and let $b_2 = b_1 \in \mathbb{Q} \setminus A$, and if $c_1 \in \mathbb{Q} \setminus A$, let $a_2 = a_1 \in A$ and let $b_2 = c_1 \in \mathbb{Q} \setminus A$. Thus $a_2 \in A$, $b_2 \in B$, and $a \leq a_2$. Moreover, if $\delta_2 = b_2 - a_2 = \frac{1}{2}\delta_1 \in \mathbb{Q}$, then $\delta_2 > 0$.

By repeating the above recursively, we obtain a sequence $(a_n)_{n\geq}$ of elements of of A, a sequence $(b_n)_{n\geq 1}$ of elements of $\mathbb{Q} \setminus A$, and a sequence $(\delta_n)_{n\geq 1}$ of elements of $\mathbb{Q}$ such that $a \leq a_n$ for all $n \in \mathbb{N}$ and $\delta_n = b_n - a_n = \frac{1}{2^{n-1}}\delta_1 > 0$ for all $n \in \mathbb{N}$.

Since $\epsilon, \delta_1 \in \mathbb{Q}$ and $\epsilon, \delta_1 > 0$, there exists $m, k, \ell, d \in \mathbb{N}$ such that $\epsilon = \frac{k}{m}$ and $\delta_1 = \frac{\ell}{d}$. Notice, for $n \in \mathbb{N}$, $\frac{1}{2^{n-1}}\delta_1 < \epsilon$ if and only if $m\ell < 2^{n-1}kd$. Moreover, since $m\ell \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $m\ell < 2^{N-1}$ and thus $m\ell < 2^{N_1}kd$ by Peano's Axioms (Definition 1.1.1). Hence, with $a' = a_N \in A$ and $\delta = \frac{1}{2^{N-1}}\delta_1 \in \mathbb{Q}$, we see that $a' \in A$, $a' + \delta = b_N \in \mathbb{Q} \setminus A$, $a \leq a'$, and $0 < \delta < \epsilon$ as desired. ∎

To demonstrate that every element of $\mathbb{R}$ has an additive inverse, consider a Dedekind cut A. Our goal is to find a Dedekind cut B such that $A + B = O$. In particular, we want B to have exactly the elements that yield a number less than 0 when added to an element of A. Since elements of $\mathbb{Q} \setminus A$ are upper bounds for A, the following turns out to be the correct set to consider.

Lemma B.3.14. *Let $A \in \mathbb{R}$. If*

$$X = \{q - r \mid r \in \mathbb{Q} \setminus A, q \in \mathbb{Q}, \text{ and } q < 0\},$$

then $X \in \mathbb{R}$ and $A + X = O$.

Hence X is the additive inverse of A so every element of $\mathbb{R}$ has an additive inverse.

Proof. To see that $X \in \mathbb{R}$, we will show that X is a Dededkind cut by verify the defining four properties of a Dedekind cut from Definition B.3.1.

$X \neq \emptyset$. To see that $X \neq \emptyset$, recall since A is a Dedekind cut that $A \neq \mathbb{Q}$. Therefore, there exists an $r \in \mathbb{Q} \setminus A$. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, we obtain that $(-1) - r \in X$ by definition so $X \neq \emptyset$ as desired.

$X \neq \mathbb{Q}$. To see that $X \neq \mathbb{Q}$, recall since A is a Dedekind cut that $A \neq \emptyset$. Therefore there exists a $a \in A$. Moreover, since A is a Dedekind cut, A is downward closed so $r > a$ for all $r \in \mathbb{Q} \setminus A$. Therefore for all $q \in Q$ with $q < 0$ and for all $r \in \mathbb{Q} \setminus A$, we have that

$$q - r < q - a < -a.$$

Hence, by the definition of X, we have that $x < -a$ for all $x \in X$. Therefore $-a \notin X$ so $X \neq \mathbb{Q}$ as desired.

X is downward closed. To see that X is downward closed, assume $x \in X$ and $q \in \mathbb{Q}$ are such that $q \leq x$. By the definition of X there exists a $r \in \mathbb{Q} \setminus A$ and a $q' \in \mathbb{Q}$ such that $q < 0$ and $x = q' - r$. Since $q \leq x = q' - r$, we obtain that $r \leq q' + q$. Therefore, since A is a Dedekind cut, Lemma B.3.2 implies that $q' + q \in \mathbb{Q} \setminus A$. Hence there exists an $r' \in \mathbb{Q} \setminus A$ so that $r' = q' + q$. Therefore $q = q' - r'$ with $r' \in \mathbb{Q} \setminus A$ and $q' \in \mathbb{Q}$ with $q' < 0$ so $q \in X$ as desired.

No element of X is an upper bound for X. To see this, suppose for the sake of a contradiction that there exists an $y \in X$ such that y is an upper bound for X. By the definition of X there exists an $r' \in \mathbb{Q} \setminus A$ and a $q' \in \mathbb{Q}$ such that $q' < 0$ and $y = q' - r'$. Since $x \leq q' - r'$ for all $x \in X$, it follows that $q - r \leq q' - r'$ for all $r \in \mathbb{Q} \setminus A$ and $q \in \mathbb{Q}$ with $q < 0$. Thus, since $r' \in \mathbb{Q} \setminus A$, it follows that $q - r' \leq q' - r'$ for all $q \in \mathbb{Q}$ with $q < 0$ so that $q \leq q'$ for all $q \in \mathbb{Q}$ with $q < 0$. Therefore, since $q' < 0$, see that if $q = \frac{1}{2}q'$, then $q \in \mathbb{Q}$, $q < 0$, but $q > q'$. Hence we have a contradiction. Therefore no element of X is an upper bound of X as desired.

Hence X is a Dedekind cut by Definition B.3.1 so $X \in \mathbb{R}$. To see that $A + X = O$, we will demonstrate that $A + X \subseteq O$ and $O \subseteq A + X$.

To see that $A + X \subseteq O$, let $z \in A + X$ be arbitrary. Hence there exists a $q \in O$ and an $r \in \mathbb{Q} \setminus A$ such that $x = q - r$ and there exists an $a \in A$ such that $z = a + x = a + q - r$. Since $a \in A$, since $r \in \mathbb{Q} \setminus A$, and since A is a Dedekind cut and thus downward closed, we have that $a < r$. Therefore, since $q \in O$ so that $q < 0$, we have that

$$z = q + (a - r) < 0 + 0 = 0$$

and thus $z \in O$. Hence, since $z \in X + A$ was arbitrary, $X + A \subseteq O$.

To see that $O \subseteq A + X$, let $z \in O$ be arbitrary. Thus $z < 0$ so if $\epsilon = -\frac{1}{2}z$, then $\epsilon > 0$. Since is a Dedekind cut and thus $A \neq \emptyset$, Lemma B.3.13 implies that there exists an $a' \in A$ and a $\delta \in \mathbb{Q}$ such that $a' + \delta \in \mathbb{Q} \setminus A$ and $0 < \delta < \epsilon$. Let $r = a' + \delta \in \mathbb{Q} \setminus A$. Notice that

$$z + r - a' < z + \delta < z + \epsilon = \frac{1}{2}z < 0.$$

Therefore, if $q = z + r - a' \in \mathbb{Q}$, then $q < 0$ so $q \in O$. Moreover, we see that

$$z = q - (r - a') = a' + (q - r) \in A + X$$

as desired. Therefore, since $z \in O$ was arbitrary, $O \subseteq A + X$.

Hence $X \in \mathbb{R}$ and $A + X = O$ as desired. ∎

Remark B.3.15. By Lemma B.3.14, we know that if $A \in \mathbb{R}$, then $-A$, the additive inverse of A, is the set

$$-A = \{q - r \mid r \in \mathbb{Q} \setminus A, q \in \mathbb{Q}, \text{ and } q < 0\}.$$

It is important to know that this is different that what one might consider $-A$ in the remainder of the course. Specially, if $c \in \mathbb{Q}$ and $A \subseteq \mathbb{Q}$, one normally defines

$$cA = \{ca \mid a \in A\}.$$

So, in this context, $-A$ (the additive inverse of A) is not $(-1)A$ (multiplying elements of the set A by -1).

Before we proceed to define the multiplication operation on $\mathbb{R}$, we note the following properties of the additive inverse we will require. Note the proof of this result only requires the field properties we have already demonstrated.

Corollary B.3.16. *The following properties of the additive inverse hold:*

a) If $A \in \mathbb{R}$ and $X \in \mathbb{R}$ is such that $A + X = O$, then $X = -A$.

b) For all $A \in \mathbb{R}$, $-(-A) = A$.

c) For all $A, B \in \mathbb{R}$, $-(A + B) = (-A) + (-B)$.

Proof. a) Notice that

$$\begin{aligned} X &= X + O && \text{by Lemma B.3.10, part c)} \\ &= X + (A + (-A)) && \text{by Lemma B.3.14} \\ &= (X + A) + (-A) && \text{by Lemma B.3.10, part b)} \\ &= (A + X) + (-A) && \text{by Lemma B.3.10, part a)} \\ &= O + (-A) && \text{by assumption} \\ &= (-A) + O && \text{by Lemma B.3.10, part a)} \\ &= -A && \text{by Lemma B.3.10, part c)} \end{aligned}$$

as desired.

b) Notice by Lemma B.3.10, part c) that

$$O = A + (-A) = (-A) + A.$$

Hence $A = -(-A)$ by part a) of this proof.

c) Notice by using Lemma B.3.10 parts a) and b) multiple times that

$$(A + B) + ((-A) + (-B)) = (A + (-A)) + (B + (-B)) = O + O = O.$$

Hence $-(A + B) = (-A) + (-B)$ by part a) of this proof. ∎

To define the multiplication operator on $\mathbb{R}$ and demonstrate it has the desired properties if highly non-trivial and technical. Given $A, B \in \mathbb{R}$, one may simply want to multiply all the elements of A by all the elements of B together to get $A \cdot B$. However, this clearly does not work since A and B are Dedekind cuts and thus contain very negative numbers that when multiplied together give very large positive numbers.

To proceed, we first begin with the case that $O \leq A$ and $O \leq B$ so that either $A = O$, $B = O$, or A and B have positive numbers. In the latter case, we simply need to include the products of these positive numbers and add in all the negative numbers.

Lemma B.3.17. *Let $A, B \in \mathbb{R}$ be such that $O \leq A$ and $O \leq B$. If*

$$X = \{ab \mid a \in A, b \in B, \text{ and } a, b \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\},$$

then $X \in \mathbb{R}$.

Proof. Let A and B be Dedekind cuts of $\mathbb{Q}$ such that $O \leq A$ and $O \leq B$. Hence $O \subseteq A \cap B$ so if $q \in \mathbb{Q}$ and $q < 0$ then $q \in A \cap B$.

To see that X is a Dededkind cut, we will verify the defining four properties of a Dedekind cut from Definition B.3.1.

$X \neq \emptyset$. To see that $X \neq \emptyset$, note $-1 \in X$ by construction. Hence $X \neq \emptyset$ as desired.

$X \neq \mathbb{Q}$. To see that $X \neq \mathbb{Q}$, recall since A and B are Dedekind cuts that $A \neq \mathbb{Q}$ and $B \neq \mathbb{Q}$. Therefore there exists a $y \in \mathbb{Q} \setminus A$ and a $z \in \mathbb{Q} \setminus B$. Moreover, since A and B are Dedekind cuts and $O \subseteq A \cap B$, it follows that $y \geq 0$ and $z \geq 0$. Since $y + 1 > y$ and $z + 1 > z$, we have by Lemma B.3.2 that $y' = y + 1$ and $z' = z + 1$ are such that $y' \in \mathbb{Q} \setminus A$, $z' \in \mathbb{Q} \setminus B$, and $y', z' \geq 1$.

We claim that $yz \notin X$. To see this, note for all $a \in A$ and $b \in B$ such that $a \geq 0$ and $b \geq 0$ that $a < y$ and $b < z$ by Lemma B.3.2 and thus $ab \leq az' < y'z'$. Therefore, since $y'z' > 0$, we obtain that $yz \notin X$. Hence $X \neq \mathbb{Q}$ as desired.

X is downward closed. To see that X is downward closed, assume $x \in X$ and $q \in \mathbb{Q}$ are such that $q \leq x$. Clearly if $q < 0$ then $q \in X$ by construction. Therefore, we may assume without loss of generality that $q \geq 0$. Hence, by considering the elements of X, there must exists $a \in A$ and $b \in B$ such that $a \geq 0$, $b \geq 0$, and $q \leq ab$. At this point, we need to divide the proof into two cases.

Case 1: $q = 0$. Since $a \in A$ and $b \in B$ such that $a \geq 0$ and $b \geq 0$, and since A and B are Dedekind cuts and thus downward closed, $0 \in A \cap B$. Hence $q = 0 = 0(0) \in X$ as desired.

Case 2: $q > 0$. Since $a \geq 0$, $b \geq 0$, and $q \leq ab$, it follows that $a > 0$ and $b > 0$. Therefore have that $0 < \frac{q}{b} \leq a$. Therefore, since A is a Dedekind cut and thus downward closed, there exists an $a' \in A$ such that $a' = \frac{q}{b} > 0$. Hence $q = a'b$ where $a' \in A$ and $b \in B$ are such that $a' > 0$ and $b' > 0$. Hence $q \in X$.

Therefore, as we have covered all possible cases, X is downward closed as desired.

No element of X is an upper bound for X. To see this, suppose for the sake of a contradiction that there exists an $y \in X$ such that y is an upper bound for X. To obtain our contradiction, we will divide the proof into three cases.

Case 1: $y < 0$. Assume $y < 0$. Let $q = \frac{1}{2}y$. Clearly $q < 0$ so $q \in X$. However, since $y < q < 0$, we have a contradiction to the fact that y is an upper bound for X.

Case 2: $y > 0$. Assume $y > 0$. By the description of X, it follows that $y = ab$ for some $a \in A$ and $b \in B$ with $a \geq 0$ and $b \geq 0$. Since $y > 0$, it follows that $a > 0$ and $b > 0$. Since A and B are Dedekind cuts and thus contain none of their upper bounds, it follows that a and b are not upper bounds for A and B respectively. Hence there exists $a' \in A$ and $b' \in B$ with

$a' > a$ and $b' > b$. Therefore $a'b' \in X$ and $yab < a'b'$, which contradicts the fact that y is an upper bound for X.

Case 3: $y = 0$. Assume $y = 0$. By the description of X, it follows that $y = ab$ for some $a \in A$ and $b \in B$ with $a \geq 0$, $b \geq 0$, and either $a = 0$ or $b = 0$. Since A and B are Dedekind cuts and thus contain none of their upper bounds, by the same idea as used in Case 2 it follows that there exists $a' \in A$ and $b \in B$ with $a', b' > 0$. Therefore $a'b' \in X$ and $y = 0 < a'b'$, which contradicts the fact that y is an upper bound for X.

Therefore, as we have covered all possible cases, we have obtained our contradiction. Hence no element of X is an upper bound of X as desired.

Therefore X is a Dedekind cut by Definition B.3.1. ∎

Given $A, B \in \mathbb{R}$, we are now ready to define $A \cdot B$ based on whether A and B are individually larger or smaller than zero. Note in the following we have included overlapping cases as it will ease future arguments in this section. Note the overlapping cases in this definition occur when $A = O$ or $B = O$. Since in Lemma B.3.17 one can see that if $A = O$ or $B = O$ then $X = O$, and since $O = -O$ by Corollary B.3.16, in all of the overlapping cases one obtains $A \cdot B = O$.

Definition B.3.18. The operation $\cdot$ on $\mathbb{R}$ is defined as follows: for $A, B \in \mathbb{R}$,

(I) if $O \leq A$ and $O \leq B$, then

$$A \cdot B = \{ab \mid a \in A, b \in B, \text{ and } a, b \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}$$

(II) If $O \leq A$ and $B \leq O$, then $A \cdot B$ is defined to be $-(A \cdot (-B))$ via (I).

(III) If $A \leq O$ and $O \leq B$, then $A \cdot B$ is defined to be $-((-A) \cdot B)$ via (I).

(IV) If $A \leq O$ and $B \leq O$, then $A \cdot B$ is defined to be $(-A) \cdot (-B)$ via (I).

Now that the multiplication operation on $\mathbb{R}$ has been defined, we need only verify the remaining properties to ensure that $\mathbb{R}$ is a totally ordered field. However, this is by far the most incredibly technical thing in this notes and astonishingly annoying. The only nice thing is that the multiplicative property of $\leq$ is quite simple.

Lemma B.3.19. *The total ordering $\leq$ on $\mathbb{R}$ has the multiplicative property. That is, if $A, B \in \mathbb{R}$ are such that $O \leq A$ and $O \leq B$, then $O \leq A \cdot B$.*

Proof. Assume $O \leq A$ and $O \leq B$. Then, by definition,

$$A \cdot B = \{ab \mid a \in A, b \in B, \text{ and } a, b > 0\} \cup O.$$

Hence $O \subseteq A \cdot B$ so $O \leq A \cdot B$ as desired. ∎

We now move onto demonstrating the multiplication operation on $\mathbb{R}$ has the necessary field properties. We begin with commutativity. Luckily this is not too bad although it involves four case. In general, the proofs that the multiplication operation on $\mathbb{R}$ has the necessary field properties will proceed by first checking the properties for non-negative elements and then extending them to all elements using Definition B.3.18.

Lemma B.3.20. *The multiplication operator on the real numbers $\mathbb{R}$ is commutative. That is $A \cdot B = B \cdot A$ for all $A, B \in \mathbb{R}$.*

Proof. Let $A, B \in \mathbb{R}$. Based on the definition of multiplication in $\mathbb{R}$, we divide the proof into four cases.

Case 1: $O \leq A$ and $O \leq B$. In this case, we see that

$$\begin{aligned} A \cdot B &= \{ab \mid a \in A, b \in B, \text{ and } a, b \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\ &= \{ba \mid b \in B, a \in A, \text{ and } b, a \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\ &= B \cdot A \end{aligned}$$

as desired.

Case 2: $O \leq A$ and $B \leq O$. In this case, we see that

$$\begin{aligned} A \cdot B &= -(A \cdot (-B)) \\ &= -((-B) \cdot A) && \text{by Case 1 as } A, -B \geq O \\ &= B \cdot A \end{aligned}$$

as desired.

Case 3: $A \leq O$ and $O \leq B$. In this case, we see that

$$\begin{aligned} A \cdot B &= -((-A) \cdot B) \\ &= -(B \cdot (-A)) && \text{by Case 1 as } -A, B \geq O \\ &= B \cdot A \end{aligned}$$

as desired.

Case 4: $A \leq O$ and $B \leq O$. In this case, we see that

$$\begin{aligned} A \cdot B &= (-A) \cdot (-B) \\ &= (-B) \cdot (-A) && \text{by Case 1 as } -A, -B \geq O \\ &= B \cdot A \end{aligned}$$

as desired.

Since we have covered all possible cases, the result follows. ∎

Next we move onto associativity. However, since we need to divide the argument based on when each of the three elements involved are non-negative or negative, there are $2^3 = 8$ cases. Yes... eight.... In addition, one also needs to keep careful track of Definition B.3.18 in each of these cases. It is useful to note that if $A \in \mathbb{R}$ and $O \leq A$, then $O + (-A) \leq A + (-A)$ by the additive property of $\leq$ (Lemma B.3.9) and thus $-A \leq O$.

Lemma B.3.21. *The multiplication operator on the real numbers* $\mathbb{R}$ *is associative. That is* $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ *for all* $A, B, C \in \mathbb{R}$.

Proof. Let $A, B, C \in \mathbb{R}$. Based on the definition of multiplication in $\mathbb{R}$, we need to divide the proof into eight cases.

Case 1: $O \leq A$, $O \leq B$, $O \leq C$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= (\{ab \mid a \in A, b \in B, \text{ and } a, b \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}) \cdot C \\ &= \{(ab)c \mid a \in A, b \in B, c \in C \text{ and } a, b, c \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\ &= \{a(bc) \mid a \in A, b \in B, c \in C \text{ and } a, b, c \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\ &= A \cdot (\{bc \mid b \in B, c \in C, \text{ and } b, c \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 2: $O \leq A$, $O \leq B$, $C \leq O$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= -((A \cdot B) \cdot (-C)) \\ &= -(A \cdot (B \cdot (-C))) && \text{by Case 1} \\ &= A \cdot (-(B \cdot (-C))) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 3: $O \leq A$, $B \leq O$, $O \leq C$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= (-(A \cdot (-B)) \cdot C \\ &= -((-(-(-(A \cdot (-B)))) \cdot C) \\ &= -((A \cdot (-B)) \cdot C) && \text{by Corollary B.3.16} \\ &= -(A \cdot ((-B) \cdot C)) && \text{by Case 1} \\ &= A \cdot (-((-B) \cdot C)) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 4: $O \leq A$, $B \leq O$, $C \leq O$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= (-(A \cdot (-B)) \cdot C \\ &= (-(-(A \cdot (-B)))) \cdot (-C) \\ &= (A \cdot (-B)) \cdot (-C) && \text{by Corollary B.3.16} \\ &= A \cdot ((-B) \cdot (-C)) && \text{by Case 1} \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 5: $A \leq O$, $O \leq B$, $O \leq C$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= (-((-A) \cdot B)) \cdot C \\ &= -((-(-((-A) \cdot B)))) \cdot C) \\ &= -(((-A) \cdot B) \cdot C && \text{by Corollary B.3.16} \\ &= -((-A) \cdot (B \cdot C) && \text{by Case 1} \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 6: $A \leq O$, $O \leq B$, $C \leq O$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= (-((-A) \cdot B)) \cdot C \\ &= (-(-((-A) \cdot B))) \cdot (-C) \\ &= ((-A) \cdot B) \cdot (-C) && \text{by Corollary B.3.16} \\ &= (-A) \cdot (B \cdot (-C)) && \text{by Case 1} \\ &= (-A) \cdot (-(-(B \cdot (-C)))) && \text{by Corollary B.3.16} \\ &= A \cdot (-(B \cdot (-C))) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 7: $A \leq O$, $B \leq O$, $O \leq C$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= ((-A) \cdot (-B)) \cdot C \\ &= (-A) \cdot ((-B) \cdot C) && \text{by Case 1} \\ &= (-A) \cdot (-(-((-B) \cdot C))) && \text{by Corollary B.3.16} \\ &= A \cdot (-((-B) \cdot C)) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Case 8: $A \leq O$, $B \leq O$, $C \leq O$. In this case, we see that

$$\begin{aligned}(A \cdot B) \cdot C &= ((-A) \cdot (-B)) \cdot C \\ &= -(((-A) \cdot (-B)) \cdot (-C) \\ &= -((-A) \cdot ((-B) \cdot (-C))) && \text{by Case 1} \\ &= A \cdot ((-B) \cdot (-C)) \\ &= A \cdot (B \cdot C)\end{aligned}$$

as desired.

Since we have exhaustively covered all possible cases, the result follows. ■

Next we move onto obtaining the multiplicative unit for $\mathbb{R}$. Based on the additive unit in $\mathbb{R}$, the description of the multiplicative unit is not surprising.

Lemma B.3.22. *The multiplication operator on the real numbers* $\mathbb{R}$ *has a multiplicative unit. That is, if* $I = \{q \in \mathbb{Q} \mid q < 1\}$*, then* $I \in \mathbb{R}$*,* $I \neq O$*, and* $I \cdot A = A$ *for all* $A \in \mathbb{R}$*.*

Proof. To begin, to show that $I \in \mathbb{R}$ we must show that I is a Dedekind cut. Clearly $I \neq \emptyset$ and $I \neq \mathbb{Q}$ by construction. Specifically, $\frac{1}{2} \in I$ and $\frac{1}{2} \notin O$ so $I \neq O$.

To see that I is downward closed let $q \in \mathbb{Q}$ and $r \in I$ be such that $q \leq r$. Since $r \in I$, we have that $r < 1$. Hence $q \leq r < 1$ so $q < 1$. Thus $q \in I$ as desired.

To see that I does not contain any of its upper bounds, suppose for the sake of a contradiction that there exists a $q \in I$ such that q is an upper bound for I. Since $q \in I$, we know that $q < 1$. Therefore $\frac{1+q}{2} < 1$ so $\frac{1+q}{2} \in I$. However since $q < I$ and since $\frac{1+q}{2} \not\leq q$ as $q < 1$, we have a contradiction to the fact that q is an upper bound for I. Hence I does not contain any of its upper bounds. Hence I is a Dedekind cut by Definition B.3.1 so $I \in \mathbb{R}$.

To complete the proof, let $A \in \mathbb{R}$ be arbitrary. To see that $I \cdot A = A$, we will divide the proof into two cases.

<u>Case 1: $O \leq A$.</u> Assume $O \leq A$. Hence

$$I \cdot A = \{ab \mid a \in A, b \in \mathbb{Q}, a \geq 0, 0 \leq b < 1\} \cup \{q \in \mathbb{Q} \mid q < 0\}.$$

To see that $I \cdot A = A$, we will demonstrate that $I \cdot A \subseteq A$ and $A \subseteq I \cdot A$.

To see that $I \cdot A \subseteq A$, let $x \in I \cdot A$ be arbitrary. If $x < 0$ then note since $O \leq A$ that $x \in O \subseteq A$ as desired. Otherwise, by the description of $I \cdot A$, there exists an $a \in A$ and a $b \in \mathbb{Q}$ such that $a \geq 0$, $0 \leq b < 1$, and $x = ab$. Since $0 \leq b < 1$, it follows that $x = ab \leq a$. Therefore, since A is a Dedekind cut and thus downward closed, this implies that $x \in A$. Hence, since $x \in I \cdot A$ was arbitrary, $I \cdot A \subseteq A$.

To see that $A \subseteq I \cdot A$, let $a \in A$ be arbitrary. If $a < 0$, then $a \in I \cdot A$ by definition. Therefore, we may assume without loss of generality that $a \geq 0$. Since A is a Dedekind cut and thus does not contain any of its upper bounds, a is not an upper bound of A. Therefore, there exists an $a' \in A$ such that $0 \leq a < a'$. Thus $b = \frac{a}{a'} \in \mathbb{Q}$ is such that $0 \leq b < 1$ and $a = a'b$. Hence $a \in I \cdot A$ as desired. Therefore, since $a \in A$ was arbitrary, $A \subseteq I \cdot A$.

Therefore $I \cdot A = A$ in this case.

<u>Case 2: $A < O$.</u> Assume $A < O$. Therefore, since $O \leq I$, we have that

$$\begin{aligned} I \cdot A &= -(I \cdot (-A)) & \\ &= -(-A) & \text{by Case 1} \\ &= A & \text{by Corollary B.3.16, part b)} \end{aligned}$$

as desired.

Therefore, since the above cover all possible case, we obtain that $I \in \mathbb{R}$ and $I \cdot A = A$ as desired. ∎

Remark B.3.23. For clarification, we will use $I = \{q \in \mathbb{Q} \mid q < 1\}$ for the multiplicative unit (i.e. one element) for $\mathbb{R}$ throughout this section whereas we will reserve 1 for the unit of $\mathbb{Q}$.

To demonstrate every non-zero element of $\mathbb{R}$ has a multiplicative unit, we will focus on $A \in \mathbb{R}$ with $A > O$ as the case $A < O$ will follow similar to how we have been handling such cases. To show that A has a multiplicative inverse, we need to construct a Dedekind cut X such that $A \cdot X = I$. Recalling that since $A > O$, we would expect $X > O$ in which case we need only multiply the non-negative elements of A and X together and add in the negative elements of $\mathbb{Q}$ afterwards. So we want the non-negative elements of X to be precise the elements of $\mathbb{Q}$ that when multiplied by non-negative elements of A yield exactly the elements of $\mathbb{Q}$ between 0 and 1. Since each element of $\mathbb{Q} \setminus A$ is an upper bound of A, the following is the correct set.

Lemma B.3.24. *Let $A \in \mathbb{R}$ be such that $O < A$. If*

$$X = \left\{ \frac{q}{r} \,\middle|\, r \in \mathbb{Q} \setminus A, q \in \mathbb{Q}, q < 1 \right\},$$

then $X \in \mathbb{R}$, $X > 0$, and $A \cdot X = I$.

Hence X is the multiplicative inverse of A so every element of $\mathbb{R}$ that is larger than O has an multiplicative inverse.

Proof. Assume $A \in \mathbb{R}$ is such that $O < A$. Thus $O \subsetneq A$ so A contains a positive element a_0 of $\mathbb{Q}$ and contains all negative elements of $\mathbb{Q}$. Therefore, since A is a Dedekind cut and thus downward closed, we know that if $r \in \mathbb{Q} \setminus A$ then $0 < a_0 < r$. Therefore $r \neq 0$ for all $r \in \mathbb{Q} \setminus A$ so that X is a well-defined set.

To see that $X \in \mathbb{R}$, we will show that X is a Dededkind cut by verify the defining four properties of a Dedekind cut from Definition B.3.1.

$\underline{X \neq \emptyset}$. To see that $X \neq \emptyset$, recall since A is a Dedekind cut that $A \neq \mathbb{Q}$. Therefore, there exists an $r \in \mathbb{Q} \setminus A$. By the start of this proof, we know that $r > 0$. Therefore, $0 \in \mathbb{Q}$ and $0 < 1$, we obtain that $\frac{0}{r} \in X$ by definition so $X \neq \emptyset$ as desired.

$\underline{X \neq \mathbb{Q}}$. To see that $X \neq \mathbb{Q}$, recall from the start of this proof that there exists an $a_0 \in A$ such that $0 < a_0 < r$ for all $r \in \mathbb{Q} \setminus A$. Therefore, for all $r \in \mathbb{Q} \setminus A$, we have that

$$\frac{q}{r} \leq \frac{1}{a_0}.$$

Therefore $\frac{2}{a_0} \notin X$ so $X \neq \mathbb{Q}$ as desired.

$\underline{X \text{ is downward closed.}}$ To see that X is downward closed, assume $x \in X$ and $y \in \mathbb{Q}$ are such that $y \leq x$. By the definition of X there exists a $r \in \mathbb{Q} \setminus A$ and a $q \in \mathbb{Q}$ such that $q < 2$ and $x = \frac{q}{r}$. Therefore $y \leq \frac{q}{r}$. Recall by start of this proof that $r > 0$. Therefore we have that $yr \leq q < 1$. Let $q' = yr$.

Therefore $q' \in \mathbb{Q}$, $q' < 1$, and $y = \frac{q'}{r}$ where $r \in \mathbb{Q} \setminus A$. Hence $y \in X$ by the definition of X as desired.

No element of X is an upper bound for X. To see this, suppose for the sake of a contradiction that there exists an $y \in X$ such that y is an upper bound for X. By the definition of X there exists an $r \in \mathbb{Q} \setminus A$ and a $q \in \mathbb{Q}$ such that $q < 1$ and $y = \frac{q}{r}$. Let $q' = \frac{1+q}{2}$ and let $x = \frac{q'}{r}$. Then $q' \in \mathbb{Q}$ and $q < q' < 1$ so $x \in X$. Moreover, since we know that $r > 0$ by the start of this proof, we obtain that

$$y = \frac{q}{r} < \frac{q'}{r} = x.$$

Since this contradicts the fact that y was an upper bound for X, we have a contradiction. Therefore no element of X is an upper bound of X as desired.

Hence X is a Dedekind cut by Definition B.3.1 so $X \in \mathbb{R}$.

To see that $O < X$, note by the proof that $X \neq \mathbb{Q}$ that there exists an $a_0 > 0$ such that $\frac{2}{a_0} \in X$. Therefore, since $\frac{2}{a_0} \notin O$, and since $X \leq O$ or $X > O$ by Lemma B.3.5 so $X \subseteq O$ or $O \subsetneq X$, it follows that $O \subsetneq X$ so $O < X$.

To see that $A \cdot X = I$, first note since $A > O$ and $X > O$ that

$$A \cdot X = \{ax \mid a \in A, x \in X, \text{ and } a, x \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}$$

To see that $A \cdot X = I$, we will demonstrate that $A \cdot X \subseteq I$ and $I \subseteq A \cdot X$.

To see that $A \cdot X \subseteq I$, let $z \in A \cdot X$ be arbitrary. If $z < 0$, then $z \in I$ by definition. Therefore, without loss of generality, we may assume that $z \geq 0$. By the definition of $A \cdot X$, there exists an $a \in A$ and a $x \in X$ such that $a \geq 0$, $x \geq 0$, and $z = ax$. By the definition of X, there exists a $r \in \mathbb{Q} \setminus A$ and a $q \in \mathbb{Q}$ such that $q < 1$ and $x = \frac{q}{r}$. Hence $z = q\frac{a}{r}$. Since A is a Dedekind cut and thus downward closed, $r \in \mathbb{Q} \setminus A$ and $a \in A$ imply that $a < r$. Since $r > 0$ by the start of this proof, and since $a \geq 0$, we have that $0 \leq \frac{a}{r} < 1$. Therefore, since $q < 1$, we obtain that $z = q\frac{a}{r} < 1$. Hence $z \in I$ by the definition of I. Therefore, since $z \in A \cdot X$ was arbitrary, $A \cdot X \subseteq I$.

To see that $I \subseteq A \cdot X$, let $z \in I$ be arbitrary. Hence $z < 1$. Since $A > O$ and $X > O$, there exist $a \in A$ and $x \in X$ such that $a, x > 0$. Hence $ax \in A \cdot X$. Therefore, since $ax > 0$ and since $A \cdot X$ is a Dedekind cut and thus downward closed, we have that

$$\{q \in \mathbb{Q} \mid q \leq 0\} \subseteq A \cdot X.$$

Therefore, if $z \leq 0$ then $z \in A \cdot X$. Hence we may assume without loss of generality that $0 < z < 1$.

Recall from the start of this proof that there exists an $a_0 \in A$ such that $0 < a_0$. Since $0 < z < 1$, we have that $\epsilon = \frac{a_0}{z}(1 - z) \in \mathbb{Q}$ is well-defined. Moreover, since $0 < z < 1$ and $a_0 > 0$, we have that $\epsilon > 0$.

By Lemma B.3.13 there exists an $a' \in A$ and a $\delta \in \mathbb{Q}$ such that $a + \delta \in \mathbb{Q} \setminus A$, $0 < a_0 \leq a'$, and $0 < \delta < \epsilon$. Let $r' = a' + \delta \in \mathbb{Q} \setminus A$. Therefore $r' > 0$ and we have that

$$\begin{aligned} z\frac{r'}{a'} &= z\frac{a'+\delta}{a'} \\ &= z + z\frac{\delta}{a'} \\ &< z + z\frac{\epsilon}{a_0} \\ &= z + \frac{z}{a_0}\epsilon \\ &= z + \frac{z}{a_0}\left(\frac{a_0}{z}(1-z)\right) \\ &= z + (1-z) = 1. \end{aligned}$$

Therefore $q = z\frac{r'}{a'} \in \mathbb{Q}$ is such that $0 < q < 1$. Moreover, we have that $z = a'\frac{q}{r'}$ since $a', r' > 0$. Let $x = \frac{q}{r'}$ so that $z = a'x$. Since $0 < q < 1$, $r' \in \mathbb{Q} \setminus A$, and $r' > 0$, we have that $x \in X$ and $x > 0$. Therefore, since $z = a'x$, $a' \in A$, $x \in X$, and $a', x > 0$, obtain that $z \in A \cdot X$. Therefore, since $z \in I$ was arbitrary, $I \subseteq A \cdot X$.

Hence $X \in \mathbb{R}$ and $A \cdot X = I$ as desired. ∎

When $A \in \mathbb{R}$ and $A > O$, we will use A^{-1} to denote the multiplicative inverse of A from Lemma B.3.24. Using this, we can demonstrate that negative elements of $\mathbb{R}$ have multiplicative inverses.

Lemma B.3.25. *Let $A \in \mathbb{R}$ be such that $A < O$. Then $O < -A$ so if $X = -((-A)^{-1})$, then X is a well-defined element of $\mathbb{R}$. Moreover $X < O$ and $A \cdot X = I$.*

Hence X is the multiplicative inverse of A so every element of $\mathbb{R}$ that is less than O has an multiplicative inverse.

Proof. Assume $A < O$. Thus Lemma B.3.9 implies that $A+(-A) < O+(-A)$ and thus $O < -A$. Therefore, Lemma B.3.24 implies that $(-A)^{-1}$ is well-defined and thus $X = -((-A)^{-1})$ is well-defined.

To see that $X < O$, notice since $-A > O$ that $(-A)^{-1} > O$ by Lemma B.3.24. Therefore, by applying Lemma B.3.9 again, we obtain that $X = -(-A)^{-1} < O$ as desired.

Finally since $X < O$ and $A < O$, we have that

$$\begin{aligned} A \cdot X &= (-A) \cdot (-X) && \text{by definition} \\ &= (-A) \cdot (-(-((-A)^{-1}))) \\ &= (-A) \cdot (-A)^{-1} && \text{by Lemma B.3.10} \\ &= I && \text{by Lemma B.3.24} \end{aligned}$$

as desired. ∎

Corollary B.3.26. *Every element of* $\mathbb{R} \setminus \{O\}$ *has a multiplicative inverse.*

Proof. This result immediately follows from Lemmata B.3.24 and B.3.25. ∎

The only remaining property of $\mathbb{R}$ that we need to demonstrate is that addition and multiplication satisfy the distributive property. This is a colossal pain. Indeed, given $A, B, C \in \mathbb{R}$, we desire to prove an equation involving $A \cdot (B + C)$, $A \cdot B$, and $A \cdot C$. Since the definition of the multiplication operation depends on whether the elements are non-negative or non-positive, this could be a lot of complicated case work. The following lemma will aid us in bypassing a lot of this casework.

Lemma B.3.27. *For all* $A \in \mathbb{R}$, $-A = A \cdot (-I)$.

Proof. Notice since $O \subseteq I$ that $O \leq I$. Therefore Lemma B.3.9 implies that $O + (-I) \leq I + (-I)$ and thus $-I \leq O$.

Let $A \in \mathbb{R}$. To see that $-A = A \cdot (-I)$, we will divide the proof into two cases.

Case 1: $A \geq O$. Notice that

$$\begin{aligned} A \cdot (-I) &= -(A \cdot (-(-I))) && \text{by definition} \\ &= -(A \cdot I) && \text{by Corollary B.3.16} \\ &= -A \end{aligned}$$

as desired.

Case 2: $A < O$. Notice that

$$\begin{aligned} A \cdot (-I) &= (-A) \cdot (-(-I)) && \text{by definition} \\ &= (-A) \cdot I && \text{by Corollary B.3.16} \\ &= -A \end{aligned}$$

as desired. ∎

Onto the proof of the distributive property.

Lemma B.3.28. *The addition and multiplication operations on* $\mathbb{R}$ *are distributive. That is, if* $A, B, C \in \mathbb{R}$*, then*

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C).$$

Proof. Let $A, B, C \in \mathbb{R}$. To prove this result, we are required to divide the proof into several cases.

Case 1: $A \geq O$, $B \geq O$, $C \geq O$. Since $B \geq O$ and $C \geq O$, we know that $B + C \geq O$. Notice that

$$\begin{aligned}
&A \cdot (B + C) \\
&= A \cdot (\{b + c \mid b \in B, c \in C\}) \\
&= \{a(b + c) \mid a \in A, b \in B, c \in C, a, b + c \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\} \\
&= \{a(b + c) \mid a \in A, b \in B, c \in C, a, b, c \geq 0\} \\
&\quad \cup \{a(b + q) \mid a \in A, b \in B, q \in \mathbb{Q}, a, b \geq 0, q < 0\} \\
&\quad \cup \{a(q + c) \mid a \in A, q \in \mathbb{Q}, c \in C, a, c \geq 0, q < 0\} \\
&\quad \cup \{q \in \mathbb{Q} \mid q < 0\}
\end{aligned}$$

whereas

$$\begin{aligned}
(A \cdot B) + (A \cdot C) &= (\{ab \mid a \in A, b \in B, a, b \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}) \\
&\quad + (\{ac \mid a \in A, c \in C, a, c \geq 0\} \cup \{q \in \mathbb{Q} \mid q < 0\}) \\
&= \{ab + ac \mid a \in A, b \in B, c \in C, a, b, c \geq 0\} \\
&\quad \cup \{ab + q \mid a \in A, b \in B, q \in \mathbb{Q}, a, b \geq 0, q < 0\} \\
&\quad \cup \{q + ac \mid a \in A, q \in \mathbb{Q}, c \in C, a, c \geq 0, q < 0\} \\
&\quad \cup \{q_1 + q_2 \in \mathbb{Q} \mid q_1, q_2 < 0\}.
\end{aligned}$$

If one looks hard enough at these sets and notes the first of the four sets in each of these two decompositions contains 0, one can observe that these sets are the same. Hence the result holds in this case.

Case 2: $A \geq O$, $B < O$, $C < O$. Since $B < O$ and $C < O$, we know that $B + C < O + C < O + O = O$. Therefore

$$\begin{aligned}
A \cdot (B + C) &= -(A \cdot (-(B + C))) \\
&= -(A \cdot ((-B) + (-C))) && \text{by Corollary B.3.16} \\
&= -((A \cdot (-B)) + (A \cdot (-C))) && \text{by Case 1} \\
&= (-(A \cdot (-B))) + (-(A \cdot (-C))) && \text{by Corollary B.3.16} \\
&= (A \cdot B) + (A \cdot C) && \text{by definition}
\end{aligned}$$

as desired.

Case 3: $A \geq O$, $B \geq O$, $C < O$, $B + C > 0$. Since $C < O$, Lemma B.3.9 implies that $C + (-C) < O + (-C)$ and thus $O < -C$. Therefore

$$\begin{aligned}
(A \cdot (B + C)) + (A \cdot (-C)) &= A \cdot ((B + C) + (-C)) && \text{by Case 1} \\
&= A \cdot (B + (C + (-C))) \\
&= A \cdot (B + O) = A \cdot B.
\end{aligned}$$

Hence

$$\begin{aligned}
A \cdot (B+C) &= (A \cdot (B+C)) + O \\
&= (A \cdot (B+C)) + ((A \cdot (-C)) + (-(A \cdot (-C))) \\
&= ((A \cdot (B+C)) + (A \cdot (-C))) + (-(A \cdot (-C))) \\
&= (A \cdot B) + (-(A \cdot (-C))) \\
&= (A \cdot B) + (A \cdot C)
\end{aligned}$$

as desired.

Case 4: $A \geq O$, $B \geq O$, $C < O$, $B + C < 0$. Since $B + C < O$, Lemma B.3.9 implies that $(B + C) + (-(B + C)) < O + (-(B + C))$ and thus $O < -(B + C)$. Therefore by Case 1 we have that

$$\begin{aligned}
(A \cdot (-(B+C))) + (A \cdot B) &= A \cdot ((-(B+C)) + B) && \text{by Case 1} \\
&= A \cdot (((-B) + (-C)) + B) \\
&= A \cdot (((-C) + (-B)) + B) \\
&= A \cdot ((-C) + ((-B) + B)) \\
&= A \cdot ((-C) + (B + (-B))) \\
&= A \cdot ((-C) + O) \\
&= A \cdot (-C).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A \cdot (-(B+C)) &= (A \cdot (-(B+C))) + O \\
&= (A \cdot (-(B+C))) + ((A \cdot B) + (-(A \cdot B))) \\
&= ((A \cdot (-(B+C))) + (A \cdot B)) + (-(A \cdot B)) \\
&= (A \cdot (-C)) + (-(A \cdot B)) \\
&= (-(A \cdot B)) + (A \cdot (-C)).
\end{aligned}$$

Therefore

$$\begin{aligned}
A \cdot (B+C) &= -(A \cdot (-(B+C))) && \text{by definition} \\
&= -((-(A \cdot B)) + (A \cdot (-C))) \\
&= (-(-(A \cdot B))) + (-(A \cdot (-C))) && \text{by Corollary B.3.16} \\
&= (A \cdot B) + (A \cdot C) && \text{by Corollary B.3.16 and definition}
\end{aligned}$$

as desired.

Case 5: $A \geq O$, $B < O$, $C \geq O$, $B + C > 0$. This case follows from Case 3 by interchaning B and C and using the commutativity of addition from Lemma B.3.10.

Case 6: $A \geq O$, $B < O$, $C \geq O$, $B + C < 0$. This case follows from Case 4 by interchaning B and C and using the commutativity of addition from Lemma B.3.10.

Notice that we obtain the result holds when $A \geq O$ by combining the above six cases. Therefore, there is only one additional case we need to consider.

Case 7: $A < O$. Since $-(-A) = A$ by Corollary B.3.16, Lemma B.3.27 that $A = (-A) \cdot (-I)$. Therefore, we have that

$$\begin{aligned} A \cdot (B + C) &= ((-A) \cdot (-I)) \cdot (B + C) \\ &= (-A) \cdot ((-I) \cdot (B + C)) \\ &= (-A) \cdot ((B + C) \cdot (-I)) \\ &= (-A) \cdot (-(B + C)) && \text{by Lemma B.3.27} \\ &= (-A) \cdot ((-B) + (-C)) && \text{by Corollary B.3.16} \end{aligned}$$

and

$$\begin{aligned} &(A \cdot B) + (A \cdot C) \\ &= (((-A) \cdot (-I)) \cdot B) + (((-A) \cdot (-I)) \cdot C) \\ &= ((-A) \cdot ((-I) \cdot B)) + ((-A) \cdot ((-I) \cdot C)) \\ &= ((-A) \cdot (B \cdot (-I))) + ((-A) \cdot (C \cdot (-I))) \\ &= ((-A) \cdot (-B)) + ((-A) \cdot (-C)) && \text{by Corollary B.3.16.} \end{aligned}$$

Note that $A < O$ implies $A + (-A) < O + (-A)$ by Lemma B.3.9 and thus $-A > O$. Therefore, by applying one of the first six cases, we obtain that

$$A \cdot (B + C) = (A \cdot B) + (A \cdot C).$$

Therefore, as we have covered all possible cases, the result holds. ∎

Theorem B.3.29. *The real numbers $\mathbb{R}$ are a totally ordered field with the Least Upper Bound Property.*

Proof. First, note that Lemma B.3.10, Lemma B.3.14, Lemma B.3.20, Lemma B.3.21, Lemma B.3.22, Corollary B.3.26, and Lemma B.3.28 together imply that $\mathbb{R}$ is a field with the operations $+$ and $\cdot$. Subsequently Lemma B.3.5, Lemma B.3.9, and Lemma B.3.19 imply that $\leq$ is a total ordering on $\mathbb{R}$ with the additive and multiplicative properties. Therefore $\mathbb{R}$ is a totally ordered field. Finally Theorem B.3.6 implies that $\mathbb{R}$ has the least upper bound property as desired. ∎

B.4 Real Numbers via Cauchy Sequences

We now turn to our second method for constructing the real numbers. By Section B.2, we know the rational numbers have all the necessary properties required in this section. In addition, we will use the absolute value function on $\mathbb{Q}$. Note parts (a), (a), and (a) of Lemma 1.3.12 and the triangle inequality (Proposition 1.3.13) hold in $\mathbb{Q}$ by the same proofs.

The technique for constructing the real numbers from the rational numbers in this section is analytic in nature. Specifically, we had constructed the real numbers and thus had Proposition 1.3.8, then every real number would be the limit of a sequence of rational numbers. Thus, we can describe the real numbers as limits of sequences of rational numbers. However, the issue with this lies in that we cannot discuss convergence of a sequence without knowing that the limit of the sequence. To bypass this, we turn to the ideas of Section 2.5; namely Cauchy sequences. By considering Cauchy sequences of rational numbers, we can complete the rational numbers and obtain the real numbers. The benefits of this approach are that it is generally viewed to be as an easier approach, it can be generalized to other objects (see MATH 4011), and it is much more thematic with this course on real analysis as it utilizes many of the techniques and ideas learnt throughout the course. However, verifying the Least Upper Bound Property is a bit of a challenge.

Thus we being by discussion Cauchy sequences of rational numbers. Since we do not have the real numbers, we cannot use the definition and properties in Section 2.5 as the definition requires 'for all $\epsilon > 0$' meaning all real numbers ϵ greater than 0. To bypass this, we simply need to restrict our ϵ to be rational numbers. Thus, for notational purposes, throughout this section we will denote the positive rational numbers by $\mathbb{Q}_+$; that is,

$$\mathbb{Q}_+ = \{q \in \mathbb{Q} \mid q > 0\}.$$

Definition B.4.1. A sequence $(q_n)_{n\geq 1}$ of rational numbers is said to be *Cauchy in* $\mathbb{Q}$ if for all $\epsilon \in \mathbb{Q}_+$ there exists an $N \in \mathbb{N}$ such that $|q_n - q_m| \leq \epsilon$ for all $n, m \geq N$. The set of all Cauchy sequences in $\mathbb{Q}$ is will be denoted $\mathcal{R}$.

Example B.4.2. For each $q \in \mathbb{Q}$, the constant sequence $(q)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{Q}$. Indeed, for all $\epsilon \in \mathbb{Q}_+$ we have that $|q_n - q_m| = |q - q| = 0 \leq \epsilon$ for all $n, m \in \mathbb{N}$.

In fact, the constant sequences of rational numbers will be how the rational numbers embed into the real numbers.

Before we proceed to defining the real numbers via $\mathcal{R}$, we will use several of the analytic techniques from the course, plus a few additional ones. Moreover, we will need to know that Cauchy sequences of rational numbers are bounded. This does not directly follow from Section 2.5 due to the change of definition of what it means to be Cauchy, but the proof remains the same.

Lemma B.4.3. *If $(q_n)_{n\geq 1} \in \mathcal{R}$, then $(q_n)_{n\geq 1}$ is bounded; that is, there exists an $M \in \mathbb{Q}_+$ such that $|q_n| \leq M$ for all $n \in \mathbb{N}$.*

Proof. Let $(q_n)_{n\geq 1} \in \mathcal{R}$. Since $(q_n)_{n\geq 1}$ is Cauchy, there exists an $N \in \mathbb{N}$ such that $|q_n - q_m| \leq 1$ for all $n, m \geq N$. Hence, by letting $m = N$, we obtain that $|q_n| \leq |q_N| + 1$ for all $n \geq N$ by the Triangle Inequality.

Let $M = \max\{|q_1|, |q_2|, \ldots, |q_{N-1}|, |q_N|+1\}$. Since $|q_n| \in \mathbb{Q}$ for all $n \in \mathbb{N}$, we obtain that $M \in \mathbb{Q}$. Moreover, if $n \leq N$ then $|q_n| \leq M$ whereas if $n \geq N$ then $|q_n| \leq |q_N| + 1 \leq M$ by the above paragraph. Hence $|q_n| \leq M$ for all $n \in \mathbb{N}$ so $(q_n)_{n\geq 1}$ is bounded as desired. ∎

To construct the real numbers, recall our motivation was that each real number could be described as the limit of rational numbers. However, there could be many sequences of rational numbers that converge to the same real number. Therefore, to distinguish real numbers via sequences of rational numbers, we need an equivalence relation on $\mathcal{R}$.

Lemma B.4.4. *Consider the relation $\sim$ on $\mathcal{R}$ defined as follows: for all $(q_n)_{n\geq 1}, (r_n)_{n\geq 1} \in \mathcal{R}$, $(q_n)_{n\geq 1} \sim (r_n)_{n\geq 1}$ if and only if for all $\epsilon \in \mathbb{Q}_+$, there exists an $N \in \mathbb{N}$ such that $|q_n - r_n| \leq \epsilon$ for all $n \geq N$. Then $\sim$ is an equivalence relation.*

Proof. To see that $\sim$ is an equivalence relation, we need to show that $\sim_{\mathbb{Q}}$ is reflexive, symmetric, and transitive.

Reflexive: To see that $\sim$ is reflexive, let $(q_n)_{n\geq 1} \in \mathcal{R}$. Since for all $\epsilon \in \mathbb{Q}_+$ we have that $|q_n - q_n| = 0 \leq \epsilon$ for all $n \in \mathbb{N}$, we see that $(q_n)_{n\geq 1} \sim (q_n)_{n\geq 1}$ by the definition of $\sim$ as desired.

Symmetric: To see that $\sim$ is symmetric, let $(q_n)_{n\geq 1}, (r_n)_{n\geq 1} \in \mathcal{R}$ be such that $(q_n)_{n\geq 1} \sim (r_n)_{n\geq 1}$. Hence for all $\epsilon \in \mathbb{Q}_+$, there exists an $N \in \mathbb{N}$ such that $|q_n - r_n| \leq \epsilon$ for all $n \geq N$. Therefore, since $|r_n - q_n| = |q_n - r_n|$ for all $n \in \mathbb{N}$, it follows that for all $\epsilon \in \mathbb{Q}_+$, there exists an $N \in \mathbb{N}$ such that $|r_n - q_n| \leq \epsilon$ for all $n \geq N$. Hence $(r_n)_{n\geq 1} \sim (q_n)_{n\geq 1}$ by the definition of $\sim$. Thus $\sim$ is symmetric.

Transitive: To see that $\sim$ is transitive, let $(q_n)_{n\geq 1}, (r_n)_{n\geq 1}, (s_n)_{n\geq 1} \in \mathcal{R}$ be such that $(q_n)_{n\geq 1} \sim (r_n)_{n\geq 1}$ and $(r_n)_{n\geq 1} \sim (s_n)_{n\geq 1}$. To see that $(q_n)_{n\geq 1} \sim (s_n)_{n\geq 1}$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $(q_n)_{n\geq 1} \sim (r_n)_{n\geq 1}$ and since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - r_n| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_1$. Similarly, since $(r_n)_{n\geq 1} \sim (s_n)_{n\geq 1}$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - s_n| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$|q_n - s_n| \leq |q_n - r_n| + |r_n - s_n| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, $(q_n)_{n\geq 1} \sim (s_n)_{n\geq 1}$ by the definition of $\sim$. Hence $\sim$ is transitive.

Therefore, since all three properties have been verified, $\sim$ is an equivalence relation. ∎

Thus, using equivalence classes, we obtain the following definition of the real numbers.

Definition B.4.5. The *real numbers*, denoted $\mathbb{R}$, is the set

$$\mathbb{R} = \{[(q_n)_{n\geq 1}] \mid (q_n)_{n\geq 1} \in \mathcal{R}\}.$$

That is, $\mathbb{R}$ is the set of all equivalent classes of Cauchy sequences in $\mathbb{Q}$.

Luckily, unlike with Dedekind cuts, defining the operations of addition and multiplication via $\mathcal{R}$ is much simpler. To being, we must show that if we add and multiply elements of $\mathcal{R}$ term-wise then we obtain elements of $\mathcal{R}$. The proofs of these facts are very similar to those used in Theorem 2.3.1.

Lemma B.4.6. *If $(q_n)_{n\geq 1}, (r_n)_{n\geq 1} \in \mathcal{R}$, then the following hold:*

a) $(q_n + r_n)_{n\geq 1} \in \mathcal{R}$.

b) $(q_n r_n)_{n\geq 1} \in \mathcal{R}$.

Proof. a) To see that $(q_n+r_n)_{n\geq 1} \in \mathcal{R}$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$ and since $(q_n)_{n\geq 1} \in \mathcal{R}$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - q_m| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_1$. Similarly $(r_n)_{n\geq 1} \in \mathcal{R}$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - r_m| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$\begin{aligned}|(q_n + r_n) - (q_m + r_m)| &= |(q_n - q_m) + (r_n - r_m)| \\ &\leq |q_n - q_m| + |r_n - r_m| \\ &\frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $(q_n + r_n)_{n\geq 1} \in \mathcal{R}$ by definition as desired.

b) To see that $(q_n r_n)_{n\geq 1} \in \mathcal{R}$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $(q_n)_{n\geq 1} \in \mathcal{R}$, Lemma B.4.3 implies that there exists an $M_1 \in \mathbb{Q}$ with $M_1 > 0$ such that $|q_n| \leq M_1$ for all $n \in \mathbb{N}$. Similarly, $(r_n)_{n\geq 1} \in \mathcal{R}$, Lemma B.4.3 implies that there exists an $M_2 \in \mathbb{Q}$ with $M_2 > 0$ such that $|r_n| \leq M_2$ for all $n \in \mathbb{N}$.

Since $\frac{\epsilon}{2M_2} \in \mathbb{Q}_+$ and since $(q_n)_{n\geq 1} \in \mathcal{R}$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - q_m| \leq \frac{\epsilon}{2M_2}$ for all $n, m \geq N_1$. Similarly, since $\frac{\epsilon}{2M_1} \in \mathbb{Q}_+$ and since $(r_n)_{n\geq 1} \in \mathcal{R}$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - r_m| \leq \frac{\epsilon}{2M_1}$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$\begin{aligned}|q_n r_n - q_m r_m| &= |(q_n r_n - q_m r_n) + (q_m r_n - q_m r_m)| \\ &\leq |q_n - q_m||r_n| + |q_m||r_n - r_m| \\ &\leq \frac{\epsilon}{2M_1} M_1 + M_2 \frac{\epsilon}{2M_2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $(q_n r_n)_{n\geq 1} \in \mathcal{R}$ by definition as desired. ∎

Although Lemma B.4.6 shows that the term-wise sum and product of elements of $\mathcal{R}$ are elements of $\mathcal{R}$, we must deal with the issue of different representatives of equivalence classes before we can define the operations of addition and multiplication on $\mathbb{R}$. The following lemma is all we need.

Lemma B.4.7. *Let $(q_n)_{n\geq 1}, (r_n)_{n\geq 1}, (q'_n)_{n\geq 1}, (r'_n)_{n\geq 1} \in \mathcal{R}$ be such that*

$$(q_n)_{n\geq 1} \sim (q'_n)_{n\geq 1} \quad \text{and} \quad (r_n)_{n\geq 1} \sim (r'_n)_{n\geq 1}.$$

Then the following hold:

a) $(q_n + r_n)_{n\geq 1} \sim (q'_n + r'_n)_{n\geq 1}$.

b) $(q_n r_n)_{n\geq 1} \sim (q'_n r'_n)_{n\geq 1}$.

Proof. a) To see that $(q_n + r_n)_{n\geq 1} \sim (q'_n + r'_n)_{n\geq 1}$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$ and since $(q_n)_{n\geq 1} \sim (q'_n)_{n\geq 1}$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - q'_n| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_1$. Similarly $(r_n)_{n\geq 1} \sim (r'_n)_{n\geq 1}$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - r'_n| \leq \frac{1}{2}\epsilon$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$\begin{aligned} |(q_n + r_n) - (q'_n + r'_n)| &= |(q_n - q'_n) + (r_n - r'_n)| \\ &\leq |q_n - q'_n| + |r_n - r'_n| \\ &\frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $(q_n + r_n)_{n\geq 1} \sim (q'_n + r'_n)_{n\geq 1}$ by definition as desired.

b) To see that $(q_n r_n)_{n\geq 1} \sim (q'_n r'_n)_{n\geq 1}$ let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $(q'_n)_{n\geq 1} \in \mathcal{R}$, Lemma B.4.3 implies that there exists an $M_1 \in \mathbb{Q}_+$ such that $|q'_n| \leq M_1$ for all $n \in \mathbb{N}$. Similarly, $(r_n)_{n\geq 1} \in \mathcal{R}$, Lemma B.4.3 implies that there exists an $M_2 \in \mathbb{Q}_+$ such that $|r_n| \leq M_2$ for all $n \in \mathbb{N}$.

Since $\frac{\epsilon}{2M_2} \in \mathbb{Q}_+$ and since $(q_n)_{n\geq 1} \sim (q'_n)_{n\geq 1}$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - q'_n| \leq \frac{\epsilon}{2M_2}$ for all $n, m \geq N_1$. Similarly, since $\frac{\epsilon}{2M_1} \in \mathbb{Q}_+$ and since $(r_n)_{n\geq 1} \sim (r'_n)_{n\geq 1}$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - r'_n| \leq \frac{\epsilon}{2M_1}$ for all $n, m \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$\begin{aligned} |q_n r_n - q'_n r'_n| &= |(q_n r_n - q'_n r_n) + (q'_n r_n - q'_n r'_n)| \\ &\leq |q_n - q'_n||r_n| + |q'_n||r_n - r'_n| \\ &\leq \frac{\epsilon}{2M_1} M_1 + M_2 \frac{\epsilon}{2M_2} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $(q_n r_n)_{n\geq 1} \sim (q'_n r'_n)_{n\geq 1} \in \mathcal{R}$ by definition as desired. ∎

By Lemmata B.4.6 and B.4.7, the following operations on $\mathbb{R}$ are well-defined.

Definition B.4.8. The operations of $+$ and $\cdot$ on $\mathbb{R}$ are defined as follows: for all $(q_n)_{n\geq 1}, (r_n)_{n\geq 1} \in \mathcal{R}$,

$$[(q_n)_{n\geq 1}] + [(r_n)_{n\geq 1}] = [(q_n + r_n)_{n\geq 1}]$$
$$[(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] = [(q_n r_n)_{n\geq 1}].$$

Luckily proving that these operations on this definition of $\mathbb{R}$ yields a field is much simpler than it was for Dedekind cuts. Before we get to the proof, we demonstrate the following technical result that is needed to prove the existence of multiplicative inverses.

Lemma B.4.9. *If $(q_n)_{n\geq 1} \in \mathcal{R}$ is such that $(q_n)_{n\geq 1} \not\sim (0)_{n\geq 1}$, then there exists a $K \in \mathbb{Q}_+$ and a $N \in \mathbb{N}$ such that $|q_n| \geq K$ for all $n \geq N$.*

Proof. Assume $(q_n)_{n\geq 1} \not\sim (0)_{n\geq 1}$. Thus there exists an $\epsilon \in \mathbb{Q}_+$ such that for all $N \in \mathbb{N}$ there exists an $n_N \geq N$ such that $|q_{n_N}| = |q_{n_N} - 0| > \epsilon$. Let $K = \frac{1}{2}\epsilon$. Clearly $K \in \mathbb{Q}_+$ by construction.

To see that K has the desired property, note since $(q_n)_{n\geq 1} \in \mathcal{R}$ that there exists an $N \in \mathbb{N}$ such that $|q_n - q_m| \leq K$ for all $n, m \geq N$. Therefore, since $n_N \geq N$, by letting $m = N$ we obtain that $|q_n - q_{n_N}| \leq K$ for all $n \geq N$. Hence for all $n \geq N$ we have by the reverse triangle inequality that

$$|q_n| = |q_{n_N} + (q_n - q_{n_N})| \geq |q_{n_N}| - |q_n - q_{n_N}| > \epsilon_0 - K = 2K - K = K$$

as desired. ∎

With Lemma B.4.9 complete, demonstrating most of the field properties hold for this definition of $\mathbb{R}$ simply follow by applying the field properties of $\mathbb{Q}$ term-wise.

Lemma B.4.10. *The real numbers $\mathbb{R}$ are a field with the above operations. That is, for all $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}], [(s_n)_{n\geq 1}] \in \mathbb{R}$,*

a) (Commutativity of $+$) $[(q_n)_{n\geq 1}] + [(r_n)_{n\geq 1}] = [(r_n)_{n\geq 1}] + [(q_n)_{n\geq 1}]$.

b) (Commutativity of $\cdot$) $[(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] = [(r_n)_{n\geq 1}] \cdot [(q_n)_{n\geq 1}]$.

c) (Associativity of $+$) $([(q_n)_{n\geq 1}] + [(r_n)_{n\geq 1}]) + [(s_n)_{n\geq 1}] = [(q_n)_{n\geq 1}] + ([(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}])$.

d) (Associativity of $\cdot$) $([(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]) \cdot [(s_n)_{n\geq 1}] = [(q_n)_{n\geq 1}] \cdot ([(r_n)_{n\geq 1}] \cdot [(s_n)_{n\geq 1}])$.

e) (Distributivity) $[(q_n)_{n\geq 1}] \cdot ([(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}]) = ([(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]) + (([q_n)_{n\geq 1}] \cdot [(s_n)_{n\geq 1}])$.

f) (Additive Unit) $O = [(0)_{n\geq 1}] \in \mathbb{R}$ *and* $[(q_n)_{n\geq 1}] + O = [(q_n)_{n\geq 1}]$.

g) (Multiplicative Unit) $I = [(1)_{n\geq 1}] \in \mathbb{R}$, $I \neq O$, *and* $[(q_n)_{n\geq 1}] \cdot I = [(q_n)_{n\geq 1}]$.

h) (Additive Inverse) $[(-q_n)_{n\geq 1}] \in \mathbb{R}$ *and* $[(q_n)_{n\geq 1}] + [(-q_n)_{n\geq 1}] = [(0, 1)]$.

i) (Multiplicative Inverse) if $[(q_n)_{n\geq 1}] \neq O$, *then there exists a* $[(t_n)_{n\geq 1}] \in \mathbb{R}$ *such that* $[(q_n)_{n\geq 1}] \cdot [(t_n)_{n\geq 1}] = I$.

Proof. a) Notice that

$$\begin{aligned} [(q_n)_{n\geq 1}] + [(r_n)_{n\geq 1}] &= [(q_n + r_n)_{n\geq 1}] \\ &= [(r_n + q_n)_{n\geq 1}] \\ &= [(r_n)_{n\geq 1}] + [(q_n)_{n\geq 1}] \end{aligned}$$

as desired.

b) Notice that

$$\begin{aligned} [(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] &= [(q_n r_n)_{n\geq 1}] \\ &= [(r_q r_n)_{n\geq 1}] \\ &= [(r_n)_{n\geq 1}] \cdot [(q_n)_{n\geq 1}] \end{aligned}$$

as desired.

c) Notice that

$$\begin{aligned} ([(q_n)_{n\geq 1}] + [(r_n)_{n\geq 1}]) + [(s_n)_{n\geq 1}] &= [(q_n + r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}] \\ &= [((q_n + r_n) + s_n)_{n\geq 1}] \\ &= [(q_n + (r_n + s_n)_{n\geq 1}] \\ &= [(q_n)_{n\geq 1}] + [(r_n + s_n)_{n\geq 1}] \\ &= [(q_n)_{n\geq 1}] + ([(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}]) \end{aligned}$$

as desired.

d) Notice that

$$\begin{aligned} ([(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]) + [(s_n)_{n\geq 1}] &= [(q_n r_n)_{n\geq 1}] \cdot [(s_n)_{n\geq 1}] \\ &= [((q_n r_n) s_n)_{n\geq 1}] \\ &= [(q_n (r_n s_n)_{n\geq 1}] \\ &= [(q_n)_{n\geq 1}] \cdot [(r_n s_n)_{n\geq 1}] \\ &= [(q_n)_{n\geq 1}] \cdot ([(r_n)_{n\geq 1}] \cdot [(s_n)_{n\geq 1}]) \end{aligned}$$

as desired.

e) Notice that

$$\begin{aligned}
&[(q_n)_{n\geq 1}] \cdot ([(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}]) \\
&= [(q_n)_{n\geq 1}] \cdot [(r_n + s_n)_{n\geq 1}] \\
&= [(q_n(r_n + s_n))_{n\geq 1}] \\
&= [((q_n r_n) + (q_n s_n))_{n\geq 1}] \\
&= [(q_n r_n)_{n\geq 1}] + [(q_n s_n)_{n\geq 1}] \\
&= ([(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]) + (([q_n)_{n\geq 1}] \cdot [(s_n)_{n\geq 1}])
\end{aligned}$$

as desired.

f) Clearly $(0)_{n\geq 1} \in \mathcal{R}$ by Example B.4.2 so $O \in \mathbb{R}$. Moreover, since $|1-0| = 1 > \frac{1}{2}$, we see that $(1)_{n\geq 1} \nsim (0)_{n\geq 1}$ so $I \neq O$. Finally, notice that

$$[(q_n)_{n\geq 1}] + O = [(q_n + 0)_{n\geq 1}] = [(q_n)_{n\geq 1}]$$

as desired

g) Clearly $(1)_{n\geq 1} \in \mathcal{R}$ by Example B.4.2 so $I \in \mathbb{R}$. Moreover

$$[(q_n)_{n\geq 1}] \cdot I = [(q_n(1))_{n\geq 1}] = [(q_n)_{n\geq 1}]$$

as desired.

h) Since the constant sequence $(-1)_{n\geq 1} \in \mathcal{R}$ by Example B.4.2, we have that

$$[(-q_n)_{n\geq 1}] = [(-1)_{n\geq 1}] \cdot [(q_n)_{n\geq 1}] \in \mathbb{R}.$$

Moreover, notice that

$$[(q_n)_{n\geq 1}] + [(-q_n)_{n\geq 1}] = [(q_n + (-q_n))_{n\geq 1}] = [(0)_{n\geq 1}] = O$$

as desired.

i) Note the proof of this part is very similar to a corresponding part of Lemma 2.3.1.

Assume $[(q_n)_{n\geq 1}] \neq O$. By Lemma B.4.9 there exists a $K \in \mathbb{Q}_+$ and an $N_1 \in \mathbb{N}$ such that $|q_n| \geq K$ for all $n \geq N_1$. For each $n \in \mathbb{N}$, let

$$t_n = \begin{cases} 0 & \text{if } n < N_1 \\ \frac{1}{q_n} & \text{if } n \geq N_1 \end{cases}.$$

Since $|q_n| \geq K > 0$ for all $n \geq N_1$, we see that t_n is well-defined and $t_n \in \mathbb{Q}$ for all $n \in \mathbb{N}$.

We claim that $(t_n)_{n\geq 1} \in \mathcal{R}$. To see this, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $(q_n)_{n\geq 1} \in \mathcal{R}$ and since $\epsilon K^2 \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $|q_n - q_m| \leq \epsilon K^2$ for all $n, m \geq N_2$.

Let $N \max\{N_1, N_2\}$. Therefore, $N \in \mathbb{N}$ and we have for all $n, m \geq N$ that

$$\begin{aligned} |t_n - t_m| &= \left| \frac{1}{q_n} - \frac{1}{q_m} \right| \\ &= \frac{|q_m - q_n|}{|q_n||q_m|} \\ &\leq \frac{\epsilon K^2}{|q_n||q_m|} \\ &\leq \frac{\epsilon K^2}{K(K)} \\ &= \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, $(t_n)_{n\geq 1} \in \mathcal{R}$ by definition.

Finally, notice since for all $\epsilon \in \mathbb{Q}_+$ we have for all $n \geq N_1$ that

$$|q_n t_n - 1| = |1 - 1| = 0 \leq \epsilon,$$

we have that $(q_n t_n)_{n\geq 1} \sim I$ and thus

$$[(q_n)_{n\geq 1}] \cdot [(t_n)_{n\geq 1}] = [(q_n t_n)_{n\geq 1}] = I$$

as desired. ∎

Now that $\mathbb{R}$ has been demonstrated to be a field, we must define the partial ordering on $\mathbb{R}$. Since we are dealing with equivalence relations, the following is exactly what we need to ensure the partial order is well-defined.

Lemma B.4.11. *Let $(q_n)_{n\geq 1}, (r_n)_{n\geq 1}, (q'_n)_{n\geq 1}, (r'_n)_{n\geq 1} \in \mathcal{R}$ be such that*

$$(q_n)_{n\geq 1} \sim (q'_n)_{n\geq 1} \quad \text{and} \quad (r_n)_{n\geq 1} \sim (r'_n)_{n\geq 1}.$$

Then the following statements are equivalent:

(i) For all $\epsilon \in \mathbb{Q}_+$ there exists an $N \in \mathbb{N}$ such that $q_n \leq r_n + \epsilon$ for all $n \geq N$.

(ii) For all $\epsilon \in \mathbb{Q}_+$ there exists an $N \in \mathbb{N}$ such that $q'_n \leq r'_n + \epsilon$ for all $n \geq N$.

Proof. We will only demonstrate that (i) implies (ii) as the proof that (ii) implies (i) will follow by symmetry of the argument and symmetry of the equivalence relation.

Assume (i) holds. To see that (ii) holds, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. We note three things:

- Since $\frac{1}{3}\epsilon \in \mathbb{Q}_+$, by the assumption that (i) holds there exists an $N_1 \in \mathbb{N}$ such that $q_n \leq r_n + \epsilon$ for all $n \geq N_1$.

- Since $(q_n)_{n\geq 1} \sim (q'_n)_{n\geq 1}$ and since $\frac{1}{3}\epsilon \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $|q_n - q'_n| \leq \frac{1}{3}\epsilon$ for all $n \geq N_2$. Hence the triangle inequality implies $q'_n \leq q_n + \frac{1}{3}\epsilon$ for all $n \geq N_2$.

- Since $(r_n)_{n\geq 1} \sim (r'_n)_{n\geq 1}$ and since $\frac{1}{3}\epsilon \in \mathbb{Q}_+$, there exists an $N_3 \in \mathbb{N}$ such that $|r_n - r'_n| \leq \frac{1}{3}\epsilon$ for all $n \geq N_3$. Hence the triangle inequality implies $r_n \leq r'_n + \frac{1}{3}\epsilon$ for all $n \geq N_3$.

Let $N = \max\{N_1, N_2, N_3\}$. Hence $N \in \mathbb{N}$ and we have for all $n \geq N$ that

$$\begin{aligned} q'_n &\leq q_n + \frac{1}{3}\epsilon \\ &\leq \left(r_n + \frac{1}{3}\epsilon\right) + \frac{1}{3}\epsilon \\ &= r_n + \frac{2}{3}\epsilon \\ &\leq \left(r'_n + \frac{1}{3}\epsilon\right) + \frac{2}{3}\epsilon \\ &= r'_n + \epsilon. \end{aligned}$$

Therefore, since $\epsilon > 0$ was arbitrary, (ii) holds as desired. ■

By Lemma B.4.11, the following is well-defined.

Definition B.4.12. The relation $\leq$ on $\mathbb{R}$ is defined as follows: for all $(q_n)_{n\geq 1}, (r_n)_{n\geq 1} \in \mathcal{R}$, $(q_n)_{n\geq 1} \leq (q_n)_{n\geq 1}$ if and only if for all $\epsilon \in \mathbb{Q}_+$ there exists an $N \in \mathbb{N}$ such that $q_n \leq r_n + \epsilon$ for all $n \geq N$.

Showing that $\leq$ is a partial ordering on $\mathbb{R}$ is a fairly straightforward task once one is comfortable enough with the type of analytic arguments used in this course.

Lemma B.4.13. *The relation of $\leq$ on $\mathbb{R}$ is a partial ordering on $\mathbb{R}$. For all $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}], [(s_n)_{n\geq 1}] \in \mathbb{R}$,*

a) (Reflexivity) $[(q_n)_{n\geq 1}] \leq [(q_n)_{n\geq 1}]$.

b) (Antisymmetry) If $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and $[(r_n)_{n\geq 1}] \leq [(q_n)_{n\geq 1}]$, then $[(q_n)_{n\geq 1}] = [(r_n)_{n\geq 1}]$.

c) (Transitivity) If $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and $[(r_n)_{n\geq 1}] \leq [(s_n)_{n\geq 1}]$, then $[(q_n)_{n\geq 1}] \leq [(s_n)_{n\geq 1}]$.

Proof. a) Let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $q_n \leq q_n + \epsilon$ for all $n \in \mathbb{N}$, it follows that $[(q_n)_{n\geq 1}] \leq [(q_n)_{n\geq 1}]$ by definition. Hence $\leq$ is reflexive.

b) Assume $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and $[(r_n)_{n\geq 1}] \leq [(q_n)_{n\geq 1}]$. To see that $[(q_n)_{n\geq 1}] = [(r_n)_{n\geq 1}]$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$

and since $\epsilon \in \mathbb{Q}_+$, there exists an $N_1 \in \mathbb{N}$ such that $q_n \leq r_n + \epsilon$ for all $n \geq N_1$. Similarly, since $[(r_n)_{n\geq 1}] \leq [(q_n)_{n\geq 1}]$ and since $\epsilon \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $r_n \leq q_n + \epsilon$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Hence $N \in \mathbb{N}$ and we have for all $n \geq N$ that

$$q_n \leq r_n + \epsilon \qquad \text{and} \qquad r_n \leq q_n + \epsilon$$

so that

$$q_n - r_n \leq \epsilon \qquad \text{and} \qquad r_n - q_n \leq \epsilon$$

and thus $|q_n - r_n| \leq \epsilon$. Therefore, since $\epsilon > 0$ was arbitrary, $[(q_n)_{n\geq 1}] = [(r_n)_{n\geq 1}]$ by definition. Hence $\leq$ is antisymmetric.

c) Assume $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and $[(r_n)_{n\geq 1}] \leq [(s_n)_{n\geq 1}]$. To see that $[(q_n)_{n\geq 1}] \leq [(s_n)_{n\geq 1}]$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$, there exists an $N_1 \in \mathbb{N}$ such that $q_n \leq r_n + \frac{1}{2}\epsilon$ for all $n \geq N_1$. Similarly, since $[(r_n)_{n\geq 1}] \leq [(s_n)_{n\geq 1}]$ and since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $r_n \leq s_n + \frac{1}{2}\epsilon$ for all $n \geq N_2$.

Let $N = \max\{N_1, N_2\}$. Hence $N \in \mathbb{N}$ and we have for all $n \geq N$ that

$$q_n \leq r_n + \frac{1}{2}\epsilon \leq s_n + \epsilon.$$

Therefore, since $\epsilon > 0$ was arbitrary, $[(q_n)_{n\geq 1}] =\leq [(s_n)_{n\geq 1}]$ by definition. Hence $\leq$ is transitive. ■

Showing that the partial ordering on this definition of $\mathbb{R}$ is a total ordering is a little delicate.

Lemma B.4.14. *The partial ordering $\leq$ on $\mathbb{R}$ is a total ordering.*

Proof. To see that $\leq$ is a total ordering, suppose for the stake of a contradiction that there exists $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}] \in \mathbb{R}$ such that $[(q_n)_{n\geq 1}] \not\leq [(r_n)_{n\geq 1}]$ or $[(r_n)_{n\geq 1}] \not\leq [(q_n)_{n\geq 1}]$. Note

- Since $[(q_n)_{n\geq 1}] \not\leq [(r_n)_{n\geq 1}]$, there exists an $\epsilon_1 \in \mathbb{Q}_+$ such that for all $N \in \mathbb{N}$ there exists a $n \geq N$ such that $q_n > r_n + \epsilon_1$.

- Since $[(r_n)_{n\geq 1}] \not\leq [(q_n)_{n\geq 1}]$, there exists an $\epsilon_2 \in \mathbb{Q}_+$ such that for all $N \in \mathbb{N}$ there exists a $n \geq N$ such that $r_n > q_n + \epsilon_2$.

Since the ordering on $\mathbb{Q}$ is a total ordering, there exists an $\epsilon_0 \in \mathbb{Q}_+$ such that $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$. Since $(q_n)_{n\geq 1} \in \mathcal{R}$ and since $\frac{1}{2}\epsilon_0 \in \mathbb{Q}_+$, there exists an $N_1 \in \mathbb{N}$ such that $|q_n - q_m| \leq \frac{1}{2}\epsilon_0$ for all $n, m \geq N_1$. Similarly, $(r_n)_{n\geq 1} \in \mathcal{R}$ and since $\frac{1}{2}\epsilon_0 \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $|r_n - r_m| \leq \frac{1}{2}\epsilon_0$ for all $n, m \geq N_2$.

Let $N_0 = \max\{N_1, N_2\}$. Clearly $N_0 \in \mathbb{N}$. Moreover, we have that

- $|q_n - q_m| \leq \frac{1}{2}\epsilon_0$ for all $n, m \geq N_0$,

- $|r_n - r_m| \leq \frac{1}{2}\epsilon_0$ for all $n, m \geq N_0$,
- there exists an $n_1 \in \mathbb{N}$ such that $n_1 \geq N_0$ and $q_{n_1} > r_{n_1} + \epsilon_0$, and
- there exists an $n_2 \in \mathbb{N}$ such that $n_2 \geq N_0$ and $r_{n_2} > q_{n_2} + \epsilon_0$.

Hence

$$\begin{aligned} q_{n_2} + \epsilon_0 &< r_{n_2} \\ &\leq r_{n_1} + \frac{1}{2}\epsilon_0 \\ &< (q_{n_1} - \epsilon) + \frac{1}{2}\epsilon_0 \\ &= q_{n_1} - \frac{1}{2}\epsilon_0 \\ &\leq \left(q_{n_2} + \frac{1}{2}\epsilon_0\right) - \frac{1}{2}\epsilon_0 \\ &= q_{n_2}. \end{aligned}$$

Since this implies $\epsilon_0 < 0$ which clearly contradicts the fact that $\epsilon_0 \in \mathbb{Q}_+$, we have a contradiction. Hence the partial ordering $\leq$ on $\mathbb{R}$ is a total ordering. ∎

Checking that this definition of $\mathbb{R}$ is an ordered field is not difficult, but requires a bit of casework.

Lemma B.4.15. *The partial ordering $\leq$ on $\mathbb{R}$ has the following additional properties:*

a) $\leq$ has the additive property; that is, if $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}], [(s_n)_{n\geq 1}] \in \mathbb{R}$ and $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ then $[(q_n)_{n\geq 1}]+[(s_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]+[(s_n)_{n\geq 1}]$.

b) $\leq$ has the multiplicative property; that is, if $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}] \in \mathbb{R}$ are such that $O \leq [(q_n)_{n\geq 1}]$ and $O \leq [(r_n)_{n\geq 1}]$, then $O \leq [(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]$.

Proof. a) Assume $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}], [(s_n)_{n\geq 1}] \in \mathbb{R}$ and $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$. To see that $[(q_n)_{n\geq 1}] + [(s_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}]$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $[(q_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}]$ and since $\epsilon \in \mathbb{Q}_+$, there exists an $N \in \mathbb{N}$ such that $q_n \leq r_n + \epsilon$ for all $n \geq N$. Therefore $q_n + s_n \leq (r_n + s_n) + \epsilon$ for all $n \geq N$. Therefore, since $\epsilon \in \mathbb{Q}_+$ was arbitrary, $[(q_n)_{n\geq 1}] + [(s_n)_{n\geq 1}] \leq [(r_n)_{n\geq 1}] + [(s_n)_{n\geq 1}]$ by definition as desired.

b) Assume $[(q_n)_{n\geq 1}], [(r_n)_{n\geq 1}] \in \mathbb{R}$ are such that $O \leq [(q_n)_{n\geq 1}]$ and $O \leq [(r_n)_{n\geq 1}]$. To proceed, we will divide the proof into three cases:

Case 1: $[(q_n)_{n\geq 1}] = O$. In this case we see that

$$[(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] = [(0)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] = [(0r_n)_{n\geq 1}] = [(0)]_{n\geq 1} = O \geq O$$

as desired.

Case 2: $[(r_n)_{n\geq 1}] = O$. In this case we see that

$$[(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}] = [(q_n)_{n\geq 1}] \cdot [(0)_{n\geq 1}] = [(q_n(0))_{n\geq 1}] = [(0)]_{n\geq 1} = O \geq O$$

as desired.

Case 3: $O < [(q_n)_{n\geq 1}]$ and $O < [(r_n)_{n\geq 1}]$. By Lemma B.4.9 there exists a $K_1 \in \mathbb{Q}_+$ and an $N_1 \in \mathbb{N}$ such that $|q_n| \geq K_1$ for all $n \geq N_1$. Since $O \leq [(q_n)_{n\geq 1}]$ and since $\frac{1}{2}K_1 \in \mathbb{Q}_+$, there exists an $N_2 \in \mathbb{N}$ such that $0 \leq q_n + \frac{1}{2}K_1$ for all $n \geq N_2$. Let $N_3 = \max\{N_1, N_2\}$. Clearly $N_3 \in \mathbb{N}$ and for all $n \geq N_3$ we have that $|q_n| \geq K_1$ and $-\frac{1}{2}K_1 \leq q_n$. Hence $q_n \geq K_1$ for all $n \geq N_3$.

By similar arguments, there exists an $K_2 \in \mathbb{Q}_+$ and an $N_4 \in \mathbb{N}$ such that $r_n \geq K_2$ for all $n \geq N_4$. Let $N_5 = \max\{N_3, N_4\}$. Therefore, for all $n \geq N_5$ we have that

$$q_n r_n \geq K_1 K_2 > 0.$$

Therefore, it follows by the definition of $\leq$ that $O \leq [(q_n)_{n\geq 1}] \cdot [(r_n)_{n\geq 1}]$.

Therefore, as we have covered all possible cases, the result follows. ∎

To complete the proof that $\mathbb{R}$ has the desired properties, only verifying $\mathbb{R}$ has the Least Upper Bound remains. To do this, we first need the following technical lemma to ensure that powers of $\frac{1}{2}$ are sufficiently small.

Lemma B.4.16. *For all $\epsilon \in \mathbb{Q}_+$ there exists an $N \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$ for all $n \geq N$.*

Proof. Note since $\frac{1}{2^{n+1}} \leq \frac{1}{2^n}$ for all $n \in \mathbb{N}$, it suffices to show that there exists an $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$.

Since $\epsilon \in \mathbb{Q}_+$, there exists $a, b \in \mathbb{N}$ such that $\epsilon = \frac{a}{b}$. By considering Peano's Axioms (Definition 1.1.1), there exists an $N \in \mathbb{N}$ such that $b < 2^N a$. Hence there exists an $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$ as desired. ∎

With Lemma B.4.16 in hand, we can verify that this definition of $\mathbb{R}$ has the Least Upper Bound Property. Note this is the most difficult aspect of the proof and much more difficult then verifying the Dedekind cut definition of $\mathbb{R}$ has the Least Upper Bound Property. Of course, overall verifying this definition of $\mathbb{R}$ had the desired properties was far simpler then verifying the Dedekind cut version had the desired properties (e.g. 15 pages to 22 pages).

Theorem B.4.17. *The real numbers $\mathbb{R}$ are a totally ordered field with the Least Upper Bound Property.*

Proof. To begin, note by Definition B.4.8 and Lemma B.4.10 that $\mathbb{R}$ is a field. Moreover, by Definition B.4.12 and Lemmata B.4.13, B.4.14, and B.4.13, $\mathbb{R}$ is a totally ordered field.

To see that $\mathbb{R}$ has the Least Upper Bound Property, let $A \subseteq \mathbb{R}$ be a non-empty subset that is bounded above. Our goal is to show that A has a least upper bound.

Let $[(q_n)_{n\geq 1}]$ be an upper bound for A. By Lemma B.4.3, there exists an $M \in \mathbb{Q}_+$ such that $|q_n| \leq M$ for all $n \in \mathbb{N}$. Therefore, since $q_n \leq M$ for all $n \in \mathbb{N}$ and since $[(M)_{n\geq 1}] \in \mathbb{R}$ by Example B.4.2, it follows by the definition of $\leq$ that $[(q_n)_{n\geq 1}] \leq [(M)_{n\geq 1}]$. Therefore, since $[(q_n)_{n\geq 1}]$ be an upper bound for A, $[(M)_{n\geq 1}]$ is an upper bound for A.

Since A is non-empty, there exists an $[(a_n)_{n\geq 1}] \in A$. Since $(a_n)_{n\geq 1} \in \mathcal{R}$, there exists an $N \in \mathbb{N}$ such that $|a_n - a_m| \leq 1$ for all $n, m \geq N_0$. Let $K = a_{N_0} - 1$ so that $K \in \mathbb{Q}$. Moreover, since $K \leq a_n$ for all $n \geq N_0$, it follows that $[(K)_{n\geq 1}] \leq [(a_n)_{n\geq 1}]$.

We will now construct two sequences of rational numbers $(u_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ that will aid us in showing that A has a least upper bound. In particular, we desire to construct $(u_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ such that $l_n \leq l_{n+1} \leq u_{n+1} \leq u_n$ for all $n \in \mathbb{N}$, the constant sequence $(u_k)_{n\geq 1}$ is an upper bound for A for all $k \in \mathbb{N}$, the constant sequence $(l_k)_{n\geq 1}$ is not an upper bound for A for all $k \in \mathbb{N}$, and $|u_n - l_n| \leq \frac{1}{2^{n-1}}|u_1 - l_1|$ for all $n \in \mathbb{N}$.

To begin, let $u_1 = M$ and $l_1 = K$ so that $u_1, l_1 \in \mathbb{Q}$, $l_1 \leq u_1$, the constant sequence $(u_1)_{n\geq 1}$ is an upper bound for A, and the constant sequence $(l_1)_{n\geq 1}$ is not an upper bound for A.

To construct u_2 and l_2, let $c_1 = \frac{u_1+l_1}{2} \in \mathbb{Q}$. Based on c_1, we will define u_2 and l_2 as follows: If the constant sequence $(c_1)_{n\geq 1}$ is a upper bound for A, we define $u_2 = c_1$ and $l_2 = l_1$. Otherwise the constant sequence $(c_1)_{n\geq 1}$ is not a upper bound for A, we define $u_2 = u_1$ and $l_2 = c_1$. In either case, we have that $u_2, l_2 \in \mathbb{Q}$, $l_1 \leq l_2 \leq u_2 \leq u_1$, the constant sequence $(u_2)_{n\geq 1}$ is an upper bound for A, the constant sequence $(l_2)_{n\geq 1}$ is not an upper bound for A, and $|u_2 - l_2| = \frac{1}{2}|u_1 - l_1|$.

To continue this recursive process ad infinitum, assume for some $N \in \mathbb{N}$ we have define $u_1, \ldots, u_N, l_1, \ldots, l_n \in \mathbb{Q}$ with the desired properties. Let $c_N = \frac{u_N+l_N}{2} \in \mathbb{Q}$. Based on we will define u_{N+1} and l_{N+1} as follows: If the constant sequence $(c_N)_{n\geq 1}$ is a upper bound for A, we define $u_{N+1} = c_N$ and $l_{N+1} = l_N$. Otherwise the constant sequence $(c_N)_{n\geq 1}$ is not a upper bound for A, we define $u_{N+1} = u_N$ and $l_{N+1} = c_N$. In either case, we have that $u_{N+1}, l_{N+1} \in \mathbb{Q}$, $l_N \leq l_{N+1} \leq u_{N+1} \leq u_N$, the constant sequence $(u_N)_{n\geq 1}$ is an upper bound for A, the constant sequence $(l_N)_{n\geq 1}$ is not an upper bound for A, and $|u_{N+1} - l_{N+1}| = \frac{1}{2}|u_N - l_N| = \frac{1}{2^N}|u_1 - l_1|$ as desired.

We now claim the the sequences $(u_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ as defined above are Cauchy sequences in $\mathbb{Q}$. To see this, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. By Lemma B.4.16 there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2^n} \leq \frac{\epsilon}{2(|u_1 - l_1| + 1)}$$

for all $n \geq N$. Notice since $l_n \leq l_{n+1} \leq u_{n+1} \leq u_n$ for all $n \in \mathbb{N}$ and since $|u_n - l_n| \leq \frac{1}{2^{n-1}}|u_1 - l_1|$ for all $n \in \mathbb{N}$ that

$$|u_{n+1} - u_n| \leq \frac{1}{2^n}|u_1 - l_1| \quad \text{and} \quad |l_{n+1} - l_n| \leq \frac{1}{2^n}|u_1 - l_1|$$

for all $n \in \mathbb{N}$. Therefore, we have for all $n \geq m \geq N$ that

$$\begin{aligned} |u_n - u_m| &= \left| \sum_{k=m}^{n-1} u_{k+1} - u_k \right| \\ &\leq \sum_{k=m}^{n-1} |u_{k+1} - u_k| \\ &\leq \sum_{k=m}^{n-1} \frac{1}{2^k} |u_1 - l_1| \\ &= \frac{1}{2^m} \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} |u_1 - l_1| \\ &\leq \frac{1}{2^m} (2) |u_1 - l_1| \\ &\leq \frac{\epsilon}{2(|u_1 - l_1| + 1)} (2) |u_1 - l_1| \\ &< \epsilon. \end{aligned}$$

Furthermore, a similar computation shows that $|u_n - u_m| \leq \epsilon$ for all $n \geq m \geq N$. Therefore, since $\epsilon > 0$ was arbitrary, $(u_n)_{n\geq 1}$ and $(l_n)_{n\geq 1}$ are Cauchy sequences in $\mathbb{Q}$ by definition. Hence $(u_n)_{n\geq 1}, (l_n)_{n\geq 1} \in \mathcal{R}$.

We claim that $[(u_n)_{n\geq 1}] = [(l_n)_{n\geq 1}]$. To see this, let $\epsilon > 0$ be arbitrary. By Lemma B.4.16 there exists an $N \in \mathbb{N}$ such that

$$\frac{1}{2^n} \leq \frac{\epsilon}{(|u_1 - l_1| + 1)}$$

for all $n \geq N$. Since $|u_n - l_n| \leq \frac{1}{2^{n-1}} |u_1 - l_1|$ for all $n \in \mathbb{N}$, we have for all $n \geq N$ that

$$|u_n - l_n| \leq \frac{1}{2^{n-1}} |u_1 - l_1| \leq \frac{\epsilon}{(|u_1 - l_1| + 1)} |u_1 - l_1| < \epsilon.$$

Therefore, as $\epsilon > 0$ was arbitrary, $[(u_n)_{n\geq 1}] = [(l_n)_{n\geq 1}]$ by definition.

We claim $[(u_n)_{n\geq 1}]$ is an upper bound for A. To see this, let $[(a_n)_{n\geq 1}] \in A$ be arbitrary. To see that $[(a_n)_{n\geq 1}] \leq [(u_n)_{n\geq 1}]$, let $\epsilon \in \mathbb{Q}_+$ be arbitrary. Since $(u_n)_{n\geq 1} \in \mathcal{R}$ and since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$, there exists an $N_1 \in \mathbb{N}$ such that $|u_n - u_m| < \frac{1}{2}\epsilon$ for all $n, m \geq N$. Moreover, since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$ and since the constant sequence $(u_N)_{n\geq 1}$ is an upper bound for A so that $[(a_n)_{n\geq 1}] \leq [(u_N))n \geq 1]$, there exists an $N' \in \mathbb{N}$ such that $a_n \leq u_N + \frac{1}{2}\epsilon$ for all $n \geq N'$. Hence for all $n \geq N'$ we have that

$$a_n \leq u_N + \frac{1}{2}\epsilon \leq \left(u_n + \frac{1}{2}\epsilon \right) + \frac{1}{2}\epsilon = u_n + \epsilon.$$

Therefore, since ϵ was arbitrary, $[(a_n)_{n\geq 1}] \leq [(u_n)_{n\geq 1}]$. Therefore, since $[(a_n)_{n\geq 1}]$ was arbitrary, $[(u_n)_{n\geq 1}]$ is an upper bound for A.

Finally, we claim that $[(u_n)_{n\geq 1}]$ is a least upper bound for A. To see this, let $[(b_n)_{n\geq 1}] \in \mathbb{R}$ be such that $[(b_n)_{n\geq 1}]$ is an upper bound for A and $[(b_n)_{n\geq 1}] \leq [(u_n)_{n\geq 1}]$. We desire to prove that $(b_n)_{n\geq 1} \sim (u_n)_{n\geq 1}$. To see this, let $\epsilon \in \mathbb{Q}_+$ be arbitrary.

Since $\frac{1}{2}\epsilon \in \mathbb{Q}_+$ and since $(u_n)_{n\geq 1} \sim (l_n)_{n\geq 1}$, there exists an $N_1 \in \mathbb{N}$ such that $|u_n - l_n| \leq \frac{1}{2}\epsilon$ for all $n \geq N_1$. Since $\frac{1}{4}\epsilon \in \mathbb{Q}_+$ and since $[(b_n)_{n\geq 1}] \leq [(u_n)_{n\geq 1}]$, there exists an $N_2 \in \mathbb{N}$ such that $b_n \leq u_n + \frac{1}{4}\epsilon$ for all $n \geq N_2$.

Recall that the constant sequence $(l_{N_1})_{n\geq 1}$ is not an upper bound for A. Thus there exists an $[(a_n)_{n\geq 1}] \in A$ such that $[(l_{N_1})_{n\geq 1}] \leq [(a_n)_{n\geq 1}]$. Since $[(b_n)_{n\geq 1}]$ is an upper bound for A we know that $[(a_n)_{n\geq 1}] \leq [(b_n)_{n\geq 1}]$ and thus $[(l_{N_1})_{n\geq 1}] \leq [(b_n)_{n\geq 1}]$. Therefore, since $\frac{1}{4}\epsilon \in \mathbb{Q}_+$, there exists an $N_3 \in \mathbb{N}$ such that $l_{N_1} \leq b_n + \frac{1}{4}\epsilon$ for all $n \geq N_3$.

Let $N_4 = \max\{N_1, N_2, N_3\}$. Therefore, by the above and the fact that $u_{n+1} \leq u_n$ for all $n \in \mathbb{N}$, we obtain for all $n \geq N_4$ that

$$l_{N_1} \leq b_n + \frac{1}{4}\epsilon \leq \left(u_n + \frac{1}{4}\epsilon\right) + \frac{1}{4}\epsilon = u_n + \frac{1}{2}\epsilon \leq u_{N_1} + \frac{1}{2}\epsilon.$$

Moreover, since $l_n \leq l_{n+1} \leq u_{n+1} \leq u_n$ for all $n \in \mathbb{N}$, we obtain for all $n \geq N_4$ that

$$l_{N_1} \leq u_n \leq u_{N_1}.$$

Therefore, we see for all $n \geq N_4$ that

$$|b_n - u_n| \leq (u_{N_1} + \frac{1}{2}\epsilon) - l_{N_1} = |u_{N_1} - l_{N_1}| + \frac{1}{2}\epsilon \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Therefore, since $\epsilon \in \mathbb{Q}_+$ was arbitrary, $(b_n)_{n\geq 1} \sim (u_n)_{n\geq 1}$. Hence $[(b_n)_{n\geq 1}] = [(u_n)]_{n\geq 1}$. Hence $[(u_n)_{n\geq 1}]$ is a least upper bound for A.

Since A was an arbitrary subset of $\mathbb{R}$ that was bounded above and we have demonstrated that A has a least upper bound, $\mathbb{R}$ has the least upper bound property as desired. ∎

B.5 Uniqueness of Real Numbers

In this section, we will demonstrate that all totally ordered field with the Least Upper Bound Property are 'isomorphic' in the appropriate sense; that is, there is a bijective map between them that preserves the totally ordered field properties and least upper bounds. This . In particular, this means there is a unique totally ordered field with the Least Upper Bound Property, which is what we call the real numbers.

To prove the desired result, we will first show that every totally ordered field with the Least Upper Bound Property contain the rational numbers (with the same ordering). To do this, we will first show the natural numbers lie inside every totally ordered field.

Lemma B.5.1. *If $\mathcal{R}$ is a totally ordered field, then there exists a map $f : \mathbb{N} \to \mathcal{R}$ such that*

- *f is injective,*
- *$f(1)$ is the multiplicative unit of $\mathcal{R}$,*
- *$f(n+m) = f(n) + f(m)$ for all $n, m \in \mathbb{N}$,*
- *$f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{N}$, and*
- *for $n, m \in \mathbb{N}$, $n < m$ if and only if $f(n) < f(m)$.*

Proof. Let I denote the multiplicative unit of $\mathcal{R}$. Define $f : \mathbb{N} \to \mathcal{R}$ recursively as follows: Let $f(1) = I$. If we have defined $f(n)$ for $n \in \mathbb{N}$, define $f(n+1) = f(n) + I$.

Clearly $f(1)$ is the multiplicative unit of $\mathcal{R}$, and $f(n+m) = f(n) + f(m)$ and $f(nm) = f(n)f(m)$ for all $n \in \mathbb{N}$ due to the definitions of addition and multiplication on $\mathbb{N}$.

Since $\mathcal{R}$ is an ordered field, I is positive by Lemma 1.2.16. Therefore, by the additive property implies that $f(n) \leq f(n) + I = f(n+1)$ for all $n \in \mathbb{N}$. If $f(n) = f(n) + I$, then by adding $-f(n)$ to both sides we would obtain that I is the zero element which contradicts the fact that $\mathcal{R}$ is a field. Hence $f(n) < f(n+1)$ for all $n \in \mathbb{N}$. Therefore, induction implies for $n, m \in \mathbb{N}$, $n < m$ if and only if $f(n) < f(m)$. Note this implies f is injective thereby completing the proof. ∎

Next, we can upgrade Lemma B.5.1 to show that the integers lie inside of every totally ordered field. Although the existence of Section B.1 may seem odd, it really aids us in the proof of the following.

Lemma B.5.2. *If $\mathcal{R}$ is a totally ordered field, then there exists a map $f : \mathbb{Z} \to \mathcal{R}$ such that*

- *f is injective,*
- *$f(1)$ is the multiplicative unit of $\mathcal{R}$,*
- *$f(0)$ is the additive unit of $\mathcal{R}$,*
- *$f(n+m) = f(n) + f(m)$ for all $n, m \in \mathbb{Z}$,*
- *$f(nm) = f(n)f(m)$ for all $n, m \in \mathbb{Z}$, and*
- *for $n, m \in \mathbb{N}$, $n \leq m$ if and only if $f(n) \leq f(m)$.*

Proof. By Lemma B.5.1, we can assume $\mathbb{N} \subseteq \mathcal{R}$ with the operations, units, and ordering of $\mathcal{R}$ giving the natural operations, units, and ordering of $\mathbb{N}$.

Notice if $m_1, n_1, m_2, n_2 \in \mathbb{N} \subseteq \mathcal{R}$ are such that $m_1 + n_2 = m_2 + n_1$, then the field properties of $\mathcal{R}$ imply that $m_1 + (-n_1) = m_2 + (-n_2)$. Therefore, by using the equivalence class characterization of $\mathbb{Z}$ from Section B.1, the map $f : \mathbb{Z} \to \mathcal{R}$ defined by

$$f([(m, n)]) = m + (-n)$$

is well-defined.

To see that f is injective, assume $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ are such that $f([(m_1, n_1)]) = f([(m_2, n_2)])$. Therefore

$$m_1 + (-n_1) = m_2 + (-n_2).$$

By adding $n_1 + n_2$ to both sides, the field properties of $\mathcal{R}$ imply that $m_1 + n_2 = m_2 + n_1$ and thus $[(m_1, n_1)] = [(m_2, n_2)]$ by definition. Hence f is injective.

Since $f([(2, 1)]) = 2 + (-1) = 1$, f sends the multiplicative unit of $\mathbb{Z}$ to the multiplicative unit of $\mathcal{R}$. Moreover, since $f([(1, 1)]) = 1 + (-1) = 0$, f sends the additive unit of $\mathbb{Z}$ to the multiplicative unit of $\mathcal{R}$.

Next, notice for all $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ that

$$\begin{aligned}
&f([(m_1, n_1)] + [(m_2, n_2)]) \\
&= f([(m_1 + m_2, n_1 + n_2)]) \\
&= (m_1 + m_2) + (-(n_1 + n_2)) \\
&= (m_1 + (-n_1)) + (m_2 + (-n_2)) \qquad \text{by the field properties of } \mathcal{R} \\
&= f([(m_1, n_1)]) + f([(m_2, n_2)])
\end{aligned}$$

and

$$\begin{aligned}
&f([(m_1, n_1)] \cdot [(m_2, n_2)]) \\
&= f([(m_1 m_2 + n_1 n_2, m_1 n_2 + n_2 m_1)]) \\
&= (m_1 m_2 + n_1 n_2) + (-(m_1 n_2 + n_2 m_1)) \\
&= (m_1 + (-n_1))(m_2 + (-n_2)) \qquad \text{by the field properties of } \mathcal{R} \\
&= f([(m_1, n_1)]) \cdot f([(m_2, n_2)])
\end{aligned}$$

as desired.

Finally, notice for $[(m_1, n_1)], [(m_2, n_2)] \in \mathbb{Z}$ that $[(m_1, n_1)] \leq [(m_2, n_2)]$ if and only if $m_1 + n_2 \leq m_2 + n_1$ in $\mathbb{N}$ if and only if $m_1 + n_2 \leq m_2 + n_1$ in $\mathcal{R}$ if and only if $m_1 + (-n_1) \leq m_2 + (-n_2)$ (by the field properties of $\mathcal{R}$ and the additive property of $\leq$ in $\mathcal{R}$) if and only if $f([(m_1, n_1)]) \leq f([(m_2, n_2)])$ as desired. ■

Next, we can upgrade Lemma B.5.2 to show that the rational numbers lie inside of every totally ordered field. Again, the mathematically precise description of the rational numbers from Section B.2 are quite useful to prove the following.

Lemma B.5.3. *If $\mathcal{R}$ is a totally ordered field, then there exists a map $f : \mathbb{Q} \to \mathcal{R}$ such that*

- *f is injective,*
- *$f(1)$ is the multiplicative unit of $\mathcal{R}$,*
- *$f(0)$ is the additive unit of $\mathcal{R}$,*
- *$f(q + r) = f(q) + f(r)$ for all $q, r \in \mathbb{Q}$,*
- *$f(qr) = f(q)f(r)$ for all $q, r \in \mathbb{Q}$, and*
- *for all $q, r \in \mathbb{Q}$, $q \leq r$ if and only if $f(q) \leq f(r)$.*

Proof. By Lemma B.5.1, we can assume $\mathbb{Z} \subseteq \mathcal{R}$ with the operations, units, and ordering of $\mathcal{R}$ giving the natural operations, units, and ordering of $\mathbb{Z}$.

Notice if $a_1, b_1, a_2, b_2 \in \mathbb{Z} \subseteq \mathcal{R}$ are such that $b_1 \neq 0$, $b_2 \neq 0$, and $a_1b_2 = a_2b_1$, then the field properties of $\mathcal{R}$ imply that $a_1b_1^{-1} = a_2b_2^{-1}$. Therefore, by using the equivalence class characterization of $\mathbb{Z}$ from Section B.2, the map $f : \mathbb{Q} \to \mathcal{R}$ defined by

$$f([(a, b)]) = ab^{-1}$$

is well-defined.

To see that f is injective, assume $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$ are such that $f([(a_1, b_1)]) = f([(a_2, b_2)])$. Therefore

$$a_1b_1^{-1} = a_2b_2^{-1}.$$

By multiplying both sides by b_1b_2, the field properties of $\mathcal{R}$ imply that $a_1b_2 = a_2b_1$ and thus $[(a_1, b_1)] = [(a_2, b_2)]$ by definition. Hence f is injective.

Since $f([(1, 1)]) = 1(1)^{-1} = 1$, f sends the multiplicative unit of $\mathbb{Q}$ to the multiplicative unit of $\mathcal{R}$. Moreover, since $f([(0, 1)]) = 0(1)^{-1} = 0$, f sends the additive unit of $\mathbb{Q}$ to the multiplicative unit of $\mathcal{R}$.

Next, notice for all $[(a_1, b_1)], [(a_2, b_2)] \in \mathbb{Q}$ that

$$\begin{aligned} f([(a_1, b_1)] + [(a_2, b_2)]) &= f([(a_1b_2 + a_2b_1, b_1b_2)]) \\ &= (a_1b_2 + a_2b_1)(b_1b_2)^{-1} \\ &= (a_1b_2 + a_2b_1)b_1^{-1}b_2^{-1} && \text{by the field properties of } \mathcal{R} \\ &= a_1b_1^{-1} + a_2b_2^{-1} && \text{by the field properties of } \mathcal{R} \\ &= f([(a_1, b_1)]) + f([(a_2, b_2)]) \end{aligned}$$

and

$$\begin{aligned}
&f\left([(a_1,b_1)]\cdot[(a_2,nb2)]\right) \\
&= f\left([(a_1a_2,b_1b_2)]\right) \\
&= (a_1a_2)(b_1b_2)^{-1} \\
&= (a_1b_1^{-1})(a_2b_2^{-1}) && \text{by the field properties of } \mathcal{R} \\
&= f\left([(a_1,b_1)]\right)\cdot f\left([(a_2,b_2)]\right)
\end{aligned}$$

as desired.

Finally, notice for $[(a_1,b_1)],[(a_2,b_2)]\in\mathbb{Q}$ with $b_1,b_2>0$ that $[(a_1,b_1)]\leq[(a_2,b_2)]$ if and only if $a_1b_2\leq a_2b_1$ in $\mathbb{Z}$ if and only if $a_1b_2\leq a_2b_1$ in $\mathcal{R}$ if and only if $a_1b_1^{-1}\leq a_2b_2^{-1}$ (by Lemma 1.2.18 since $\mathcal{R}$ is an ordered field) if and only if $f\left([(a_1,b_1)]\right)\leq f\left([(a_2,b_2)]\right)$ as desired. ∎

Finally, we can prove that if we have two totally ordered fields with the Least Upper Bound Property, then there is a bijective map between them that preserves all of the desired properties. To do this, we will use the fact that every totally ordered field contains the rational numbers. We will then use the fact that the rational numbers and least upper bounds completely describe the elements of the totally ordered field. This was the same motivation that was used to develop the Dedekind cut approach to the real numbers discussed in Section B.3.

Theorem B.5.4. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are totally ordered fields with the Least Upper Bound Property, then there exists a map $f:\mathcal{R}_1\to\mathcal{R}_2$ such that*

- *f is bijective,*
- *if I_k is the multiplicative unit of $\mathcal{R}_k$ for $k=1,2$, then $f(I_1)=I_2$,*
- *if O_k is the additive unit of $\mathcal{R}_k$ for $k=1,2$, then $f(O_1)=O_2$,*
- *$f(x_1+x_2)=f(x_1)+f(x_2)$ for all $x_1,x_2\in\mathcal{R}_1$,*
- *$f(x_1x_2)=f(x_1)f(x_2)$ for all $x_1,x_2\in\mathcal{R}_1$,*
- *for all $x_1,x_2\in\mathcal{R}_1$, $x_1\leq x_2$ if and only if $f(x_1)\leq f(x_2)$,*
- *if $A\subseteq\mathcal{R}_1$ is non-empty, A is bounded above if and only if $f(A)$ is bounded above, and*
- *for all $A\subseteq\mathcal{R}_1$ such that A is bounded above, $\mathrm{lub}(f(A))=f(\mathrm{lub}(A))$.*

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be totally ordered fields with the Least Upper Bound Property. By Lemma B.5.3, we can assume $\mathbb{Q}\subseteq\mathcal{R}_1$ and $\mathbb{Q}\subseteq\mathcal{R}_2$ with the operations, units, and ordering of $\mathcal{R}_1$ and $\mathcal{R}_2$ giving the natural operations, units, and ordering of $\mathbb{Q}$.

Existence of f. For all $\alpha \in \mathcal{R}_1$ and $\beta \in \mathcal{R}_2$, let

$$A_{1,\alpha} = \{q \in \mathbb{Q} \mid q < \alpha\}$$
$$A_{2,\beta} = \{q \in \mathbb{Q} \mid q < \beta\}.$$

We claim that $\alpha = \text{lub}_{\mathcal{R}_1}(A_{1,\alpha})$ and $\beta = \text{lub}_{\mathcal{R}_2}(A_{2,\beta})$. To see the former, note that α is clearly an upper bound for $A_{1,\alpha}$. To see $\alpha = \text{lub}_{\mathcal{R}_1}(A_{1,\alpha})$, suppose for the sake of a contradiction that there exists an $\gamma \in \mathcal{R}_1$ such that $\gamma < \alpha$ and $q \leq \gamma$ for all $q \in A_{1,\alpha}$. Since $\mathcal{R}_1$ is a totally ordered field with the Least Upper Bound Property, Proposition 1.3.8 holds for $\mathcal{R}_1$ so there exists a $q \in \mathbb{Q}$ such that $\gamma < q < \alpha$, which is a contradiction. Hence $\alpha = \text{lub}_{\mathcal{R}_1}(A_{1,\alpha})$. By similar arguments $\beta = \text{lub}_{\mathcal{R}_2}(A_{2,\beta})$.

Clearly $A_{1,\alpha}$ is bounded above by α in $\mathcal{R}_1$ and $A_{2,\beta}$ is bounded above by β in $\mathcal{R}_2$. Next we claim that $A_{1,\alpha}$ is bounded above in $\mathcal{R}_2$ and $A_{2,\beta}$ is bounded above in $\mathcal{R}_1$ for all $\alpha \in \mathcal{R}_1$ and $\beta \in \mathcal{R}_2$. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are totally ordered fields with the Least Upper Bound Property, Theorem 1.3.6 holds so there exists $M_1, M_2 \in \mathbb{N}$ such that $\alpha < M_1$ and $\beta < M_2$. Therefore, since the orderings on $\mathcal{R}_1$ and $\mathcal{R}_2$ restrict to the natural ordering on $\mathbb{Q}$ and since $A_{1,\alpha}, A_{2,\beta} \subseteq \mathbb{Q}$, M_1 is an upper bound for $A_{1,\alpha}$ in $\mathcal{R}_2$ and M_2 is an upper bound for $A_{2,\alpha}$ in $\mathcal{R}_1$.

Consider the functions $f : \mathcal{R}_1 \to \mathcal{R}_2$ and $g : \mathcal{R}_2 \to \mathcal{R}_1$ defined for all $\alpha \in \mathcal{R}_1$ and $\beta \in \mathcal{R}_2$ by

$$f(\alpha) = \text{lub}_{\mathcal{R}_2}(A_{1,\alpha})$$
$$g(\beta) = \text{lub}_{\mathcal{R}_1}(A_{2,\beta}).$$

By the above, f and g are well-defined functions.

We claim that $g(f(\alpha)) = \alpha$ for all $\alpha \in \mathcal{R}_1$. To see this, let $\alpha \in \mathcal{R}_1$. Since $f(\alpha) = \text{lub}_{\mathcal{R}_2}(A_{1,\alpha})$, we know that $f(\alpha) > q$ for all $q \in A_{1,\alpha}$. Therefore

$$A_{1,\alpha} \subseteq \{q \in \mathbb{Q} \mid q < f(\alpha)\} = A_{2,f(\alpha)}.$$

We claim that the above set inclusion is actually an equality. To see this, suppose for the sake of a contradiction that there exists a

$$q_0 \in \{q \in \mathbb{Q} \mid q < f(\alpha)\} \setminus A_{1,\alpha}.$$

Since $q_0 \notin A_{1,\alpha}$, we know that $q_0 > \alpha$. Hence for all $q \in A_{1,\alpha}$ we have that $q < \alpha < q_0$. Therefore q_0 is an upper bound for $A_{1,\alpha}$ in $\mathbb{Q}$. Therefore q_0 is an upper bound for $A_{1,\alpha}$ in $\mathcal{R}_2$. However, $q_0 < f(\alpha)$ by definition. Therefore, since q_0 is an upper bound for $A_{1,\alpha}$ in $\mathcal{R}_2$ and since $f(\alpha) = \text{lub}_{\mathcal{R}_2}(A_{1,\alpha})$, we have a contradiction.

Hence
$$A_{1,\alpha} = \{q \in \mathbb{Q} \mid q < f(\alpha)\} = A_{2,f(\alpha)}.$$
so that
$$g(f(\alpha)) = \text{lub}_{\mathcal{R}_1}(A_{2,f(\alpha)}) = \text{lub}_{\mathcal{R}_1}(A_{1,\alpha}) = \alpha$$

as desired.

A similar argument shows that $f(g(\beta)) = \beta$ for all $\beta \in \mathcal{R}_2$. Therefore f and g are invertible functions so $f : \mathcal{R}_1 \to \mathcal{R}_2$ is a bijection.

To see that f has the desired properties, we will need to prove a lot.

$f(I_1) = I_2$ and $f(O_1) - O_2$. Note for all $q \in \mathbb{Q}$ that $A_{1,q} = A_{2,q}$. Therefore

$$f(q) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,q}) = \mathrm{lub}_{\mathcal{R}_2}(A_{2,q}) = q$$

for all $q \in \mathbb{Q}$. Hence $f(1) = 1$ and $f(0) = 0$. Therefore f takes the multiplicative and additive units of $\mathcal{R}_1$ to the multiplicative and additive units of $\mathcal{R}_2$ respectively.

$f(x_1 + x_2) = f(x_1) + f(x_2)$. Let $x_1, x_2 \in \mathcal{R}_1$. Notice that if $q_1, q_2 \in \mathbb{Q}$ are such that $q_1 < f(x_1)$ and $q_2 < x_2$, then $q_1 + q_2 < x_1 + x_2$. Therefore

$$\begin{aligned} A_{1,x_1} + A_{1,x_2} &= \{q_1 + q_2 \mid q_1 \in A_{1,x_1}, q_2 \in A_{1,x_2}\} \\ &\subseteq A_{1,x_1+x_2}. \end{aligned}$$

We claim that the above set inclusion is an equality. To see this, assume $q \in \mathbb{Q}$ is such that $q < x_1 + x_2$. Thus $q - x_1 < x_2$. Since $\mathcal{R}_1$ is a totally ordered field with the Least Upper Bound Property, Proposition 1.3.8 holds for $\mathcal{R}_1$ so there exists a $q_2 \in \mathbb{Q}$ such that $q - x_1 < q_2 < x_2$. Hence if $q_1 = q - q_2$, then $q_1 < x_1$, $q_2 < x_2$, and $q_1 + q_2 = q$. Therefore, since $q \in \mathbb{Q}$ was arbitrary, we obtain that

$$A_{1,x_1} + A_{1,x_2} = A_{1,x_1+x_2}.$$

Returning to showing that $f(x_1+x_2) = f(x_1)+f(x_2)$, note that we have

$$\begin{aligned} f(x_1 + x_2) &= \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1+x_2}) \\ &= \mathrm{lub}_{\mathcal{R}_2}\left(A_{1,x_1} + A_{1,x_2}\right) \end{aligned}$$

and

$$f(x_1) + f(x_2) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

Therefore, to show that $f(x_1 + x_2) = f(x_1) + f(x_2)$, it suffices to show that

$$\mathrm{lub}_{\mathcal{R}_2}\left(A_{1,x_1} + A_{1,x_2}\right) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

To see this, notice $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) \geq q_1$ for all $q_1 \in A_{1,x_1}$ and $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}) \geq q_2$ for all $q_2 \in A_{1,x_2}$ and thus

$$\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}) \geq q_1 + q_2$$

for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$. Therefore $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2})$ is an upper bound for $A_{1,x_1} + A_{1,x_2}$ in $\mathcal{R}_2$ so

$$\mathrm{lub}_{\mathcal{R}_2}\left(A_{1,x_1} + A_{1,x_2}\right) \leq \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

To obtain the equality, let $\gamma \in \mathcal{R}_2$ be an upper bound of $A_{1,x_1} + A_{1,x_2}$ in $\mathcal{R}_2$ such that

$$\gamma \leq \text{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \text{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

Since γ is an upper bound of $A_{1,x_1} + A_{1,x_2}$, we have that $q_1 + q_2 < \gamma$ for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$. Hence $q_1 < \gamma - q_2$ for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$. Therefore $\gamma - q_2$ is an upper bound of A_{1,x_1} in $\mathcal{R}_2$ for all $q_2 \in A_{1,x_2}$ so

$$\text{lub}_{\mathcal{R}_2}(A_{1,x_1}) leq \gamma - q_2$$

for all $q_2 \in A_{1,x_2}$. However, this implies

$$q_2 \leq \gamma - \text{lub}_{\mathcal{R}_2}(A_{1,x_1})$$

for all $q_2 \in A_{1,x_2}$ and thus $\gamma - \text{lub}_{\mathcal{R}_2}(A_{1,x_1})$ is an upper bound for A_{1,x_2} so

$$\text{lub}_{\mathcal{R}_2}(A_{1,x_1}) \leq \gamma - \text{lub}_{\mathcal{R}_2}(A_{1,x_1}).$$

Hence $\text{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \text{lub}_{\mathcal{R}_2}(A_{1,x_2}) \leq \gamma$. Thus $\gamma = \text{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \text{lub}_{\mathcal{R}_2}(A_{1,x_2})$ so that

$$\text{lub}_{\mathcal{R}_2}(A_{1,x_1} + A_{1,x_2}) = \text{lub}_{\mathcal{R}_2}(A_{1,x_1}) + \text{lub}_{\mathcal{R}_2}(A_{1,x_2})$$

thereby completing the proof that $f(x_1 + x_2) = f(x_1) + f(x_2)$.

$f(x_1x_2) = f(x_1)f(x_2)$. To prove this, we will divide the proof into a few cases.

Case 1: $x_1 = 0$ or $x_2 = 0$. In this case, by the properties of fields, $x_1x_2 = 0$ and either $f(x_1) = 0$ so $f(x_1)f(x_2) = 0$ or $f(x_2) = 0$ so $f(x_1)f(x_2) = 0$. Hence $f(x_1x_2) = 0 = f(x_1)f(x_2)$.

Case 2: $x_1 > 0$ and $x_2 > 0$. Assume $x_1 > 0$ and $x_2 > 0$. Since $\mathcal{R}_1$ is a totally ordered field with the Least Upper Bound Property, Proposition 1.3.8 holds for $\mathcal{R}_1$ so there exists a $q_1, q_2 \in \mathbb{Q}$ such that $0 < q_1 < x_1$ and $0 < q_2 < x_2$. Hence A_{1,x_1} and A_{1,x_2} contain positive elements.

Let

$$X = \{q_1q_2 \mid q_1 \in A_{1,x_1}, q_2 \in A_{1,x_2}, q_1, q_2 > 0\} \cup \{q \in \mathbb{Q} \mid, q \leq 0\}$$

We claim that

$$A_{1,x_1x_2} = X.$$

To see this, first note by the properties of a total order field that $x_1x_2 > 0$. Thus

$$\{q \in \mathbb{Q} \mid, q \leq 0\} \subseteq A_{1,x_1x_2}$$

by definition. Moreover, if $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$ are such that $q_1, q_2 > 0$, then $0 < q_1 < x_1$ and $0 < q_2 < x_2$ so by the properties of a total order field we have that $0 < q_1q_2 < x_1x_2$ so $q_1q_2 \in A_{1,x_1x_2}$. Hence

$$A_{1,x_1x_2} \supseteq X.$$

To see the other set inclusion, let $q_0 \in A_{1,x_1x_2}$ be arbitrary. Clearly if $q_0 \leq 0$ then $q_0 \in X$. Thus, we may assume without loss of generality that $q_0 > 0$.

Thus $0 < q_0 < x_1x_2$. Since $x_2 > 0$, the properties of a total ordered field imply that $0 < q_0x_2^{-1} < x_1$. Since $\mathcal{R}_1$ is a totally ordered field with the Least Upper Bound Property, Proposition 1.3.8 holds for $\mathcal{R}_1$ so there exists a $q_1 \in \mathbb{Q}$ such that $0 < q_0x_2^{-1} < q_1 < x_1$. Thus the properties of a total ordered field imply $0 < q_0q_1^{-1} < x_2$. Therefore, if $q_2 = q_0q_1^{-1} \in \mathbb{Q}$, then $0 < q_1 < x_1$, $0 < q_2 < x_2$, and $q = q_1q_2$. Hence

$$A_{1,x_1x_2} \supseteq X$$

as claimed.

Returning to showing that $f(x_1x_2) = f(x_1)f(x_2)$ in this case, notice that

$$f(x_1x_2) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1x_2}) = \mathrm{lub}_{\mathcal{R}_2}(X)$$

whereas

$$f(x_1)f(x_2) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

Therefore, to show that $f(x_1x_2) = f(x_1)f(x_2)$, it suffices to show that

$$\mathrm{lub}_{\mathcal{R}_2}(X) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

To see this, notice $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) \geq q_1$ for all $q_1 \in A_{1,x_1}$ and $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}) \geq q_2$ for all $q_2 \in A_{1,x_2}$. Therefore, for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$ such that $q_1, q_2 > 0$, we have that

$$\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}) \geq q_1q_2 > 0.$$

Therefore, since X contains a positive element as A_{1,x_1} and A_{1,x_2} contain positive elements, $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2})$ is an upper bound for X in $\mathcal{R}_2$ so

$$\mathrm{lub}_{\mathcal{R}_2}(X) \leq \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

To obtain the equality, let $\gamma \in \mathcal{R}_2$ be an upper bound of X in $\mathcal{R}_2$ such that

$$\gamma \leq \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}).$$

Note $\gamma > 0$ since X contains a positive element. Since γ is an upper bound of X, we have that $q_1q_2 < \gamma$ for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$ with $q_1, q_2 > 0$. Hence, by the properties of a totally ordered field, we have that $0 < q_1 < \gamma q_2^{-1}$ for all $q_1 \in A_{1,x_1}$ and $q_2 \in A_{1,x_2}$ with $q_1, q_2 > 0$. Therefore γq_2^{-1} is an upper bound of A_{1,x_1} in $\mathcal{R}_2$ for all $q_2 \in A_{1,x_2}$ so

$$0 < \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) leq \gamma q_2^{-1}$$

for all $q_2 \in A_{1,x_2}$ (where the first inequality comes since A_{1,x_1} has a positive element). However, the properties of a totally ordered field then implies

$$0 < q_2 \leq \gamma \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})^{-1}$$

for all $q_2 \in A_{1,x_2}$ with $q_2 > 0$ and thus $\gamma \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})^{-1}$ is an upper bound for A_{1,x_2} so

$$0 < \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) \leq \gamma \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})^{-1}.$$

Hence $\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2}) \leq \gamma$. Thus $\gamma = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2})$ so that

$$\mathrm{lub}_{\mathcal{R}_2}\left(A_{1,x_1} + A_{1,x_2}\right) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1})\mathrm{lub}_{\mathcal{R}_2}(A_{1,x_2})$$

thereby completing the proof that $f(x_1x_2) = f(x_1)f(x_2)$ in this case.

Case 3: $x_1 = -1$ and $x_2 > 0$. First note for all $x \in \mathcal{R}_1$ that

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x)$$

by the additive property of f. Therefore, by the properties of fields (see Lemma 1.2.1), it follows that $f(-x) = -f(x)$ for all $x \in \mathcal{R}_1$.

In particular, we see that $f(-1) = -f(1) = -1$. Therefore, by the properties of fields, we have that

$$f(x_1x_2) = f((-1)x_2) = f(-x_2) = -f(x_2) = (-1)f(x_2) = f(x_1)f(x_2)$$

as desired.

Case 4: $x_1 < 0$ and $x_2 > 0$. In this case, note that $-x_1 > 0$ by the properties of a totally ordered field. Therefore

$$\begin{aligned}
f(x_1x_2) &= f((-1)((-x_1)x_2)) && \text{by Lemma 1.2.1}\\
&= f(-1)f((-x_1)x_2) && \text{by Case 3}\\
&= (-1)f(-x_1)f(x_2) && \text{by Case 2}\\
&= f((-1)(-x_1))f(x_2) && \text{by Case 3}\\
&= f(x_1)f(x_2) && \text{by Lemma 1.2.1}
\end{aligned}$$

as desired.

Case 5: $x_1 > 0$ and $x_2 < 0$. In this case, note that $-x_2 > 0$ by the properties of a totally ordered field. Therefore

$$\begin{aligned}
f(x_1x_2) &= f((-1)(x_1(-x_2))) && \text{by Lemma 1.2.1}\\
&= f(-1)f(x_1(-x_2)) && \text{by Case 3}\\
&= (-1)f(x_1)f(-x_2) && \text{by Case 2}\\
&= f(x_1)f((-1)(-x_2)) && \text{by Case 3}\\
&= f(x_1)f(x_2) && \text{by Lemma 1.2.1}
\end{aligned}$$

as desired.

Case 6: $x_1 < 0$ and $x_2 < 0$. In this case, note that $-x_1 > 0$ and $-x_2 > 0$ by the properties of a totally ordered field. Therefore

$$
\begin{aligned}
f(x_1x_2) &= f((-x_1)(-x_2)) && \text{by Lemma 1.2.1}\\
&= f(-x_1)f(-x_2) && \text{by Case 3}\\
&= f((-1)x_1)f((-1)x_2) && \text{by Case 2}\\
&= (-1)^2 f(x_1)f(x_2) && \text{by Case 3}\\
&= f(x_1)f(x_2) && \text{by Lemma 1.2.1}
\end{aligned}
$$

as desired.

Since the above cases cover all possibilities, the proof that $f(x_1x_2) = f(x_1)f(x_2)$ for all $x_1, x_2 \in \mathcal{R}_1$ is complete.

f is order preserving. First assume $x \in \mathcal{R}_1$ is such that $x > 0$. Since $\mathcal{R}_1$ is a totally ordered field with the Least Upper Bound Property, Proposition 1.3.8 holds for $\mathcal{R}_1$ so there exists a $q \in \mathbb{Q}$ such that $0 < q < x$. Therefore $q \in A_{1,x_1}$ so

$$f(x) = \mathrm{lub}_{\mathcal{R}_2}(A_{1,x_1}) \geq q > 0.$$

To see the desired property, assume $x_1, x_2 \in \mathcal{R}_1$ are such that $x_1 \leq x_2$. If $x_1 = x_2$ then clearly $f(x_1) \leq f(x_2)$. Otherwise, if $x_1 < x_2$ then $0 < x_2 + (-x_1)$ by the properties of a totally ordered field. Thus we have that

$$0 < f(x_2 + (-x_1)) = f(x_2) + f(-x_1) = f(x_2) + (-f(x_1))$$

so $f(x_1) < f(x_2)$ by the properties of a totally ordered field.

Conversely, assume $x_1, x_2 \in \mathcal{R}_1$ are such that $f(x_1) \leq f(x_2)$. Since everything we have proved thus far for f must also hold for g by symmetry, we have that

$$x_1 = g(f(x_1)) \leq g(f(x_2)) = x_2$$

as desired.

f preserves being bounded above. Let $A \subseteq \mathcal{R}_1$ be non-empty. To see the desired property, first assume α is an upper bound of A. Thus $a \leq \alpha$ for all $a \in A$. Therefore, since f preserves the ordering, this implies that $f(a) \leq f(\alpha)$ for all $a \in A$ and thus $f(\alpha)$ is an upper bound of $f(A)$.

For the other direction, note by symmetry and the fact that $f(A) \neq \emptyset$, we have that if β is an upper bound of $f(A)$ then $g(\beta)$ is an upper bound for $g(f(A)) = A$. Hence A is bounded above if and only if $f(A)$ is bounded above.

f preserves least upper bounds. Let $A \subseteq \mathcal{R}_1$ be non-empty and bounded above. The proof of the previous fact shows that α is an upper bound for A if and only if $f(\alpha)$ is an upper bound for $f(A)$. Therefore $f(\mathrm{lub}_{\mathcal{R}_1}(A))$ is an upper bound for $f(A)$ so

$$f(\mathrm{lub}_{\mathcal{R}_1}(A)) \geq \mathrm{lub}_{\mathcal{R}_2}(f(A)).$$

By symmetry, we have that

$$g(\mathrm{lub}_{\mathcal{R}_2}(f(A))) \geq \mathrm{lub}_{\mathcal{R}_1}(g(f(A))) = \mathrm{lub}_{\mathcal{R}_1}(A).$$

Therefore, since f preserves the ordering, we obtain that

$$f(\mathrm{lub}_{\mathcal{R}_1}(A)) \leq f\left(g(\mathrm{lub}_{\mathcal{R}_2}(f(A)))\right) = mathrmlub_{\mathcal{R}_2}(f(A)).$$

Hence

$$f(\mathrm{lub}_{\mathcal{R}_1}(A)) = \mathrm{lub}_{\mathcal{R}_2}(f(A))$$

as desired.

By combining all of the above, the proof is complete. ∎

Corollary B.5.5. *There is a unique totally ordered field with the Least Upper Bound Property. Said field is called the real numbers and denoted* $\mathbb{R}$.

Proof. Since Theorem B.5.4 shows that any two totally ordered fields with the Least Upper Bound Property are the same upto relabelling the elements, the proof is complete. ∎

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